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Orbits of computably enumerable sets: low sets can avoid an upper cone

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Abstract

We investigate the orbit of a low computably enumerable (c.e.) set under automorphisms of the partial order $\mathscr{E}$ of c.e. sets under inclusion. Given an arbitrary low c.e. set A and an arbitrary noncomputable c.e. set C, we use the New Extension Theorem of Soare to construct an automorphism of $\mathscr{E}$ mapping A to a set B such that $C \not\leqslant_T B$. Thus, the orbit in $\mathscr{E}$ of the low set A cannot be contained in the upper cone above C. This complements a result of Harrington, who showed that the orbit of a noncomputable c.e. set cannot be contained in the lower cone below any incomplete c.e. set. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

The computably enumerable (c.e.) sets form an upper semi-lattice under Turing reducibility. Under set inclusion, they form a lattice $\mathscr{E}$, as first noted by Myhill in [14], and the properties of a c.e. set as an element of $\mathscr{E}$ often help determine its properties under Turing reducibility and vice versa. The $\mathscr{E}$-definable property of maximality, for instance, enabled Martin to characterize the high c.e. degrees as those which contained a maximal set [12], and other $\mathscr{E}$-definable properties discovered by Harrington and Soare imply Turing completeness, Turing incompleteness, and nonlowness (see [8,10]).

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The study of $\mathscr{E}$ often focusses on automorphisms of the lattice and the orbits of c.e. sets under those automorphisms. We say that two c.e. sets are *automorphic* if they lie in the same orbit. Again, the Turing-degree properties of a set often yield insight into the orbit of the set. Harrington and Soare have shown (in [9]) that the orbit of a noncomputable c.e. set must contain a set of high degree, and the same paper proves Harrington's theorem that the orbit of a noncomputable c.e. set cannot be contained in the lower cone $\{B \in \mathscr{E} : B \leqslant_T A\}$ below any c.e. set A (unless A is Turing-complete, of course). On the other hand, Wald showed in [20] that the orbit of a low c.e. set must intersect the lower cone below any given promptly simple set C. (This result fails to hold for certain nonprompt sets C, however, by a result of Downey and Harrington.)

In this paper we use the Turing-definable property of lowness to avoid an upper cone. Specifically, our main theorem is:

Theorem 1. *For every low c.e. set A and every noncomputable c.e. set C, there exists an automorphism of $\mathscr{E}$ mapping A to a set B such that $C \nleqslant_T B$.*

Thus, the orbit of A cannot be contained in the upper cone above C. The assumption of lowness of A will be essential for our construction to succeed, as discussed in Section 2.4. (Also, the theorem clearly would fail for a computable set C.)

The main tool for proving this result is the New Extension Theorem of Soare, as stated in [19]. The lowness of A allows us to predict with fair certainty (i.e. with only finitely many incorrect guesses) which elements of any given c.e. set will eventually enter A and which will stay in its complement $\bar{A}$.

Much of the machinery in this paper is identical to that used in [9,19,20]. We have deliberately tried to keep our notation and intuitions the same as in those papers whenever possible, in order that readers familiar with the constructions in those papers will find it easier to follow this one. Often we will refer to [20] instead of repeating definitions and machinery explained there. One noticeable distinction is the use of $\mathscr{K}_\alpha$, which was defined in [9] (Eq. (14), p. 625) to mean precisely the opposite of its meaning in [19,20], and the present paper. *Caveat lector!*

All sets mentioned in this paper will be c.e. unless specifically stated otherwise. (Complements, of course, need not be c.e.)

2. Construction

2.1. Defining *e*-states on a tree

To prove Theorem 1, we must construct an automorphism of $\mathscr{E}$. By a result of Soare (in [18, XV.2.6]), it suffices to construct an automorphism of $\mathscr{E}^*$, the quotient of $\mathscr{E}$ by the ideal of finite sets. Thus, we must map every c.e. set U_e to some other c.e. set $\hat{U}_e$ in such a way that unions and intersections are preserved up to finitely many elements. Ordinarily, we would employ an e-state construction for this purpose, where by the e-state of an element x at stage s we simply mean

$$\{i < e : x \in U_{i,s}\}$$

and the general e-state of x is

$$\{i < e : x \in U_i\}.$$

(Thus, for instance, the 4-state 0101 indicates that an element lies in U_1 and U_3, but not in U_0 or U_2.) The corresponding e-state for sets $\hat{U}_e$ in the range of the automorphism would be defined in exactly the same way. We have a copy of ω, denoted by $\hat{\omega}$, containing the elements of sets in the range, and we write $\hat{x}$ to stand for such an element.

To ensure that the map be onto, we would use a second enumeration $V_0, V_1, \ldots$ of all c.e. sets and make sure that for each e there is a c.e. set $\hat{V}_e$ which maps to V_e. This gives rise to an additional e-state, that with respect to the sets $\hat{V}_e$, and the full e-state of x would be $\langle \sigma, \tau \rangle$ where σ is the e-state relative to the sets U_e and τ is the e-state relative to the sets $\hat{V}_e$. We would then need to create our automorphism in such a way that for every full e-state (relative to sets in the domain) which contained infinitely many elements x, the corresponding full e-state (relative to the sets $\hat{U}_e$ and V_e in the range) contained infinitely many elements $\hat{x}$, and conversely. (For details, see [18, XV 4.3].)

In the present theorem, however, we have additional negative requirements $\mathscr{Q}_e$ to ensure that the image of A under the automorphism does not lie in the upper cone above C. These requirements follow the *Sacks preservation strategy* for B, the image of A, and are stated below, after we define the necessary machinery. (A description of the Sacks preservation strategy in a simpler situation is given in [18, VII.3.1].)

In order to construct the automorphism while respecting the negative requirements, we must make guesses about which e-states really do contain infinitely many elements. Since elements can move from one e-state to another between stages, the number of elements in a given e-state fluctuates. Some e-states accumulate more and more elements, and wind up in the end with infinitely many; we say that such e-states are *well-resided*. For other e-states, there are infinitely many elements which enter that state at some stage but only finitely many which remain there for good. These e-states are *well-visited*, but not well-resided. (The well-resided states are also considered to be well-visited.) Finally, an e-state which is not well-visited has only finitely many elements that ever enter that state. We write $\mathscr{K}$ to represent the set of well-resided e-states, $\mathscr{M}$ to represent the set of well-visited e-states, and $\mathscr{N}$ to represent the set of e-states which are well-visited but not well-resided. Thus $\mathscr{K} = \mathscr{M} - \mathscr{N}$.

Our guesses about these possibilities for each e-state lead us to employ a tree construction. Each node α of the tree T at level e will represent a guess about which e-states are well-visited and which of those are well-resided. Indeed, the c.e. sets we build will depend on our guesses: for each $\alpha \in T$ with $|\alpha| \equiv 1 \pmod 5$, we will have a set U_α. Therefore, we will not speak of e-states, but rather of α-states, which are just e-states relative to the sets $U_{\alpha\restriction 1}, U_{\alpha\restriction 6}, U_{\alpha\restriction 11}, \ldots, U_\alpha$. The *true path* f through T will correspond to the correct guesses, and the collection $\{U_\alpha : \alpha \subset f \ \& \ |\alpha| \equiv 1 \pmod 5\}$ will include every c.e. set W_e (up to finite difference).

We will use $\mathscr{M}_\alpha$ to denote the set of α-states which α believes to be well-visited. The set containing those states which α believes to be well-visited but not well-resided will be partitioned into two subsets $\mathscr{B}_\alpha \sqcup \mathscr{R}_\alpha$, according to the method which α believes is used to remove elements from those states. Also, for each α we write

$$e_\alpha = \max\{k \in \omega\colon 5k < |\alpha|\}.$$

Therefore, if $\beta = \alpha^-$ is the immediate predecessor of α in T, then the set U_α is defined if and only if $e_\alpha > e_\beta$. We also have the sets V_α on the $\hat{\omega}$-side which ensure that the automorphism is onto. Then $\hat{e}_\alpha$ is defined by

$$\hat{e}_\alpha = \max\{k \in \omega\colon 5k + 1 < |\alpha|\}$$

and the set V_α is defined if and only if $\hat{e}_\alpha > \hat{e}_\beta$. (For the purposes of this paper, we could use a modulus smaller than 5, but we will adhere to the usage in previous papers.)

T will contain a unique node ρ of length 1, and we will ensure that $U_\rho = A$. The set $\hat{U}_\rho$ which we build will be the image of A under the automorphism, so this is the set B for which we must worry about the negative restraints. We will often speak of $\bar{A}$-states and $\bar{B}$-states. These terms refer to full α-states which exclude U_ρ and $\hat{U}_\rho$, respectively. If x is in an $\bar{A}$-state at stage s, then $x \notin A_s$, and if $\hat{x}$ is in a $\bar{B}$-state at stage s, then $\hat{x} \notin B_s$.

We think of the sets U_α as being "red" sets, containing elements $x \in \omega$, by which we mean that the elements x are enumerated in these sets by a player called "RED". The other player in the game, "BLUE", tries to match the moves of RED by moving his own elements $\hat{x}$ (from the other copy $\hat{\omega}$ of ω) among the sets $\hat{U}_\alpha$, so that the map taking U_α to $\hat{U}_\alpha$ will be an automorphism. Again, to ensure surjectivity of this map, RED will also play sets V_α containing the elements $\hat{x} \in \hat{\omega}$, so that every computably enumerable set is represented (up to finite difference) by at least one V_α along the true path, and it will be up to BLUE to build corresponding sets $\hat{V}_\alpha$ of the elements $x \in \omega$. Ultimately, BLUE's goal is that each full α-state on the ω-side should contain infinitely many elements x if and only if the corresponding full α-state on the $\hat{\omega}$-side contains infinitely many elements $\hat{x}$.

In light of this RED/BLUE dichotomy, the class $\mathscr{N}_\alpha$ of α-states which are well-visited but not well-resided will be partitioned into disjoint subclasses $\mathscr{R}_\alpha$ and $\mathscr{B}_\alpha$. The latter contains every state which is emptied out by BLUE, i.e. such that cofinitely many of the elements which enter that state eventually leave the state because they are enumerated into some other blue set. (Here we include B as a blue set.) $\mathscr{R}_\alpha$ contains every state which is emptied out by RED. Of course, an α-state v can be emptied out by both players, since there could be infinitely many elements enumerated into a red set and infinitely many others enumerated into a blue set. Such states are assigned to either $\mathscr{R}_\alpha$ or $\mathscr{B}_\alpha$ (but not both!) according to which player empties out the corresponding γ-state, where $\gamma \subseteq \alpha$ is the least predecessor of α such that the γ-state corresponding to v is not well-resided.

2.2. *Definitions*

To the extent possible, we take our definitions straight from [9,20]. One change is the use of the superscript 0, so that (for instance) $\mathscr{M}_\alpha^0$ and $\hat{\mathscr{M}}_\alpha^0$ will replace $\mathscr{M}_\alpha^A$ and $\hat{\mathscr{M}}_\alpha^{\bar{B}}$.

To define the tree T, we need the formal definition of an α-state.

Definition 2. An α-*state* is a triple $\langle\alpha,\sigma,\tau\rangle$ where $\sigma\subseteq\{0,\dots,e_\alpha\}$ and $\tau\subseteq\{0,\dots,\hat{e}_\alpha\}$. The only λ-state is $v_{-1}=\langle\lambda,\emptyset,\emptyset\rangle$. If $0\notin\sigma$, then we call the state an $\bar{A}$-state or a $\bar{B}$-state.

