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On the Classification of Inductive Limits of Sequences of Semisimple Finite-Dimensional Algebras

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The problem of classifying C^*-algebras which are inductive limits of sequences of finite-dimensional C^*-algebras has been considered, in special cases, by several authors. Glimm in [7] and Dixmier in [6] gave a complete description of the various possibilities arising when the finite-dimensional C^*-algebras are restricted to be simple. In this case the inductive limit C^*-algebras are also simple. In [2], Bratteli showed that there are simple inductive limit C^*-algebras which do not arise in this way. The present paper is the result of an attempt to classify these algebras.

As well as leading to concrete nonisomorphism theorems, and, perhaps surprisingly, isomorphism theorems for simple C^*-algebras, the method developed in this paper makes possible a simplification and extension of the work of Behnke and Leptin in [4] and [5], work which in particular gave an enumeration of those separable C^*-algebras with spectrum consisting of two points. Behnke and Leptin did not show that such a C^*-algebra is an inductive limit of finite-dimensional C^*-algebras, but this follows easily from their preliminary work (viz., Lemma 1 of [4]).

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Bratteli showed in Theorem 2.7 of [2] that if a C^*-algebra has a dense subalgebra which is the union of an increasing sequence of finite-dimensional sub-C^*-algebras then this dense subalgebra is unique as an involutive algebra, and determines the isomorphism type of the C^*-algebra. (This is also implicit in [6] and [7] for the cases studied there.)

The classification problem is thus reduced to that for involutive algebras which are inductive limits of sequences of finite-dimensional C^*-algebras.

It is not difficult to see that it is enough to consider algebra isomorphisms. (A proof is outlined in an appendix.) The problem is, then, that of classifying the algebras described in the title (with complex scalars).

The approach to be described below is valid with any algebraically closed field as the scalars, and this seems to be the natural context in which to proceed. It turns out that the classification does not depend on the field; it is combinatorial in nature. (If the field were not algebraically closed, or if one considered algebras of matrices over a ring which is not a field, this would not necessarily be the case.)

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The invariant that seems to be appropriate for the problem described in the title was introduced implicitly by Dixmier in [6], but must now be approached in a more abstract manner, even if the scalars are complex. (It happens that for the algebra introduced by Bratteli, defined by the diagram on [2, p. 213], the invariant can be defined as in [6], using a unique normalized trace. This is not possible, however, for the algebra defined by a slightly different diagram—see 6.5 below—, although this algebra also has a unique normalized trace.)

Briefly, the invariant that we shall consider is the ring invariant consisting of the set of equivalence classes of idempotents, together with addition where defined. Here, equivalence of idempotents e and f in a given ring means (as is customary; see [8, p. 22]) the existence of elements x and y with $xy = e$, $yx = f$. This relation will be denoted by $e \sim f$. It is easily verified to be reflexive, symmetric and transitive, and, moreover, additive; that is, if $e_1 \sim e_2$, $f_1 \sim f_2$ and e_i is orthogonal to f_i ($e_i f_i = f_i e_i = 0$), then $e_1 + f_1 \sim e_2 + f_2$. The operation of addition of orthogonal idempotents therefore induces a partially defined binary relation in the set of equivalence classes, which we shall call addition.

The invariant thus defined, which is what may be called an abelian local semigroup, proves to be useful in classifying the algebras described in the title (over a fixed algebraically closed field), for three reasons:

(i) The invariant is complete. (Hence, the classification is independent of the field.)

(ii) Those local semigroups which appear as values of the invariant may be characterized in a fairly simple way (see 5.5 below).

(iii) Study of the structure of the invariant in various cases yields certain nonisomorphism theorems (see 6.4, 6.5, and 6.6) and also isomorphism theorems (see 6.2 and 6.6) which it does not seem clear how to obtain by other means.

In analogy with [6], we shall refer to this invariant as the range of the dimension on the ring (the dimension, naturally, being the map of idempotents into equivalence classes).

4. Completeness of the Invariant

4.1 Lemma. *Let A be the inductive limit of a sequence of semisimple finite-dimensional algebras, and let B and C be isomorphic semisimple finite-dimensional subalgebras of A. Let φ be an isomorphism from B onto C which preserves dimension, that is, takes each idempotent into an idempotent in the same equivalence class (with respect to A). Then φ extends to an automorphism of A which preserves dimension.*

Proof. Let $\{e_1, ..., e_p\}$ be a maximal set of orthogonal minimal idempotents of B. We shall first construct x and y in A such that

$$xy = \sum e_i, \qquad yx = \varphi \sum e_i,$$
$$\varphi b = ybx, \qquad b \in B.$$

For this it is enough to consider the case that B and C are simple.

