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Bounds on the chromatic number of intersection graphs of sets in the plane

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Abstract

In this paper we show that the chromatic number of intersection graphs of congruent geometric figures obtained by translations of a fixed figure in the plane is bounded by the clique number. Further, in the paper we prove that the triangle-free intersection graph of a finite number of compact connected sets with piecewise differentiable Jordan curve boundaries is planar and hence, is 3-colorable.

Keywords: Coloring; Intersection graphs; Clique number

1. Introduction

An interesting problem of estimating the maximum chromatic number of intersection graphs on the plane was started by Asplund and Grűnbaum [2]. They proved that the chromatic number of every intersection graph of boxes, i.e. rectangles with sides parallel to the coordinate axes on the plane, with clique number at most k, is at most $4k^2$. When $k=2$, this maximum chromatic number is 6. Later this result was improved to $4k^2-3k$. In [3], the problem of coloring the intersection graphs of boxes with given girth was considered. We proved that intersection graphs of boxes on the plane with girth greater than five can be colored with less than six colors. Unlike the two-dimensional case, for three-dimensional boxes, Burling [3] constructed a series of triangle-free intersection graphs with arbitrarily high chromatic number. A question of natural interest which stimulated the work in this paper is to consider intersection

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graphs of geometric figures other than boxes. The main result of this paper is the fact that the intersection graph of convex polygons obtained by translations of a fixed polygon with clique number at most k is $(8k-7)$-colorable. The upper bounds are stronger for the intersection graphs of circles, triangles and parallelograms.

(1) The intersection graph of equal circles or translates of a fixed triangle with clique number at most k is $(3k-2)$-colorable.
(2) The chromatic number of the intersection graph of translates of a fixed d-dimensional parallelepiped with clique number at most k is at most $2^{d-1}(k-1)+1$. As a corollary of result 2 we have that the chromatic number of the intersection graph of parallelograms obtained by the parallel transformations of some parallelogram on the plane is $(2k-1)$-colorable.

Also, we prove the following result: Let G be a triangle-free intersection graph of a finite number of compact connected sets A_i with boundaries that are piecewise differentiable Jordan curves and for every $i \neq j$ let $A_i \backslash A_j$ be nonempty and connected. Then G is planar and hence 3-colorable.

Concepts and notation not defined in this paper will be used as in standard textbooks. All graphs considered are finite, undirected and without loops and multiple edges. The set of vertices, the set of edges and the degree of vertex x in G are denoted by $V(G)$, $E(G)$ and $\deg_G(x)$, respectively. The *chromatic number* of G denoted by $\chi(G)$ is the least number k for which G is k-colorable. A set X is called a *clique* in G if the subgraph induced by X is complete. The largest number of vertices in G inducing a clique is denoted by $\omega(G)$. For a family F of sets S_i, its *intersection graph* G_F is the undirected graph with vertex set F, such that for $S_a, S_b \in V(G_F)$, $(a,b) \in E(G_F)$ is equivalent to $S_a \cap S_b \neq \emptyset$. A graph G is called *k-degenerate* if for every $H \subset G$ there exists a vertex $v \in V(H)$ such that $\deg_H(v) \leqslant k$. It is known [5] that the chromatic number of any k-degenerate graph is at most $k+1$.

2. The intersection graphs of translates on the plane

Let P be the Cartesian plane. We consider finite sets of translates on P. In a set of geometric objects, we use the term the *highest object* to mean a figure which contains a point with the maximum y coordinate.

Lemma 1. *Let $\{A_i\}$, $1 \leqslant i \leqslant n$, be a set of convex geometric objects obtained by translations of some convex geometric figure. For every i choose a point $x_i \in A_i$ so that x_i maps to x_j as A_i maps onto A_j by some translation. If for some i,j,k, $x_i \in A_k$ and $x_j \in A_k$ then A_i intersects A_j* (Fig. 1).

Proof. Let i,j, and k be chosen such that $x_i \in A_k$ and $x_j \in A_k$. By the condition of the lemma, $A_i = A_k + \vec{a}$ and $A_j = A_k + \vec{b}$, where $\vec{a} = (\overrightarrow{x_k, x_i})$ and $\vec{b} = (\overrightarrow{x_k, x_j})$. Hence $x_i + \vec{b} =: x_k^1 \in A_j$ and $x_j + \vec{a} =: x_k^2 \in A_i$. Points x_k^1 and x_k^2 coincide because $x_k^1 = x_k + \vec{a} + \vec{b}$ and $x_k^2 = x_k + \vec{b} + \vec{a}$. Therefore, A_i intersects A_j. $\square$

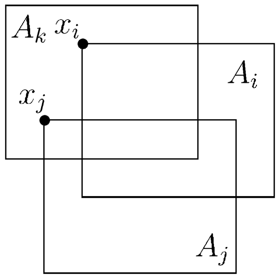

Fig. 1.

