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# Wiener Processes

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## INTRODUCTION

By a Wiener process we shall mean a smooth generalized stochastic process with independent increments. We give a precise definition in Section 1. Our choice of the name is motivated by the similarity of these processes to those given by Wiener himself in [*6*]. Our purpose in the present paper is to develop the theory of Wiener processes with a view to applying them to continuously splittable distributions on Hilbert space. These were introduced by us in [*5*], as a special cases of weak distributions in Hilbert space. The latter were introduced by Segal in [*3*] and [*4*]. A definition for splittable distributions is given in Section 2. This section also contains our main technical result; namely: the proof that certain classes of random variables associated with Wiener processes are infinitely divisible. As corollaries we derive similar results for integration in Hilbert space. In Section 3 we analyze the infinitesimal structure of Wiener Processes in a way that makes precise the analog between them and the rather nebulous concept of continuous direct products of probability measure spaces. In Section 4 we develop the theory of stochastic integrals, and as examples introduce the normal, the Poisson, and the Cauchy Wiener processes. We discuss the connection between splittability and stochastic integrals for Poisson Wiener Processes in Section 5 and for normal Wiener processes in Section 6 together with their applications to integration on Hilbert space.

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## 1. Wiener Processes

By a *Wiener process* we shall mean a countably addivitive may $\psi$ from a Boolean $\sigma$-algebra $\mathfrak{M}$ of subsets of a space $M$ to random variables that has independent values on disjoint subsets. Here and throughout this paper, whenever we refer implicitly to a topology on random variables as we have just done in the phrase "countably additive map," we shall always mean the topology of convergence in measure. We shall say that $\psi$ is a *continuous* Wiener process if $\psi(A) = 0$ whenever $\psi(A')$ is either 0 or $\psi(A)$, for all elements $A'$ of $\mathfrak{M}$ contained in $A$.

It is sometimes of technical convenience to form the measure ring of a Wiener process. To do this we let $\mathfrak{N}$ be the family of all elements $A$ of $\mathfrak{M}$ such that $\psi$ vanishes on all elements of $\mathfrak{M}$ contained in $A$. It is obvious that $\mathfrak{N}$ is a $\sigma$-ideal in $\mathfrak{M}$, that $\mathfrak{M}/\mathfrak{N}$ is a Boolean $\sigma$-algebra, and that $\psi$ induces a countably additive function on $\mathfrak{M}/\mathfrak{N}$. Actually, it follows from Lemma 1 below that $\mathfrak{M}/\mathfrak{N}$ is a complete Boolean algebra and $\psi$ is completely additive on it for the same reason that this is so in the more familiar case of the measure ring of a probability space. If $\psi$ is a continuous Wiener process, then the Boolean algebra $\mathfrak{M}/\mathfrak{N}$ has no atoms.

Lemma 1. *Let $\{X(\gamma) : \gamma \in \Gamma\}$ be an indexed family of independent random variables. Suppose $\Sigma X(\gamma_n)$ converges in measure for all non-repeating sequences $\{\gamma_n\}$ from $\Gamma$. Then $X(\gamma) = 0$ for all but countably many $\gamma$ in $\Gamma$.*

*Proof.* Let $X'(\gamma)$ coincide with $X(\gamma)$ when $|X(\gamma)| \leqslant 1$ and vanish otherwise. We apply the Kolmogoroff three-series theorem (see [*2*], p. 307, or [*1*], p. 111) to all nonrepeating sequences $\{\gamma_n\}$ from $\Gamma$. Since $\Sigma X(\gamma_n)$ converges in measure, it follows that

$$\Sigma P[|X(\gamma_n)| > \epsilon] \tag{1}$$

$$\Sigma \text{ Variance } [X'(\gamma_n)] \tag{2}$$

and

$$\Sigma E(X'(\gamma_n)) \tag{3}$$

all converge. Since (1) always converges, $|X(\gamma)| \leqslant \epsilon$ for all but countably many $\gamma$ in $\Gamma$. Since (2) always converges, $X'(\gamma)$ is constant for all but countably many $\gamma$ in $\Gamma$. Since (3) and all its rearrangements converge; that is, since (3) always converges absolutely, $E(X'(\gamma)) = 0$ for all but countably many $\gamma$ in $\Gamma$. Consequently, $X(\gamma)$ vanishes for all but countably many $\gamma$ in $\Gamma$.

## 2. Infinite Divisibility

It is classical that the random variables of a suitably smooth stochastic process with independent increments have infinitely divisible probability distributions (see [*2*], p. 308, or [*1*], p. 420). A probability distribution on a finite-dimensional real vector space is defined to be *infinitely divisible* if for all positive integers $n$ it is the $n$-fold convolution of some probability distribution with itself. A sufficient condition for a finite-dimensional vector-valued random variable to have an infinitely divisible probability distribution is that for all $\epsilon > 0$ it be a constant plus a finite sum of random vectors $Y$ with $P(| Y | > \epsilon) < \epsilon$. (See [*2*], p. 550.) We wish to show that a continuous Wiener process has values with infinitely divisible probability distributions. However, we formulate the following lemma a little more generally than is necessary for this in order to handle related cases of interest.

Lemma 2. *Let $V$ be the set of all random variables on a probability space with values in a real vector space of finite dimension. Denote by $C$ the set of all constants in $V$, and form $V/C$ topologized by convergence in measure. Let $\psi$ be a countably additive map from a complete Boolean algebra $\mathfrak{B}$ to $V/C$ such that $\psi$ has independent values on disjoint elements of $\mathfrak{B}$. If $\mathfrak{B}$ has no atoms, then the values of $\psi$ consist of random vectors with infinitely divisible distributions.*

*Proof.* First we observe that $\psi$ is actually completely additive. The reason for this is the fact if $\mathscr{F}$ is a disjoint family of elements of $\mathfrak{B}$, then, for all but countably many $\beta$ in $\mathscr{F}$, $\psi(\beta)$ is the 0 element of $V/C$ (i.e. the coset of constants). This fact almost follows directly from Lemma 1. The thing that makes lemma 1 rigorously applicable is a standard result about the existence of absolute centering constants (see [*1*], p. 112). This result implies that one may select from each coset $\sigma$ in $V/C$ a representative random vector $v(\sigma)$ such that if a series $\Sigma\sigma_n$ of independent elements of $V/C$ converges $V/C$, then $v(\sigma_n)$ converges in measure.

