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Note

A note on flow polynomials of graphs

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Abstract

Using the decomposition theory of modular and integral flow polynomials, we answer a problem of Beck and Zaslavsky, by providing a general situation in which the integral flow polynomial is a multiple of the modular flow polynomial.

Keywords: Totally cyclic orientation; Eulerian-equivalence class; Integral flow polynomial; Modular flow polynomial; Isomorphism

In this note we present an answer to an open problem proposed by Beck and Zaslavsky [1] in the decomposition theory of flow polynomials, by showing a general situation in which the integral flow polynomial is a multiple of the modular flow polynomial.

Let us start with reviewing some definitions and notations. By a *graph* $G = (V(G), E(G))$, we mean that G has finite vertex set $V(G)$ and edge set $E(G)$. Loops and multiple edges are allowed. Assume that all graphs considered here are connected. Each edge $e \in E(G)$ is incident with two vertices $u, v \in V(G)$, and it can be assigned a direction either from u to v or from v to u, but not both. In particular, a loop has two directions from a vertex to itself. If each edge of G has a direction, we say that G is *oriented*. Let $\mathcal{O}(G)$ denote the set of all orientations of G.

Fix an orientation $\varepsilon \in \mathcal{O}(G)$. By a *circle* of G we mean a 2-regular connected subgraph in G. A *directed cycle* is a circle in which all edges have a consistent direction with respect to ε. We say that ε is a *totally cyclic orientation* if every edge belongs to a directed cycle. Denote the set of totally cyclic orientations by $\mathcal{CO}(G)$. Given two orientations $\varepsilon_1, \varepsilon_2 \in \mathcal{O}(G)$, let $E(\varepsilon_1 \neq \varepsilon_2)$ denote the subset of $E(G)$ composed of the edges having opposite directions with respect to ε_1 and ε_2. We say that $\varepsilon_1, \varepsilon_2$ are *Eulerian-equivalent* if $E(\varepsilon_1 \neq \varepsilon_2)$ can be written as a disjoint union of directed cycles. As shown in [3,4], Eulerian-equivalence relation is indeed an equivalence relation on $\mathcal{O}(G)$ and induces an equivalence relation on $\mathcal{CO}(G)$. Let $[\varepsilon]$ be the Eulerian-equivalence class of ε. Let $[\mathcal{CO}(G)]$ be the set of all Eulerian-equivalence classes in $\mathcal{CO}(G)$.

For a fixed orientation $\varepsilon \in \mathcal{O}(G)$ and a given vertex $v \in V(G)$, let $E^+(v, \varepsilon)$ be the set of edges taking v as the head and $E^-(v, \varepsilon)$ the set of edges taking v as the tail. A *nowhere-zero flow* of G is a function $f : E(G) \to A$ such that

$$\sum_{e \in E^+(v,\varepsilon)} f(e) = \sum_{e \in E^-(v,\varepsilon)} f(e), \quad \forall v \in V(G)$$

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holds, where A is an abelian group and f never takes the value 0. Taking $A = \mathbb{Z}$, an integral nowhere-zero flow is called a *nowhere-zero k-flow* if $|f(e)| < k$ for every edge $e \in E(G)$. It was known that both the number of nowhere-zero flows with values in a finite abelian group and the number of nowhere-zero k-flows are independent of the chosen orientation and the actual group structure [3,4,6,7]. The former one is a polynomial function of the order of the finite abelian group A, and is called the *modular flow polynomial*, denoted $\phi(G, k)$. The latter one is a polynomial function of k, and is called the *integral flow polynomial*, denoted $\phi_{\mathbb{Z}}(G, k)$. For each orientation $\varepsilon \in \mathcal{O}(G)$, the number of nowhere-zero integral flows with values in $\{0, 1, \dots, k-1\}$ is also a polynomial function of k [3,4], denoted $\phi_{\mathbb{Z}}(G, \varepsilon, k)$.