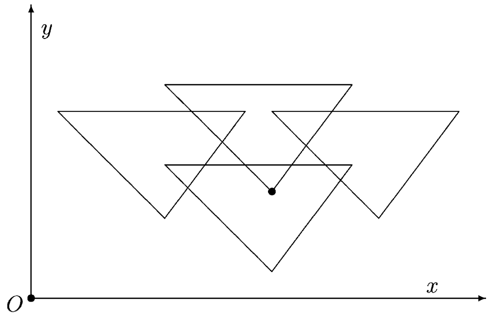

Fig. 2.

Proposition 2. *Let G_t be the intersection graph of a set of triangles obtained by translations of some triangle and $\omega(G_t) \leqslant k$. Then $\chi(G_t) \leqslant 3k-2$.*

Proof. Select a base side for the triangles which is parallel to the x-axis and call the other two sides adjoint. Orient all triangles so that the base is on the top. Every two triangles containing the vertex opposite to the base of the highest object intersect. By Lemma 1, every two triangles intersecting an adjoint side of the highest object and not containing the vertex opposite to the base, intersect each other (Fig. 2). Therefore, in any subgraph of G_t, the vertex corresponding to the highest object can have degree at most $3k-3$, and statement about degeneracy follows. Hence $\chi(G_t) \leqslant 3k-2$. □

Proposition 3. *Let G_c be the intersection graph of a set of equal circles and $\omega(G_c) \leqslant k$. Then $\chi(G_c) \leqslant 3k-2$.*

Proof. Let $\{O(x_i, y_i)\colon 1 \leqslant i \leqslant n\}$ be a set of circles of a given radius r centered at (x_i, y_i). Choose $O(x_k, y_k)$ such that for all i, $y_k \geqslant y_i$. Observe that the centers of all circles $O(x_i, y_i)$ such that $O(x_i, y_i) \cap O(x_k, y_k) \neq \emptyset$ lie on a semi-disc Q, where Q is an intersection of the disc of radius $2r$ centered at (x_k, y_k) with the semi-plane $y - y_k \leqslant 0$. Divide the semi-disc Q into three equal sectors C_ξ, for $\xi = 1, 2, 3$. We will show that

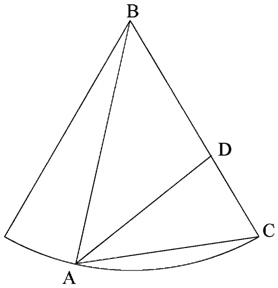

Fig. 3.

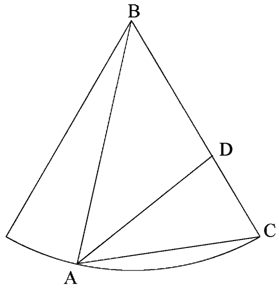

Fig. 4.

all circles with centers in one of these sectors intersect. Then repeating the argument used in previous proof we will obtain that G_c is $(3k-3)$-degenerate

Case 1: Let the points (x_i, y_i) and (x_j, y_j) belong to an arc of some sector C_ξ. The maximum distance between points (x_i, y_i) and (x_j, y_j) occurs when they are at the ends of an arc of a sector with angle 60° on the semi-disc $\partial Q \backslash \{(x, y) \mid y \leqslant y_k\}$. In this case, the two centers are $2r$ apart.

We will use the following simple fact: Suppose that in a triangle the lengths of two sides are at most R. Then the distance from the point common to the two sides to any point on the opposite side is at most R.

Case 2: Let $A = (x_i, y_i)$ belong to an arc of a sector and $D = (x_j, y_j)$ belong to the side of the same sector (Fig. 3). By the construction, $AB = 2r$. By case 1, $AC \leqslant 2r$. Hence $AD \leqslant 2r$.

Case 3: Let $A = (x_i, y_i)$ and $D = (x_j, y_j)$ belong to different sides of the same sector (Fig. 4). By the construction, $AB \leqslant 2r$. By case 2, $AC \leqslant 2r$. Hence $AD \leqslant 2r$.