By hypothesis, there exist x_0 and y_0 in A such that

$$x_0 y_0 = e_1, \qquad y_0 x_0 = \varphi e_1.$$

There exist $e_{21}, ..., e_{p1}$ and $e_{12}, ..., e_{1p}$ in B such that $e_{21}e_{12} = e_2, ..., e_{p1}e_{1p} = e_p$, $e_{12}e_{21} = \cdots = e_{1p}e_{p1} = e_1$. Set

$$\sum_{i=1}^{p} e_{i1} x_0 \varphi e_{1i} = x, \qquad \sum_{i=1}^{p} (\varphi e_{i1}) y_0 e_{1i} = y,$$

where $e_{11} = e_1$. It is easily verified that x and y satisfy the conditions in the preceding paragraph.

To prove the lemma, we may assume that A has a unit (adjoining one if necessary). Then the hypothesis on A implies that there exist x' and y' in A such that

$$x'y' = 1 - \sum e_i, \qquad y'x' = 1 - \varphi \sum e_i.$$

(x' and y' may be chosen in any semisimple finite-dimensional subalgebra of A containing 1 and x and y.) The elements $x + x'$ and $y + y'$ are invertible elements of T, $x + x'$ is the inverse of $y + y'$, and φ is the restriction to B of the similarity defined by $y + y'$.

4.2 LEMMA. *Let A and A' be algebras, and let A_1' be a semisimple finite-dimensional subalgebra of A'. Denote by d (resp. d') the dimension on A (resp. A'), that is, the map of idempotents into equivalence classes (see 3), and denote by E (resp. E', E_1') the set of idempotents in A (resp. A', A_1'). Suppose that $d'(E_1')$ is isomorphic, as a local semigroup, to a sub local semigroup of $d(E)$. Then there exists a subalgebra A_1 of A and an isomorphism φ from A_1 onto A_1' such that*

$$d' \circ \varphi = d \mid A_1 .$$

Proof. Identify $d'(E_1')$ with a sub local semigroup of $d(E)$. Let $\{e_1',..., e_p'\}$ be a maximal set of orthogonal minimal idempotents in A_1'. Then there exist orthogonal idempotents $e_1 ,..., e_p$ in A such that

$$d(e_1) = d'(e_1'),..., d(e_p) = d'(e_p').$$

Hence, if the sum of certain of the idempotents $e_1',..., e_p'$ is the unit of a minimal two-sided ideal of A', then the corresponding e's are equivalent, and are therefore contained in a subalgebra of A isomorphic to this ideal. The sum of such subalgebras of A, one for each minimal two-sided ideal of A', is a direct sum, and this algebra, together with a corresponding direct sum of isomorphisms, satisfies the requirements of the lemma.

4.3 THEOREM. (cf. [6, 2.14]). *Let A and A' be inductive limits of sequences of semisimple finite-dimensional algebras. Suppose that the range of the dimension on A, that is, the set of equivalence classes of idempotents together with addition where defined (see 3), is isomorphic, as a local semigroup, to the range of the dimension on A'. Then A is isomorphic to A'.*

Proof. There exists an increasing sequence $A_1 \subset A_2 \subset \cdots$ of semisimple finite-dimensional subalgebras of A with union A, and an analogous sequence $A_1' \subset A_2' \subset \cdots$ in A'. We shall construct sequences $m_1 < m_2 < \cdots$ and $n_1 < n_2 < \cdots$, subalgebras $B_1 , B_2 ,...$ of A such that

$$A_{m_1} \subset B_1 \subset A_{m_2} \subset B_2 \subset \cdots,$$

and isomorphisms ψ_i from B_i onto A'_{n_i} , $i = 1, 2,...$, such that ψ_{i+1} extends ψ_i . The ψ's then have a unique common extension to A, an isomorphism from A onto B.

Denote by E (resp. E', E_i , E_i') the set of idempotents of A (resp. A', A_i , A_i'), and by d (resp. d') the dimension on A (resp. A'), that is, the map of idempotents into equivalence classes. By hypothesis, $d(E)$ and $d'(E')$ may be identified as local semigroups.

In particular, $d(E_1)$ is contained in $d'(E')$, and hence by 4.2 there exists a subalgebra B_1' of A' and an isomorphism φ_1 of A_1 onto B_1' such that

$$d' \circ \varphi_1 = d \mid A_1 .$$

Choose $n_1 \geqslant 1$ such that A'_{n_1} contains B_1'. Since $d'(E'_{n_1})$ is contained in $d(E)$, by 4.2 there exists a subalgebra B_1 of A and an isomorphism ψ_1 of B_1 onto A'_{n_1} such that

$$d' \circ \psi_1 = d \mid B_1 .$$

Consider the map $\psi_1^{-1}\varphi_1$; it is an isomorphism of A_1 onto a subalgebra of A, and by the dimension-preserving properties of φ_1 and ψ_1, it preserves the dimension in A:

