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# Full Sign-Invertibility and Symplectic Matrices

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ABSTRACT

An $n \times n$ sign pattern $H$ is said to be *sign-invertible* if there exists a sign pattern $H^{-1}$ (called the *sign inverse* of $H$) such that, for all matrices $A \in Q(H)$, $A^{-1}$ exists and $A^{-1} \in Q(H^{-1})$. If, in addition, $H^{-1}$ is sign-invertible [implying $(H^{-1})^{-1} = H$], $H$ is said to be *fully sign-invertible* and $(H, H^{-1})$ is called a sign-invertible pair. Given an $n \times n$ sign pattern $H$, a *symplectic pair* in $Q(H)$ is a pair of matrices $(A, D)$ such that $A \in Q(H), D \in Q(H)$, and $A^T D = I$. (Symplectic pairs are a pattern generalization of orthogonal matrices which arise from a special symplectic matrix found in $n$-body problems in celestial mechanics [1].)

We discuss the digraphical relationship between a sign-invertible pattern $H$ and its sign inverse $H^{-1}$, and use this to cast a necessary condition for full sign-invertibility of $H$. We proceed to develop sufficient conditions for $H$'s full sign-invertibility in terms of allowed paths and cycles in the digraph of $H$, and conclude with a complete characterization of those sign patterns that require symplectic pairs.

## 1. INTRODUCTION

A *sign pattern* is an $m \times n$ matrix $H$ whose entries are 0, 1, or $-1$. We say that an $m \times n$ matrix $A \in Q(H)$ (the sign class of $H$) if sgn $A_{ij} =$

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sgn $H_{ij}$ for all pairs of indices $(i, j)$, where

$$\operatorname{sgn} x = \begin{cases} -1, & x < 0, \\ 0, & x = 0, \\ 1, & x > 0. \end{cases}$$

Equivalently, we sometimes write sgn $A$ = sgn $H$ in this case. The *signed digraph* of the $n \times n$ pattern $H$ is the directed graph $D(H)$ with $n$ vertices and the arc set $\{(i, j) : i \neq j,\ H_{ij} \neq 0\}$, with each arc assigned the weight sgn $H_{ij}$. Since it will not cause confusion to do so, we will refer to the signed digraph of a sign pattern $H$ simply as the *digraph* of $H$. An $n \times n$ sign pattern $H$ is *sign-nonsingular* if det $A \neq 0$ for all matrices $A \in Q(H)$; $H$ is *conbinatorially singular* if det $A = 0$ for all $A \in Q(H)$ [2].

The matrix obtained by eliminating the $i$th row and the $j$th column of the matrix $A$ is called the $(i, j)$ *minor matrix* of $A$ and is denoted $A[i, j]$. Its determinant is called the $(i, j)$ *minor* of $A$ and is denoted $K_A[i, j]$. An $n \times n$ sign pattern $H$ is said to be *sign-invertible* if there exists a sign-pattern $H^{-1}$ (called the *sign inverse* of $H$) such that for all $A \in Q(H)$, $A^{-1}$ exists and $A^{-1} \in Q(H^{-1})$. It should be noted that the sign inverse of $H$ is not the same as the matrix inverse of $H$, as the latter is not in general a $(0, \pm 1)$ matrix; however, the matrix inverse of $H$ is always in $Q(H^{-1})$.

Because the inverse of a matrix depends on its adjoint matrix, it is evident that a pattern $H$ is sign-invertible if and only if $H$ is sign-nonsingular and each of its minor matrices is either sign-nonsingular or combinatorially singular. Sign-invertibility has been investigated previously by Bassett, Maybee, and Quirk [3] and by Lady and Maybee [4]. In addition, Thomassen [5] developed a polynomial-time algorithm using strict parity digraphs to test a pattern for sign-invertibility.