Denote the minimum and maximum of $\mathfrak{B}$ by 0 and 1 respectively. It will be sufficient to establish that $\psi(1)$ is infinitely divisible. Let $\mathscr{L}$ be a maximal chain in $\mathfrak{B}$. Then $\mathscr{L}$ must contain 0 and 1 also the supremum and infimum of any nonempty subset of $\mathscr{L}$. Consequently $\mathscr{L}$ is a complete chain, and $\psi$ is continuous on $\mathscr{L}$ in the order topology of $\mathscr{L}$. But $\mathscr{L}$, being complete, is compact; and, therefore, $\mathscr{L}$ has a unique uniform structure that induces its topology; viz., the family of all neighborhoods of the diagonal in $\mathscr{L} \times \mathscr{L}$. For any neighborhood $\mathfrak{N}$ of 0 in $V/C$, because of the uniform continuity of $\psi$, we can find a neighborhood $U$ of the diagonal in $\mathscr{L} \times \mathscr{L}$ such that $(\psi)\gamma - \psi(\gamma')$ is in $\mathfrak{N}$ whenever $(\gamma, \gamma')$ is in $U$. Since the chain $\mathscr{L}$ has no gaps because $\mathfrak{B}$ has no atoms, it is an exercise to produce a finite number $0 = \gamma_0 < \gamma_1 < \cdots < \gamma_n = 1$ of points in $\mathscr{L}$ such that $(\gamma_0, \gamma_1)$, $(\gamma_1, \gamma_2),\ldots, (\gamma_{n-1}, \gamma_n)$ are all in $U$. Then the equation $\psi(1) = \Sigma[\psi(\gamma_j) - \psi(\psi_{j-1})]$ establishes the infinite divisibility of $\psi(1)$ according to the sufficient condition for infinite divisibility given above.

By applying this lemma to the measure ring of a Wiener process we get the following corollary.

COROLLARY 1. *The values of a continuous Wiener process have infinitely divisible distributions.*

Lemma 2 is also applicable to continuously splittable distributions on a real Hilbert space $H$ (see [5]). A *distribution* on $H$ is a linear map to random variables from a linear manifold of continuous linear functionals on $H$ that is dense in the dual $H^*$ of $H$. With each projection $P$ on $H$ there is associated a Boolean $\sigma$-algebra $\mathscr{S}(P)$ of sets; namely, $\mathscr{S}(P)$ is the smallest Boolean $\sigma$-algebra with respect to which all the random variables corresponding to linear functionals of the form $\xi \circ P$ with $\xi$ in $H^*$ are measurable. We say that $P$ *splits* the distribution if the domain of the distribution is invariant under $P$ and the Boolean $\sigma$-algebras $\mathscr{S}(P)$ and $\mathscr{S}(I - P)$ are stochastically independent.

DEFINITION. A distribution on $H$ is said to be *continuously splittable* if there exists a complete non-atomic Boolean algebra $\mathfrak{B}$ of projections on $H$ satisfying the following conditions:

(1) Every projection in $\mathfrak{B}$ splits the distribution.

(2) For every linear functional $\xi$ in the domain of the distribution, the map that sends $P$ into the random variable associated with $\xi \circ P$ is countably additive on $\mathfrak{B}$.

We may apply Lemma 2 to the map in (2) above the get the following corollary.

COROLLARY 2. *The random variable associated with a continuous linear functional by a continuously splittable distribution on a real Hilbert space is infinitely divisible.*

The random variable of Corollary 2 is simply a special case of a splittable random variable, which we define as follows. Suppose we have a continuously splittable distribution on a real Hilbert space, $\mathfrak{B}$ being the relevant complete nonatomic Boolean algebra of projections; in this context, we term a random variable $X$ *splittable* if, for each $P$ in $\mathfrak{B}$, we can write $X$ as $X_P + X_{I-P}$ where $X_P$ is $\mathscr{S}(P)$-measurable and $X_{I-P}$ is $\mathscr{S}(I-P)$-measurable. So the following corollary, in fact, includes corollary 2.

COROLLARY 3. *In the context of a continuously splittable distribution on a real Hilbert space, every splittable random variable is infinitely divisible.*

*Proof.* Let $\mathfrak{B}$ be the relevant complete nonatomic Boolean algebra of projections, and let $X$ be the splittable random variable. For $P$ in $\mathfrak{B}$, in the decomposition $X = X_P + X_{I-P}$ given above, the random variable $X_P$ is actually uniquely determined up to an additive constant; for the difference of two versions of $X_P$ would be both $\mathscr{S}(P)$-measurable and $\mathscr{S}(I-P)$-measurable and, therefore, constant. Consequently, we get a well-defined map $\psi$ on $\mathfrak{B}$ by taking $\psi(P)$ to be $X_P$ modulo constants. Lemma 2 can be applied to the map $\psi$ to yield the present corollary as soon as we have verified that $\psi$ is countably additive. Suppose that $\{P_n\}$ is a disjoint sequence of elements of $\mathfrak{B}$ with $P = \Sigma P_n$. We wish to show that $\psi(P) = \Sigma\psi(P_n)$. Actually $\Sigma\psi(P_n)$ converges as we shall demonstrate below by Lemma 3. Consider $\Delta = \psi(P) - \Sigma\psi(P_n)$; it is stochastically independent of all $\mathscr{S}(P_n)$. But for every linear functional $\xi$ in the domain of the distribution, item (2) in the definition of "continuously splittable" says that the random variable associated with $\xi \circ P$ is the sum of the series of random variables associated with $\xi \circ P_n$. It follows that $\mathscr{S}(P)$ is the smallest Boolean $\sigma$-algebra containing all the $\mathscr{S}(P_n)$. Hence $\Delta$ is independent of $\mathscr{S}(P)$. Being also independent of $\mathscr{S}(I-P)$, $\Delta$ must be the coset of constants.