Lemma 1 (*[3,4]*). *If ε and ε' are two Eulerian-equivalent orientations, then*

$$\phi_{\mathbb{Z}}(G, \varepsilon, k) = \phi_{\mathbb{Z}}(G, \varepsilon', k). \tag{1}$$

Beck and Zaslavsky noticed that both $\phi_{\mathbb{Z}}(3K_2, x)$ ($3K_2$ is the graph of 3 parallel links) and $\phi_{\mathbb{Z}}(K_4, x)$ (K_4 is the complete graph of 4 vertices) have integral coefficients, and moreover $\phi_{\mathbb{Z}}(3K_2, x) = 3\phi(3K_2, x)$, $\phi_{\mathbb{Z}}(K_4, x) = 4\phi(K_4, x)$; see [1, Example 3.4]. Then, they proposed the following problem [1, Problem 3.5]:

Is there any general reason why for $3K_2$ and K_4 both of the integral flow polynomials have integral coefficients and the integral flow polynomial is a multiple of the modular polynomial?

Now we can answer the above problem. Note that the modular flow polynomials always have integral coefficients [1,3]. Therefore, we focus on the reason why $\phi_{\mathbb{Z}}(G, x)$ is a multiple of $\phi(G, x)$ for some graph G. In fact, the answer is implied in the following theorem due to Kochol [4, Equations (1) and (2)] and Chen and Stanley [3, Theorems 4.4 and 5.6], which shows that the modular and integral flow polynomials admit a nice decomposition.

Proposition 2 (*[3,4]*). *Given a graph G, let $\phi(G, x)$ be the modular flow polynomial and $\phi_{\mathbb{Z}}(G, x)$ the integral flow polynomial. Then*

$$\phi(G, x) = \sum_{[\varepsilon] \in [\mathcal{CO}(G)]} \phi_{\mathbb{Z}}(G, \varepsilon, x) \tag{2}$$

$$\phi_{\mathbb{Z}}(G, x) = \sum_{\varepsilon \in \mathcal{CO}(G)} \phi_{\mathbb{Z}}(G, \varepsilon, x). \tag{3}$$

Note that similar results hold for integral and modular tension polynomials of graphs, see [2,5]. Our main result is an immediate consequence of Lemma 1 and Proposition 2.

Theorem 3. *If each equivalence class of $[\mathcal{CO}(G)]$ has the same number of totally cyclic orientations, say m, then we have*

$$\phi_{\mathbb{Z}}(G, x) = m\phi(G, x). \tag{4}$$

Before answering the problem of Beck and Zaslavsky, let us recall the definition of isomorphism between two directed multigraphs. Suppose that $D = (V, E)$ and $D' = (V', E')$ are directed multigraphs. We say that D is *isomorphic* to D' if there is a bijection $\theta : V \to V'$ such that for all vertices u, v in V the number of edges from u to v in D is the same as the number of edges from $\theta(u)$ to $\theta(v)$ in D'. Then θ is called an *isomorphism* from D to D'. Given a graph G, note that each orientation $\varepsilon \in \mathcal{O}(G)$ determines a unique directed graph D_ε. We say that two orientations $\varepsilon_1, \varepsilon_2 \in \mathcal{O}(G)$ are *isomorphic* if D_{ε_1} and D_{ε_2} are isomorphic to each other. By Theorem 3, we obtain the following result.

Corollary 4. *If G is a graph such that any two non-Eulerian equivalent orientations in $\mathcal{CO}(G)$ are isomorphic, then the integral flow polynomial $\phi_{\mathbb{Z}}(G, x)$ has integral coefficients and is a multiple of the modular flow polynomial $\phi(G, x)$.*

Proof. If $[\mathcal{CO}(G)]$ has only one Eulerian-equivalence class, then the result clearly holds. Otherwise, suppose that ε_1 and ε_2 are any two totally cyclic orientations which are not Eulerian-equivalent, and we will show that $[\varepsilon_1]$ and $[\varepsilon_2]$ have the same number of totally cyclic orientations. Let θ be the isomorphism from D_{ε_1} to D_{ε_2}. For any orientation

