Kan Extensions: An Exploration

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Introduction:

In his book *Categories for the Working Mathematician* Saunders MacLane entitles an entire section "All Concepts are Kan Extensions." He then proceeds to say, "The notion of Kan extensions subsumes all other fundamental concepts of category theory." Kan extensions are named after mathematician Daniel Kan, a professor at MIT who is perhaps most well known for his work in homotopy theory. In the 1960's, Kan used these extensions to further categorify the categorical definition of a limit. Accordingly, Kan extensions are particularly discussed in defining the notion of a limit and also provide very strong criteria for the existence of adjoints. This paper specifically explores limits.

Before starting with any definitions, it will help to motivate an intuitive sense of what an "extension" is in mathematics. At a high level of generality an extension of some structure is another structure which contains that which has been extended. This talk more specifically explores a Kan extension, which in terms of the previous definition can be thought to be extension of a functor, or more specifically an extension of the diagram generated by a functor in its codomain. We do this by considering a set of other functors equipped with very specific natural transformations into the one we are extending.

Definition:

It actually turns out that we have all the language to understand the definition of a Kan extension without any additional discussion, so I will start by simply stating it.

Definition (Kan Extension)

Let K $:\mathbf{M}\rightarrow\mathbf{C}$ and T:M$\rightarrow\mathbf{A}$ be functors. The <u>right</u> Kan extension of T along K is a pair (R, ε) with a functor R: $\mathbf{C}\rightarrow\mathbf{A}$ and a natural transformation ε: $RK\rightarrow_N T$ which satisfies the following universal property:

If (H, δ) is another pair with a functor H: $\mathbf{C}\rightarrow\mathbf{A}$ and a natural transformation δ: $HK\rightarrow_N T$ then there exists a unique natural transformation σ: $H\rightarrow_N R$ such that $\delta = \varepsilon \circ (\sigma * K)$: $HK\rightarrow_N T$.

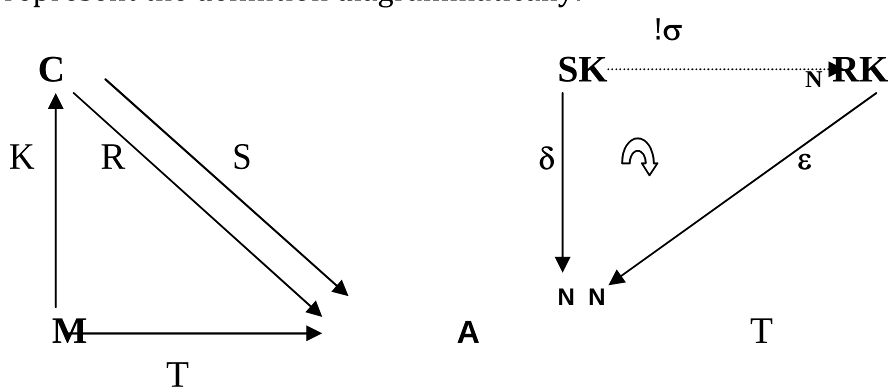
We denote the functor R as R = $Ran_K T$.

A note about convention: an arrow followed by a subscript N ($\rightarrow_N$) refers to a natural transformation.

An interesting point:
Consider the two sets Nat(H, R) and Nat(HK, T). That is, the set of natural transformations between the functors H and R and the set of natural transformations from HK to T. Note by the universal property of the definition of a right Kan extension above, we have that for every δ in Nat(HK, T) there exists a unique σ in Nat(H, R). This implies that there exists a bijective map between these two sets. So we write:

$$\mathrm{Nat}(H, R) \cong \mathrm{Nat}(HK, T)$$

Now we represent the definition diagrammatically:



(Note this diagram need not commute.) (This diagram must commute.)

In closing, we are of course left wondering whether there exists such a thing as a left Kan extension. As we expect, it turns out there does indeed exist a left extension. It is simply the dual notion of the right Kan extension.

We write the left Kan extension of a functor T along K as $Lan_K T$.