By exploiting the vector-valued feature of the statement of Lemma 2, we can extend corollary 3 to finite sets of spliitable random variables.

COROLLARY 4. *In the context of a continuously splittable distribution on a real Hilbert space, any finite number of splittable random variables have an infinitely divisible joint probability distribution.*

The following lemma was used above to help verify the hypothesis of countable additivity in an application of Lemma 2.

LEMMA 3. *Let $V$ be the set of all real random variables on a probability space. Denote by $C$ the set of all constants in $V$, and form $V/C$ topologized by convergence in measure. Suppose $\psi$ is a finitely additive map from a Boolean algebra $\mathfrak{B}$ to $V/C$ such that $\psi$ has independent values on disjoint elements of $\mathfrak{B}$. If $\{\beta_n\}$ is any disjoint sequence of elements of $\mathfrak{B}$, then $\Sigma\psi(\beta_n)$ converges in $V/C$.*

*Proof.* Suppose that $\{X_n\}$ is a sequence of independent random variables. A sufficient condition for there to exist a sequence $\{c_n\}$ of constants such that $\Sigma(X_n - c_n)$ converges in measure is that there exist a random variable $X$ such that $X - \sum_{i+1}^{n} X_i$ is independent of $X_1, X_2, \ldots, X_n$ for all $n$ (see [*1*], pp. 119, 120). If we take $X_n$ to be a representative of $\psi(\beta_n)$, and $X$ to be a representative of the value of $\psi$ on the maximum of $\mathfrak{B}$, the Lemma follows.

## 3. INFINITESIMAL STRUCTURE

Let $\mathfrak{M}$ be a $\sigma$-algebra of subsets of a space $M$ and let $\psi$ be a continuous Wiener process on $\mathfrak{M}$. For each set $A$ in $\mathfrak{M}$, let $\phi(A, t)$ be the logarithm of the characteristic function of $\psi(A)$. That is,

$$\phi(A, t) = \log(E(\exp(it\psi(A))))$$

Since $\psi(A)$ is infinitely divisible we may apply the Lévy-Khintchine decomposition to it. (See for example [*1*] p. 130.) We obtain $\phi(A, t) = i\gamma(A)t + \int K(\lambda, t)\, d\mu(A, \lambda)$ where $K(\lambda, t)$ is the kernal function $\{\exp(i\lambda t) - 1 - i\lambda t/(1 + \lambda^2)\}(1 + \lambda^2)/\lambda^2$ if $\lambda \neq 0$ and $-\frac{1}{2}t^2$ if $\lambda = 0$, $\mu(A, \cdot)$ is a finite Borel measure on $R$, $\gamma(A)$ is a real number and the integral is taken over $R$. On taking $\beta$ to be a fixed Borel set in $R$ and applying the uniqueness of the Lévy-Khintchine decomposition together with the countable additivity of the process $\psi$ we see that $\mu(\cdot, \beta)$ is a totally finite measure on $\mathfrak{M}$, and that $\gamma$ is a countably additive signed measure on $\mathfrak{M}$.

If $x$ is a rational number or infinity, let $\beta_x$ denote the interval $(-\infty, x]$. Now choose a totally finite measure $\rho$ on $\mathfrak{M}$ so that $|\gamma|$, (the toal variation of $\gamma$), and all the measures $\mu(\cdot, \beta_x)$ are absolutely continuous with regard to it. Let $f(\alpha, x)$ and $\eta(\alpha)$ denote the Radon–Nikodym derivatives of $\mu(\cdot, \beta_x)$ and $\gamma$ repsectively with regard to $\rho$. Since for $x$ and $x'$ rational with $x \leqslant x'$, and any set $A$ in $\mathfrak{M}$ we have $0 \leqslant \mu(A, \beta_x) \leqslant \mu(A, \beta_{x'}) \leqslant \mu(A, \beta_\infty)$, it follows after changing the $f(\alpha, x)$ on a null set of $\mathfrak{M}$ that for each $\alpha$, $f(\alpha, x)$ is nonnegative, monotone increasing and bounded as a function of $x$. For each $\alpha$ in $M$

we may therefore use $f(\alpha, x)$ to construct a Borel measure $\nu(\alpha, \cdot)$ on $R$ so that $f(\alpha, x) = \nu(\alpha, \beta_x)$ for $x$ rational or infinite. Finally let $F(\alpha, t)$ be the function defined by:

$$F(\alpha, t) = i\eta(\alpha)t + \int_R K(\lambda, t)\, d\nu(\alpha, \lambda).$$

We can now state:

PROPOSITION 1. *Let $\mathfrak{M}$ be a $\sigma$-algebra of subsets of a space $M$. Let $\psi$ be a continuous Wiener process on $\mathfrak{M}$. If $\phi(A, t)$ is the logarithm of the characteristic function of $\psi$ and $F(\alpha, t)$ and $\rho$ are as described above then for each $\alpha$ in $M$,* $\exp(F(\alpha, t))$ *is the characteristic function of an infinitely divisible distribution on $R$ and*

$$\phi(A, t) = \int_A F(\alpha, t)\, d\rho(\alpha)$$

To prove the proposition it remains only establish the last equation above. This amounts to showing that

$$\int_A \left(\int_R g(\lambda)\, d\nu(\alpha, \lambda)\right) d\rho(\alpha) = \int_R g(\lambda)\, d\mu(A, \lambda)$$

whenever $A$ is a set in $\mathfrak{M}$ and $g(\lambda) = K(\lambda, t)$ for any $t$. Consequently, it is enough to show that the equation holds when $g(\lambda)$ is any bounded Baire function. It certainly holds if $g(\lambda)$ is the characteristic function of the interval $(-\infty, x]$ with $x$ rational and hence when $g(\lambda)$ is in the linear span $V$ of these characteristic functions. However, the set of bounded Baire functions is the smallest set containing $V$ and closed under bounded pointwise limits. Our result now follows on applying the Lebesgue-dominated convergence theorem.