Therefore, we have shown that the distance between any two circle centers in one sector is at most $2r$. As a result, all circles with centers in one of the three sectors intersect. Thus G_c is $(3k-3)$-degenerate and as a consequence $\chi(G_c) \leqslant 3k-2$. □

Proposition 4. *Let G_p^d be the intersection graph of parallelepipeds in d-dimensional Euclidean space obtained by the parallel transformations of a fixed parallelepiped and $\omega(G_p^d) \leqslant k$. Then $\chi(G_p^d) \leqslant 2^{d-1}(k-1)+1$.*

Proof. Let $x_1, \ldots, x_d$ be a Cartesian coordinate system. Let one of the faces of all parallelepipeds be parallel to the $x_1, \ldots, x_{d-1}$-plane. We choose the highest parallelepiped (the figure with the maximum x_d coordinate of the face parallel to the $x_1, \ldots, x_{d-1}$-plane.) Every d-dimensional parallelepiped intersecting the highest figure contains at least one vertex of the $(d-1)$-dimensional parallelepiped which is a base of the d-dimensional figure. It is clear that parallelepipeds containing the same vertex intersect. The number of vertices in the base is 2^{d-1}. Since $\omega(G_p^d) \leqslant k$ we have that the graph G_p^d is $2^{d-1}(k-1)$-degenerate. Hence $\chi(G_p^d) \leqslant 2^{d-1}(k-1)+1$. □

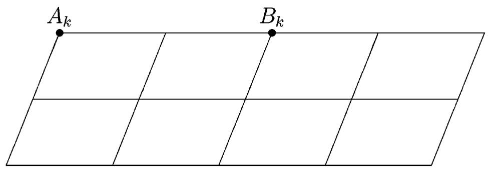

Fig. 5.

Corollary 5. *The intersection graph of parallelograms obtained by parallel transformations of a fixed parallelogram on the plane with clique number at most k is $2k-1$-colorable.*

Proposition 6. *Let G_k be the intersection graph of convex polygons obtained by the parallel transformation of a fixed polygon and $\omega(G_k)\leqslant k$. Then graph G_k is $(8k-7)$-colorable.*

Proof. Let xy be the Cartesian plane. $\mathcal{M}=\{M_i\}$ is a set of polygons with properties described above. As it was proved in [4, Section 7] there exists an affine transformation $\mathcal{T}$ and parallelograms P_{1i} and P_{2i} such that $\mathcal{T}(M_i)\subset P_{1i}$ and $P_{2i}\subset\mathcal{T}(M_i)$ for each i. Moreover, corresponding sides of P_{1i} and P_{2i} are parallel and the ratio of their lengths is $1:2$. Consider an affine transformation $\mathcal{T}'\circ\mathcal{T}$ on the plane, where $\mathcal{T}'$ is an xy-plane rotation which makes one of the sides of parallelograms be parallel to the x-axis. Obviously, the intersection property is invariant with respect to $\mathcal{T}'\circ\mathcal{T}$. Let $\mathcal{M}'=\{M_i'=\mathcal{T}'\circ\mathcal{T}(M_i)\,|\,M_i\in\mathcal{M}\}$, $P_{1i}'=\mathcal{T}'(P_{1i})$, $P_{2i}'=\mathcal{T}'(P_{2i})$. Denote by $\{A_iB_i\}$ upper sides of parallelograms P_{1i}'. Among all figures we consider a figure with the maximum y coordinate of the A_iB_i side. Denote this figure by P_{1k}'. Let $\hat{P}_{1k}'$ be a parallelogram obtained by the parallel transformation of P_{1k}' which places A_k on B_k. A set bounded by $P_{1k}'\cup\hat{P}_{1k}'$ has the following property: if any figure M_l' from $\mathcal{M}'$ intersects $P_1'k$ then the point B_l belongs to this set. Divide $P=P_{1k}'\cup\hat{P}_{1k}'$ into eight equal parts $K_i i=1,\dots,8$ as shown on a Fig. 5. Using Lemma 1 we obtain that if some chosen points of two inscribed parallelograms belong to the same K_i set then these two parallelograms and hence the circumscribed polygons intersect. Thus, the vertex corresponding to the highest polygon participates in eight different complete subgraphs, each of which can be of order at most k. Consequently, G_k is $8(k-1)$-degenerate and G_k is $(8k-7)$-colorable. □

3. Planar intersection graphs

Theorem 7. *Let G be the triangle-free intersection graph of finite number of compact connected sets A_i with boundaries that are piecewise differentiable Jordan curves. For every i and j, let $A_i\backslash A_j$ be nonempty and connected. Then G is a plane graph.*

Proof. The proof is based on the following fact: Given two curves ϕ_1, ϕ_2 which connect points A, B and A, C, respectively, and intersect in a point $D \neq A$ we can find curves $\hat{\phi}_1, \hat{\phi}_2$, which connect points A, B and C, D, respectively, such that $\hat{\phi}_1 \cap \hat{\phi}_2 = \{A\}$ and $\hat{\phi}_1 \cup \hat{\phi}_2$ differs from $\phi_1 \cup \phi_2$ only in a small neighborhood of point D.