As in [20], we define our tree T with a specific node ρ at level 1, since the corresponding c.e. sets U_0 and $\hat{U}_0$ are A and B. Also, here we specify the sets U_i and V_i. Pick some i such that $W_i=A$, and define

$$U_{0,s} = W_{i,s},$$

$$U_{e,s} = W_{e,s} \quad \text{for all } e>0,$$

$$V_{e,s} = W_{e,s} \quad \text{for all } e.$$

Definition 3. We define the tree T as follows:

Let the empty node λ be the root of T and ρ the unique node at level 1, defined as follows:

$$\mathscr{M}^0_\lambda=\hat{\mathscr{M}}^0_\lambda=\emptyset, \qquad \mathscr{M}^0_\rho=\hat{\mathscr{M}}^0_\rho=\{\langle\rho,\emptyset,\emptyset\rangle,\langle\rho,\{0\},\emptyset\rangle\},$$

$$\mathscr{R}^0_\lambda=\mathscr{B}^0_\lambda=\emptyset, \qquad \mathscr{R}^0_\rho=\mathscr{B}^0_\rho=\emptyset,$$

$$k_\lambda=-1, \qquad k_\rho=-1,$$

$$e_\lambda=-1, \qquad e_\rho=0,$$

$$\hat{e}_\lambda=-1, \qquad \hat{e}_\rho=-1.$$

For every $\beta\in T$ with $\beta\neq\lambda$, we define the immediate successors of β exactly as in Definition 2.11 of [9], only using $\mathscr{M}^0_\alpha$, $\mathscr{R}^0_\alpha$, and $\mathscr{B}^0_\alpha$ in place of $\mathscr{M}_\alpha$, $\mathscr{R}_\alpha$, and $\mathscr{B}_\alpha$, and similarly on the $\hat{\omega}$-side. (The consistency required by part (i) above is defined exactly as in Definitions 2.3 and 2.4 of [20]. Notice that we can compute uniformly for any α whether it is consistent or not, since there are only finitely many α-states.)

We identify the finite object $\langle\mathscr{M}^0_\alpha,\mathscr{R}^0_\alpha,\mathscr{B}^0_\alpha,k_\alpha\rangle$ with an integer under some effective coding, so that we may regard T as a subtree of $\omega^{<\omega}$. Therefore the partial order on T will be denoted by $\subseteq$. We write $\alpha<_L\gamma$ to denote that α is to the left of γ on the tree, i.e. that there exists $\delta\in T$ and $m<n$ in ω with $\delta\hat{\ }m\subseteq\alpha$ and $\delta\hat{\ }n\subseteq\gamma$.

The superscript "0" in $\mathscr{M}^0_\alpha$, etc., is intended to make clear that we are only concerned with $\bar{A}$-states (and $\bar{B}$-states, in the dual). After all, $U_0=A$, so any $\bar{A}$-state $v=\langle\alpha,\sigma,\tau\rangle$ will have $\sigma(0)=0$ (as defined below). Similarly, $\hat{\sigma}(0)=0$ for $\bar{B}$-states $\hat{v}$. (These superscripts did not appear in [9], since that construction was also concerned with A-states and B-states. Theorem 4, proven later, made it unnecessary to consider such states.)

In Section 2.5 we will approximate the true path f through T by a uniformly computable sequence of nodes $\{f_s\}_{s\in\omega}$. A node α will lie on f if and only if α is the leftmost node at level $|\alpha|$ in T such that $\alpha\subseteq f_s$ for infinitely many s. The nodes of the true path are the only nodes for which we ultimately need the construction to work, but since all we have is an approximation to the true path, we must follow the dictates of that approximation at each stage. Each element x ($\hat{x}$) will be assigned to a given node $\alpha(x,s)$ ($\alpha(\hat{x},s)$) at each stage. The node $\alpha(x,s)$ may be redefined at stage $s+1$ to equal an immediate successor of $\alpha(x,s)$. Moreover, if the true path moves to the left of $\alpha(x,s)$, then $\alpha(x,s)$ may be redefined so that $\alpha(x,s+1) <_L \alpha(x,s)$ or so that $\alpha(x,s+1)$ is a predecessor of $\alpha(x,s)$. However, $\alpha(x,s+1)$ will never move back to the right of $\alpha(x,s)$. The construction will ensure that $\alpha(x) = \lim_s \alpha(x,s)$ exists and that cofinitely many x wind up being assigned to nodes on f, with the finitely many remaining ones all being assigned to nodes to the left of f.

We use the elements assigned to node α and its successors at stage s to help build U_α, defining S_α, R_α, and Y_α exactly as in Section 2.3 of [9]. Also, our formal definition of the α-state of an element $x \in \omega$ or $\hat{x} \in \hat{\omega}$ is exactly Definition 2.2 of [20].

For each $\alpha \in T$ we define the following classes of $\bar{A}$–α-states:

$$\mathscr{E}^0_\alpha = \left\{ v : (\exists^\infty x)(\exists s) \left[x \in \bar{A}_s \cap \left(S_{\alpha,s} - \bigcup_{t<s} S_{\alpha,t} \right) \ \& \ v(\alpha,x,s) = v \right] \right\},$$

$$\mathscr{F}^0_\alpha = \{ v : (\exists^\infty x)(\exists s)[x \in R_{\alpha,s} \ \& \ v(\alpha,x,s) = v \ \& \ x \notin A_s]\}.$$

Thus $\mathscr{E}^0_\alpha$ consists of states well visited by elements x when they first *enter* S_α and $\mathscr{F}_\alpha{}^0$ of those states well-visited by elements at some stage while they *remain* in R_α, so $\mathscr{E}^0_\alpha \subseteq \mathscr{F}_\alpha{}^0$. For each $\alpha \in T$, $\mathscr{M}^0_\alpha$ represents α's "guess" at the true $\mathscr{F}_\alpha{}^0$ such that if $\alpha \subset f$ then $\mathscr{M}^0_\alpha = \mathscr{F}_\alpha{}^0$. For $\alpha \subset f$ we shall achieve $\mathscr{M}^0_\alpha = \mathscr{F}_\alpha{}^0$ by ensuring that Properties (6)–(12) of [9] all hold for $\mathscr{M}^0_\alpha$ and $\hat{\mathscr{M}}^0_\alpha$.

Having said that every $\alpha \in T$ should have an associated set $\mathscr{M}^0_\alpha$ such that $\mathscr{M}^0_\alpha = \mathscr{F}_\alpha{}^0$ if $\alpha \subset f$, we note that although this is the *property* we want $\mathscr{M}^0_\alpha$ to have, we cannot simply *define* $\mathscr{M}^0_\alpha$ to be α's guess at $\mathscr{F}_\alpha{}^0$ because that definition would be circular. (The definition of $\mathscr{F}_\alpha{}^0$ depends on U_α, and the construction of U_α in Section 2.5 will depend on $\mathscr{M}^0_\alpha$.) Rather we must define here a certain set $\mathscr{F}_\beta{}^{0+}$ which depends only on β, and then let $\mathscr{M}^0_\alpha$ be α's guess at $\mathscr{F}_\beta{}^{0+}$ so that $\mathscr{M}^0_\alpha = \mathscr{F}_\beta{}^{0+}$ $(= \mathscr{F}_\alpha{}^0)$ for $\alpha \subset f$. We define the set Z_{e_α} exactly as in [9], to satisfy Eqs. (38)–(41) of that paper (with $\bar{A}$- and $\bar{B}$-states, as usual). We also use their subsequent definition of *provable incorrectness* of a node α.

To construct an automorphism we must show for $\alpha \subset f$ that

$$\hat{\mathscr{K}}^0_\alpha = \{\hat{v} : v \in \mathscr{K}^0_\alpha\}. \tag{1}$$

To achieve (1) note that unlike $\mathscr{E}^0_\alpha$ and $\mathscr{F}_\alpha{}^0$, $\mathscr{K}_\alpha{}^0$ is Π^0_3 not Π^0_2 so α cannot guess at $\mathscr{K}_\alpha{}^0$ directly but only at a certain Σ^0_2 approximation $\mathscr{N}_\alpha{}^0$ to $\mathscr{M}^0_\alpha - \mathscr{K}_\alpha{}^0$. We divide $\mathscr{N}_\alpha{}^0$ into the disjoint union of sets $\mathscr{R}^0_\alpha$ and $\mathscr{B}^0_\alpha$ which correspond to those $v \in \mathscr{N}_\alpha{}^0$ which α believes are being emptied by RED and BLUE, respectively. To discuss the phenomenon of a player emptying out a state, we define the relations $\leqslant_R$ and $\leqslant_B$ as in Definition 2.6 of [9], and the sets $\mathscr{R}^0_\alpha$ and $\mathscr{B}^0_\alpha$ as in its Eqs. (16)–(24). The intuition is that if x is in α-state $v_0 = v(\alpha,x,s)$ and $v_0 <_R v_1$ ($v_0 <_B v_1$) then RED (BLUE) can enumerate x in the necessary U sets ($\hat{V}$ sets) causing $v_1 = v(\alpha,x,s+1)$. For $\hat{v}_0$ and $\hat{v}_1$

the role of σ and τ is reversed because on the $\hat{\omega}$-side BLUE (RED) plays the $\hat{U}$ sets (V sets), and hence

$$[v_0 <_R v_1 \Leftrightarrow \hat{v}_0 <_B \hat{v}_1] \; \& \; [v_0 <_B v_1 \Leftrightarrow \hat{v}_0 <_R \hat{v}_1]. \tag{2}$$

If $\alpha \subset f$ then $v \in \mathscr{R}^0_\alpha$ implies $F(\alpha^-, v)$ and hence

$$(\forall v \in \mathscr{R}^0_\alpha)(\forall x \in Y_\alpha)(\forall s)[v(\alpha,x,s) = v \Rightarrow (\exists t > s)[v(\alpha,x,t) \neq v]]. \tag{3}$$

It will be BLUE's responsibility to change the α-state of x if $v(\alpha, x, s) \in \mathscr{B}^0_\alpha$ and $x \in R_\alpha$. However, $\mathscr{B}^0_\alpha \cap \mathscr{R}^0_\alpha = \emptyset$ so if $v(\alpha, x, s) = v \in \mathscr{R}^0_\alpha$ then BLUE can wait for RED to change the α-state of each x to meet (3), by restraining x from entering any blue set until we reach a stage $t > s$ such that $v(\alpha, x, s) <_R v(\alpha, x, t)$.

Finally, we define the restriction of an α-state v to a node $\beta \subset \alpha$ exactly as in Definition 2.7 of [9].

2.3. The New Extension Theorem

Soare developed his New Extension Theorem (NET) to simplify the process of constructing automorphisms. Using the NET, one can divide the construction into three distinct parts and concentrate on each separately, rather than having to satisfy all three simultaneously. The idea is that in building an automorphism which maps A to B, at each stage $s + 1$ we can consider three classes of elements of ω: those elements x which are still in $\bar{A}_{s+1}$; those x which enter A at stage $s + 1$; and those x which were already in A_s. (On the $\hat{\omega}$ side, we have the same three classes: $\hat{x} \in \bar{B}_{s+1}$, $\hat{x} \in B_{s+1} - B_s$, and $\hat{x} \in B_s$.) Indeed, the NET constructs the automorphism on the third class itself, leaving only two types of elements for us to worry about.

In the construction of the tree T in the preceding section, we defined the sets $\mathscr{M}^0_\alpha$, $\hat{\mathscr{M}}^0_\alpha$, etc., for each $\alpha \in T$. In [9], a similar construction required the inclusion of A-states as well as $\bar{A}$-states in $\mathscr{M}_\alpha$. With the New Extension Theorem, however, we need only consider $\bar{A}$- and $\bar{B}$-states. The NET requires that for each α on the true path, $\mathscr{M}^0_\alpha = \hat{\mathscr{M}}^0_\alpha$ and $\mathscr{N}_\alpha{}^0 = \hat{\mathscr{N}}^0_\alpha$. Together, these will guarantee that

$$\mathscr{K}^0_\alpha = \mathscr{M}^0_\alpha - \mathscr{N}^0_\alpha = \hat{\mathscr{M}}^0_\alpha - \hat{\mathscr{N}}^0_\alpha = \hat{\mathscr{K}}^0_\alpha$$

so that the well-resided $\bar{A}$–α-states correspond precisely to the well-resided $\bar{B}$–α-states.

The second class of elements contains those x which enter A at stage $s + 1$, and those $\hat{x}$ entering B_{s+1}. The New Extension Theorem requires us to record the α-state of each such x at stage s, as a sort of snapshot of its status at the moment it enters A, and similarly for each $\hat{x}$ that enters B. We define for each α:

$$\mathscr{G}^A_\alpha = \{v \in \mathscr{M}^0_\alpha : (\exists^\infty x)(\exists s)[x \in A_{s+1} - A_s \; \& \; v(\alpha,x,s) = v]\},$$
$$\hat{\mathscr{G}}^B_\alpha = \{\hat{v} \in \hat{\mathscr{M}}^0_\alpha : (\exists^\infty \hat{x})(\exists s)[\hat{x} \in B_{s+1} - B_s \; \& \; \hat{v}(\alpha,\hat{x},s) = \hat{v}]\}.$$

Thus $\mathscr{G}^A_\alpha$ contains those $\bar{A}$–α-states such that infinitely many elements x are in that state at the moment of entering A, and similarly for $\hat{\mathscr{G}}^B_\alpha$. The NET then requires that for each α on the true path, the α-states in $\mathscr{G}_\alpha^A$ must correspond precisely to those in $\hat{\mathscr{G}}_\alpha^B$.