$$d \circ \psi_1^{-1}\varphi_1 = d \mid A_1 .$$

Hence, by 4.1, $\psi_1^{-1}\varphi_1$ is the restriction to A_1 of a dimension-preserving automorphism of A, say ρ. Replacing B_1 by $\rho^{-1}B_1$ and ψ_1 by $\psi_1\rho$, we may suppose that $\psi_1^{-1}\varphi_1$ is the identity on A_1; that is, that B_1 contains A_1 and ψ_1 extends φ_1. Thus, with $m_1 = 1$, we have

$$A_{m_1} \subset B_1 ,$$

and ψ_1 is an isomorphism from B_1 onto A'_{n_1} such that

$$d' \circ \psi_1 = d \mid B_1 .$$

Next, choose $m_2 > m_1$ such that A_{m_2} contains B_1. Since $d(E_{m_2})$ is contained in $d'(E')$, by 4.2 there exists a subalgebra B_2' of A' and an isomorphism φ_2 of A_{m_2} onto B_2' such that

$$d' \circ \varphi_2 = d \mid A_{m_2} .$$

As in the preceding paragraph, considering the map $\varphi_2\psi_1^{-1}$, and replacing φ_2 by $\sigma^{-1}\varphi_2$ where σ is an extension of $\varphi_2\psi_1^{-1}$ to a dimension-preserving automorphism of A' (4.1), we may suppose that B_2' contains A'_{n_1} and φ_2 extends ψ_1.

Choose $n_2 > n_1$ such that A'_{n_2} contains B_2'. Again by 4.2, there exists a subalgebra B_2 of A and an isomorphism ψ_2 from B_2 onto A'_{n_2} such that

$$d' \circ \psi_2 = d \mid B_2 .$$

Again using 4.1, we may choose B_2 and ψ_2 so that B_2 contains A_{m_2} and ψ_2 extends φ_2. Since φ_2 extends ψ_1, so also does ψ_2.

Summarizing, we have

$$A_{m_1} \subset B_1 \subset A_{m_2} \subset B_2 ,$$

and we have isomorphisms

$$\psi_1 : B_1 \to A'_{n_1} , \qquad \psi_2 : B_2 \to A'_{n_2} ,$$

with ψ_2 an extension of ψ_1. Since, also, ψ_2 is dimension-preserving (i.e., $d' \circ \psi_2 = d \mid B_2$), we may continue the construction.

4.4. *Remark.* Using 4.3, it is possible to give a proof of the result of Bratteli referred to in 2 (Theorem 2.7 of [2]). By Lemmas 1.6 and 1.8 of [7], every projection in a C^*-algebra with a dense locally finite-dimensional sub involutive algebra is equivalent to a projection in the dense subalgebra. It is known that every idempotent in a C^*-algebra is equivalent to a projection (see [8, p. 34]). This shows that the range of the dimension is the same on the C^*-algebra and on the dense subalgebra.

One consequence of this argument is that if two C^*-algebras in the class considered are isomorphic as rings they are isomorphic as algebras. This fact is trivial for algebras with unit, but, since a ring isomorphism is not obviously continuous, it does not seem to be trivial otherwise.

5. General Structure of the Invariant

5.1 Theorem. *Let A be the inductive limit of a sequence of semisimple finite-dimensional algebras. The range of the dimension on A, that is, the local semigroup of equivalence classes of idempotents in A (see 3), is isomorphic to a generating upward directed hereditary subset of the positive cone of a countable ordered abelian group which is the inductive limit of a sequence of finitely generated ordered abelian groups with simplicial positive cones.*

Proof. If A is finite-dimensional, the conclusion is clear.

In the general case, suppose that A is the inductive limit of the sequence

$$A_1 \to A_2 \to \cdots$$

of semisimple finite-dimensional algebras. Denote by $D(A_i)$ the range of the dimension on A_i; there is a canonically induced sequence

$$D(A_1) \to D(A_2) \to \cdots.$$

For each i there exists an embedding

$$D(A_i) \to G_i$$

of $D(A_i)$ as an upward directed hereditary subset generating the positive cone of an ordered abelian group G_i with simplicial positive cone, and the sequence $D(A_1) \to D(A_2) \to \cdots$ determines a sequence

$$G_1 \to G_2 \to \cdots.$$

We shall show that $D(A)$ may be embedded as an upward directed hereditary subset generating the positive cone of the inductive limit of the sequence of ordered groups $G_1 \to G_2 \to \cdots$. Clearly it is enough to show that $D(A)$ is the inductive limit of the sequence $D(A_1) \to D(A_2) \to \cdots$. This follows from the fact that if two idempotents of A are equivalent in A then they are the images of two equivalent idempotents in some A_i.