An $n \times n$ sign pattern $H$ is *negative-diagonal* if $H_{ii} = -1$ for $i = 1, \dots, n$. A square (0, 1) matrix with exactly one zero in each row and column is called a *permutation matrix*. A *signature matrix* is a diagonal matrix whose diagonal entries belong to $\{-1, 1\}$. Two sign patterns $G$ and $H$ are *sign-equivalent* (written $G \sim H$) if $G$ can be obtained from $H$ by negating some of its rows and columns and then permuting the rows and columns. Note that $G$ and $H$ are sign-equivalent if and only if there exist permutation matrices $P$ and $Q$ and signature matrices $X$ and $Y$ such that $G = PXHYQ$. Note also that for a permutation matrix $P$, $P^{-1} = P^T$ is also a permutation matrix, and that for a signature matrix $X$, $X^{-1} = X$.

It is easy to show that sign-equivalence is an equivalence relation that preserves sign-invertibility, and it is well known that any sign-nonsingular (and hence any sign-invertible) pattern is sign-equivalent to some negative-diagonal pattern. The following lemmas establish some connections between sign-equivalence and sign-invertibility, which are used later.

Lemma 1.1 is a direct consequence of the results of Bassett, Maybee, and Quirk [3], and is explicitly stated and proven by Lady and Maybee in [4]:

LEMMA 1.1. *Let $H$ be a negative-diagonal sign-invertible pattern with sign inverse $H^{-1}$. Then $H^{-1}$ is negative-diagonal.*

LEMMA 1.2. *Suppose $H$ is sign-invertible, $H'$ is negative-diagonal, and $H \sim H'$. Then $H'$ is sign-invertible, $(H')^{-1}$ is negative-diagonal, and $H^{-1} \sim (H')^{-1}$.*

*Proof.* Since sign-invertibility is an equivalence-class property, $H'$ is sign-invertible, and $(H')^{-1}$ is negative-diagonal by Lemma 1.1. Since $H \sim H'$, there exist permutation matrices $P$ and $Q$ and signature matrices $X$ and $Y$ such that $H = PXH'YQ$. Since $H$ is sign-invertible, $H^{-1} = (PXH'YQ)^{-1} = Q^T Y(H')^{-1}XP^T$. Thus $H^{-1} \sim (\mathrm{H}')^{-1}$. ■

In investigating the relationship between the digraphs of a sign-invertible pattern and its sign inverse, it is thus sufficient (in the sense of Lemma 1.2) to consider only negative-diagonal patterns.

A *simple k-path* in an $n \times n$ matrix $A$ is a product of entries from $A$ of the form

$$A_{i_1, i_2} A_{i_2, i_3} \cdots A_{i_k, i_{k+1}},$$

where the set $\{i_1, i_2, \ldots, i_{k+1}\}$ consists of distinct indices. A *simple k-cycle* in an $n \times n$ matrix $A$ is a product of entries from $A$ of the form

$$A_{i_1, i_2} A_{i_2, i_3} \ldots A_{i_k, i_1},$$

where the set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices. A nonzero simple $k$-path ($k$-cycle) in a sign pattern $H$ corresponds to a simple $k$-path ($k$-cycle) in the digraph of $H$. The *sign* of a path or cycle in $D(H)$ is simply the product of the signs of the arcs that constitute that path or cycle. The *complementary cycle* for a cycle $A_{i_1, i_2} A_{i_2, i_3} \ldots A_{i_k, i_1}$ is the cycle $A_{i_k, i_{k-1}} A_{i_{k-1}, i_{k-2}} \ldots A_{i_2, i_1} A_{i_1, i_k}$.

In Section 2, we recall some important results involving sign-invertibility and digraphs. The first of these is a digraphical characterization due to Bassett, Maybee, and Quirk [3]. Next, we adapt several previous results to give a digraphical characterization of the relationship between a sign-invertible pattern $H$ and its sign inverse $H^{-1}$. The *transitive closure* of an unsigned digraph $D(H)$ is a digraph with the vertex set of $D(H)$ whose arc set includes arc $(i, j)$ whenever $D(H)$ contains a simple $(i, j)$ path. $D(H)$ is called *transitively closed* if the transitive closure of $D(H)$ is $D(H)$ itself. We conclude that if $H$ is sign-invertible, then the digraph $D(H^{-1})$ is the transitive closure of $D(H)$, with the common arcs in the two digraphs having opposite signs.