If we can accomplish these two conditions, then the New Extension Theorem guarantees that the third part of the automorphism construction can be carried out as well, and therefore that there exists an automorphism mapping each U_α ($\alpha \subset f$) to the corresponding $\hat{U}_\alpha$.

Theorem 4 (New Extension Theorem, Soare [19]). *Assume that T is a computable priority tree as defined above, with infinite true path f, and suppose that each of the collections $\{U_\alpha\}_{\alpha\subset f}$ and $\{V_\alpha\}_{\alpha\subset f}$ contains every computably enumerable set, up to finite difference. If for each $\alpha \subset f$ we have:*
(T1) $\mathscr{K}_\alpha^0 = \hat{\mathscr{K}}_\alpha^0$, *and*
(T2) $\mathscr{G}_\alpha^A = \hat{\mathscr{G}}_\alpha^B$,
then there exists an automorphism of $\mathscr{E}$ mapping U_α to $\hat{U}_\alpha$ for each $\alpha \subset f$.

It is left to us to satisfy our own requirements for U_ρ and $\hat{U}_\rho$, namely that $U_\rho = A$ (which we have already ensured, simply by arranging our enumeration of the c.e. sets to begin with A) and that $\hat{U}_\rho$ does not lie in the upper cone above C (which is the hard part).

Definition 5. The *true path* $f \in [T]$ is defined by induction on n. Let $\beta = f \restriction n$ be consistent. Then $f \restriction (n+1)$ is the $<_L$-least $\alpha \in T$, $\alpha \supset \beta$, of length $m = n+1$ such that:

(i) $m \equiv 1 \bmod 5 \Rightarrow \mathscr{M}_\alpha^0 = \mathscr{F}_\beta^{0+}$ & $k_\alpha = k_\beta^+$,

(ii) $m \equiv 2 \bmod 5 \Rightarrow \hat{\mathscr{M}}_\alpha^0 = \hat{\mathscr{F}}_\beta^{0+}$ & $k_\alpha = k_\beta^+$,

(iii)
$$m \equiv 3 \bmod 5 \Rightarrow$$
$$[\mathscr{R}_\alpha^\alpha = \{v : v \in \mathscr{M}_\alpha^0 - (\mathscr{R}_\alpha^{<\alpha} \cap \mathscr{B}_\alpha^{<\alpha}) \text{ \& } F(\beta, v)\}$$
$$\text{\& } \hat{\mathscr{B}}_\alpha^\alpha = \{\hat{v} : v \in \mathscr{R}_\alpha^\alpha\}],$$

(iv)
$$m \equiv 4 \bmod 5 \Rightarrow$$
$$[\hat{\mathscr{R}}_\alpha^\alpha = \{\hat{v} : \hat{v} \in \hat{\mathscr{M}}_\alpha^0 - (\hat{\mathscr{R}}_\alpha^{<\alpha} \cup \hat{\mathscr{B}}_\alpha^{<\alpha}) \text{ \& } \hat{F}(\beta, v)\}$$
$$\text{\& } \mathscr{B}_\alpha^\alpha = \{v : \hat{v} \in \hat{\mathscr{R}}_\alpha^\alpha\}],$$

(v) unless otherwise specified in (i)–(iv), $\mathscr{M}_\alpha^0$, $\mathscr{R}_\alpha^0$, $\mathscr{B}_\alpha^0$, k_α, and their duals take the values $\mathscr{M}_\beta^0$, $\mathscr{R}_\beta^0$, $\mathscr{B}_\beta^0$, k_β, and their duals, respectively.

(If β were inconsistent, it would be a terminal node and the true path would end at β. We will show in Lemmas 9 and 11, however, that this cannot be the case.)

For a consistent $\beta = f \restriction n$, $\mathscr{F}_\beta^{0+}$ is just a finite set of states and k_β^+ is an integer, so clearly α exists. Note that each of the conditions in Definition 5 is Π_2^0. Hence, there is a computable collection of c.e. sets $\{D_\alpha\}_{\alpha\in T}$ such that $\alpha \subset f$ iff $|D_\alpha| = \infty$. Fix a simultaneous computable enumeration $\{D_{\alpha,s}\}_{\alpha\in T,\, s\in\omega}$.

We impose the following positive requirements, for all $\alpha \in T$, all α-states v, and all $i \in \omega$, to ensure that $\mathscr{G}^A = \hat{\mathscr{G}}^B$:

$$\mathscr{P}_{\langle \alpha,v,i \rangle} : v \in \mathscr{G}^A_\alpha \Rightarrow |\{\hat{x} : (\exists s)[\hat{x} \in B_{\text{at}\, s+1} \cap \hat{Y}_{\alpha,s} \ \&\ \hat{v}(\alpha,\hat{x},s) = v]\}| \geqslant i.$$

Clearly each $\mathscr{P}_{\langle \alpha,v,i \rangle}$ will only put finitely many elements into B. Indeed, since $\mathscr{P}_{\langle \alpha,v,i-1 \rangle}$ has higher priority than $\mathscr{P}_{\langle \alpha,v,i \rangle}$, each $\mathscr{P}_{\langle \alpha,v,i \rangle}$ will only require that a single element enter B.

The negative requirements $\mathscr{Q}_e$ are the standard ones for the Sacks strategy for avoiding an upper cone:

$$\mathscr{Q}_e : C \neq \{e\}^B.$$

To satisfy these, we define the length functions $l(e,s)$ and restraint functions $r(e,s)$ (as in [18, VII.3.1]):

$$l(e,s) = \max\{x : (\forall y < x)[\{e\}^{B_s}_s(y) \downarrow = C_s(y)]\},$$
$$r(e,s) = \max\{u(B_s; e,x,s) : x \leqslant l(e,s)\}.$$

In the construction, we will restrain (with priority e) all elements $< r(e,s)$ from entering B at stage s. Thus, we will preserve the computation $\{e\}^B(y)$ for every $y \leqslant l(e,s)$, including $y = l(e,s)$ itself. If $\lim_s l(e,s) = \infty$, then C would be computable, contrary to hypothesis. Moreover, for each e, $l(e,s)$ will be nondecreasing as a function of s, except at the finitely many stages s at which $\mathscr{N}_e$ is injured, i.e. at which $B_{s+1} \upharpoonright (r(e,s) + 1) \neq B_s \upharpoonright (r(e,s) + 1)$. Therefore, there exists a finite limit $l(e) = \lim_s l(e,s)$. Then the computation $\{e\}^B(l(e))$ must either diverge or converge to a value distinct from $C(l(e))$. Hence $\mathscr{Q}_e$ will be satisfied.

2.4. Intuition for the lowness of A

If A were an arbitrary set, then it would be extremely difficult, perhaps impossible, to satisfy the requirements $\mathscr{Q}_e$. It remains an open question whether Theorem 1 holds without the assumption that A is low. Without lowness, the difficulty is that if all the elements x in some $\bar{A}$–α-state v enter A, then we have to put all but finitely many of the elements $\hat{x}$ from the corresponding $\bar{B}$–α-state $\hat{v}$ into B, probably violating some requirement $\mathscr{Q}_e$ in the process. Whenever an element x enters state α, we put a corresponding element in the corresponding $\bar{B}$-state, in order to make $\mathscr{M}^A = \hat{\mathscr{M}}^B$. If x later enters A, then we need to have the corresponding element enter B, to ensure that $\mathscr{G}^A = \hat{\mathscr{G}}^B$ and $\mathscr{N}^A = \hat{\mathscr{N}}^B$. If this happens infinitely often, then $\mathscr{Q}_e$ would be injured infinitely often, and would not be satisfied. To avoid injuring $\mathscr{Q}_e$, we must allow all but finitely many elements $\hat{x}$ to remain in $\hat{v}$ rather than entering B, and then $\hat{v}$ will be a well-resided state and v will not be.

The assumption that A is low allows us to avoid this difficulty, by guaranteeing that we will only have to make the choice between $\mathscr{Q}_e$ and emptying out $\hat{v}$ finitely many times. (At those times we obey $\mathscr{Q}_e$, since it is acceptable for finitely many elements to remain permanently in $\hat{v}$.) We use a variation of Robinson's Trick (see

[15]), as expressed in Soare's Lowness Lemma in [19], to predict which elements x in the $\bar{A}$–α-state v will eventually enter A. Each element x in state v is assigned a marker $\Gamma^{\alpha}_{v,i}$: the first element gets marker $\Gamma^{\alpha}_{v,0}$; the next gets $\Gamma^{\alpha}_{v,1}$, and so on. At the same time, when x enters state α, we check the Robinson Trick prediction for its marker number and also check subsequent stages of the enumeration of A. At some such stage, these must agree: either x is not yet in A and is predicted to stay in $\bar{A}$ (in which case the marker stays assigned to it), or x enters A (in which case we immediately put it into our own enumeration of A and make its marker unassigned).

If only finitely many elements remain permanently in the α-state v, then there must be some marker $\Gamma^{\alpha}_{v,n}$ such that every element to which it is assigned eventually enters A. Pick the least such n; then there is a stage s such that at all subsequent stages the prediction function for $\Gamma^{\alpha}_{v,n}$ always predicts (correctly!) that any element x to which we try to assign it will enter A. When we check ahead, we confirm this prediction, so we enumerate x into our enumeration of A immediately and leave $\Gamma^{\alpha}_{v,n}$ unassigned. Thus, after stage s, every x which enters v immediately gets enumerated into A, so we can match up α with the corresponding $\bar{B}$-state without doing any further injury to the $\mathscr{Q}_e$-requirements. Only finitely many elements will remain in the $\bar{B}$-state, and only finitely many in state α.

Recall from Definitions 2.3 and 2.4 of [20] that a node $\alpha \in T$ is *$\mathscr{R}$-consistent* if it satisfies both of the following:

$$(\forall v_0 \in \mathscr{R}^0_{\alpha})(\exists v_1)[v_0 <_R v_1 \ \& \ v_1 \in \mathscr{M}^0_{\alpha}], \tag{4}$$

$$(\forall \hat{v}_0 \in \hat{\mathscr{R}}^0_{\alpha})(\exists \hat{v}_1)[\hat{v}_0 <_R \hat{v}_1 \ \& \ \hat{v}_1 \in \hat{\mathscr{M}}^0_{\alpha}]. \tag{5}$$

Lowness of A allows us to ensure that every α on the true path is $\mathscr{R}$-consistent. Without lowness, the equation for $\mathscr{R}^0$ would be impossible, since states could be emptied out into A with no advance warning to us. BLUE will make every $\alpha \subset f$ $\mathscr{R}$-consistent by waiting to enumerate x in any blue sets until RED has enumerated x in some red set, exactly as described on pp. 626–627 of [9].

2.5. Construction

To parallel the construction in [9], the steps presented in this section will be denoted as Steps 0–5 and $\hat{0}$–$\hat{5}$ for the construction, with final Steps 10, $\widehat{10}$, and 11 at which we define f_{s+1} and other necessary items. (In the construction in [9], Steps 10 and $\widehat{10}$ were substeps of Step 11. We have separated the two because the actions in our Step 11 must be performed at every stage, whereas the action in our Steps 10 and $\widehat{10}$ must not be performed unless the preceding steps do not apply.) Steps $\hat{1}$–$\hat{5}$ and $\widehat{10}$ are the obvious duals to Steps 1–5 and 10, and will not be stated. There is no dual of Step 11.

Our construction is as follows:

Stage $s=0$: For all $\alpha \in T$ define $U_{\alpha,0}=V_{\alpha,0}=\hat{U}_{\alpha,0}=\hat{V}_{\alpha,0}=\emptyset$, and define $m(\alpha,0)=0$. Define $Y_{\lambda,0}=\hat{Y}_{\lambda,0}=\emptyset$, and $f_0=\rho$. Define every $Q^{\alpha}_{v,i,0}=\emptyset$ and every marker $\Gamma^{\alpha}_{v,i,0}$ to be unassigned. Define $A_0=B_0=\emptyset$. Let $l(e,0)=r(e,0)=0$ for every e.

