NOTES AND EXERCISES ON ∞-CATEGORIES

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Abstract. These notes are designed to accompany the lectures given by the author in the Geometric Langlands Seminar at the University of Chicago in the autumn quarter of 2007. The goal of the lectures was to give a brief introduction to J. Lurie's work on ∞-categories. In the notes we provide some additional background from category theory and homotopy theory, as well as an extensive collection of exercises that may help the reader digest the material.

Contents

`Preliminary version, comments are welcome!`

Introduction

In these notes we provide an introduction to the language of ∞-categories developed in J. Lurie's book [T]. It would be difficult to improve upon Lurie's own treatment of the subject, and we made no attempt to do that. Instead, our goal was to ease the transition into the world of ∞-categories for the reader who only has a modest background in classical category theory. The material covered in our notes corresponds merely to a portion of Sections 1.1 and 1.2 of [T]. Although the notes can, in principle, be read independently of [T], we do not recommend doing so.

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Organization of the text. The rather extensive first section provides an overview and an informal discussion of the main notions of higher category theory that will be explained in these notes. For now we only mention that three sorts of higher categories will appear in this text: ∞-*categories*[1] (in the terminology of [T]), which are also called *quasi-categories* by A. Joyal [Joy] and *weak Kan complexes* by J.M. Boardman and R.M. Vogt; *simplicial categories*; and *topological categories*.

Sections 2–5 give some background from ordinary category theory and the (homotopy) theory of simplicial sets needed to understand Lurie's approach to higher category theory. The background is discussed in more detail below.

Section 6 is rather technical, and discusses the "simplicial nerve functor" $\mathfrak{N}$ and its left adjoint, $\mathfrak{C}$. These functors are the analogues for simplicial categories of the usual nerve functor for ordinary categories and of its left adjoint, taking a simplicial set to its "fundamental (or Poincaré) category".

The basic language of ∞-categories is introduced in Section 7. To the reader who is familiar with classical category theory and homotopy theory, we recommend proceeding from Section 1 directly to Section 7 and referring back to Sections 2–6 as needed. Section 8 discusses the important notion of a homotopy (co)limit of a functor from an ordinary (small) category to an ∞-category.

Background for Lurie's work. Lurie's book [T] is very nearly self-contained. In particular, the appendix to it provides all the background from category theory, the theory of simplicial sets and the theory of model categories, that is required to read the book. Moreover, in the main body of [T], the introduction of every concept of higher category theory is accompanied with a concise review of its classical (category-theoretical or homotopy-theoretical) analogue. Nevertheless, as we only cover about 3% of [T] in these notes, we found it appropriate to supply the reader with a little bit of background as well. We assume the knowledge of elementary category theory, including the notions of adjoint pairs of functors and enriched categories. Some standard terminology is reviewed in Section 2, while in Section 3 we introduce a convenient technical device for proving general category-theoretical results. The notion of a simplicial set and Quillen's model category structure on the category thereof are reviewed in Section 4. Finally, Section 5 recalls the definition of the nerve of a small category, and describes the left adjoint to the nerve functor.

A few words about the exercises. Section 9 of these notes contains a list of exercises that are meant to help the reader digest the material explained in the notes. We would like to thank Vladimir Drinfeld and Jacob Lurie for suggesting some of these exercises. Many other exercises are either standard facts from category theory and homotopy theory, or results taken from Lurie's book [T] whose proof only requires a knowledge of the material presented in these notes.

[1] As explained in Section 1, a more precise term is "$(\infty, 1)$-categories".

The reader is strongly advised to at least attempt all of the *non-starred* exercises that we gave. (The starred ones are optional for the first reading.) Many of the exercises ask for proofs of various results stated in the text; usually it is better to find the proofs on your own than to look them up somewhere. As a general rule, we recommend solving the exercises in the order in which they are stated.

Acknowledgements. Jacob Lurie helped us immensely by quickly and concisely answering numerous question about his works (most of these questions were asked by Drinfeld, who also communicated the answers to me). I would also like to thank Vladimir Drinfeld for valuable advices (on preparing both my lectures and these notes), Peter May for his comments on the notes, and Peter May and Michael Shulman for their help with questions on algebraic topology and category theory.

1. Overview of the main notions

This section is an informal introduction to the notions of higher category theory that we discuss in these notes. We will be as brief as possible with the motivational comments, because we can say nothing that is not already contained in [T], and the reader will surely find Lurie's book to be very informative and inspirational (in particular, §§1.1.1–1.1.2 thereof carefully explain the motivation behind the approach to higher category theory introduced by Joyal and explored by Lurie).