A sign-invertible pattern $H$ whose sign inverse $H^{-1}$ is also sign-invertible [implying that $(H^{-1})^{-1} = H$] is said to be *fully sign-invertible*, and $(H, H^{-1})$ is called a *sign-invertible pair* in this case. In Section 3, we show that a necessary but not sufficient condition for the full sign-invertibility of a negative-diagonal pattern $H$ is that $D(H^{-1})$ and $D(H)$ have the same arc set, with each arc in $D(H^{-1})$ having the opposite sign from its associated arc in $D(H)$. In this case, we denote the relationship between the patterns $H$ and $H^{-1}$ by $H^{-1} = \operatorname{neg} H$. Finally, we use the transitive closure relationship between $D(H)$ and $D(H^{-1})$ to express this necessary condition in terms of $D(H)$ only.

Let $H$ be a sign pattern. A path or cycle in $D(H)$ is said to be *even* if it contains an even number of arcs; otherwise, it is said to be *odd*. A path or cycle has *even (odd) positive-arc parity* if it contains an even (odd) number of positive arcs. A similar statement defines *even (odd) negative-arc parity*. In Section 4 we obtain sufficient conditions for full sign-invertibility of a negative-diagonal pattern $H$. These conditions make considerable use of the structural similarity between the digraphs of $H$ and $H^{-1}$ that is necessary for full sign-invertibility. We first demonstrate that the sign inverse $H^{-1}$ of a sign-invertible pattern $H$ for which $H^{-1} = \operatorname{neg} H$ is always sign-nonsingular by showing that all cycles in $D(H)$ must be even. We then cast the path-sign uniformity condition necessary for the sign-invertibility of $H^{-1}$ in terms of the positive-arc parity of simple paths in $D(H)$.

We now give a brief introduction to symplectic matrices from the standpoint of $n$-body problems [1]. Consider the $2n \times 2n$ block matrix

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix},$$

where $I_n$ is the $n \times n$ identity matrix, and $0$ is the $n \times n$ zero matrix. A $2n \times 2n$ block matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A$, $B$, $C$, and $D$ are $n \times n$ matrices, is called *symplectic* if it satisfies the symplectic condition $M^T J M = J$, which is equivalent to both of the matrices $A^T C$ and $D^T B$ being symmetric, together with the condition

$A^T D - C^T B = I_{2n}$. The $2n \times 2n$ symplectic matrices form a group (which we shall call $S_{2n}$) under matrix multiplication. We shall be concerned specifically with the subgroup of $S_{2n}$ for which $B = C = 0$, each element of which corresponds to an uncoupled linear canonical transformation in an $n$-body problem. For such matrices, the symplectic condition becomes $A^T D = I_{2n}$.

It is with these ideas in mind that we define a symplectic pair. Given an $n \times n$ sign pattern $H$, a *symplectic pair* in $Q(H)$ is a pair of matrices $(A, D)$ such that $A \in Q(H)$, $D \in Q(H)$, and $A^T D = I$. (The requirement that $A$ and $D$ share the same sign pattern characterizes transformations that yield the so-called Jacobi coordinates, which have historically been useful in solving certain $n$-body problems in celestial mechanics.) Symplectic pairs are a pattern generalization of orthogonal matrices [6] in the sense that for a symplectic pair $(A, D)$ in $Q(H)$, $D^{-1} \in Q(H^T)$.

Using terminology from [7], let $P$ be a property that a real, square matrix may or may not possess. The sign pattern $H$ is said to *allow* property $P$ if there exists a matrix $A \in Q(H)$ such that $A$ has property $P$. $H$ is said to *require* $P$ if all matrices $A$ in $Q(H)$ have property $P$. As an example, the sign pattern $H$ requires symplectic pairs if, for all $A \in Q(H)$, there exists some $D \in Q(H)$ such that $A^T D = I$.

In Section 5, we first relate sign-invertibility to requirement of symplectic pairs, and note that sign-invertibility provides a connection between those patterns that require symplectic pairs and those that allow symplectic pairs. We then demonstrate that a negative-diagonal sign pattern requires symplectic pairs if and only if its digraph is the disjoint union of isolated vertices and negative 2-cycles.