In a *formal* way proposition 1 may be thought of as decompositing the probability measure space associated with the random variables $\{\psi(A)\}$ into a *continuous* direct product with one factor for each $\alpha$ in $M$, so that any $\psi(A)$ is a continuous direct sum of stochasticly independent random variables with one for each $\alpha$ in $A$. It is important to notice that the random variable over $\alpha$ does not appear and probably *does not exist*. The corresponding characteristic function $\exp(F(\alpha, t))$ does exist however and our proposition says that the characteristic function of $\psi(A)$ is just a continuous product of these functions as $\alpha$ ranges over $A$.

Below we shall need an estimate of $F(\alpha, t)$. Since

$$| K(\lambda, t)| \leqslant At^2 + B,$$

it may be seen from the definition of $F(\alpha, t)$ that $|F(\alpha, t)| \leqslant Q(|t|)$ where $Q(t)$ is a quadratic polynomeal in $t$ whose coefficients are nonnegative functions on $M$ which are integrable with regard to $\rho$.

## 4. Stochastic Integrals

The structure described above lets us discuss stochastic integrals for continuous Wiener processes in a relatively closed form. To this end let $\mathfrak{M}$ be a $\sigma$-algebra of subsets of a space $M$ and let $\psi$ be a continuous Wiener process on $\mathfrak{M}$. Let $f$ denote a function on $M$ which is measurable with regard to $\mathfrak{M}$. When the stochastic integral of $f$ with regard to $d\psi$ exists, we shall write it as $\int f d\psi$. The function $\phi(f, t)$ will denote the logarithm of the corresponding characteristic function. That is $\phi(f, t) = \log(E(\exp(\text{it}\int f d\psi)))$.

First let $f$ be a step function $\lambda_1 S_1 + \cdots + {}_n S_n$ where $S_1, \ldots, S_n$ are the characteristic functions of disjoint sets $A_1, \ldots, A_n$ in $\mathfrak{M}$. Then $\int f d\psi = \lambda_1 \psi(A_1) + \cdots + \lambda_n \psi(A_n)$. A computation shows that $\phi(f, t) = \int F(\alpha, f(\alpha)\, t)\, d\rho(\alpha)$ where $F(\alpha, t)$ and $\rho$ are described in Proposition 1.

Next suppose that $f$ is bounded. Let $f_1, f_2, \ldots,$ be a uniformly bounded sequence of step functions converging to $f$ pointwise. Let $X_n = \int f_n d\psi$. Then $X_1, X_2, \ldots,$ converges in measure to a limit independent of the particular choice of $f_1, f_2, \ldots,$ so that we can write the limit as $\int f d\psi$. To show the convergence of the $X_n$ it is sufficient to show that as $n$ and $m$ go to infinity, the characteristic function of $X_n - X_m$ converges to 1 pointwise in $t$. This amounts to showing that $\int F(\alpha, (f_n(\alpha) - f_m(\alpha)t)\, d\rho(\alpha)$ converges to zero as $m$ and $n$ go to infinity; which in turn follows from the Lebesgue-dominated convergence theorem using the estimate at the end of Section 3.

The independence of $\int f d\psi$ from the particular sequence $\{f_n\}$ used in its definition is clear. Finally we observe that $\phi(f, t) = \lim \phi(f_n, t) = \int F(\alpha, f(\alpha)t)\, d\rho(\alpha)$

Now suppose that $f$ is any measurable function on $M$. For $n = 1, 2, 3, \ldots$, let $g_n(\alpha) = f(\alpha)$ if $n - 1 \leqslant |f(\alpha)| < n$ and let $g_n(\alpha) = 0$ otherwise. Let $f_n = g_1 + \cdots + g_n$. We will call $f$ integrable with regard to $d\psi$ if for each set $A$ in $\mathfrak{M}$, the elements $\int_A f_n\, d\psi$ converge in measure as $n$ goes to infinity. Since the terms of the sequence $\int_A g_1\, d\psi$, $\int_A g_2\, d\psi, \ldots$ are stochasticly independent random variables, a necessary and sufficient condition for convergence, is the convergence of infinite product of the corresponding characteristic functions. (See [*1*], p. 115.) This, in turn, amounts to the convergence for each $t$ of $\int_A F(\alpha, f_n(\alpha)\, t)\, d\rho(\alpha)$ as $n$ goes to infinity. Hence in terms of our definition, $f$ is integrable with regard to $d\psi$ if and only if $F(\alpha, f(\alpha)\, t)$ is integrable with regard to $\rho$ for each $t$. Then we have $\phi(f, t) = \int F(\alpha, f(\alpha)\, t)\, d\rho(\alpha)$.

Let $\psi$ be a continuous Wiener process on the Boolean $\sigma$-algebra $\mathfrak{M}$ of subsets of a space $M$. There are three very striking illustrations of the foregoing theory of stochastic integrals. We shall say that the Wiener process is:

(N) *centered normal*,

(P) *Poisson*,

or

(C) *Cauchy*,

respectively, provided the probability distributions of all the values of $\psi$ are as described. To be specific for $A$ in $\mathfrak{M}$, $\psi(A)$ is distributed, in each case respectively, as follows:

(N) with density $[2\pi\rho(A)]^{-1} \exp[-x^2/2\rho(A)]$;

(P) concentrated on the nonnegative integers with $n$ receiving probability $\exp(-\rho(A))[\rho(A)]^n/n!$;

(C) with density $\{\pi\rho(A)[1 + (x/\rho(A))^2]\}^{-1}$.