Note that the real crux of the definition of a right Kan extension above is existence of the natural transformations described. It is for this reason that the dual of a right extension does not affect diagram I above, but simply reverses all of the arrows in diagram II. That is:

$$\begin{tikzcd}
SK & & RK \arrow[ll, dotted, "!\sigma_N"'] \\
T \arrow[u, "\delta_N"] \arrow[urr, "\varepsilon_N"']
\end{tikzcd}$$

Examples (Limits):
Example 1: Products

Consider the following diagram:

$$\begin{tikzcd}
\mathbf{1} \arrow[drr, "R"] & & \\
\mathbf{M} \arrow[u, "K"] \arrow[rr, "T"'] & & \underline{\mathbf{Set}}
\end{tikzcd}$$

First we ask what the categories and functors do here.

The categories:
-1: The category with one object, call it "*" and one morphism, id_*.
-**Set**: The category of small sets, where maps are functions between them.
-**M**: The discrete category {1,2} with two objects 1 and 2, and 2 morphisms, simply id_1 and id_2.

The functors:
-T takes the two objects of **M** to two arbitrary sets. Since T is a functor, it only takes the identity morphisms of the two objects {1,2} to the identity morphisms on the two arbitrary sets, call them A and B.
-K takes both elements of **M** to *.
-R takes * to some set in **Set,** call it C.

We want to find $\mathrm{Ran}_K T$ if it exists.

If it exists, then the **ε** we are looking for simply represents two morphisms in **Set** from the object C produced by the composite functor RK, into the objects A and B mapped to by the functor T. Since there is no morphism between A and B mapped to by T, we have generated a cone on T without any questions of commutativity.

That is, a diagram which looks like this:

$$\begin{tikzcd} & C \arrow[dl, "\varepsilon_\alpha"'] \arrow[dr, "\varepsilon_\beta"] & \\ A & & B \end{tikzcd}$$

Now, suppose there exists another S: **1→Set** going to a set D ϵ ob(**Set**) coupled with a natural transformation δ: SK$\rightarrow_N$T. This will also generate a cone whose vertex is D. (It is identical to the diagram immediately above, except the natural transformations will be labeled differently.)

The natural transformation **σ**: S$\rightarrow_N$R is then a morphism from D to C.

But we want this **σ** to be unique and we have to cones in **Set:**

$$\begin{tikzcd} D \arrow[r, dotted, "! \sigma"] \arrow[dr, "\delta_\alpha"'] \arrow[drr, "\delta_\beta" near start] & C \arrow[d, "\varepsilon_\alpha"'] \arrow[dr, "\varepsilon_\beta"] & \\ & A & B \end{tikzcd}$$

This diagram should look highly familiar. It is simply the limit of two objects without morphisms in between one another, better known as a product. And since **Set** is a category with products, we know this **σ** to exist.

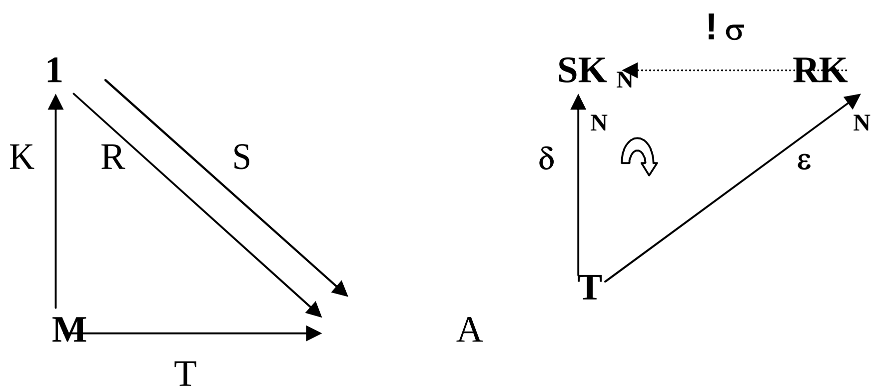
We have just defined the notion of a product on a functor using the existence of a right Kan extension! As you correctly suspect, coproducts can be defined similarly using a left Kan extension. This can be seen by using the dual universal property in the definition of a Kan extension to reverse the limit diagram immediately above.

The next example will continue to work in the dual by using a left Kan extension.