In Section 6, we extend our characterization of patterns that require symplectic pairs beyond the negative-diagonal case. For this extension, it is important to establish that the symplectic-pair property is a sign-equivalence-class property. (This fact separates the symplectic-pair property from related properties which are not satisfied on a sign-equivalence-class basis; see, for example, the discussion on self-inverse sign patterns in [7].) We use this result in conjunction with our results from Section 5 to characterize requirement of symplectic pairs for arbitrary sign patterns.

## 2. THE DIGRAPHICAL RELATIONSHIP BETWEEN $H$ AND $H^{-1}$

In this section we discuss the digraphical relationship between a sign-invertible pattern $H$ and its sign inverse $H^{-1}$.

The following is a digraphical equivalent for sign-invertibility which was proven by Bassett, Maybee, and Quirk in [3]. Thomassen, in [5], used this characterization in his algorithm to test for sign-invertibility.

PROPOSITION 2.1. *Let $H$ be a negative-diagonal sign pattern. Then $H$ is sign-invertible if and only if both of the following conditions are satisfied*:

C1. *$H$ is sign-nonsingular.*

C2. *Given any pair of distinct vertices $i$ and $j$ in $D(H)$, all simple paths from $i$ to $j$ have the same sign.*

This result will be used later in developing conditions on $D(H)$ for the sign-invertibility of $H^{-1}$.

Proposition 2.2 summarizes some important results due to Bassett, Maybee, and Quirk which follow from Proposition 2.1. Most of this previous work was done for irreducible matrices (see [3] and [4] for stronger results thus obtained). Since the assumption of irreducibility is too restrictive for most of the work to follow, these results have been adapted to include the reducible case as well.

PROPOSITION 2.2. *Let $H$ be a negative-diagonal, sign-invertible sign pattern. Then $D(H^{-1})$ is the transitive closure of $D(H)$. In addition, if $i \neq j$ and $H_{ij} \neq 0$, then* $\operatorname{sgn} H_{ij}^{-1} = -\operatorname{sgn} H_{ij}$.

REMARK. In words, the proposition states that if an arc is present in the digraph of $H$, it will be present with opposite sign in the digraph of $H^{-1}$, and if a simple path from $i$ to $j$ occurs in $D(H)$, then the arc $(i, j)$ will occur in $D(H^{-1})$. In the special case where $H$ is irreducible [for which $D(H)$ is strongly connected], $D(H^{-1})$ will be a complete digraph.

EXAMPLE 2.3. Consider the $3 \times 3$ sign pattern

$$H = \begin{pmatrix} -1 & 0 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}.$$

Direct computation shows that $H$ is sign-invertible, and that its sign inverse is

$$H^{-1} = \begin{pmatrix} -1 & 1 & 1 \\ -1 & -1 & 1 \\ -1 & -1 & -1 \end{pmatrix}.$$

The digraphs of $H$ and $H^{-1}$ are shown in Figure 1; it is clear that each arc from $D(H)$ is preserved, with opposite sign, in $D(H^{-1})$, although additional arcs in $D(H^{-1})$ due to paths in $D(H)$ are generated as well.

## 3. FULL SIGN-INVERTIBILITY: A NECESSARY CONDITION

It should be noted that, unlike the analogous matrix result, the sign-invertibility of a sign pattern does not imply that its sign inverse will be sign-invertible; in fact, Example 2.3 provides an instance of a sign-invertible pattern $H$ whose sign inverse $H^{-1}$ is not even sign-nonsingular. Thus the fully sign-invertible patterns are a proper subclass of the sign-invertible patterns.

The following corollary to Proposition 2.2 gives a necessary condition on $H$ for full sign-invertibility.