1.1. **General remarks on higher categories.** Roughly speaking, the theory of higher categories aims to find a generalization of classical category theory where, in addition to the ordinary objects and morphisms, one also has "morphisms between morphisms", usually called 2-morphisms; 3-morphisms; and so on. In developing this approach one can either stop at a finite level and obtain the notion of an n-category, where $n \in \mathbb{N}$, or continue indefinitely and obtain the notion of an ∞-category.

However, working with compositions of n-morphisms immediately presents the following problem. If one assumes that all compositions are *strictly associative* (i.e., associative in the usual sense of this word), higher categories become easy to define by induction[2], but for $n \geq 3$ one obtains a notion which has an extremely limited scope of applications[3]. To get the correct notion one must only require compositions of n-morphisms to be associative up to certain $(n+1)$-morphisms, for all $n \geq 1$, but carefully writing out all the necessary coherence axioms is very difficult in practice. For instance, the precise definition of a 3-category along these lines takes many pages to spell out, and to the best of our knowledge, there exists no place in the literature where the full definition of a 4-category is written down.

[2] One regards the collection of (small) strict n-categories, $n-\mathcal{C}at_{strict}$, as an ordinary category, and one defines a strict $(n+1)$-category to be a category enriched over $n-\mathcal{C}at_{strict}$.

[3] Most 3-categories that occur in practice are not equivalent to strict 3-categories.

The approach to higher category theory used by Joyal and Lurie is different. They define the notion of an ∞-category directly (without first defining n-categories for $n \geq 2$), but they restrict attention to a special class thereof, namely, to ∞-categories where n-morphisms are invertible for all $n \geq 2$. A more precise name for this sort of categories is $(\infty, 1)$-*categories*[4]. More generally, an (∞, k)-category is an ∞-category where n-morphisms are invertible for all $n > k$. In principle, the theory developed in [T] allows one to give a definition of an (∞, k)-category by induction on k. However, for the most part, Lurie restricts attention to the cases $k = 0$ and $k = 1$.

1.2. **∞-groupoids and topological categories.** By convention [T], the two terms "$(\infty, 0)$-category" and "∞-groupoid" have the same meaning, even though neither of them has a unique definition. One of the most common choices is to define an ∞-groupoid to be simply a topological space. Now, if $\mathcal{C}$ is an (∞, k)-category in any reasonable sense, then the collection of morphisms (i.e., 1-morphisms) between two objects of $\mathcal{C}$ should form an $(\infty, k-1)$-category (more or less by definition). In particular, taking $k = 1$, these remarks immediately lead to one candidate of the notion of an ∞-category. Namely, one could try to develop the theory of ∞-categories in the framework of *topological categories*, which are, by definition, categories enriched over the category of topological spaces.

In many places his book [T], Lurie explains how to define various higher categorical notions in the setting of topological categories, but he also explains why this setting is technically rather difficult to manage. Thus, for the development of the general theory, he uses a different "model" of higher categories. This is not to say that topological categories, as well as their "combinatorial cousins", simplicial categories (see §1.5), should be completely disregarded. On the contrary, they are used as important technical tools in Lurie's work.

1.3. **Quasi-categories, a.k.a. ∞-categories.** The term "quasi-category" was introduced by André Joyal. By definition, a *quasi-category* is a simplicial set X with the property that for all integers $0 < i < n$, every morphism from the "inner horn" $\Lambda^n_i \xrightarrow{f} X$ extends, possibly non-uniquely, to a morphism $\Delta^n \xrightarrow{\tilde{f}} X$. Here, Δ^n denotes the standard n-simplex, viewed as a simplicial set in the usual way, and Λ^n_i is obtained from Δ^n by removing its interior and the interior of the fact opposite to the i-th vertex of Δ^n. It turns out that quasi-categories provide the most technically convenient model of the notion of an ∞-category.

Originally, quasi-categories were introduced under the name *weak Kan complexes* by Boardman and Vogt. (The terminology is motivated by the fact that if we require the extension property for all $0 \leq i \leq n$ in the definition above, we arrive at the notion of a usual Kan simplicial set.) However, Joyal was the first person to realize that this notion should play an important role in higher category theory [Joy]. In

[4]Thus general (even strict) 2-categories are not ∞-categories in Lurie's sense.

these notes, in order to be consistent with the terminology used in [T], we will use the term ∞-*category* in place of quasi-category.