Stage $s+1$: Find the least $n<11$ such that Step n applies to some $x\in Y_{\alpha,s}$ and perform the intended action. If there is no such n, then find the least $n<11$ such that Step $\hat{n}$ applies to some $\hat{x}\in\hat{Y}_{\alpha,s}$, and perform the indicated action. Having completed that, apply Step 11, and go to stage $s+2$.

(In Steps 0–5 and $\hat{0}$–$\hat{5}$ we let $\alpha\in T$, $\alpha\neq\lambda$, be arbitrary, let $\beta=\alpha^-$, and let $x\in Y_{\lambda,s}$ ($\hat{x}\in\hat{Y}_{\lambda,s}$) be arbitrary.)

The sets $\{\tilde{A}_s\}_{s\in\omega}$ represent a given computable enumeration of A, from which we will derive our own enumeration $\{A_s\}_{s\in\omega}$ to satisfy the New Extension Theorem. Being low, the set A has semi-low complement, and we let h be a computable function (as in XI.3.5 of [18]) such that $\lim_t h(j,t)$ is the characteristic function of the set

$$\{j: W_j\cap\bar{A}\neq\emptyset\}.$$

(Semi-lowness of $\bar{A}$ implies that this set is computable in $\emptyset'$. Indeed, the following construction requires only semi-lowness of $\bar{A}$, not actual lowness of A.)

Step 0: *Moving elements into A*.

Substep 0.1: *Enumerated elements.* If $x\in(Y_{\lambda,s}\cap\tilde{A}_{s+1})-(Y_{\lambda,s-1}\cap\tilde{A}_s)$,

(0.1.1) where $v(\alpha(x,s),x,s)=v$, add to $\mathscr{L}^{\mathscr{G}}$ a new pair $\langle\beta,\hat{v}\restriction\beta\rangle$ for every $\beta\subseteq\alpha(x,s)$,

(0.1.2) enumerate x into A_{s+1}, and

(0.1.3) designate every Γ-marker attached to x as unassigned.

Substep 0.2: *Assigning a Γ-marker to an x believed not to go into A*: In the following, to *challenge* x with regard to marker type j $(=1,2,3)$ and α-node v means to do the following:

(i) Where i is the least number such that the marker $\Gamma_{v,i}^{j,\alpha}$ is currently unassigned, enumerate x into $Q_{v,i}^{j,\alpha}$.

(ii) Find the least t such that either
  (a) $h(q_{v,i}^{j,\alpha},t)\downarrow=1$ or
  (b) $x\in\tilde{A}_t$.
  In case (a), assign marker $\Gamma_{v,i}^{j,\alpha}$ to x and ignore (iii)–(v). In case (b),

(iii) if $j=1$ or 2, add to $\mathscr{L}^{\mathscr{G}}$ a pair $\langle\beta,\hat{v}\restriction\beta\rangle$ for every $\beta\subseteq\alpha(x,s)$; if $j=3$, add a pair $\langle\beta,\hat{v}\restriction\beta\rangle$ for every $\beta\subsetneq\alpha(x,s)$;

(iv) enumerate x into A_{s+1} immediately; and

(v) designate every Γ-marker attached to x as unassigned.

Then Substep (0.2) consists of repeating the following three instructions:

(0.2.1) If some element x is to be moved into some Y_α in $\bar{A}$-state v by Step 1 or 2, then challenge x with regard to marker type 1 and α-state v.

(0.2.2) If some element x is to be put into $\bar{A}$-state v by one of Steps 1–5 or 11C, then challenge x with regard to marker type 2 and α-state v.

(0.2.3) If there is some element x such that, as a result of x being enumerated into U_{e_α} and/or the action of Steps 1–5 or 11C, $v^+(x,\alpha)$ will become equal to $\bar{A}$-state v, then challenge x with regard to marker type 3 and α-state v.

We repeat these instructions until none of these three challenges described enters case (b) (that is, none of them causes an element to enter A_{s+1}).

Step $\hat{0}$: *Moving elements into B*.

Find the first unmarked pair $\langle \alpha, \hat{v}_0 \rangle$ in $\mathscr{L}^{\mathscr{G}}$ satisfying all of the following:

($\hat{0}$.1) for some k, $\mathscr{P}_{\langle \alpha, v_0, k \rangle}$ is not satisfied;

($\hat{0}$.2) α is consistent;

($\hat{0}$.3) there exist elements $\hat{y}_0 < \hat{y}_1 < \hat{y}_2 < \cdots < \hat{y}_{2k}$ such that for each $i \leqslant 2k$, both of the following hold:

$$(\exists t \leqslant s)[\hat{y}_i \in R_{\alpha,t} \ \& \ \hat{v}(\alpha, \hat{y}_i, t) = \hat{v}_0]$$

and

$$\hat{y}_i \notin B_s \text{ or } (\exists t < s)[\hat{y}_i \in B_{\mathrm{at}\, t+1} \ \& \ \hat{v}(\alpha, \hat{y}_i, t) \neq \hat{v}_0];$$

($\hat{0}$.4) $\hat{y}_{2k} > 2 \cdot \langle \alpha, v_0, k \rangle$;

($\hat{0}$.5) $\hat{y}_{2k} > r(e, s)$ for every $e \leqslant \langle \alpha, v_0, k \rangle$;

($\hat{0}$.6) $\hat{v}(\alpha, \hat{y}_{2k}, s) = \hat{v}_0$.

Action: Enumerate $\hat{y}_{2k}$ into B_{s+1}. (Notice that by ($\hat{0}$.6), $\hat{y}_{2k} \notin B_s$.) Also, mark the first unmarked copy of $\langle \alpha, \hat{v}_0 \rangle$ on $\mathscr{L}^{\mathscr{G}}$.

Step 1: *Prompt pulling of x from R_β to S_α to ensure $\mathscr{M}_\alpha^0 \subseteq \mathscr{E}_\alpha^0$.* Suppose $\langle \alpha, v_1 \rangle$ is the first unmarked entry on the list $\mathscr{L}_s$ such that the following conditions hold for some x, where $v_1 = \langle \alpha, \sigma_1, \tau_1 \rangle$,

(1.1) $x \in R_{\beta,s} - Y_{\alpha,s}$, and α is $\mathscr{R}$-consistent;

(1.2) $x > k_\alpha$ and $x > |\alpha|$;

(1.3) x is α-eligible (i.e., $\neg(\exists t)[x \leqslant t \leqslant s \ \& \ f_t < \alpha]$);

(1.4) $\neg[\alpha(x,s) <_L \alpha]$;

(1.5) $x > m(\alpha, s)$;

(1.6) $v(\beta, x, s) = v_1 \restriction \beta$;

(1.7) $e_\alpha > e_\beta \Rightarrow v^+(\alpha, x, s) = v_1$.

Action: Choose the least x corresponding to $\langle \alpha, v_1 \rangle$, and do the following.

(1.8) Mark the α-entry $\langle \alpha, v_1 \rangle$ on $\mathscr{L}_s$.

(1.9) Move x to S_α.

(1.10) If $e_\alpha > e_\beta$ and $e_\alpha \in \sigma_1$ then enumerate x in $U_{\alpha,s+1}$.

(1.11) If $\hat{e}_\alpha > \hat{e}_\beta$ and $\hat{e}_\alpha \in \tau_1$ then enumerate x in $\hat{V}_{\alpha,s+1}$. (Hence, $v(\alpha, x, s+1) = v_1$. Also $v_1 \in \mathscr{M}_\alpha^0$ because $\langle \alpha, v_1 \rangle \in \mathscr{L}$ implies $v_1 \in \mathscr{M}_\alpha^0$.)

Step 2: *Move x from S_β to S_α so $Y_\alpha =^* \omega$.* Suppose there is an x such that

(2.1) $x \in S_{\beta,s}$,

(2.2) $x > |\alpha|$ and $x > k_\alpha$.

(2.3) x is α-eligible,

(2.4) $x < m(\alpha, s)$,

(2.5) α is the $<_L$-least $\gamma \in T$ with $\gamma^- = \beta$ satisfying (2.1)–(2.4).

Action: Choose the least pair $\langle \alpha, x \rangle$ and

(2.6) move x from S_β to S_α.

(In Step 2 we need (2.4) so Y_α will not grow while α is waiting for another prompt pulling under Step 1.)

Step 3: *For α $\mathscr{M}$-inconsistent to ensure $\alpha \not\subset f$.* Suppose for $\alpha \in T$ there exists $x > k_\alpha$ such that,

(3.1) $e_\alpha > e_\beta$,

(3.2) $x \in S_{\alpha,s}$,
(3.3) $v(\alpha,x,s) = v_0 \in \mathscr{M}^0_\alpha$,
(3.4) $(\exists v_1)[v_0 <_{\mathrm{B}} v_1 \ \&\ v_1 \upharpoonright \beta \in \mathscr{M}^0_\beta \ \&\ v_1 \notin \mathscr{M}^0_\alpha]$.
Action: Choose the least such pair $\langle \alpha, x\rangle$ and,
(3.5) enumerate x in $\hat{V}_{\delta,s+1}$ for all $\delta \subset \alpha$ such that $e_\delta \in \tau_1$. (This action causes $v(\alpha,x,s+1) = v_1$. Hence, α is provably incorrect at all stages $t \geqslant s+1$ so $\alpha \not\subset f$.)
Step 4: *Delayed* RED *enumeration into* U_α. Suppose $x \in R_{\alpha,s}$ and
(4.1) $e_\alpha > e_\beta$,
(4.2) $x \notin U_{\alpha,s}$,
(4.3) $x \in Z_{e_\alpha,s} =_{\mathrm{dfn}} U_{e_\alpha,s} \cap Y_{\beta,s-1}$.
Action: Choose the least such pair $\langle \alpha, x\rangle$ and,
(4.4) enumerate x in $U_{\alpha,s+1}$.
Step 5: BLUE *emptying of state* $v_0 \in \mathscr{B}^0_\alpha$. Suppose for $\alpha \in T$ there exists x such that either Case 1 or Case 2 holds.
Case 1: Suppose
(5.1) $v(\alpha,x,s) = v_0 \in \mathscr{B}^0_\alpha$, say $v_0 = \langle \alpha, \sigma_0, \tau_0\rangle$,
(5.2) $x \in S_{\alpha,s}$,
(5.3) α is a consistent node.
Action: Choose the least such pair $\langle \alpha, x\rangle$. Let $v_1 = h_\alpha(v_0) >_B v_0$, where h_α is a target function satisfying Eq. (32) of [9]. Write $v_1 = \langle \alpha, \sigma_1, \tau_1\rangle$.
(5.4) Enumerate x into $\hat{V}_\delta$ for all $\delta \subseteq \alpha$ such that $\hat{e}_\delta > \hat{e}_{\delta^-}$ and also $e_\delta \in \tau_1 - \tau_0$. (Hence, $v(\alpha,x,s+1) = v_1$.)
Case 2: Suppose that (5.1) holds and
(5.5) $x \in S_{\gamma,s}$ where $\gamma^- = \alpha$, and
(5.6) γ is not a consistent node.
Action: Perform the same action as in Case 1 to achieve $v(\alpha,x,s+1) = v_1$.
(In (5.6) note that $\gamma \in T$ implies (5.3) for $\alpha = \gamma^-$ since inconsistent nodes are terminal, so h_α exists in Case 2. Note in Step 5 Case 2 that the enumeration may not be γ-legal, since possibly $v(\gamma,x,s+1) \notin \mathscr{M}^0_\gamma$, but this will not matter because we shall prove that $\gamma \not\subset f$ if γ is inconsistent. Hence, it only matters that the enumeration is α-legal, i.e. $v(\alpha,x,s) \in \mathscr{M}^0_\alpha$.)
Step 10: Filling Y_λ. Choose the least $x < s$ such that $x \notin Y_{\lambda,s}$. Put x in S_λ.
Step 11: *Defining* f_{s+1}, $m(\alpha,s+1)$, $\mathscr{L}_{s+1}$, $Y_{\lambda,s+1}$, *and* B_{s+1}.
Substep 11A: *Defining* f_{s+1}. First, we define δ_t by induction on t for $t \leqslant s+1$. Let $\delta_0 = \rho$ (as given in Definition 3, the definition of T). Given δ_t, let $v \leqslant s$ be maximal such that $\delta_t \subseteq f_v$ if such a v exists, or let $v = 0$ otherwise. (Let $\{D_{\gamma,v}\}_{\gamma \in T, v \in \omega}$ be the simultaneous recursive enumeration specified on p. 10.) Choose the $\leqslant_L$-least $\alpha \in T$ such that $\alpha^- = \delta_t$ and $D_{\alpha,s+1} \neq D_{\alpha,v}$ if α exists, and define $\delta_{t+1} = \alpha$. If α does not exist, define $\delta_{t+1} = \delta_t$. Finally, define $f_{s+1} = \delta_{s+1}$.
Substep 11B: *Defining* $m(\alpha,s+1)$, $\mathscr{L}_{s+1}$, *and their duals*. For each $\alpha \subseteq f_{s+1}$, if every α-entry $\langle \alpha, v\rangle$ on $\mathscr{L}_s$ and every α-entry $\langle \alpha, \hat{v}\rangle$ on $\hat{\mathscr{L}}_s$ is marked we say that the lists are α-*marked* and we
(11.1) define $m(\alpha,s+1) = m(\alpha,s)+1$, and
(11.2) add to the bottom of list $\mathscr{L}_s$ ($\hat{\mathscr{L}}_s$) a new (unmarked) α-entry $\langle \alpha, v\rangle$ ($\langle \alpha, \hat{v}\rangle$) for every such α and every $v \in \mathscr{M}^0_\alpha$. Let $\mathscr{L}_{s+1}$($\hat{\mathscr{L}}_{s+1}$) be the resulting list.