COROLLARY 3.1. *Suppose $H$ is negative-diagonal and fully sign-invertible. Then*

$$\operatorname{sgn} H_{ij}^{-1} = \begin{cases} -\operatorname{sgn} H_{ij}, & i \neq j, \\ \operatorname{sgn} H_{ij}, & i = j. \end{cases}$$

REMARK. This relationship, which we denote by $H^{-1} = \operatorname{neg} H$, can be interpreted in digraphical terms as follows. If $H$ is negative-diagonal and fully sign-invertible, $D(H)$ and $D(H^{-1})$ have identical arc sets, where each arc in

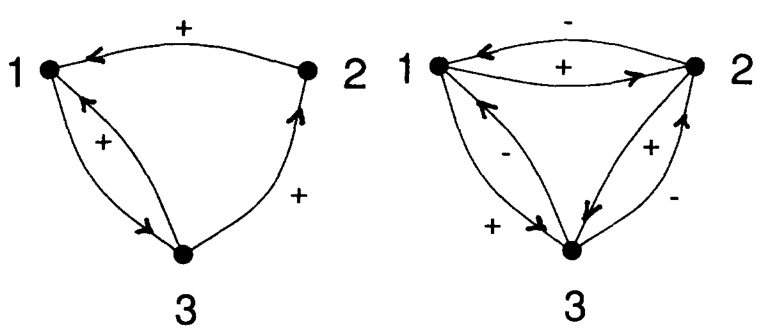


FIG. 1.

one has opposite sign from its counterpart in the other. It should be noted that for $n > 2$, the digraph corresponding to a sign-nonsingular pattern cannot be complete; as a result, irreducible sign-invertible patterns cannot be fully sign-invertible.

*Proof.* By Lemma 1.1, the sign inverse of a negative-diagonal sign-invertible pattern is also negative-diagonal, so Proposition 2.2 implies that

$$\operatorname{sgn} H_{ij}^{-1} = \begin{cases} -\operatorname{sgn} H_{ij}, & i \neq j,\ H_{ij} \neq 0, \\ \operatorname{sgn} H_{ij}, & i = j. \end{cases}$$

Suppose $i \neq j$ and $H_{ij} = 0$, but $H_{ij}^{-1} \neq 0$. Since $H$ is fully sign-invertible, Proposition 2.2 applies to $H^{-1}$, with $(H^{-1})^{-1} = H$, so

$$\operatorname{sgn} H_{ij} = -\operatorname{sgn} H_{ij}^{-1} \neq 0,$$

which is a contradiction. Thus $H_{ij} = 0$ implies $H_{ij}^{-1} = 0$, and so $H^{-1} = \operatorname{neg} H$. ∎

While $H^{-1} = \operatorname{neg} H$ is a necessary condition for the full sign-invertibility of $H$, it involves the digraphs of both $H$ and $H^{-1}$, requiring the computation of $H^{-1}$, and is thus undesirable. The next proposition, which follows directly from Proposition 2.2, casts the condition $H^{-1} = \operatorname{neg} H$ in terms only of $D(H)$.

PROPOSITION 3.2. *Suppose $H$ is negative-diagonal and sign-invertible, so that $H^{-1}$ exists. Then $H^{-1} = \operatorname{neg} H$ if and only if $D(H)$ is transitively closed.*

Stated alternatively, Proposition 3.2 says that $H^{-1} = \operatorname{neg} H$ if and only if the existence of a path from $i$ to $j$ in $D(H)$ implies the existence in $D(H)$ of the arc $(i, j)$.

## 4. FULL SIGN-INVERTIBILITY: SUFFICIENT CONDITIONS

The condition $H^{-1} = \operatorname{neg} H$, while necessary for the full sign-invertibility of $H$, is not sufficient, as the following example shows.

EXAMPLE 4.1. Consider the $3 \times 3$ sign pattern

$$H = \begin{pmatrix} -1 & 0 & -1 \\ -1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Direct computation shows that $H$ is sign-invertible, and that its sign inverse is

$$H^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & -1 \\ 0 & 0 & -1 \end{pmatrix},$$

so that $H^{-1} = \operatorname{neg} H$ is satisfied. However, $H^{-1}$ is not sign-invertible, since the minor $K_{H^{-1}}[3,2]$ is ambiguously signed. In this section we develop sufficient conditions for the full sign-invertibility of $H$.