In each case, $\rho$ is a finite measure on $M$; in case $N$, $\rho(A)$ = variance of $\psi(A)$; in case $P$, $\rho(A) = E(\psi(A))$ = variance of $\psi(A)$. Then $\phi(A, t)$ the logarithm of the characteristic function of $\psi(A)$ is given, in each case respectively, by:

$$\text{(N)}\quad \phi(A, t) = -\tfrac{1}{2}\rho(A)t^2;$$
$$\text{(P)}\quad \phi(A, t) = \rho(A)(e^{it} - 1);$$
$$\text{(C)}\quad \phi(A, t) = -\rho(A)|\, t\,|.$$

Taking $\rho$ to be nonatomic makes $\psi$ continuous. On taking $\rho$ to be the measure described in proposition 1 we get the structure function $F(\alpha, t)$ to be. (N): $-1/2t^2$; (P): $e^{it} - 1$; (C): $-|\, t\,|$. Since the variable $\alpha$ over $M$ does not appear explicitly we refer to these processes as homogeneous.

Consequently, a necessary and sufficient condition for $f$ to be stochasticly integrable with respect to $d\psi$ is, in each case, respectively:

(N) that $f$ be in $L_2(M, \mathfrak{M}, \rho)$;

(P) that $f$ be $\sigma$-measurable;

(C) that $f$ be in $L_1(M, \mathfrak{M}, \rho)$.

If $\psi$ is the Poisson process (P) and $\psi'$ is the translate defined by $\psi'(A) = \psi(A) + \rho(A)$. Then the structure function $F(\alpha, t)$ for $\psi'$ is just $\exp(it) - 1 - it$. Hence a necessary and sufficient condition that $f$ be integrable with regard to $d\psi'$ is that $f$ be in $L_1(M, \mathfrak{M}, \rho)$ which is quite different from what we get for $d\psi$ itself.

This suggests defining stochastic integrals only up to additive constants. Consequently, returning to the notation of the first part of this section, if $\psi$ is any Wiener process we will say that $f$ has a *renormalized stochastic integral* if there exists constants $c_n$ so that $\int f_n \, d\psi - c_n$ converges in measure. We will write the limit class of random variables as $\int^r f d\psi$. A necessary and sufficient condition for the existance of $\int^r f d\psi$ is the convergence of the product of *absolute values* of the characteristic functions of $\int_M g_n \, d\psi$. (See [1], p. 115.) It is straightforward that this happens if and only if $\mathrm{Re}(F(\alpha, f(\alpha)\, t)$ is integrable with regard to $\rho$ for all $t$ and then settingRe $\phi(f, t) = \mathrm{Re}(\log(\mathrm{E}(\exp(it \int^r f d\psi))))$ we have $\mathrm{Re}\, \phi(f, t) = \int \mathrm{Re}\, F(\alpha, f(\alpha) t) \, d\rho(\alpha)$.

## 5. Stochastic Integrals and Splittability

Let $\psi$ be a continuous Wiener process on the Boolean $\sigma$-algebra $\mathfrak{M}$ of subsets of the space $M$. For each $A$ in $\mathfrak{M}$, let $\mathscr{S}(A)$ be the smallest Boolean $\sigma$-algebra of sets with respect to which all values of $\psi$ on subsets of $A$ are measurable. In this context a random variable $X$ will be called *splittable* if for each $A$ in $\mathfrak{M}$ we can write $X = X_A + X_{M-A}$, where $X_A$ is $\mathscr{S}(A)$-measurable, and $X_{M-A}$ is $\mathscr{S}(M - A)$-measurable. Any stochastic integral or, indeed, any renormalized stochastic integral is splittable. It might be conjectured that every splittable random variable is a renormalized stochastic integral. This conjecture is false in general as the counterexample at the end of this section shows. However, in certain special cases, the conjecture is true. The case *os* a centered normal Wiener process is dealt with in the next section. The case of a Poisson Wiener process is covered in the following proposition. In proving the proposition it is convenient to have the following trivial probabalistic lemma.

LEMMA 4. *Let $\mathscr{S}$ be a Boolean $\sigma$-algebra of events of a probability space, and let $S$ be an event in $\mathscr{S}$. Suppose that $T$ is an event of positive probability that is independent of $\mathscr{S}$. If $X$ is an $\mathscr{S}$-measurable random variable, then its restriction to $S$ is determined by its restriction to $S \cap T$.*

*Proof.* Suppose $X_1$ and $X_2$ are two $\mathscr{S}$-measurable random variables that coincide on $S \cap T$. Let $U$ be the event that $X_1 = X_2$. Then $P(S)\,P(T) = P(S \cap T) = P(U \cap S \cap T) = P(U \cap S)\,P(T)$. Since $P(T) \neq 0$, it follows that $P(S) = P(U \cap S)$. In other words, $X_1$ coincides with $X_2$ on $S$.

PROPOSITION 2. *Consider a continuous Poisson Wiener process. In this context every splittable random variable is a constant plus a stochastic integral.*

*Proof.* Let $\psi$ be a continuous Poisson Wiener process on the Boolean $\sigma$-algebra $\mathfrak{M}$ of subsets of the space $M$. Let $\rho(A) = E(\psi(A))$ for all $A$ in $\mathfrak{M}$. We shall need to form a sequence of partititions of the measure space $(M, \mathfrak{M}, \rho)$ as follows. For $n = 1, 2,...$ choose disjoint sets $M(n, 1), M(n, 2),..., M(n, 2^n)$ in $\mathfrak{M}$ such that $\rho(M(n. i)) = 2^{-n}\rho(M)$ for $i = 1,..., 2^n$. Do this in such a way that the partition

$$\{M(n + 1, 1) ..., M(n + 1, 2^{n+1})\}$$

is a refinement of the partion $\{M(n, 1),..., M(n, 2^n)\}$ for all $n$.