5.2 *Remark.* Any countable torsion-free abelian group can be made into an ordered group which is the inductive limit of a sequence of finitely generated ordered abelian groups with simplicial positive cones. The preceding theorem raises the question of classifying such ordered groups, and the upward directed hereditary subsets of their positive cones. The following theorem, together with 4.3, shows that it is precisely this question which concerns this paper.

5.3 LEMMA. *Let $G_1 \to G_2$ be a morphism of finitely generated ordered abelian groups with simplicial positive cones. Let D_1 be a singly generated hereditary subset of G_1^+, and let D_2 be a singly generated hereditary subset of G_2^+ containing the image of D_1. Then there exists a morphism $A_1 \to A_2$ of semisimple finite-dimensional algebras such that the induced morphism of the ranges of the dimensions of A_1 and A_2 is isomorphic to $D_1 \to D_2$.*

Proof. The invariants of either kind of morphism are easily visualized, and seen to be the same. They can be described by a column vector x, a matrix P multiplying x, and a column vector y, all with entries in $\mathbf{Z}^+$, such that the entries of y are greater than those of Px. The entries of x and the corresponding columns of P are determined up to a permutation, as are the entries of y and the corresponding rows of P.

5.4 LEMMA. *Let A_1 and A_2 be semisimple finite-dimensional algebras, and denote by $D(A_1)$ (resp. $D(A_2)$) the range of the dimension on A_1 (resp. A_2), an abelian local semigroup (see 3). Let $D(A_1) \to D(A_2)$ be a morphism. Then this morphism is induced by a morphism $A_1 \to A_2$.*

Proof. This is seen in the same way as 5.3.

5.5 THEOREM. *Let G be a countable ordered abelian group which is the inductive limit of a sequence of finitely generated ordered abelian groups with simplicial positive cones, and let D be an upward directed hereditary subset of G^+. Then D, with addition where defined, is isomorphic to the range of the dimension on an algebra which is the inductive limit of a sequence of semisimple finite-dimensional subalgebras. (By 4.3, such an algebra is unique.) We shall call such an ordered group a dimension group.*

Proof. Suppose that G is the inductive limit, as an ordered group, of the sequence

$$G_1 \to G_2 \to \cdots$$

where each G_i is a finitely generated ordered abelian group with simplicial positive cone (in other words, an ordered group isomorphic to the direct sum $\mathbf{Z}^n$ for some n), and the morphisms are positive (but not necessarily injective!).

Since D is an upward directed hereditary subset, and therefore an increasing union of singly generated hereditary subsets, we may choose a singly generated hereditary subset D_i of G_i^+ for each $i = 1, 2, \ldots$ such that D_{i+1} contains the image of D_i and such that the union of the images of the D_i in G is equal to D. By 5.3 there exist morphisms $A_1 \to A_2$, $A_2' \to A_3$, $A_3' \to A_4$, ... of semisimple finite-dimensional algebras such that the induced morphisms $D(A_1) \to D(A_2)$, $D(A_2') \to D(A_3)$, $D(A_3') \to D(A_4)$,... are isomorphic to $D_1 \to D_2$, $D_2 \to D_3$, $D_3 \to D_4$,... . These isomorphisms in particular determine isomorphisms $D(A_2) \to D(A_2')$, $D(A_3) \to D(A_3')$,..., which by 5.4 are induced by morphisms $A_2 \to A_2'$, $A_3 \to A_3'$,... . Consider the composed sequence

$$A_1 \to A_2 \to A_2' \to A_3 \to A_3' \to A_4 \to \cdots;$$

the subsequence

$$A_1 \to A_2 \to A_3 \to A_4 \to \cdots$$

induces a sequence

$$D(A_1) \to D(A_2) \to D(A_3) \to D(A_4) \to \cdots$$

isomorphic to

$$D_1 \to D_2 \to D_3 \to D_4 \to \cdots.$$

Denote by A the inductive limit of the sequence

$$A_1 \to A_2 \to \cdots.$$

As remarked in the proof of 5.1, $D(A)$ is the inductive limit of the induced sequence

$$D(A_1) \to D(A_2) \to \cdots.$$

Since this by construction is isomorphic to the sequence

$$D_1 \to D_2 \to \cdots,$$

we need only show that the inductive limit of this sequence is D. This follows immediately from the fact that the inductive limit of the sequence

$$G_1 \to G_2 \to \cdots$$

is G.

5.6 THEOREM. *Let G be a dimension group, that is, a countable ordered abelian group with the property stipulated in* 5.1 *and* 5.5. *Let D be an upward* *directed hereditary subset of* G^+. *Then any morphism from* D *into a group extends* (*uniquely*) *to the subgroup of* G *generated by* D.

Proof. Choose $G_1 \to G_2 \to \cdots$ with inductive limit G, and $D_1 \to D_2 \to \cdots$ with inductive limit D as in the first two sentences of the proof of 5.5.