To ensure the full sign-invertibility of $H$, we require that conditions C1 and C2 from Proposition 2.1 hold for $H^{-1}$; in addition, we prefer to phrase these conditions on $H^{-1}$ in terms of $D(H)$, making use of the structural similarities between $D(H)$ and $D(H^{-1})$ that have been established.

The following lemma relates cycle parity in $D(H)$ to sign-nonsingularity of $H^{-1}$.

LEMMA 4.2. *Suppose $H$ is negative-diagonal and sign-invertible, and $H^{-1} = \operatorname{neg} H$. Then $H^{-1}$ is sign-nonsingular if and only if all cycles in $D(H)$ are even.*

*Proof.* We make use in the proof of the well-known result due to Bassett, Maybee, and Quirk (see [3]) that a negative-diagonal pattern $H$ is sign-nonsingular if and only if its digraph contains no positive cycles. First suppose all cycles in $D(H)$ are even. Since $H$ is sign-nonsingular, any cycle in $D(H)$ is negative and hence has odd negative-arc parity, so any cycle in $D(H)$, being even, has odd positive-arc parity as well. Since $H^{-1} = \operatorname{neg} H$, it follows that any cycle in $D(H^{-1})$ has odd negative-arc parity, so that all cycles in $D(H^{-1})$ are negative and $H^{-1}$ is sign-nonsingular.

Now suppose that $H^{-1}$ is sign-nonsingular, so that all cycles in $D(H^{-1})$ are negative and hence have odd negative-arc parity. Since $H^{-1} = \operatorname{neg} H$, it follows that all cycles in $D(H)$ have odd positive-arc parity. Since $H$ is sign-nonsingular, and since all cycles in $D(H)$ are therefore negative, all cycles in $D(H)$ must have odd negative-arc parity, and are therefore even. ■

The next lemma shows that no cycles of length greater than two can occur in sign-invertible patterns for which $H^{-1} = \operatorname{neg} H$.

LEMMA 4.3. *Suppose $H$ is negative-diagonal and sign-invertible, and $H^{-1} = \operatorname{neg} H$. Then any cycle in $D(H)$ has length two.*

*Proof.* Since $H^{-1} = \operatorname{neg} H$, $D(H)$ is transitively closed by Proposition 3.2. Thus if $D(H)$ contains a cycle of length $p > 2$, it follows that $D(H)$ contains a complete subdigraph of order $p$. Such a subdigraph contains a double 3-cycle, which guarantees the existence of a positive cycle in $D(H)$, contradicting the sign-nonsingularity of $H$. Thus all cycles in $D(H)$ have length two. ∎

We use these lemmas to show that the sign-nonsingularity of $H^{-1}$ follows directly from the condition $H^{-1} = \operatorname{neg} H$.

PROPOSITION 4.4. *Suppose $H$ is negative-diagonal and sign-invertible, and $H^{-1} = \operatorname{neg} H$. Then $H^{-1}$ is sign-nonsingular.*

*Proof.* By Lemma 4.3, all cycles in $D(H)$ have length two, and are therefore even. So by Lemma 4.2, $H^{-1}$ is sign-nonsingular. ∎

This shows that condition C1 from Proposition 2.1 is satisfied for $H^{-1}$ as long as $H^{-1} = \operatorname{neg} H$. We now develop conditions on $D(H)$ that will ensure that condition C2 will be satisfied for $H^{-1}$.

PROPOSITION 4.5. *Suppose $H$ is negative-diagonal and sign-invertible, and $H^{-1} = \operatorname{neg} H$. Then $H^{-1}$ satisfies condition C2 if and only if for each pair of distinct vertices $(i, j)$ in $D(H)$, all paths from $i$ to $j$ in $D(H)$ have the same positive-arc parity.*

*Proof.* Let $(i, j)$ be an arbitrary pair of distinct vertices from $D(H)$, and suppose that all paths from $i$ to $j$ in $D(H)$ have the same positive-arc parity. Since $H^{-1} = \operatorname{neg} H$, all paths from $i$ to $j$ in $D(H^{-1})$ have the same negative-arc parity and hence the same sign, fulfilling condition C2 for $H^{-1}$. Now suppose C2 holds for $H^{-1}$, i.e., that all paths from $i$ to $j$ in $D(H^{-1})$ have the same sign and hence the same negative-arc parity. Since $H^{-1} = \operatorname{neg} H$, all paths from $i$ to $j$ in $D(H)$ have the same positive-arc parity. ∎

The foregoing results are summarized in the following theorem.