If the lists are not both α-marked then let $m(\alpha, s+1) = m(\alpha, s)$, $\mathscr{L}_{s+1} = \mathscr{L}_s$, and $\hat{\mathscr{L}}_{s+1} = \hat{\mathscr{L}}_s$.

Substep 11C: *Emptying* R_α *to the right of* f_{s+1}. For every α such that $f_{s+1} <_L \alpha$, *initialize* α, by removing every $x \in S_{\alpha,s}$ ($\hat{x} \in \hat{S}_{\alpha,s}$), and putting x in S_δ ($\hat{x}$ in $\hat{S}_\delta$) for $\delta = \alpha \cap f_{s+1}$ (where $\alpha \cap f_{s+1}$ denotes the longest γ such that $\gamma \subseteq \alpha$ and $\gamma \subseteq f_{s+1}$).

For each $x \in Y_{\lambda,s+1}$ such that $x \notin A_{s+1}$, let $\alpha(x, s+1)$ denote the unique γ such that $x \in S_{\gamma,s+1}$, and similarly for $\hat{x} \in \hat{Y}_{\lambda,s+1} - B_{s+1}$. If $x \in A_{s+1}$, then $\alpha(x, s+1)$ diverges; likewise for $\hat{x} \in B_{s+1}$.

Define the length function $l(e, s+1)$ and the restraint function $r(e, s+1)$ for stage $s+1$ as follows:

$$l(e, s+1) = \max\{x : (\forall y < x)[\{e\}_{s+1}^{B_{s+1}}(y) \downarrow = C_{s+1}(y)]\},$$

$$r(e, s+1) = \max\{u(B_{s+1}, e, x, s+1) : x \leqslant l(e, s+1)\}.$$

(Here u represents the standard use function for relative Turing machines.)

This completes stage $s+1$ and the construction.

Remark 6. Notice that the only step which can put elements into $B = \hat{U}_\rho$ is Step $\hat{0}$. All of Steps 1–5 and their duals are dedicated toward the $\bar{A}/\bar{B}$ part of the game. Steps $\hat{1}$, $\hat{3}$, and $\hat{5}$ may put elements $\hat{x}$ into certain sets $\hat{U}_\alpha$ in order to change $\hat{v}(\alpha, \hat{x}, s+1)$. In Steps $\hat{1}$ and $\hat{5}$, however, this can only happen when the desired $\hat{v}(\alpha, \hat{x}, s+1)$ is a $\bar{B}$-state, so we are not required to put $\hat{x}$ into B. Also, Step $\hat{3}$ never applies with $\beta = \lambda$ because ρ, the unique node at level 1 of T, is $\mathscr{M}$-consistent (see Definition 2.3 of [20]). Thus these steps never require any $\hat{x}$ to enter B.

3. Proof of the theorem

We now prove that the preceding construction satisfies Theorem 1. In Section 3.1, we verify the restrictions of certain tree properties to $\bar{A}$ and $\bar{B}$. In Section 3.2, we use these tree properties to verify the correctness of $\mathscr{M}^0$, $\hat{\mathscr{M}}^0$, $\mathscr{N}^0$, and $\hat{\mathscr{N}}^0$, thus guaranteeing that $\mathscr{K}^{\bar{A}} = \hat{\mathscr{K}}^{\bar{B}}$. Finally, in Section 3.3, we use this verification to check that $\mathscr{G}^A = \hat{\mathscr{G}}^B$.

3.1. Tree properties

The construction of [9] is designed to ensure that certain properties of the tree T (the *tree properties*) hold automatically for every $\alpha = \beta^+$ on the true path:

(1) $\mathscr{M}_\alpha = \mathscr{F}_\beta^+$,
(2) $\hat{\mathscr{M}}_\alpha = \hat{\mathscr{F}}_\beta^+$ and
(3) k_α is a correct guess.

Since our construction employs the New Extension Theorem, we need only verify the correctness of the restrictions of these properties to $\bar{A}$- and $\bar{B}$-states. The New Extension

Theorem takes care of the A/B aspect of the game, and we handle the $\mathscr{G}^A/\hat{\mathscr{G}}^B$ aspect in Steps 0 and $\hat{0}$, which we have added to the original construction of [9].

To help handle the $\mathscr{G}^A/\hat{\mathscr{G}}^B$ game, however, our construction defined the first level of the tree artificially, so that it contains only the node ρ. Therefore we must give special proofs of the tree properties (restricted to $\bar{A}$ and $\bar{B}$) for ρ.

(We will assume that A is infinite and coinfinite, for otherwise A would be computable and would itself be the set B the theorem requires.)

Sublemma 7. $\mathscr{M}_\rho = \mathscr{F}_\lambda^+$.

Proof. By Step 10, every element x of ω eventually enters Y_λ. (Lemma 6 below implies that Step 10 acts infinitely often. The proofs of the lemmas of Section 3.2 do not rely on the sublemmas of this subsection at all.) Every element of the infinite set A eventually enters some A_s by Step 0, and no element of the infinite set $\bar{A}$ ever does. Thus, there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \in A_s$, and there are infinitely many x such that for some s, $x \in Y_{\lambda,s}$ and $x \notin A_s$, so $\mathscr{F}_\lambda^+ = \{\langle \rho, \emptyset, \emptyset \rangle, \langle \rho, \{0\}, \emptyset \rangle\} = \mathscr{M}_\rho$. (In particular, then, $\mathscr{M}_\rho^0 = \mathscr{F}_\lambda^{0+}$.) $\square$

Sublemma 8. $\hat{\mathscr{M}}_\rho^0 = \hat{\mathscr{F}}_\lambda^{0+}$.

Proof. $\hat{\mathscr{M}}_\rho^0$ contains $\langle \rho, \emptyset, \emptyset \rangle$, which is the only possible $\bar{B}$–ρ-state. By Step 10, every element $\hat{x}$ of $\hat{\omega}$ eventually enters Y_λ. As noted in Remark 6, only Step $\hat{0}$ ever puts any elements into $\hat{U}_0$, and it waits to do so until such elements are already in Y_λ. Thus, $\langle \rho, \emptyset, \emptyset \rangle \in \hat{\mathscr{F}}_\lambda^{0+}$, so $\hat{\mathscr{F}}_\lambda^{0+} = \hat{\mathscr{M}}_\rho^0$. $\square$

Sublemma 9. *No element of $\bar{A}$ or $\bar{B}$ remains permanently in a nonwell-resided ρ-state.* (*Thus, the guess $k_\rho = -1$ is correct.*)

Proof. If $x \in \bar{A}$ ($\hat{x} \in \bar{B}$), then x ($\hat{x}$) will remain permanently in the ρ-state $v = \langle \rho, \emptyset, \emptyset \rangle$, which we have just seen is well-visited. To see that this state is well-resided, we must note that $\bar{A}$ and $\bar{B}$ are infinite. We assumed this for $\bar{A}$. For $\bar{B}$, we note that by Remark 6, Step $\hat{0}$ is the only step to put any elements into B, and for each $\langle \alpha, v, i \rangle$, it puts at most one element $\hat{y}$ into B, with $\hat{y} > 2 \cdot \langle \alpha, v, i \rangle$. Hence $\bar{B}$ must be infinite. $\square$

This completes the verification of the restricted versions of the tree properties for ρ. It remains to see that Lemmas 8 and 9 hold for all states, not just $\bar{B}$-states. This will be the very last line in the verification of Theorem 1, once we have proven that B is infinite. Since all D_α, $|\alpha| > 1$, are defined as in Definition 5, these properties hold automatically for all $\alpha \supseteq \rho$ with α on the true path f:

(1) $\mathscr{M}_\alpha^0 = \mathscr{F}_\beta^{0+}$ (where $\beta = \alpha^-$),
(2) $\hat{\mathscr{M}}_\alpha^0 = \hat{\mathscr{F}}_\beta^{0+}$, and
(3) k_α is the upper bound for the set of all $x \in \bar{A}$ and $\hat{x} \in \bar{B}$ that remain permanently in nonwell-visited α-states.

3.2. Verification that $\mathscr{M}^0=\hat{\mathscr{M}}^0$, and $\mathscr{N}^0=\hat{\mathscr{N}}^0$

For purposes of parallelism, we number our Lemmas 1–12 to match Lemmas 5.1–5.12 of [9,20]. All twelve of these lemmas have duals, which we will not state or prove except when the proof of the dual requires a distinct technique, principally in Lemma 11, which yields a nice insight into the construction and the reasons why Theorem 1 actually holds.

Of course, our lemmas hold for the $\bar{A}/\bar{B}$ game, whereas in [9] they held for the entire universe of elements. Also, our first Lemma matches Lemma 5.0 of [20].

Lemma 0. *Let $\alpha\subset f$, where f is the true path through T.*
(i) *If the $\bar{A}$-state v lies in $\mathscr{E}_\alpha^0$, then there exists an infinite set $\{x_i\}_{i\in\omega}\subseteq\bar{A}$ such that*

$$(\forall i)\left[\lim_s \Gamma^{1,\alpha}_{v,i,s}=x_i\ \&\ (\exists s)[x_i\in S_{\alpha,s}-Y_{\alpha,s-1}\ \&\ v(\alpha,x_i,s)=v]\right].$$

(ii) *If the $\bar{A}$-state v lies in $\mathscr{F}_\alpha^0$, then there exists an infinite set $\{x_i\}_{i\in\omega}\subseteq\bar{A}$ such that*

$$(\forall i)\left[\lim_s \Gamma^{2,\alpha}_{v,i,s}=x_i\ \&\ (\exists s)[x_i\in R_{\alpha,s}\ \&\ v(\alpha,x_i,s)=v]\right].$$

(iii) *If the $\bar{A}$-state v lies in $\mathscr{F}_\alpha^{0+}$, then there exists an infinite set $\{x_i\}_{i\in\omega}\subseteq\bar{A}$ such that*

$$(\forall i)\left[\lim_s \Gamma^{3,\alpha}_{v,i,s}=x_i\ \&\ (\exists s)[x\in R_{\alpha,s}\ \&\ v^+(\alpha,x_i,s)=v]\right].$$

Proof. All of these proofs are similar; the proof of (i) is given in [20] (Section 3.2, Lemma 5.0). □

The construction makes the following lemma clear. (When, e.g., Step 1 of the construction applies to a node α and an element x, we will sometimes say, "Step 1_α applies to x".)

Lemma 1. *At stage $s+1$,*
(i) *if x enters R_α, $\alpha\neq\lambda$, then Step 1 or Step 2 applies to α and x;*
(ii) *if x moves from S_α to S_δ then one of the following steps must apply to x: Step 1_δ with $\delta<_L\alpha$ or $\delta^-=\alpha$; Step 2_δ with $\delta^-=\alpha$; or Substep $11C_\alpha$, so $f_{s+1}<_L\alpha$;*
(iii) *if $x\in S_{\alpha,s}$ is enumerated in a set U_α at stage $s+1$ then Step 1 or Step 4 must apply to x;*
(iv) *if $x\in S_{\alpha,s}$ is enumerated in a set $\hat{V}_\alpha$ then Step 1, Step 3, or Step 5 must apply to x.* □

Lemma 2 (True Path Lemma). *The true path $f=\liminf_s f_s$.*

Proof. This is clear from the definition of f_s in Step 11A and from the choice of the sets D_α. □

Hereafter f will always represent the true path.