Suppose $X$ is a splittable random variable. Consider the event that $\psi(M) = 0$. On this event $\psi(A) = 0$ for all $A$ in $\mathfrak{M}$. Hence every $\mathscr{S}(M)$-measurable function is constant where $\psi(M) = 0$. We may assume without loss of generality that $X$ vanishes when $\psi(M) = 0$. We are about to demonstrate that any splittable random variable $X$ that vanishes when $\psi(M) = 0$ is determined by its restriction to the event $\psi(M) = 1$. Let $Q_n$ be the event that $\psi(M(n, i) \leqslant 1$ for $i = 1,$ $=,..., 2^n$. To show that $X$ is determined on $Q_n$, we write

$$X = X(1) + X(2) + \cdots + X(2^n)$$

where $X(i)$ is $\mathscr{S}(M(n, i))$-measurable. We may and shall take $X(i)$ to be 0 when $\psi(M) = 0$. Now $X$(i) vanishes when $\psi(M(n, i)) = 0$ and $\psi(M - M(n, i)) = 0$. Hence, by the lemma, $X(i)$ vanishes whenever $\psi(M(n, i)) = 0$. So $X(1)$ coincides with $X$ on the event $T$ that:

$$\psi(M(n, 2)) = \psi(M(n, 3)) = \cdots = \psi(M(n, 2^n)) = 0.$$

Hence $X(1)$ is determined, when $\psi(M(n, 1)) \leqslant 1$ and the event $T$ obtains, by the restriction of $X$ to the event $\psi(M) \leqslant 1$. By the lemma, therefore, $X(1)$ is determined when $\psi(M(n, 1)) \leqslant 1$ by the restriction of $X$ to the event $\psi(M) \leqslant 1$. A similar thing holds for $X(2),..., X(2^n)$.

Hence $X = \Sigma X(i)$ is determined on $Q_n$ by the restriction of $X$ to the event $\psi(M) \leqslant 1$ or, equally well, by its restriction to the event $\psi(M) = 1$, since $X$ vanishes when $\psi(M) = 0$. Our assertion now follows because the union of all the $Q_n$ is the whole probability space.

In view of what we have just observed, to prove the proposition it suffices to find a measurable function $f$ on $M$ such that $\int f d\psi$ coincides with $X$ when $\psi(M) = 1$. A convenient way to do this is by means of a specific realization of the process $\psi$. Our choice of realization is the one given by Wiener and Wintner in [7]. It also resembles the immage of the duality transform given in [3]. For $n = 0, 1, 2, \ldots$, take the measure space $(M^n, \mathfrak{M}^n, \rho^n)$ to be the $n$-fold Cartesian product of the measure space $(M, \mathfrak{M}, \rho)$ with itself with the understanding that $M^0$ consists of a single 0-triple of $\rho^0$ measure 1. Form a probability space $\Omega$ by taking $\Omega = \bigcup_{n=0}^{\infty} M^n$, declaring a subset $S$ of $\Omega$ to be measurable if each $S \cap M^n$ is in $\mathfrak{M}^n$, and defining probabilities by

$$P(S) = \exp(-\rho(M)) \sum_{n=0}^{\infty} \rho^n(S \cap M^n)/n!.$$

If we take, for each $A$ in $\mathfrak{M}$ and $\omega$ in $\Omega$, $[\psi(A)](\omega) =$ (number of coordinates of $\omega$ in $A$), then this $\psi$ is equivalent to the process we started with. The subset $M^n$ of has become the event that $\psi(M) = n$. If $f$ is a measurable function on $M$, what is the restriction of $\int f d\psi$ to the event $\psi(M) = 1$, i.e., to the set $M^1 = M$? The answer is that this restriction is just $f$ itself, because this is obviously so when $f$ has has only values 0 and 1. To make $X = \int f d\psi$, therefore, it suffices to let $f$ be the restriction of $X$ to $M^1 = M$.

COUNTEREXAMPLE. Let $(M, \mathfrak{M}, \rho)$ be a nonatomic measure space with $\rho(M)$ finite. Let $\psi_1$ and $\psi_2$ be independent Poisson Wiener processes on $\mathfrak{M}$ such that

$$\rho(A) = E(\psi_1(A)) = E(\psi_2(A)) \quad \text{for all} \quad A \text{ in } \mathfrak{M}.$$

Let $\lambda_1$ and $\lambda_2$ be distinct nonzero real numbers. Set $\psi = \lambda_1\psi_1 + \lambda_2\psi_2$. Then the random variable $\psi_1(M)$ is splittable relative to the Wiener process $\psi$, but it is not a constant plus a stochastic integral with respect to $d\psi$.

*Proof.* First, suppose $\psi_1(M) = c + \int f d\psi$. By setting $\psi = \lambda_1\psi_1 + \lambda_2\psi_2$ and sorting $\psi_1$ from $\psi_2$, we get the equation: $\int (1 - \lambda_1 f)\, d\psi_1 = c + \lambda_2 \int f d\psi_2$. Since the left- and right sides are independent, they are both constant. Hence $\int f d\psi_2$ is constant.

Therefore, $\int f d\psi_1$ is constant also. That implies the absurd claim that $\psi_1(M)$ is constant.

Next, to show $\psi_1(M)$ is splittable relative to $\psi$, it is enough to show that $\psi_1(A)$ is $\mathscr{S}(A)$-measurable for all $A$ in $\mathfrak{M}$. For this it is sufficient to establish that $\psi_1(M)$ is $\mathscr{S}(M)$-measurable. This is especially obvious if $\lambda_1$ and $\lambda_2$ are linearly independent over the rational numbers, because $\psi_1(M)$ and $\psi_2(M)$ have only integer values, and so $\psi(M)$ determines $\psi_1(M)$. But it is true in general. To express $\psi_1(M)$ in terms of $\psi$, we resort to the partitions of the measure space $(M, \mathfrak{M}, \rho)$ described in the first part of the proof of the preceding proposition. Let $\phi(\lambda_1) = 1$ and $\phi(x) = 0$ when $x \neq \lambda_1$. Then

$$\psi_1(M) = \lim_{n\to\infty} \sum_{i=1}^{2^n} \phi(\psi(M(n, i))).$$

The intuitive meaning of this equation is so simple that it ought to be mentioned. An ordinary Poisson stochastic process on an interval (see [1], Chapter VIII, Section 4) has paths that are step functions with jumps of amount 1. If one takes two independent such processes and forms $\lambda_1$ times the first plus $\lambda_2$ times the second, the resulting process has paths that are step functions with jumps of $\lambda_1$ or $\lambda_2$. From such a path one can determine how many times the first process has jumped by counting the number of jumps of $\lambda_1$.