Let $D \to G'$ be a morphism of D into a group G'. We must show that if two finite families in D have the same sum in G, then their images have the same sum in G'. It is enough to show that whenever two such families are represented in some D_i, to have the same sum in G_i, then their images have the same sum in G'. This follows from the fact that D_i contains a basis for the subgroup of G_i that it generates (so that a morphism of D_i can be extended to this subgroup).

5.7 Corollary. *In* 5.1, *the embedding is unique.*

5.8 *Remark.* It is not difficult to see that the invariant we have chosen also generalizes the topological invariant used in unpublished work of Singer and of Smith and Smith (see [1]) to recover the classification of Glimm. If the scalars are complex, then the space of quasi-invertible elements of an algebra in the class we are considering can be suitably topologized (for example, by any C^* algebra norm) so that the range of the dimension on the algebra is isomorphic to a certain subset of the first homotopy group of this space: namely, the set of homotopy classes of simple loops of the form $\{\exp i\lambda e \mid 0 \leqslant \lambda \leqslant 2\pi\}$ with e an idempotent.

6. Special Cases

6.1. *Groups of Rank One*

The dimension group of a simple finite-dimensional algebra is $\mathbf{Z}$, and is therefore of rank one—any two elements are dependent. Hence the dimension group of an algebra which is the inductive limit of simple finite-dimensional algebras is of rank one.

Conversely, every dimension group of rank one arises in this way. We shall prove this by showing that a dimension group of rank one is the inductive limit of a sequence of singly generated dimension groups. The assertion then follows by the proof of 5.5. Alternatively, the resulting list of dimension groups of rank one can be matched with the list of examples given in [6].

A torsion-free abelian group of rank one is isomorphic to a subgroup of $\mathbf{Q}$. Any cone in $\mathbf{Q}$ is contained in $\mathbf{Q}^+$, after multiplication by -1 if necessary. Suppose, then, that G is a subgroup of $\mathbf{Q}$ and that G has a dimension group ordering such that G^+ is contained in $\mathbf{Q}^+$. To show that G is the inductive limit of a sequence of singly generated dimension groups, we must show that

G^+ is the union of an increasing sequence of singly generated subcones. Given $g_1, \ldots, g_n$ in G^+, then, we must find $g \in G^+$ dividing each of $g_1, \ldots, g_n$. For g choose the greatest common divisor of $g_1, \ldots, g_n$ in $\mathbf{Q}^+$; then g is in the subgroup generated by $g_1, \ldots, g_n$ and in particular belongs to G. There exist a finitely generated ordered abelian group G' with simplicial positive cone, a morphism $G' \to G$, and elements g', g_1' in G' mapping into g, g_1 respectively, such that $g_1' \in G'^+$ and g' divides g_1'. Since G' is isomorphic to a direct sum of copies of $\mathbf{Z}$, g' lies in G'^+; hence, g belongs to G^+. This argument in fact shows that $G^+ = G \cap \mathbf{Q}^+$, so that G is a sub ordered group of $\mathbf{Q}$.

As remarked on [6, p. 114], a subgroup of $\mathbf{Q}$ containing a fixed element of $\mathbf{Q}$ is determined by the set of integers which divide that element (in the subgroup). Hence it can be seen that for any automorphism of the subgroup there exist two nonzero integers m and n dividing the subgroup such that the automorphism consists of multiplying by mn^{-1}.

A hereditary subset of the positive cone of a subgroup of $\mathbf{Q}$ may or may not be singly generated, and may or may not be proper. If there exists one which is proper and not singly generated, there exists a continuum of such (by definition of continuum!). If two hereditary subsets are isomorphic, as local semigroups, then this is due to an automorphism of the subgroup, whence by the previous paragraph, both are integral multiples of a third.

6.2. *Ordered Groups "Locally" of Rank One*

There are many types of dimension group cones in a countable torsion-free abelian group not of rank one. We shall describe here a class of cones including those arising from the C^*-algebras studied in [3], [4], and [5]. The definition and classification proceed most smoothly in a free group, which is general enough for our present purpose.

The characteristic property of a cone of the class we shall consider seems to be easiest to express in terms of the existence of a basis (i.e., an independent set of generators), although the basis is not determined uniquely by the cone unless the cone is simplicial. There should exist a basis, and a (partial) ordering of this basis, such that an element of the group is in the cone if its maximal nonzero coordinates are positive. The order of the coordinates refers to the order of the corresponding basis elements. The basis elements, in particular, lie in the cone.