Lemma 3. *For all* $\alpha \in T$,
(i) $f <_L \alpha \Rightarrow R_\alpha = \emptyset$,
(ii) $\alpha <_L f \Rightarrow Y_\alpha =^* \emptyset$,
(iii) $\alpha \subset f \Rightarrow Y_{<\alpha} =_{\mathrm{dfn}} \bigcup\{Y_\delta : \delta <_L \alpha\} =^* \emptyset$.

Proof. Part (i) must hold, because Substep 11C sets $S_{\alpha,s+1} = \emptyset$ whenever $f_{s+1} <_L \alpha$. For part (ii), if $\alpha <_L f$, pick an s such that $\alpha <_L f_t$ for all $t \geqslant s$. Then $Y_\alpha = Y_{\alpha,s} =^* \emptyset$. Finally, for part (iii), if $\alpha \subset f$, then $Y_{<\alpha} \subseteq \{0, 1, \ldots, s\}$, where s is a stage such that $f_t \not<_L \alpha$ for all $t \geqslant s$. □

In Lemma 4, since it is now possible for an element x to disappear from the game by being enumerated into A (or B, in the dual lemma), we must slightly modify the statement of (iv) from [9] by restricting x to elements of $\bar{A}$ (and $\hat{x}$ to $\bar{B}$, in the dual), as shown:

Lemma 4. *For every* $\alpha \in T$ *such that* $\alpha \neq \lambda$, *if* $\beta = \alpha^-$, *then*
(i) $Y_\alpha \backslash Y_\beta = \emptyset$ *and* $Y_\alpha \subseteq Y_\beta$,
(ii) *For each* x *there is at most one* s *such that* $x \in R_{\alpha,s+1} - R_{\alpha,s}$,
(iii) $U_\alpha \backslash Y_\alpha = \hat{V}_\alpha \backslash Y_\alpha = \emptyset$, *and*
(iv) *If* $\alpha \subset f$, *then*

$$(\exists v_\alpha)(\forall x \in \bar{A})(\forall s \geqslant v_\alpha)[x \in R_{\alpha,s} \Rightarrow (\forall t \geqslant s)[x \in R_{\alpha,t}]]$$

(*and correspondingly with* $\bar{B}$ *in the dual*).

Proof. Part (i) follows from Lemma 1(i).

For (ii), we note from Lemma 1(ii) that if $x \in R_{\alpha,t} - R_{\alpha,t+1}$, then $x \in S_{\delta,t+1}$ for some δ, and either $\delta <_L \alpha$, or α was initialized at stage $t+1$. In the former case, x can never re-enter R_α (by Lemma 1(ii), again). If α was initialized, then $\delta = f_{t+1} \subset \alpha$, and x could only return to R_α by applications of Step 1 or Step 2. However, we know that $x < t$ by Step 10 (since $x \in R_{\alpha,t}$), so restrictions (1.3) and (2.3) in Steps 1 and 2 rule out the return of x to R_α.

For (iii), any of Steps 1 and 3–5 can put an x into some $U_{\alpha,s+1}$ or $\hat{V}_{\alpha,s+1}$, but each of them either requires $x \in Y_{\alpha,s}$ or puts $x \in Y_{\alpha,s+1}$.

Finally, (iv) assumes $\alpha \subset f$, so by Lemma 3(iii), $Y_{<\alpha}$ is finite. Let v_α be a stage so large that $f_s <_L \alpha$ only if $s < v_\alpha$, and also that every $y \in Y_{<\alpha}$ never again either enters or leaves R_α. (By part (ii) of this lemma, each of the finitely many $y \in Y_{<\alpha}$ enters R_α at most once.) Lemma 1(ii) makes it clear that the only way for *any* $x \in \bar{A}$ to leave R_α at any stage is for it to enter $Y_{<\alpha}$ or for $f_{s+1} <_L \alpha$. Neither of these can occur at any stage $s > v_\alpha$, by our choice of v_α. □

Lemma 5. *For all* $x \in \bar{A}$:
(i) $\alpha(x) = \lim_s \alpha(x,s)$ *exists, and*
(ii) x *is enumerated in at most finitely many r.e. sets* U_γ, $\hat{V}_\gamma$, *and hence for* $\alpha = \alpha(x)$,

$$v(\alpha, x) \underset{\mathrm{dfn}}{=} \lim_s v(\alpha, x, s) \text{ exists.}$$

(*Similarly with* $\bar{B}$ *in the dual.*)

Proof. Lemma 1(ii) gives the conditions under which $\alpha(x,s+1) \neq \alpha(x,s)$ can occur. Let $\gamma = f \restriction x$ be the initial segment of the true path with length x, and choose $s > v_\gamma$ with $f_s \restriction x = \gamma$. Substep 11C forces either $\alpha(x,s) <_L \gamma$ or $\alpha(x,s) \subseteq \gamma$. (It is impossible for $\gamma \subsetneq \alpha(x,s)$ since $|\gamma| = x$.) Moreover, Substep 11C will never apply to x after stage s.

Now Steps 1 and 2 can only move x into S_α if $x > |\alpha|$. Also, each α has only finitely many predecessors in T, and x cannot be moved back and forth among these predecessors infinitely often because of Lemma 4(ii). Therefore, if $\alpha(x,s+1) \neq \alpha(x,s)$ occurs infinitely often, then there must be infinitely many stages at which either $\alpha(x,s+1) <_L \alpha(x,s)$. However, there is no infinite sequence $\{\delta_1 <_L \delta_2 <_L \delta_3 <_L \cdots\}$ in T with every $|\delta_i| < x$. This proves part (i).

Part (ii) follows from (i) because $\alpha(x,s)$ eventually converges to some $\alpha(x)$, and there are only finitely many possible $\alpha(x)$-states. Once x leaves some $\alpha(x)$-state, it can never return to that state, because the sets U_γ and $\hat{V}_\gamma$ which we enumerate are c.e. Moreover, x will never be enumerated in any U_γ or $\hat{V}_\gamma$ unless $\gamma \subseteq \alpha(x)$. $\square$

Lemma 6. *If the hypotheses of some Step* 0–5 *or* $\hat{0}$–$\hat{5}$ *remain satisfied, then that step eventually applies. Also, each of Steps* 10 *and* $\widehat{10}$ *applies infinitely often.*

Proof. If Steps 10 and $\widehat{10}$ never applied after some stage s_0, then there would only be finitely many elements x and $\hat{x}$ in Y_λ and $\hat{Y}_\lambda$, to which the steps preceding Step 10 would apply at every stage after s_0. Each of these steps performs some action when applied, either moving an x or an $\hat{x}$ into a new S_α or enumerating it into some U_α, V_α, $\hat{U}_\alpha$, or $\hat{V}_\alpha$. However, such actions can only occur finitely often for any given x or $\hat{x}$, by Lemma 5, so eventually Step 10 or $\widehat{10}$ must apply, providing a new element x or $\hat{x}$. In order for Step 10 or $\widehat{10}$ to apply, the hypotheses of all the other steps must be unsatisfied. This proves the lemma. $\square$

Lemma 7. *If* $\alpha \subset f$, $\rho \subsetneq \alpha$, *and* $\beta = \alpha^-$ *then*
(i) $(\forall \gamma <_L f)\ [m(\gamma) =_{\text{dfn}} \lim_s m(\gamma,s) < \infty]$,
(ii) $m(\alpha) =_{\text{dfn}} \lim_s m(\alpha, s) = \infty$,
(iii) $\mathscr{E}^0_\alpha \supseteq \mathscr{M}^0_\alpha = \mathscr{F}_\beta^{0+}$, *and*
(iv) $\hat{\mathscr{E}}^0_\alpha \supseteq \hat{\mathscr{M}}^0_\alpha = \hat{\mathscr{F}}_\beta^{0+}$.

Proof. For part (i), we note that for each $\gamma <_L f$, Substep 11B can only apply finitely often. Hence $\lim_s m(\gamma,s)$ must be finite.

Turning to (ii), we let α and β be as given in the lemma. The definition of the true path (Definition 5) yields $\mathscr{M}^0_\alpha = \mathscr{F}_\beta^{0+}$ and $\hat{\mathscr{M}}^0_\alpha = \hat{\mathscr{F}}_\beta^{0+}$. By Substep 11B, $m(\alpha, s)$ is nondecreasing as a function of s; we claim that it increases infinitely often. Otherwise there would exist a stage s_0 with $m(\alpha, s) = m(\alpha, s_0)$ for all $s \geqslant s_0$.

Claim. *Every* α*-entry* $\langle \alpha, v_1 \rangle$ *on* $\mathscr{L}$ ($\langle \alpha, \hat{v}_1 \rangle$ *on* $\hat{\mathscr{L}}$) *is eventually marked.*

As in [20], we modify the proof of this claim in the nondual case, since it is now possible for elements to leave the game before they can enter S_α. We will use

Lemma 0(iii) to guarantee a supply of elements $\{x_i\}_{i\in\omega}$ that remain in $\bar{A}$ because their Γ^3-tags are never removed.

If some entry $\langle\alpha, v_1\rangle$ on $\mathscr{L}$ were never marked, then no more α-entries would ever be added to $\mathscr{L}$ after $\langle\alpha, v_1\rangle$. Choose a stage s_1 large enough that neither any α-entries on $\mathscr{L}$ nor any entry on $\mathscr{L}$ preceding $\langle\alpha, v_1\rangle$ is ever marked after stage s_1, that $Y_{<\alpha,s_1} = Y_{<\alpha}$ (using Lemma 3), and that $Y_{\alpha,s_1}\restriction m(\alpha, s_0) = Y_\alpha\restriction m(\alpha, s_0)$. Now requirement (2.4) prevents Step 2 from enumerating any $x > m(\alpha, s_0)$ into R_α after stage s_1, and Step 1 will never again put any x into R_α because by (1.8), that would involve marking an unmarked α-entry on $\mathscr{L}$.

We have $v_1 \in \mathscr{M}^0_\alpha$ because $\langle\alpha, v_1\rangle \in \mathscr{L}$. Also $\mathscr{M}^0_\alpha = \mathscr{F}^{0+}_\beta$, since $\alpha \subset f$. Hence Lemma 0(iii) applied to β provides an infinite collection of elements $\{x_i\}_{i\in\omega} \subset \bar{A}$. By the choice of s_1 all but finitely many x_i satisfy (1.1)–(1.7). (Satisfying (1.5) uses the assumption that $\lim_s m(\alpha, s)$ is finite.) Thus, some such x_i is moved to S_α under Step 1 at some stage $s + 1 > s_1$, and the entry $\langle\alpha, v_1\rangle$ is then marked, contrary to hypothesis. This establishes the claim for $\mathscr{L}$.

With the claim, we see that Substep 11B will apply to α at some stage $s > s_1$, forcing $m(\alpha, s) > m(\alpha, s - 1)$.

(The proof of (ii) in the dual case is simpler, because we never enumerate any element of $\hat{S}_{\beta,s}$ into B.)

Condition (iii) now follows (and (iv) similarly) because for any $v_1 \in \mathscr{M}^0_\alpha$, (ii) forces infinitely many entries $\langle\alpha, v_1\rangle$ to be added to $\mathscr{L}$, and for each to enter, all previous such entries must have been marked. The only way for an entry to be marked is for an x in α-state v_1 to enter S_α, and if this happens infinitely often, then $v_1 \in \mathscr{E}^0_\alpha$. $\square$

Lemma 8. *$\alpha \subset f \Rightarrow$*
(i) *$R_\alpha =^* Y_\alpha \cap \bar{A} =^* Y_\lambda \cap \bar{A} = \bar{A}$; and*
(ii) *Y_α is infinite. (And similarly for the dual lemma, with B for A.)*

Proof. By Lemma 6(i) Step 10 must eventually put every element $x \in \omega$ into Y_λ. By induction we may assume that $R_\beta =^* Y_\beta \cap \bar{A} =^* \bar{A}$ and Y_β is infinite, for $\beta = \alpha^-$. By Lemma 7 $m(\alpha) = \infty$, and $m(\gamma) < \infty$ for all $\gamma <_L \alpha$ with $\gamma^- = \beta$.