The idea of considering the jumps in a stochastic process can be used to give counterexamples very generally. We shall give a brief indication of this use. Let $\psi$ be a continuous Wiener process on the Borel subsets of the unit interval [0, 1]. The stochastic process $t \to \psi[0, t])$ has paths with at worst jump discontinuities (see [1], Chapter VIII). If $\psi$ is not normal, then for some $\epsilon > 0$, the number of jumps of absolute value $>\epsilon$ is not the zero random variable. In fact, it has a Poisson distribution. Intuitively, it is clear that this random variable is splittable. However, in most situations, a renormalized stochastic integral could not have a Poisson distribution.

## 6. Application to the Normal Distribution

In [5] we studied splittable random variables in the context of the normal distribution on a real Hilbert space and achieved a partial result. We can now strengthen that result by applying Corollary 4 together with the known structure of infinitely divisible probability distributions.

A *normal* distribution on a real Hilbert space is a distribution that associates a normally distributed random variable (constants permitted) with every linear functional in its domain. The *standard normal* distribution on a real Hilbert space is the distribution with all continuous linear functional in its domain that associates with any linear functional a normally distributed random variable with mean 0 and standard deviation equal to the norm of the linear functional. Since the standard normal distribution has independent values on orthogonal linear functionals, it is continuously splittable relative to any nonatomic complete Boolean algebra of projections.

We shall use the following simple consequence of the Lévy–Khintchine decomposition theory of infinitely divisible probability distributions on Euclidean spaces (see [2], Chapter XVII, especially Section 11).

LEMMA 5. *Suppose $X_0, X_1, \ldots, X_n$ are random variables whose joint distribution is infinitely divisible. Suppose $X_1, \ldots, X_n$ are jointly normally distributed. Then there exists a random variable $W$ equal to a linear combination of $X_1, \ldots, X_n$ such that $X_0 - W$ is independent of $X_1, \ldots, X_n$. Furthermore, the variance of $W$ is dominated by a number depending only on the distribution of $X_0$.*

*Proof.* Let $M$ be the joint distribution of $X_0, X_1, \ldots, X_n$ on a Euclidean $(n+1)$-space with coordinates $x_0, x_1, \ldots, x_n$. It results from the Lévy–Khintchine theory that $M$ can be written as a convolution $N * P$ where $N$ is a normal distribution and $P$ is so far from being normal that any coordinate normally distributed with respect to $P$ must be constant. In our case, since $N$ is normal and $X_1, \ldots, X_n$ are jointly normally distributed, $P$ must be concentrated on a line parallel to the $x_0$-axis. Let us rephrase this in terms of random variables $Y_0, \ldots, Y_n, Z_0, \ldots, Z_n$ on a new probability space whose joint distribution is the Cartesian product $N \times P$. Then $X_0, \ldots, X_n$ have the same joint distribution as $Y_0 + Z_0, Y_1 + Z_1, \ldots, Y_n + Z_n$, and $Z_1, \ldots, Z_n$ are constants. Since $Y_0, \ldots, Y_n$ are jointly normally distributed there exists constants $c_0, c_1, \ldots, c_n$ such that the conditional expectation

$$E(Y_0 \mid Y_1, Y_2, \ldots, Y_n) = c_0 + c_1 Y_1 + c_2 Y_2 + \cdots + c_n Y_n .$$

Take $W = c_1 X_1 + c_2 X_2 + \cdots + c_n X_n$. Now the joint distribution of $X_0 - W, X_1, \ldots, X_n$ is the same as the joint distribution of $\xi_0, \xi_1, \ldots, \xi_n$, where $\xi_0 = Y_0 + Z_0 - E(Y_0 \mid Y_1, \ldots, Y_n) + c_0 - c_1 Z_1 \cdots - c_n Z_n$, $\xi_1 = Y_1 + Z_1, \ldots, \xi_n = Y_n + Z_n$. Hence to show that $X_0 - W$ is independent of $X_1, \ldots, X_n$, it is sufficient to observe that $Y_0 - E(Y_0 \mid Y_1, \ldots, Y_n)$ is independent of $Y_1, \ldots, Y_n$, an observation that follows from the joint normality of $Y_0, Y_1, \ldots, Y_n$.

To establish a bound for the variance of $W$ we appeal to the Lévy–Khintchine decomposition of the distribution of $X_0$. The canonical normal part of the distribution of $X_0$, unique up to translation, has a variance that dominates the variance of $W$. For, since $X_0 = W + (X_0 - W)$ where $W$ is normal, $X_0 - W$ is infinitely divisble, and $W$ and $X_0 - W$ are independent, it follows that the normal part of the distribution of $X_0$ must be the convolution of the distribution of $W$ with another probability measure.

It is handy to introduce a couple of notations in connection with the last lemma. If one consider two versions of the $W$ above, their difference is both a function of and independent of $X_1, \ldots, X_n$. Consequently, $W - E(W)$ is uniquely determined by $X_0, \ldots, X_n$, and we shall denote it by $P(X_0 \mid X_1, \ldots, X_n)$. Intuitively, it is a sort of conditional expectation altered by a possibly infinite constant. The number depending only on the distribution of $X_0$, that is, the variance of the normal part of $X_0$, we shall denote by $vn(X_0)$.