As an illustration we shall describe completely the cones of this kind, and their automorphisms, when the group is of rank two (and free), thus recovering the results of [4]. There are, up to permutations, only two orders on a set of two elements. If the elements are not comparable, then the ordered group is isomorphic to $\mathbf{Z}^2$ and there is only one nontrivial automorphism. If the two elements are comparable, the cone is the smallest cone in $\mathbf{Z}^2$ containing $\mathbf{Z}_+{}^2$ and invariant under the automorphism defined by the inverse of the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

All automorphisms are easily seen to be powers of this one. Hence two distinct upward directed hereditary subsets are isomorphic only if they are singly generated and their generators are related by a power of $\left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. An upward directed hereditary subset which is not singly generated is invariant under all automorphisms, and is described by its largest second coordinate.

A similar description is possible when the rank is unrestricted. The group can be represented as $\bigoplus_{i \in I} \mathbf{Z}_i$, where each $\mathbf{Z}_i$ is isomorphic to $\mathbf{Z}$ and I is the ordered basis, in such a way that the positive cone is the smallest cone containing $\bigoplus \mathbf{Z}_i^+$ and invariant under certain group automorphisms,—those determined by $I \times I$ matrices with 1 in each diagonal position and finitely many integers underneath. Every automorphism of the ordered group is the product (in a unique way) of one of these automorphisms and an automorphism which permutes the coordinates. The most notable phenomenon appearing when the rank is not finite is a restriction on the number of generating hereditary subsets. If the ordered basis I has no maximal element, then no proper hereditary subset of the positive cone can generate the group. Consequently, two algebras of the class considered in this paper, with the same dimension group determined as above by an ordered set I without maximal elements, are isomorphic.

The preceding paragraph yields a natural generalization of the classification results in [4] and [5]. It should be remarked that we have not eliminated the task of showing that a separable C^*-algebra with finite spectrum has a dense locally finite-dimensional sub involutive algebra (see [5, Section 2]).

The fact that an ordered group of the kind considered, if it has finite rank, arises from a C^*-algebra with finite spectrum can be established by constructing examples (as in [5]) which exhaust all the possibilities for the group. This may also be deduced by an analysis of the structure of the lattice of faces of the positive cone of the group, which is the same as that of the lattice of ideals of the algebra.

6.3. *Groups of Rank Two. General Remarks*

Any countable torsion-free abelian group of rank two is isomorphic to a subgroup of $\mathbf{Q}^2$. A subgroup of $\mathbf{Q}^2$ containing a fixed pair of independent elements is determined by the set of 2×2 integral matrices of rank two which divide these elements (in the subgroup). Since this set of matrices contains $SL(2, \mathbf{Z})$ if the two fixed elements are $(1, 0)$ and $(0, 1)$, it is not commutative, and a description of the possibilities in general seems difficult. On the other hand, any set of integral matrices of rank two determines a group, the smallest subgroup of $\mathbf{Q}^2$ containing $\mathbf{Z}^2$ in which $\mathbf{Z}^2$ is divisible by each of these matrices, and some progress can be made if this set of matrices is commutative. Let us consider the groups defined in this way by the powers of a single matrix.

It seems to be natural, in considering a group defined as above by a single matrix, i.e., the smallest subgroup of $\mathbf{Q}^2$ containing $\mathbf{Z}^2$ and divisible by a given integral matrix of rank two, to consider the order defined by the smallest cone containing $\mathbf{Z}_+^2$ and divisible by this matrix. To ensure that this group be a dimensional group, we shall assume that the entries of the matrix are positive.

If either off-diagonal entry of the matrix is zero, or if both diagonal entries are the same, then the ordered group is one of those considered in 6.2, if the group is free. Whether the group is free or not, its positive cone has a proper nonzero face (possibly two), and so any corresponding algebra has a proper nonzero ideal (possibly two). We shall not at this time consider this case further.

If neither off-diagonal entry is zero, or if the diagonal entries are distinct, then the positive cone is again a half-space, but now the half-space is open (except for zero); the cone has no proper nonzero faces. Such a matrix has two distinct real eigenvalues, and it is not difficult to see that the eigenspace corresponding to the eigenvalue closer to zero is the line supporting the positive cone.

We shall conclude this paper with a calculation of the automorphisms of the ordered group determined as above by the matrix

$$\begin{bmatrix} 1 & 1 \\ n & 1 \end{bmatrix},$$

in the cases $n = 2$, $n = 4$, and $n = 3$. (The positive cone is

$$\bigcup_{r=1,2,\dots} \begin{bmatrix} 1 & 1 \\ n & 1 \end{bmatrix}^{-r} \mathbf{Z}_+^2.)$$

6.4. *The Dimension Group Determined by* $\left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right]$

The matrix $\left[\begin{smallmatrix} 1 & 1 \\ 2 & 1 \end{smallmatrix}\right]$ was used by Bratteli on [2, p. 213] to construct a simple C^*-algebra with unit which has a dense locally finite-dimensional sub involutive algebra but which has no simple finite-dimensional subalgebra with the same unit. What we shall show implies that if the first row of numbers in the diagram on [2, p. 213] is changed, then unless the new first row (and hence every new row) is one of the old rows the new C^*-algebra is of a different isomorphism type.