Now by Lemma 3, $Y_{<\alpha} =^* \emptyset$. Also, cofinitely many of the elements $x \in (Y_\beta - Y_\alpha) \cap \bar{A}$ will eventually enter S_β. Therefore, cofinitely many such x will satisfy (2.1)–(2.5) at some stage, and will be moved to S_α by Step 2. Once there, cofinitely many of them will remain in R_β forever, by Lemma 4(iv).

Part (ii) follows immediately from part (i), since $\bar{A}$ is infinite (as is $\bar{B}$, in the dual case).

The proof of the dual case is nearly the same, except that to see that $\hat{Y}_\alpha$ is infinite, we must observe that since $\hat{Y}_\beta$ is infinite, infinitely many elements will enter $\hat{S}_\beta$ via Step $\hat{1}$ or $\hat{2}$. By the above reasoning, cofinitely many of these must eventually enter $\hat{S}_\alpha$. $\square$

Lemma 9. *$\alpha \subset f \Rightarrow \alpha$ is $\mathscr{M}$-consistent.*

Proof. The proof of the lemma itself is the same as that of Lemma 5.9 of [20]. In the dual, there is no need to appeal to an analogue of Lemma 0(i), since we do not need $\hat{x}\in\bar{B}$. We simply note that since $\alpha\subset f$, we have $\hat{v}_0\in\hat{\mathcal{M}}_\alpha^0=\hat{\mathcal{E}}_\alpha^0$, so there will be infinitely many $\hat{x}>k_\alpha$ and $s>\hat{v}_\alpha$ available to us with $\hat{x}\in\hat{S}_{\alpha,s+1}-\hat{S}_{\alpha,S}$ and $\hat{v}(\alpha,\hat{x},s+1)=\hat{v}_0$. As with x above, Step $\hat{3}$ must eventually move each such $\hat{x}$ into some α-state $\hat{v}_1$ with $\hat{v}_0<_B\hat{v}_1$. Since α is inconsistent, $\hat{x}$ cannot enter B at any stage $t>\hat{v}_\alpha$ so $\hat{v}_1\notin\hat{\mathcal{M}}_\alpha^0$. Again, this forces $\alpha\not\subset f$. $\square$

Lemma 10. *If* $\alpha\subset f$ *then*
(i) $\hat{\mathcal{M}}_\alpha^0=\{\hat{v}:v\in\mathcal{M}_\alpha^0\}$,
(ii) $\mathcal{M}_\alpha^0=\mathcal{F}_\alpha^0=\mathcal{E}_\alpha^0$, *and*
(iii) $\hat{\mathcal{M}}_\alpha^0=\hat{\mathcal{F}}_\alpha^0=\hat{\mathcal{E}}_\alpha^0$.

Proof. The proof is identical to that of Lemma 5.10 in [9], except that we restrict to $\bar{A}$- and $\bar{B}$-states as usual. This restriction does change Claim 5 of the proof in [9], so we give our own version here. (Note also that [9] occasionally refers to $\mathcal{E}_\beta^1$ and Y_β^1. The superscript 1 may be ignored for our purposes, as it refers to a partition of elements of Y_β which is unnecessary for our result.)

Claim 5. *If* $\hat{x}\in\hat{Y}_{\alpha,s}$, $\hat{v}_1=\hat{v}(\alpha,\hat{x},s)\in\hat{\mathcal{M}}_\alpha^0$, $s>v_\alpha$ *of Lemma* 4(iv), *and* BLUE *causes enumeration of* $\hat{x}$ *so that* $\hat{v}_2=\hat{v}(\alpha,\hat{x},s+1)$ *then either* $\hat{v}_2\in\hat{\mathcal{M}}_\alpha^0$ *or* $\hat{v}_2$ *is a B-state.*

Proof. Suppose $\hat{x}\in\hat{Y}_{\alpha,s}$ and BLUE causes this enumeration at stage $s+1$, so $\hat{v}_1<_B\hat{v}_2$. Since $s>v_\alpha$, $\hat{x}\in\hat{R}_{\alpha,s}\cap\hat{R}_{\alpha,s+1}$. Hence, either Step $\hat{1},\hat{3},\hat{5}$, or $\hat{0}$ applies to $\hat{x}$ at stage $s+1$ for some $\gamma\supseteq\alpha$. Assume that $\hat{v}_2$ is not a B-state. (Thus Step $\hat{0}$ cannot have applied.) If Step $\hat{1}_\gamma$ or $\hat{5}_\gamma$ applies then $\hat{v}_3=\hat{v}(\gamma,\hat{x},s+1)\in\hat{\mathcal{M}}_\gamma^0$ so $\hat{v}_2=\hat{v}_3\restriction\alpha\in\hat{\mathcal{M}}_\alpha^0$. (Here Step $\hat{5}_\gamma$ means Step $\hat{5}$ Case 1 for $\hat{x}\in\hat{Y}_{\gamma,s}$ or Step $\hat{5}$ Case 2 for $\hat{x}\in\hat{Y}_{\delta,s}$ where $\gamma=\delta^-$.) If Step $\hat{3}_\gamma$ applies, then $\gamma\supsetneq\alpha$ (since α is $\mathcal{M}$-consistent and γ is not) and $\hat{v}_3=\hat{v}(\gamma^-,\hat{x},s+1)\in\hat{\mathcal{M}}_{\gamma^-}^0$ by (3.4) so $\hat{v}_2=\hat{v}_3\restriction\alpha\in\hat{\mathcal{M}}_\alpha^0$. This completes the proof of Claim 5. $\square$

Lemma 11. $\alpha\subset f\Rightarrow\alpha$ *is* $\mathcal{R}$*-consistent.*

Proof. To prove $\mathcal{R}^0$-consistency of α, assume for a contradiction that $\alpha\subset f$ and α is not $\mathcal{R}^0$-consistent. Choose $v_1\in\mathcal{R}_\alpha^0$ such that for all $v_2\in\mathcal{M}_\alpha^0$, $v_1\not<_R v_2$. Being inconsistent, α is a terminal node on T, so $S_{\alpha,s}=R_{\alpha,s}$ for all s. Thus, by Lemma 4(iv), there exists a stage v_α such that $S_{\alpha,s}\cap\bar{A}\subseteq S_{\alpha,t}$ for every s and t with $t\geqslant s\geqslant v_\alpha$.

Now $\hat{v}_1\in\hat{\mathcal{R}}_\alpha^0\subseteq\hat{\mathcal{M}}_\alpha^0=\hat{\mathcal{E}}_\alpha^0$ by Lemma 10. Therefore Lemma 0(i) yields an infinite set $\{x_i\}_{i\in\omega}\subseteq\bar{A}$ such that

$$(\forall i)(\exists s)[x_i\in S_{\alpha,s+1}-Y_{\alpha,s}\ \&\ v(\alpha,x_i,s+1)=v_1].$$

Let x be any such x_i with the corresponding $s>v_\alpha$. Now Step 0 will not apply to x at any stage $t>s+1$ because $x\in\bar{A}$. Steps 1 and 2 would both remove x from S_α, which is impossible at any stage $t>v_\alpha$. By Lemma 9, α must be $\mathcal{M}$-consistent, so Step 3 will never apply. Also, Step 5 does not apply to $\mathcal{R}^0$-inconsistent nodes such as α. Therefore, if x is to be removed from state v_1 as required by $F(\beta, v_1)$, then Step 4 must act, enumerating x into some red set U_γ with $\gamma \subseteq \alpha$. Since this happens for infinitely many elements x, and there are only finitely many α-states v with $v_1 <_R v$, one of those states v must lie in $\mathscr{F}_\alpha{}^0$, hence in $\mathscr{M}_\alpha^0$, by Lemma 10(iii). This contradicts $\mathscr{R}^0$-inconsistency.

To prove $\hat{\mathscr{R}}^0$-consistency of α, assume for a contradiction that $\alpha \subset f$ and α is not $\hat{\mathscr{R}}^0$-consistent. Choose $\hat{v}_1 \in \hat{\mathscr{R}}_\alpha^0$ such that for all $\hat{v}_2 \in \mathscr{M}_\alpha^0$, $\hat{v}_1 \not<_R \hat{v}_2$. Being inconsistent, α is a terminal node on T, so $\hat{S}_{\alpha,s} = \hat{R}_{\alpha,s}$ for all s. Thus, by the dual of Lemma 4(iv), there exists a stage v_α such that $\hat{S}_{\alpha,s} \cap \bar{B} \subseteq \hat{S}_{\alpha,t}$ for every s and t with $t \geqslant s \geqslant v_\alpha$.

Now $\hat{v}_1 \in \hat{\mathscr{R}}_\alpha^0 \subseteq \hat{\mathscr{M}}_\alpha^0 = \hat{\mathscr{E}}_\alpha^0$ by the dual of Lemma 10. Therefore there exist infinitely many elements $\hat{x}$ such that

$$(\exists s)[\hat{x} \in \hat{S}_{\alpha,s+1} - (B_s \cup \hat{Y}_{\alpha,s}) \;\&\; \hat{v}(\alpha, \hat{x}, s+1) = \hat{v}_1].$$

Take any such $\hat{x} > k_\alpha$ for which the corresponding $s > v_\alpha$. Step 0 does not apply to the $\hat{\omega}$-side, and Steps $\hat{1}$ and $\hat{2}$ would both remove $\hat{x}$ from $\hat{S}_\alpha$, which is impossible at any stage $t > v_\alpha$. By the dual of Lemma 9, α must be $\mathscr{M}$-consistent, so Step $\hat{3}$ will never apply. Steps $\hat{5}$ and $\hat{0}$ do not apply to $\hat{\mathscr{R}}^0$-inconsistent nodes such as α. Therefore, if $\hat{x}$ is to be removed from state $\hat{v}_1$ as required by $\hat{F}(\beta, \hat{v}_1)$, then Step $\hat{4}$ must act, enumerating $\hat{x}$ into some red set V_γ with $\gamma \subseteq \alpha$. Since this happens for infinitely many elements $\hat{x}$, and there are only finitely many α-states $\hat{v}$ with $\hat{v}_1 <_R \hat{v}$, one of those states $\hat{v}$ must lie in $\hat{\mathscr{F}}_\alpha{}^0$, hence in $\hat{\mathscr{M}}_\alpha^0$, by the dual of Lemma 10(iii). This contradicts $\hat{\mathscr{R}}^0$-inconsistency. □

We remark that while the two halves of the preceding proof appear quite similar, the similarity is deceptive. In fact, the proof of $\mathscr{R}^0$-consistency, depends on the lowness of A, which guided the proof of Lemma 0. On the other hand, in the proof of the dual $\hat{\mathscr{R}}^0$-consistency, we used instead the fact that inconsistent nodes do not require any elements to be enumerated into any blue sets, including B itself. This works in the present situation because the only external requirements for the construction of B are negative requirements, namely the $\mathscr{Q}_e$ of the Sacks preservation strategy. (The positive requirements stem from the automorphism construction itself, not from any properties which we demand of B.) Herein lies the connection between lowness of A and the ability of A to avoid an upper cone.

Lemma 12. *If $\alpha \subset f$ and $v_1 \in \mathscr{B}_\alpha^0$, then*

$$\{x : x \in Y_\alpha \;\&\; v(\alpha, x) = v_1\} =^* \emptyset.$$

Proof. The lemma itself is proved exactly as is Lemma 5.12 of [9]. In the dual case, we note that the state $\hat{v}_2'$ of that proof might possibly be a B-state. If so, then $\hat{v}_2$ would not lie in $\hat{\mathscr{M}}_\alpha^0$. However, in that case $\hat{v}_2$ would also be a B-state, so $\hat{v}_2 \neq \hat{v}_1$. □

Lemma 13. *For every $\alpha \subset f$, $\mathscr{M}_\alpha^0 = \hat{\mathscr{M}}_\alpha^0$ and $\mathscr{N}_\alpha{}^0 = \hat{\mathscr{N}}_\alpha{}^0$.*

Proof. Lemma 10(i) gives the result for $\mathscr{M}$. Moreover, since $\alpha \subset f$, we know that $\mathscr{R}_\alpha^0 = \hat{\mathscr{B}}_\alpha^0$ and $\mathscr{B}_\alpha^0 = \hat{\mathscr{R}}_\alpha^0$ (see Definition 5). To prove $\mathscr{N}_\alpha^{\;0} = \hat{\mathscr{N}_\alpha}^0$, therefore, we need only show that for each $\bar{A}$–α-state v in $\mathscr{M}_\alpha^0$,

$$v \in \mathscr{B}_\alpha^0 \cup \mathscr{R}_\alpha^0 \iff \{x \in \omega : v(\alpha, x) = v\} \text{ is finite}$$

and similarly for $\hat{v} \in \hat{\mathscr{B}}_\alpha^0 \cup \hat{\mathscr{R}}_\alpha^0$.