THEOREM. *Consider a continuously splittable normal distribution on a real Hilbert space. In this context every splittable random variable is a constant plus an $L_2$ limit of random variables associated with linear functionals by the distribution.*

*Proof.* Let $X$ be the splittable random variable, and let $\tilde{\xi}$ be the variable associated with the linear functional $\xi$ by the distribution. For every finite subset $F = \{\xi_1, \xi_2, \ldots, \xi_n\}$ of the domain of the distribution, the joint distribution of $X, \tilde{\xi}_1, \tilde{\xi}_2, \ldots, \tilde{\xi}_n$ is infinitely divisible according to corollary 4. Hence we may form $X_F = P(X \mid \tilde{\xi}_1, \ldots, \tilde{\xi}_n)$. Along with $X_F$ we shall consider the smallest Boolean $\sigma$-algebra $\mathscr{S}_F$ with respect to which $\tilde{\xi}_1, \ldots, \tilde{\xi}_n$ are measurable. The net $\{X_F\}$ is a martingale relative to the Boolean $\sigma$-algebras $\{\mathscr{S}_F\}$. Such a martingale converges in $L_2$ if and only if its random variables are bounded in $L_2$(cf. [1], p. 319 or [2], p. 236; the fact that our indices are a directed set rather than the positive integers makes no difference). But for all $F$, $\| X_F \|_2^2 =$ variance $X_F \leqslant vn(X)$. Therefore, $\lim_F X_F$ exists in $L_2$, and $X - \lim_F X_F$, being independent of all $\tilde{\xi}$, is constant.

Since the standard normal distribution produces an isometry from continuous linear functionals to random variables in their $L_2$ norm, in this case the theorem becomes the following especially simple corollary.

COROLLARY 5. *Consider the standard normal distribution on a real Hilbert space together with a complete non-atomic Boolean algebra of projections. In this context every splittable random variable is a constant plus a random variable associated with a linear functional by the distribution.*

This corollary can be applied to normal Wiener processes to yield a result that contrasts spharply with the counterexample in Section 5.

COROLLARY 6. *Consider a continuous Wiener process whose values are normally distributed with mean* 0. *Any random variable that is splittable relative to such a process is a constant plus a stochastic integral.*

*Proof.* Suppose $\psi$ is the Wiener process on the Boolean $\sigma$-algebra $\mathfrak{M}$ of subsets of the space $M$. We get a measure $\rho$ by letting $\rho(A)$ be the variance of $\psi(A)$ for each $A$ in $\mathfrak{M}$. Let $H$ be the Hilbert space of all real square-integrable functions on $M$. The map that sends $f$ in $H$ into the stochastic integral $\int f d\psi$ gives the standard normal distribution on the dual of $H$. According to the last corollary, any splittable random variable is a constant plus such a stochastic integral.

In [5], a weaker version of Corollary 5 was used to discuss transformations of the standard normal distribution on Hilbert space. Let $N$ be the standard normal distribution on a real Hilbert space $H$. If $T$ is a continuous homogeneous linear transformation on $H$, the transform $N'$ of $N$ by $T$ is the distribution given by $N'(\xi) = N(\xi \circ T)$ for all $\xi$ in the dual $H^*$ of $H$. If $T$ is a continuous affine transformation, say $Tx = Lx + a$ where $L$ is homogeneous linear and $a$ is in $H$, then $\xi \circ T$ is not homogeneous linear since $\xi(Tx) = \xi(Lx) + \xi(a)$. Nevertheless, it makes sense to define the transform $N'$ of $N$ by $T$ as $N'(\xi) = N(\xi \circ L) + \xi(a)$. If $T$ is non-linear, it gets hard to give a direct interpretation to $N(\xi \circ T)$. What might we mean by saying that $N'$ comes from $N$ via a splittable transformation relative to a complete nonatomic Boolean algebra $\mathfrak{B}$ of projections in $H$? We are interested in assuming, in addition that $N'$ is equivalent to $N$ in the sense of mutual absolute continuity [4]. Whatever all this may mean, it must imply the following two things:

(1) the logarithm of the Radon–Nikodym derivative $dN'/dN$ is a splittable random variable.

(2) each $N'(\xi)$ with $\xi$ in $H^*$ is a splittable random variable.

If we apply corollary 5 to (1), we find that $\log(dN'/dN)$ is a constant plus $N(\alpha)$ for some $\alpha$ in $H^*$. Let a be the vector corresponding to the linear functional $\alpha$ under the canonical correspondence between $H$ and $H^*$, and let $N_a$ be the transform of $N$ under the translation: $x$ $x + a$. What we have seen is that $dN'/dN = dN_a/dN$. This implies that $N'$ and $N_a$ are essentially the same distribution in the sense that $N'(\xi)$ and $N_a(\xi)$ have the same probability distribution for all $\xi$ in $H^*$. It was wrongly stated in the introduction to [5] that $N'$ must come from $N$ via a translation, whereas any Euclidean transformation of $H$ whose orthogonal part commutes with $\mathfrak{B}$ will do. That these are the only possibilities is not hard to see from (2). Applying Corollary 5 to $N'(\xi)$ we find that $N'(\xi)$ must be a constant plus $N(S\xi)$. The constant is $E(N'(\xi)) = E(N_a(\xi)) = \xi(a)$. Since $N(S\xi)$ and $N_a(\xi)$ have the same variance it follows that $\| S\xi \| = \| \xi \|$; in other words $S$ is an isometry of $H^*$ into itself. It is not hard to see from (2) that the adjoint $T$ of $S$ on $H$ commutes with $\mathfrak{B}$. Hence $N'(\xi) = N(\xi \circ T) + \xi(a)$ where $T$ is an orthogonal transformation that commutes with $\mathfrak{B}$ and $a$ is in $H$. In case $\mathfrak{B}$ is a maximal Boolean algebra of projections, then $T = 2P - I$ for some $P$ in $\mathfrak{B}$.

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## References

1. Doob, J. L., "Stochastic Processes." Wiley, New York, 1953.
2. Feller, W., "Probability Theory," Vol. 2. Wiley, New York, 1966.
3. Segal, I. E., Tensor algebras over Hilbert spaces. *Trans. Am. Math. Soc.* **81** (1956), 106–134.
4. Segal, I. E., Distributions in Hilbert space. *Trans. Am. Math. Soc.* **88** (1958), 12–41.
5. Shale, D. and Stinespring, W. F., Continuously spilttable distributions in Hilbert space. *Illinois J. Math.* **10** (1966), 574–578.
6. Wiener, N., The homogeneous chaos. *Am. J. Math.* **60** (1938), 897–936.
7. Wiener, N. and Wintner, A., The discrete chaos.