What we shall show is that the only automorphisms of the dimension group which is determined by $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ (as described in the last paragraph of 6.3) are the powers of this matrix.

The eigenvalues of $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ are $1 \pm \sqrt{2}$, with corresponding eigenvectors $(1, \pm\sqrt{2})$. Hence the dimension group determined by $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$, which is as a group equal to $\mathbf{Z}^2$ since $\det\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right] = -1$, has as nonzero positive elements those in the open half-space of $\mathbf{R}^2$ containing $(1, \sqrt{2})$ with boundary the subspace spanned by $(1, -\sqrt{2})$. Since this subspace intersects the group only at zero, the ordered group is isomorphic, via the quotient map of $\mathbf{R}^2$ by this subspace, to the subgroup $\{p + q\sqrt{2} \mid p, q \in \mathbf{Z}\}$ of $\mathbf{R}$ with the relative order. Order automorphisms of a subgroup of $\mathbf{R}$ are continuous and hence are multiplications by nonzero positive elements of $\mathbf{R}$. If multiplication by $x \in \mathbf{R}$ is an automorphism of the subgroup $\{p + q\sqrt{2} \mid p, q \in \mathbf{Z}\}$ then both x and x^{-1} are in this subgroup. If $p + q\sqrt{2}$ and also $(p + q\sqrt{2})^{-1} = (p - q\sqrt{2})(p^2 - 2q^2)$ are in this subgroup, then we deduce first that p and q must be relatively prime, and hence that $p^2 - 2q^2 = \pm 1$. It follows from the theory of Pell's equation that $(p + q\sqrt{2})^2$ must be some power of $(1 \pm \sqrt{2})^2$. (The author is indebted to Noriko Yui for drawing this to his attention. A suitable reference is [9, p. 54].) Hence $\pm(p + q\sqrt{2})$ is a power of $1 \pm \sqrt{2}$. This happens to be the same as a power of $1 + \sqrt{2}$, which is the eigenvalue of $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ farther from zero, and therefore also the automorphism induced by $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ of the quotient of $\mathbf{R}^2$ modulo the subspace generated by $(1, -\sqrt{2})$. We have shown that an arbitrary automorphism of the ordered group determined by $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ induces in a faithful representation of this group the same automorphism as some power of $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$.

6.5. *The Dimension Group Determined by* $\left[\begin{smallmatrix}1&1\\4&1\end{smallmatrix}\right]$

We shall show that the only automorphisms of the dimension group determined by the matrix $\left[\begin{smallmatrix}1&1\\2&1\end{smallmatrix}\right]$ (as described in the last paragraph of 6.3) are the powers of the matrix $\left[\begin{smallmatrix}1&1\\4&1\end{smallmatrix}\right]$.

The eigenvalues of the matrix $\left[\begin{smallmatrix}1&1\\4&1\end{smallmatrix}\right]$ are 1 ± 2, with corresponding eigenvectors $(1, \pm 2)$. Hence the dimension group is the subgroup of $\mathbf{Q}^2$ generated by $(0, 1)$ together with the rank one subgroup divisible by 3 generated by $(1, 2)$, and the nonzero elements of the positive cone are those in the open half-space of $\mathbf{Q}^2$ containing $(1, 2)$ with boundary the subspace spanned by $(1, -2)$. It is seen that there is only one nonzero subgroup of the dimension group which is divisible by 3. Therefore this must be left invariant by any group automorphism. Moreover, an ordered group automorphism must leave invariant the intersection of the group with the boundary of the positive cone, that is, the subgroup generated by $(1, -2)$. Because of the divisibility properties of the two rank one subgroups that we have shown to be left invariant by an order automorphism, namely, the maximal rank one subgroups containing $(1, \pm 2)$, the corresponding eigenvalues of such an automorphism must be, up to sign, powers of 1 ± 2. Hence, to complete the proof of the assertion, it is enough to show that the pair of eigenvalues -1, 1 is not admissible. The corresponding automorphism of $\mathbf{Q}^2$ takes $(1, 0)$ into $(0, \frac{1}{2})$, which is easily seen to lie outside the dimension group.

We remark that the dimension group considered above is not totally ordered. This means that in the simple C^*-algebra associated with an upward directed generating hereditary subset of the positive cone of this group, comparability of projections (in the sense of Murray and von Neumann) fails. This answers a question raised by J. Dixmier at the Symposium in C^*-algebras at Bâton Rouge, Louisiana in March, 1967.

Analogously, the fact that the ordered group studied in 6.4 is totally ordered implies that projections in an associated C^*-algebra such as that considered by Bratteli on [2, p. 213] are comparable.

6.6. *The Dimension Group Determined by* $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$

We shall show that there is an automorphism of the dimension group determined by the matrix $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$ (as described in the last paragraph of 6.3) which is not a power of this matrix. Namely, we shall show that this group is divisible by 2.