THEOREM 4.6. *Suppose $H$ is negative-diagonal and sign-invertible, and $H^{-1} = \operatorname{neg} H$. Then $H$ is fully sign-invertible if and only if for each pair of distinct vertices $(i, j)$ in $D(H)$, all paths from $i$ to $j$ in $D(H)$ have the same positive-arc parity.*

*Proof.* For sufficiency, Propositions 4.4 and 4.5 respectively imply that $H^{-1}$ satisfies conditions C1 and C2. Thus by Proposition 2.1, $H^{-1}$ is sign-invertible, so that $H$ is fully sign-invertible.

For necessity, suppose that the hypotheses of the theorem hold and $H$ is fully sign-invertible, so that $H^{-1}$ is sign-invertible. Then Proposition 2.1 implies that $H^{-1}$ satisfies condition C2, and so Proposition 4.5 implies that the parity condition holds. ■

## 5. SIGN-INVERTIBILITY AND SYMPLECTIC PAIRS

In this section we establish some results linking symplectic pairs to sign-invertibility and full sign-invertibility, and characterize the negative-diagonal patterns that require symplectic pairs.

For a sign pattern $H$ to require symplectic pairs, $H$ must be sign-invertible, with the added requirement that $H^{-1} = H^T$: in other words, a pattern $H$ that allows symplectic pairs will require symplectic pairs if and only if $H$ is sign-invertible. Now $H^T$ is sign-nonsingular if and only if $H$ is; furthermore, since there is a one-to-one correspondence between the $(i, j)$ paths of $H$ and the $(j, i)$ paths of $H^T$, it follows that $H^T$ will be sign-invertible if and only if $H$ is. So for a sign pattern $H$ that allows symplectic pairs, sign-invertibility and full sign-invertibility coincide.

We now give a complete characterization of the negative-diagonal sign patterns that require symplectic pairs.

PROPOSITION 5.1. *A negative-diagonal sign pattern $H$ requires symplectic pairs if and only if $D(H)$ is the disjoint union of isolated vertices and negative 2-cycles.*

REMARK. An equivalent characterization, given in terms of the pattern $H$ itself rather than $D(H)$, is that the negative-diagonal pattern $H$ requires symplectic pairs if and only if there exists a permutation matrix $P$ such that $PHP^T$ is the direct sum of $m$ copies of the $2 \times 2$ pattern

$$J = \begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$$

with the negative identity matrix $I_{n-2m}$, where $m$ is the number of 2-cycles in $D(H)$.

*Proof.* Suppose $H$ requires symplectic pairs. Then $H$ is sign-invertible, and hence fully sign-invertible, so $H^{-1} = H^T = \operatorname{neg} H$. Thus, for any two distinct vertices $i$ and $j$ of $D(H)$, the existence of an arc $(i, j)$ in $D(H)$ implies the existence of the arc $(j, i)$ in $D(H)$ with opposite sign, inducing a negative 2-cycle. To see that no pair of 2-cycles can share a common vertex, suppose the opposite, i.e., that there exists a 2-cycle corresponding to both of the pairs $(i, j)$ and $(i, k)$. Then there is a path from $j$ to $k$, so that the arc $(j, k)$ exists by Proposition 3.2. The resulting 3-cycle contradicts Lemma 4.3. Thus $D(H)$ is the disjoint union of isolated vertices and negative 2-cycles.