Suppose $v \in \mathscr{R}_\alpha^0$. Then $F(\beta, v)$ must hold, where $\beta = \alpha^-$. Therefore, by Eq. (19) of [9], only finitely many $x \in Y_\beta$ remain permanently in the α-state v. Since $\beta \subset \alpha \subset f$, we know that $Y_\beta =^* \omega$, so $v \in \mathscr{N}_\alpha^{\;0}$. The proof for $\hat{v} \in \hat{\mathscr{R}}_\alpha^0$ is analogous.

Now let $v \in \mathscr{B}_\alpha^0$ and suppose $v(\alpha, x) = v$. We know there exists a node γ and a stage s_0 such that $x \in S_{\gamma,s}$ for all $s \geqslant s_0$. Since $\alpha \subset f$, R_α is cofinite, so we may assume that $\gamma \supseteq \alpha$. Let $v_1 = v(\gamma, x)$ be the permanent γ-state of x, and suppose that $s_1 \geqslant s_0$ is such that $v(\gamma, x, s) = v_1$ for all $s \geqslant s_1$. Then $v_1 \upharpoonright \alpha = v$, and v_1 is an $\bar{A}$-state. By part (vi) of Definition 3, $v_1 \in \mathscr{B}_\gamma^0$. If γ is a consistent node, then by Lemma 6, there will eventually be a stage $s \geqslant s_1$ at which Case 1 of Step 5 applies, so x will be moved into some other γ-state $v_2 >_B v_1$ at stage s_1. If γ is inconsistent, then again x will change γ-states at some stage $s \geqslant s_1$ at which Case 2 of Step 5 applies. In either case, this contradicts our assumption that $v(\gamma, x) = v_1$. Thus there are only finitely many x which reside permanently in the α-state v, forcing $v \in \mathscr{N}_\alpha^{\;0}$.

For $\hat{v} \in \hat{\mathscr{B}}_\alpha^0$, the dual proof holds for all $\hat{x} \in \bar{B}$. If $\hat{x} \in B$, then clearly $\hat{v}$ is not the final α-state of $\hat{x}$, since every state in $\hat{\mathscr{B}}_\alpha^0$ is a $\bar{B}$-state. Therefore again $\hat{v} \in \hat{\mathscr{N}_\alpha}^0$.

Now suppose $v \in \mathscr{N}_\alpha^{\;0}$, i.e. v is a well-visited but nonwell-resided α-state. In the construction, the only steps at which an element x may be moved out of v are Steps 0, 1, 4, and 5. (Step 3 never applies to α, by Lemmas 9 and 11.) If Step 5_γ applies (for some $\gamma \supseteq \alpha$), then $v \in \mathscr{B}_\alpha^0$, by part (vi) of Definition 3. Since $\alpha \subset f$, Step 1 can only move elements in R_α to regions S_γ, where $\alpha \subset \gamma$ (except for finitely many elements), and when it does so, it enumerates them only into U_γ or $\hat{V}_\gamma$, leaving the α-state unchanged. Step 0 could move infinitely many elements into A, but by Lemma 0, there must also be infinitely many elements from $\bar{A}$ in the state v, since $v \in \mathscr{M}_\alpha^0 = \mathscr{E}_\alpha^0$.

Therefore, suppose Step 4 changes the α-state of cofinitely many of the elements in state v. By definition of $k_\alpha = k_\beta^+$, the finitely many elements not moved can never enter Y_β. Hence $F(\beta, v)$ holds. Since $v \in \mathscr{M}_\alpha^0$ and $\alpha \subset f$, part (iii) of Definition 3 forces $v \in \mathscr{R}_\alpha^0 \cup \mathscr{B}_\alpha^0$.

Finally, for the dual case $\hat{v} \in \hat{\mathscr{N}_\alpha}^0$, the same argument holds, except that Step $\hat{0}$ could move an element out of $\hat{v}$. If cofinitely many of the elements which enter state $\hat{v}$ are so moved, then according to Step 0, cofinitely many elements in the corresponding state v on the ω-side must have entered A. This contradicts Lemma 0, so there must be infinitely many elements in $\hat{v}$ which are not moved into B by Step $\hat{0}$. $\square$

Lemma 14. *$\{U_\alpha : \alpha \subset f\}$ and $\{V_\alpha : \alpha \subset f\}$ each forms a skeleton for the collection of all c.e. sets. (That is, for every e there exist $\gamma \subset f$ and $\delta \subset f$ such that $W_e =^* U_\gamma =^* V_\delta$.)*

Proof. Steps 4 and $\hat{4}$ accomplish this, since $R_\alpha =^* \omega$ and $\hat{R}_\alpha =^* \hat{\omega}$ for all $\alpha \subset f$. The only exception is the set $A = U_0$, which is covered by Substep (0.1). $\square$

3.3. Verifying that $\mathscr{G}^A = \hat{\mathscr{G}}^B$

Our proof that $\mathscr{G}^A = \hat{\mathscr{G}}^B$ follows the same ideas as in Section 1.3.3 of [20]. First, however, we need to show that all requirements are satisfied.

Lemma 15. *Every requirement $\mathscr{Q}_e$ is satisfied. (Hence $C \not\leqslant_T B$.)*

Proof. Each positive requirement $\mathscr{P}_{\langle \alpha', v', j \rangle}$ puts at most one element into B, so (by induction) there exists a stage s_0 so large that no $\mathscr{P}_{\langle \alpha', v', j \rangle}$ with $\langle \alpha', v', j \rangle \leqslant e$ puts any elements into B at any stage $\geqslant s_0$. Notice also, by the remark at the end of the construction, that only Step $\hat{0}$ ever puts any elements into B, and that it respects all higher-priority negative requirements $\mathscr{Q}_i$ when doing so.

Now suppose that $\mathscr{Q}_e$ fails, i.e. $C = \{e\}^B$. Then $\lim_s l(e,s) = \infty$, and we can use this fact to compute C. Given x, find a stage $s \geqslant s_0$ such that $l(e,s) > x$. As in [18, Theorem VII.3.1], we must then have

$$\{e\}_s^{B_s}(x) = \{e\}^B(x) = C(x),$$

since by our choice of s_0, the initial segment of B_s used in this computation will never again be changed.

This contradicts the noncomputability of C. Hence $\mathscr{Q}_e$ must be satisfied. $\square$

Lemma 16. *For every e, $\lim_s r(e,s)$ exists and is finite.*

Proof. The proof follows the proof of Lemma 2 in [18, VII.3.1], exactly, with Lemma 15 yielding an x such that $C(x) \neq \{e\}^B(x)$. $\square$

To show that $\mathscr{G}^A = \hat{\mathscr{G}}^B$, we will prove the following two lemmas:

Lemma 17. *For any node α and α-state v_1, $\mathscr{L}^{\mathscr{G}}$ contains infinitely many pairs $\langle \alpha, \hat{v}_1 \rangle$ if and only if $v_1 \in \mathscr{G}_\alpha^A$.*

Proof. Such a pair is added to $\mathscr{L}^{\mathscr{G}}$ exactly when Step 0 enumerates some $x \in v_1$ into A. Moreover, no step except Step 0 ever puts any elements into A. Thus, $\mathscr{L}^{\mathscr{G}}$ contains infinitely many such pairs if and only if infinitely many $x \in v_1$ are enumerated into A; that is, if and only if $v_1 \in \mathscr{G}_\alpha^A$. $\square$

Lemma 18. *For any node $\alpha \subset f$ and α-state v_1, $\mathscr{L}^{\mathscr{G}}$ contains infinitely many pairs $\langle \alpha, \hat{v}_1 \rangle$ if and only if $\hat{v}_1 \in \hat{\mathscr{G}}^B$.*

Proof. To show the "if" part of this statement, we assume that infinitely many elements $\hat{x}$ enter B while in α-state $\hat{v}_1$, and observe that

(1) we do not move any element $\hat{x}$ in α-state $\hat{v}_1$ into B except when required to do so in Step $\hat{0}$ by some pair $\langle \gamma, \hat{v}_1' \rangle$ in $\mathscr{L}^{\mathscr{G}}$ with $\alpha \subseteq \gamma$ and $\hat{v}_1 = \hat{v}_1' \restriction \alpha$; and that

(2) when Step $\hat{0}$ does require such an $\hat{x}$ to enter B, we mark the corresponding pair $\langle \gamma, \hat{v}_1' \rangle$, so for infinitely many $\hat{x}$ in α-state $\hat{v}_1$ to enter B, there must be infinitely many such pairs in $\mathscr{L}^{\mathscr{G}}$; and that

(3) therefore there must be infinitely many pairs $\langle \alpha, \hat{v}_1 \rangle$ in $\mathscr{L}^{\mathscr{G}}$, since whenever we add a $\langle \gamma, \hat{v}_1' \rangle$, we also add a $\langle \alpha, \hat{v}_1 \rangle$ for each $\alpha \subseteq \gamma$.

To show the "only if" part, suppose that for a given α and v, $\mathscr{L}^{\mathscr{G}}$ contains infinitely many pairs $\langle \alpha, \hat{v} \rangle$. We claim that for every k, the requirement $\mathscr{P}_{\langle \alpha, v, k \rangle}$ is satisfied.

To see this, assume by induction that $\mathscr{P}_{\langle \alpha, v, k-1 \rangle}$ is satisfied, and notice that we can find a stage s_0 so large that $\mathscr{L}^{\mathscr{G}}$ contains at least k pairs $\langle \alpha, \hat{v} \rangle$ at stage s_0 and that for all $s \geqslant s_0$ and all $e \leqslant \langle \alpha, v, k \rangle$, $r(e,s) = r(e, s_0)$. By Lemma 17, $v \in \mathscr{G}_\alpha^A$. Therefore $v \in \mathscr{M}_\alpha^0$, and by Lemmas 13 and 10(iii), $\hat{v} \in \hat{\mathscr{M}}_\alpha^0 = \hat{\mathscr{F}}_\alpha^0$. If $\mathscr{P}_{\langle \alpha, v, k \rangle}$ remained unsatisfied forever, then the definition of $\hat{\mathscr{F}}_\alpha^0$ would guarantee that there must exist distinct elements $\hat{y}_0, \hat{y}_1, \hat{y}_2, \ldots, \hat{y}_{2k}$ and a stage $s > s_0$ at which these elements satisfy conditions $(\hat{0}.4)$ $-(\hat{0}.6)$. Now α is consistent, by Lemmas 9 and 11, and $\mathscr{P}_{\langle \alpha, v, k \rangle}$ would not be satisfied at stage s, so by Step $\hat{0}$ of the construction, the element $\hat{y}_{2k}$ would have to enter B from state $\hat{v}$ at stage $s+1$.

Since $v \in \mathscr{G}_\alpha^A$, we know that the hypothesis of $\mathscr{P}_{\langle \alpha, v, k \rangle}$ is satisfied for every k. Since the requirements themselves are all satisfied, we conclude that $\hat{v} \in \hat{\mathscr{G}}_\alpha^B$. □

With this result we can finally extend Sublemmas 8 and 9 of Section 3.1 to B-states. Since A is infinite, $\mathscr{G}_\alpha^A$ is nonempty for each $\alpha \subset f$, so $\hat{\mathscr{G}}_\alpha^B$ is also nonempty, forcing B to be infinite. Therefore, the ρ-state $\langle \rho, \{0\}, \emptyset \rangle$ is well-resided, so $\hat{\mathscr{F}}_\lambda^+ = \hat{\mathscr{M}}_\rho$. Also, since the only well-visited ρ-state is well-resided, the guess $k_\rho = -1$ is correct.

Lemmas 17 and 18 together show that $\mathscr{G}^A = \hat{\mathscr{G}}^B$. Lemma 15 shows that $C \not\leqslant_T B$. Along with Lemmas 13 and 14 and Theorem 4, this completes the proof of Theorem 1. □

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