ε_1' in $[\varepsilon_1]$, the edge set $E(\varepsilon_1 \neq \varepsilon_1')$ can be written as a disjoint union of directed cycles with respect to ε_1. Since ε_1 and ε_2 are isomorphic, then the edge set $E(\varepsilon_1 \neq \varepsilon_1')$ can also be written as a disjoint union of directed cycles with respect to ε_2. For the orientation ε_2, by reversing the direction of the edges in $E(\varepsilon_1 \neq \varepsilon_1')$ and keeping the direction of other edges, we obtain an orientation ε_2' which obviously belongs to $[\varepsilon_2]$. Note that the orientation ε_2' is uniquely determined by ε_1' when fixing $\varepsilon_1, \varepsilon_1'$ and the isomorphism from D_{ε_1} to D_{ε_2}. This implies that the cardinality of $[\varepsilon_1]$ is less than or equal to that of $[\varepsilon_2]$. Similarly, we can prove that the cardinality of $[\varepsilon_2]$ is less than or equal to that of $[\varepsilon_1]$. Therefore, each Eulerian-equivalence class of $[\mathcal{CO}(G)]$ has the same number of totally cyclic orientations. Applying Theorem 3, we complete the proof. ■

In particular, we have the following conclusion.

Corollary 5. *If G is $3K_2$ or K_4, then any two totally cyclic orientations of $\mathcal{CO}(G)$ are isomorphic. Therefore, in both cases the integral flow polynomial is a multiple of the modular flow polynomial.*

Proof. Let us first consider graph $3K_2$ with the vertex set $\{v_1, v_2\}$. For any totally cyclic orientation $\varepsilon \in \mathcal{CO}(3K_2)$, we have either $|E^+(v_1, \varepsilon)| = 2$ or $|E^+(v_1, \varepsilon)| = 1$. Given any two totally cyclic orientations ε_1 and ε_2, if $|E^+(v_1, \varepsilon_1)| = |E^+(v_1, \varepsilon_2)|$ then the identity map is an isomorphism between D_{ε_1} and D_{ε_2}. If $|E^+(v_1, \varepsilon_1)| \neq |E^+(v_1, \varepsilon_2)|$, then we take the bijection θ given by $\theta(v_1) = v_2$ and $\theta(v_2) = v_1$ as the desired isomorphism.

For any totally cyclic orientation $\varepsilon \in \mathcal{CO}(K_4)$, the equality

$$\sum_{v \in V(K_4)} |E^+(v, \varepsilon)| = \sum_{v \in V(K_4)} |E^-(v, \varepsilon)|$$

forces that there are exactly two vertices, say v_1, v_2, such that $|E^+(v_1, \varepsilon)| = |E^+(v_2, \varepsilon)| = 2$, and exactly two vertices, say v_3, v_4, such that $|E^+(v_3, \varepsilon)| = |E^+(v_4, \varepsilon)| = 1$. Without loss of generality, we may assume that for ε the edge incident with $\{v_1, v_2\}$ is directed from v_2 to v_1 and the edge incident with $\{v_2, v_3\}$ is directed from v_3 to v_2. In this case, the directions of the remained edges are uniquely determined. For any two orientations $\varepsilon_1, \varepsilon_2 \in \mathcal{CO}(K_4)$, we label the vertices of D_{ε_1} and D_{ε_2} as above. Suppose that $V(D_{\varepsilon_1}) = \{v_1, v_2, v_3, v_4\}$ and $V(D_{\varepsilon_2}) = \{v_1', v_2', v_3', v_4'\}$. Then the bijection θ with $\theta(v_i) = v_i'$ is clearly an isomorphism between D_{ε_1} and D_{ε_2}. ■

Theorem 3, Corollaries 4 and 5 actually present an answer to the problem of Beck and Zaslavsky. For the graph $3K_2$, there are 2 Eulerian-equivalence classes in $[\mathcal{CO}(3K_2)]$, and each class has 3 totally cyclic orientations. Therefore, $\phi_{\mathbb{Z}}(3K_2, x) = 3\phi(3K_2, x)$. For the graph K_4, there are 6 Eulerian-equivalence classes in $[\mathcal{CO}(K_4)]$, and each class has 4 totally cyclic orientations. Therefore, $\phi_{\mathbb{Z}}(K_4, x) = 4\phi(K_4, x)$. However, there exists some graph G for which the condition of Corollary 4 is not satisfied but its integral flow polynomial $\phi_{\mathbb{Z}}(G, x)$ is still a multiple of $\phi(G, x)$. For such a graph, the reader may consult the graph K_4 minus one edge.

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