We have

$$\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}^2 = 2\begin{bmatrix}2 & 1\\ 3 & 2\end{bmatrix}.$$

The matrix $\left[\begin{smallmatrix}2 & 1\\ 3 & 2\end{smallmatrix}\right]$ is permutable with $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$ and therefore leaves the dimension group invariant. $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]^2$ maps the dimension group onto itself, therefore so does 2 (and incidentally also $\left[\begin{smallmatrix}2 & 1\\ 3 & 2\end{smallmatrix}\right]$). This shows that the dimension group is divisible by 2. Moreover, since

$$\begin{bmatrix}1 & 1\\ 3 & 1\end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix}-1 & 1\\ 3 & -1\end{bmatrix},$$

the smallest group containing Z^2 and divisible by 2 is also divisible by $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$. This shows that the elements of the dimension group are just those elements of $\mathbf{Q}^2$ with dyadic rational coordinates.

The eigenvalues of $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$ are $1 \pm \sqrt{3}$, with corresponding eigenvectors $(1, \pm\sqrt{3})$. Hence the automorphisms of the dimension group determined by $\left[\begin{smallmatrix}1 & 1\\ 3 & 1\end{smallmatrix}\right]$ are the 2×2 matrices of rank two with dyadic rational entries which have $(1, -\sqrt{3})$ as an eigenvector and which leave both half-spaces determined by this vector invariant.

7. Appendix

Let A and A' be involutive algebras each of which is the inductive limit of a sequence of finite-dimensional C^*-algebras. Then if A and A' are isomorphic as algebras they are isomorphic as involutive algebras. (This was stated in 2.)

The proof depends on the fact that any idempotent in a finite-dimensional C^*-algebra is similar to a projection. (This is actually true in any C^*-algebra—see 4.5.)

We shall first show that if B is a finite-dimensional C^*-algebra, C is a semisimple subalgebra of B, and S is a system of matrix units for C which refines a system of partially isometric matrix units for $C \cap C^*$, then there exists a sub-C^*-algebra D of B and a similarity φ of C onto D such that φ takes S into a system of partially isometric matrix units for D, and, moreover, fixes each element of $S \cap S^*$ (and hence each element of $C \cap C^*$). We note first that we may suppose that the idempotent elements of S are selfadjoint. (The algebra generated by these idempotents is generated by its intersections with $C \cap C^*$ and with the commutant of $C \cap C^*$ inside C. The latter intersection is generated by its idempotents and hence is similar, within the commutant of $C \cap C^*$ in B, to a sub-C^*-algebra of B. Applying this similarity transformation to all of C leaves fixed each element of $C \cap C^*$, and takes each idempotent element of S into a projection.) Then the unique linear map on C which takes each element of S into its partially isometric part is an isomorphism with the required properties.

Now, by hypothesis A (resp. A') is the union of an increasing sequence $A_1 \subset A_2 \subset \cdots$ (resp. $A_1' \subset A_2' \subset \cdots$) of finite-dimensional sub involutive algebras each of which is a C^*-algebra. Denote by ψ an algebra isomorphism of A onto A'. Choose $k_1 = 1, 2,\ldots$ such that $\psi A_{k_1} \supset A_1'$, and $m_2 > 1$ such that $A'_{m_2} \supset \psi A_{k_1}$. By the preceding paragraph there exist a sub involutive algebra B_1 of A' such that $A'_{m_2} \supset B \supset A'$, and a similarity φ_1 of ψA_{k_1} onto B_1, inside A'_{m_2}, such that $\varphi_1\psi \mid A_{k_1}$ is an involutive algebra isomorphism of A_{k_1} onto B_1. Choose $k_2 > k_1$ such that $\psi A_{k_2} \supset A'_{m_2}$, and $m_3 > m_2$ such that $A'_{m_3} \supset \psi A_{k_2}$. Then by the preceding paragraph, there exist a sub involutive algebra B_2 of A' such that $A'_{m_3} \supset B_2 \supset A'_{m_2}$, and a similarity φ_2 of ψA_{k_2} onto B_2, inside A'_{m_3}, such that $\varphi_2\psi \mid A_{k_2}$ is an involutive algebra isomorphism of A_{k_2} onto B_2 extending $\varphi_1\psi \mid A_k$. Proceeding in this way, one constructs increasing sequences $k_1 < k_2 < \cdots$, $B_1 \subset B_2 \subset \cdots$, and $\varphi_1 \subset \varphi_2 \subset \cdots$ such that $\varphi_n\psi \mid A_{k_n}$ is an involutive algebra isomorphism of A_{k_n} onto B_n, and $\cup B_n = A'$. This is equivalent to constructing an involutive algebra isomorphism from A onto A'.

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