Now suppose $D(H)$ is the disjoint union of isolated vertices and negative 2-cycles, and recall the statement of Proposition 2.1. Since all cycles in $D(H)$ are negative, $H$ is sign-nonsingular, so that $H$ satisfies C1. Since, given two distinct vertices $i$ and $j$ from $D(H)$, the only possible path from $i$ to $j$ is the arc $(i, j)$, $H$ satisfies property C2, so that $H$ is sign-invertible by Proposition 2.1. In addition, the transitive closure of $D(H)$ is trivially identical to $D(H)$ in this case, so that $H^{-1} = \operatorname{neg} H$ by Proposition 3.2. Since $D(H)$ is the disjoint union of isolated vertices and negative 2-cycles, it follows that $H^T = \operatorname{neg} H$, from which we have $H^T = H^{-1}$. Thus $H$ requires symplectic pairs. ■

## 6. ARBITRARY SIGN PATTERNS AND SYMPLECTIC PAIRS

In Section 5, we characterized the negative-diagonal sign patterns $H$ that require symplectic pairs in terms of conditions on $D(H)$. We now extend these results to arbitrary sign patterns by showing that allowing symplectic pairs is a sign-equivalence-class property.

PROPOSITION 6.1. *Suppose $H$ allows symplectic pairs, and $G \sim H$. Then $G$ allows symplectic pairs.*

*Proof.* By assumption, there exist permutation matrices $P$ and $R$ and signature matrices $X$ and $Y$ such that $PXHYR = G$. Since $H$ allows symplectic pairs, there is a matrix $A \in Q(H)$ such that $A^{-1} \in Q(H^T)$. Then $PXAYR \in Q(G)$ and $(PXAYR)^{-1} = R^T YA^{-1}XP^T \in Q(R^T YA^T XP^T) = Q((PXAYR)^T) = Q(G^T)$. So $G$ allows symplectic pairs. ■

Proposition 6.1 shows that allowing symplectic pairs is a sign-equivalence-class property. The following corollary discusses the repercussions in terms of requiring symplectic pairs.

COROLLARY 6.2. *Suppose the sign pattern $H$ requires symplectic pairs and $G \sim H$. Then $G$ requires symplectic pairs.*

*Proof.* Since $H$ requires symplectic pairs, $H$ is sign-invertible, and since sign-invertibility is a sign-equivalence-class property, $G$ is sign-invertible. Since $H$ requires symplectic pairs, $H$ allows symplectic pairs, and hence $G$ allows symplectic pairs by Proposition 6.1. Thus we conclude that $G$ requires symplectic pairs. ■

Corollary 6.2 shows that our result regarding negative-diagonal sign patterns and requirement of symplectic pairs carries over to arbitrary patterns as follows. Given an arbitrary pattern $H$ that is not combinatorially singular (in which case $H$ couldn't even allow symplectic pairs), there exists a negative-diagonal pattern $G$ such that $G \sim H$. Having constructed such a $G$, we can use Proposition 5.1 to test $G$ for requirement of symplectic pairs. Corollary 6.2 then asserts that $H$ will require symplectic pairs if and only if $G$ does. Proposition 5.1 therefore provides a characterization of all patterns that require symplectic pairs.

## 7. CONCLUSIONS

To summarize, the sign-invertibility of the negative-diagonal sign pattern $H$ can be tested by checking that C1 and C2 hold for $H$ (Thomassen [5] gave a polynomial-time algorithm for performing this task). Having confirmed $H$'s sign-invertibility, we can test $H$ for full sign-invertibility graphically, using only $D(H)$, as follows. The condition $H^{-1} = \operatorname{neg} H$, which is necessary for $H$'s full sign-invertibility, can be checked using $D(H)$ and Proposition 3.2. Once this condition is confirmed, $H^{-1}$ is automatically sign-nonsingular, and full sign-invertibility of $H$ can be checked using $D(H)$ and Theorem 4.6.

The arbitrary sign pattern $H$ can be checked for requirement of symplectic pairs as follows. If $H$ is combinatorially singular, $H$ doesn't require (or, indeed, allow) symplectic pairs. If $H$ isn't combinatorially singular, there exists a negative-diagonal pattern $G$ satisfying $G \sim H$ which can be checked for requirement of symplectic pairs via Proposition 5.1. $H$ will require symplectic pairs if and only if $G$ does.

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