Example 2: Colimits

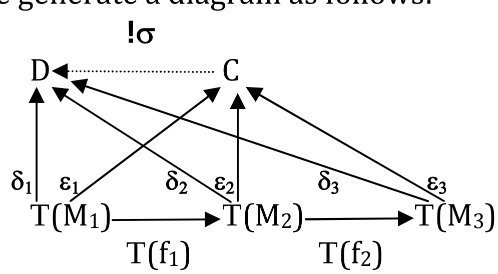
Note that in the last example, our selection of categories was quite specific. Choosing the category **Set** of small sets allowed us to guarantee the existence of a limit to the diagram generated by the functor T. Our selection of the discrete category with two objects allowed us to generate two cones without worrying about whether they commute. Indeed, it was precisely for this reason that we were able to define products using a right Kan extension. Our decision to map **M** → **1** by the functor K was even more important. Note that it allowed us to articulate two particular objects, C and D, when we post composed K with the functors R and S respectively. These objects then served as the vertices of the cones we generated.

In this example, we will again send K to the category **1,** but will allow the other categories we choose to be arbitrary. Thus, we start with the following diagrams:



For continuity, suppose RK maps to an object A in ob(A) and SK maps to an object B in ob(A).

Since we have chosen arbitrary categories **M** and **A** in this example, we suppose the functor T generates a more complicated diagram in **A**. Given the above natural transformations we generate a diagram as follows:



We want to say definitively that this diagram is comprised of two cocones with C and D as vertices. In order to do so we must show that each triangle in the diagram above commutes. It turns out that the commutativity of the above diagram follows from the naturality of δ and ε, coupled with the fact that the only morphisms from C→C and D→D are those objects' respective identities. This is clarified by the following naturality square, which can be generalized to any of the triangles in the above cocones.

$$\begin{tikzcd}
T(M_1) \arrow[r, "T(f_1)"] \arrow[d, "\varepsilon_1"'] \arrow[dr, phantom, "\circlearrowright"] & T(M_2) \arrow[d, "\varepsilon_2"] \\
RK(M_1) = C \arrow[r, "id_c"'] & C = RK(M_2)
\end{tikzcd}$$

Since the map from C→C is simply the identity of C, we know this diagram commutes. This commutative diagram implies that the following diagram also commutes.

$$\begin{tikzcd}
& C \arrow[d, phantom, "\circlearrowright"] & \\
T(M_1) \arrow[ur, "\varepsilon_1"] \arrow[rr, "T(f_1)"'] & {} & T(M_2) \arrow[ul, "\varepsilon_2"']
\end{tikzcd}$$

This diagram is simply one of the subtriangles of the cocones represented earlier.

If indeed the unique σ exists, then the left Kan extension along K has been used to define a colimit of the functor T. Conversely, suppose the colimit of the functor T exists, or the colimit of the diagram generated by T. This implies the unique existence of such σ and we can simply use the morphisms from the colimit diagram to construct a left Kan extension along the functor K, again.

Thus we have proven a crucial theorem regarding Kan extensions. It can be stated as follows.

Theorem 1: A functor T: **M→A** has a colimit if and only if it has a left Kan extension along the unique functor K_1: **M→1** and then colim(T) is the value of $\mathrm{Lan}_K T$ on the object of T.

Note, the dual version of this theorem also holds, which we expect to be the case after example 1.

Conclusion (Mentioning of another theorem:

In closing this paper we will state another crucial theorem regarding Kan extensions and the existence of adjoints. Although this paper has clearly focused on limits, we will include the theorem to suggest how Kan extensions might be applicable in other category theoretical concepts.

Theorem 2:(Formal Criteria of the Existence of an Adjoint)

A functor G: $\mathbf{A} \rightarrow \mathbf{X}$ has a left adjoint if and only if the right Kan extension $Ran_G 1_A$: $\mathbf{X} \rightarrow \mathbf{A}$ exists and is preserved by G. When this is the case this right Kan extension is a left adjoint $F = Ran_G 1_A$ for G and the counit transformation $\boldsymbol{\varepsilon}:(Ran_G 1_A)G \rightarrow_N 1_A$ for the Kan extension is the counit $\boldsymbol{\varepsilon}$: $FG \rightarrow_N 1$ of the adjunction.

Indeed, with the previously stated theorems we increasingly understand MacLane's statement that "all concepts are Kan extensions."

Sources:

(1) S. Maclane. (1998). *Categories for the working mathematician*, New York: Springer.
(2) A. Anderson. (2007). *Kan Extensions and Nonsensical Generalizations*, University of Chicago Mathematics Department.
(3) Lauda, Aaron. (2010). *Introduction to Category Theory Class Notes.*