From the definition of a Jordan curve, $\mathring{A}_i$ is connected, where $\mathring{A}_i$ is an interior of A_i. For all i, j, k $A_i \cap A_j \cap A_k = \emptyset$, since G has no triangles. Then $A_j \cup A_k$ cannot cover all of A_i for every $i = 1, \dots, n$. This means there exists a point $x_i \in \mathring{A}_i$ such that $x_i \notin \bigcup_{j \neq i} A_j$. Let I be the set of pairs (i, j), $i \neq j$, such that $A_i \cap A_j \neq \emptyset$, and let y_{ij} be some point in $A_i \cap A_j$. Assume that $y_{ij} = y_{ji}$. We will prove that for every $(i, j) \in I$ there exists a curve $\varphi_{ij} = [x_i, y_{ij}]$ with the following properties:

(i) $\varphi_{ij} \subset (\mathring{A}_i \setminus \bigcup_{i \neq j} A_j) \cup (A_i \cap A_j)$.
(ii) $\varphi_{ij} \cap A_j$ is a connected curve.

(iii) $$\varphi_{i_1 j_1} \cap \varphi_{i_2 j_2} = \begin{cases} x_{i_1} & \text{if } i_1 = i_2, j_1 \neq j_2, \\ y_{i_1 i_2} & \text{if } i_1 \neq i_2, j_1 = i_2 \text{ and } j_2 = i_1, \\ \emptyset & \text{if } i_1 \neq i_2 \text{ and either } j_1 \neq i_2 \text{ or } j_2 \neq i_1 \end{cases}$$
for all i_1, i_2, j_1, j_2.

We can always find curves $\varphi_{ij} = [x_i, y_{ij}]$ satisfying properties (i), (ii).

We say that two curves $\varphi_{i_1 i_2}$ and $\varphi_{j_1 j_2}$ *intersect* at a point z, if in a sufficiently small circle with a center at z, sets of the intersection points of this circle with $\varphi_{i_1 i_2}$ and the circle with $\varphi_{j_1 j_2}$ *separate* each other. This means that in every part of the circle between two points of one curve there is a point belonging to another curve.

We say that two curves $\varphi_{i_1 i_2}$ and $\varphi_{j_1 j_2}$ *touch* at z, if in a sufficiently small circle with a center at z, sets of the intersection points of this circle with $\varphi_{i_1 i_2}$ and the circle with $\varphi_{j_1 j_2}$ *do not separate* each other. This means that in every part of the circle between two points of one curve there are no points belonging to another curve.

Let $|I| = k$. We prove the existence of $2k$ curves φ_{ij} with the above described properties by induction on the number of φ_{ij}. The base of the induction is true by the conditions of the theorem. Suppose we have $n - 1 < 2k$ curves. We will show the existence of the nth curve. Assume that there is no curve with property (iii), and $\varphi_{i\xi}$ is the first curve intersecting φ_{ik} in $(\overrightarrow{x_i, y_{ik}})$ direction with z_ξ the first intersection point. Let $a = [x_i, z_\xi] \subset \varphi_{i\xi}$, $b = [z_\xi, y_{i\xi}] \subset \varphi_{i\xi}$, $c = [x_k, z_\xi] \subset \varphi_{ik}$, $d = [z_\xi, y_{ik}] \subset \varphi_{ik}$. Form new curves $\varphi'_{i\xi}$ and φ'_{ik} in the following way: $\varphi'_{i\xi} = c \cup b$, $\varphi'_{ik} = a \cup d$. It is easy to see that curves $\varphi'_{i\xi}$ and φ'_{ik} touch at z_ξ. Let $B(z_\xi, \delta)$ be a circle of radius $\delta > 0$ with a center at z_ξ, where δ is a sufficiently small integer. Substitute parts of the curves $\varphi'_{i\xi}$ and φ'_{ik} lying in $B(z_\xi, \delta)$ with parts of the circle connecting points of the sets $\partial B(z_\xi, \delta) \cap \varphi'_{i\xi}$ and $\partial B(z_{xi}, \delta) \cap \varphi'_{ik}$ and not containing any points of φ'_{ik} and $\varphi'_{i\xi}$, respectively. Proceeding in the same way with all other intersection points we build a system of curves φ_{ij}, for all j such that $(i, j) \in I$, satisfying all requirements.

Thus we define the intersection graph of A_i as $V(G) = \{x_i \mid x_i \in (\mathring{A}_i \setminus \bigcup_{j \neq i} A_j)\}$, $E(G) = \{\varphi_{ij} \cup \varphi_{ji}\}$. From the construction it is easy to see that G is a plane graph. $\square$

Corollary 8. *The intersection graph of A_i given in the theorem is* 3-*colorable*.

Proof. By Grötzsch theorem every planar triangle-free graph is 3-colorable. □

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