Let us observe right away that the notion of an ∞-category can be viewed as a generalization of the notion of an ordinary category. Namely, recall that if $\mathcal{C}$ is a small category, its nerve, $N(\mathcal{C})$, is a simplicial set, from which $\mathcal{C}$ can be recovered up to an isomorphism (not merely up to an equivalence). Furthermore, simplicial sets that are isomorphic to nerves of ordinary categories can be characterized as those that satisfy the extension property appearing in the definition of a quasi-category with the added requirement of *uniqueness* of the extension $\widetilde{f}$.

1.4. **Passage from topological categories to ∞-categories.** Lurie's book [T] provides ample evidence that Joyal's notion of a quasi-category is "the right one", and that it yields an extremely convenient setting for higher category theory. Thus we will not elaborate on this point any further. However, the reader may pose the natural question of how do topological categories relate to ∞-categories. In this subsection we informally explain how every (small) topological category gives rise to an ∞-category, leaving all the details for the main body of the notes.

Let $\mathcal{C}$ be a small topological category. To obtain an ∞-category from $\mathcal{C}$, we would like to imitate and generalize the usual construction of the nerve of an ordinary category. The most naive approach would be to define the nerve $N(\mathcal{C})$ in exactly the same way as in classical category theory. In other words, the 0-simplices of $N(\mathcal{C})$ would be the objects of $\mathcal{C}$, the 1-simplices of $N(\mathcal{C})$ would be the morphisms of $\mathcal{C}$, and in general, the n-simplices of $N(\mathcal{C})$ would be sequences of n composable morphisms of $\mathcal{C}$. The face and degeneracy maps of $N(\mathcal{C})$ would be given by various partial compositions and insertions of identity morphisms.

However, this construction disregards the topologies on the Hom spaces in the category $\mathcal{C}$, and thus clearly must yield an incorrect notion of a nerve. The failure of this approach can already be seen from the fact that the correct notion of an equivalence between two topological categories, $\mathcal{C}_1$ and $\mathcal{C}_2$, is a topological functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ such that the induced maps $\mathrm{Hom}_{\mathcal{C}_1}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}_2}(F(X), F(Y))$ are weak homotopy equivalences for all $X, Y \in \mathcal{C}_1$ and F is essentially surjective[5]. Nevertheless, two topological categories that are equivalent in this correct sense may have naive nerves that are rather different from each other[6].

To obtain the correct notion one must remember that usually it is wrong to require two morphisms in a topological category to be equal. Rather, one should apply the philosophy of higher category theory and require instead that the two morphisms be

[5]Also in the appropriate topological sense, which is made precise in Definition 7.10(a).

[6]For instance, if M is a contractible topological monoid, it determines a topological category $\mathcal{C}$ with one object, and $\mathcal{C}$ is topologically equivalent to a category with one object and one morphism. However, the (geometric realization of the) naive nerve of $\mathcal{C}$ is the classifying space of M viewed as a monoid with the *discrete* topology.

connected by a continuous path[7] and remember the choice of this path. Similarly, one should not require two paths between two morphisms to be equal; instead, one should require them to be homotopic, and remember the choice of a homotopy; and so forth. Thus, if $\mathcal{C}$ is a (small) topological category and we try to define the "clever" nerve, $\mathfrak{N}(\mathcal{C})$, of $\mathcal{C}$, the 0-simplices and the 1-simplices of $\mathfrak{N}(\mathcal{C})$ should be the objects and morphisms of $\mathcal{C}$, but to give a 2-simplex of $\mathfrak{N}(\mathcal{C})$ one must give objects $x_0, x_1, x_2 \in \mathcal{C}$, morphisms $x_0 \xrightarrow{f_{01}} x_1$, $x_1 \xrightarrow{f_{12}} x_2$, $x_0 \xrightarrow{f_{02}} x_2$, and a continuous path in $\operatorname{Hom}(x_0, x_2)$ joining f_{02} and $f_{12} \circ f_{01}$.

The description of the 3-simplices in $\mathfrak{N}(\mathcal{C})$ is more complicated. Let us list all the data needed to specify an element of $\mathfrak{N}(\mathcal{C})_3$:

- Four objects, x_0, x_1, x_2, x_3, of $\mathcal{C}$.
- Six morphisms, $f_{ij} : x_i \longrightarrow x_j$, in $\mathcal{C}$ for all $0 \leq i < j \leq 3$.
- Four continuous paths (in the appropriate Hom spaces), namely, a path γ_0 joining f_{13} and $f_{23} \circ f_{12}$; a path γ_1 joining f_{03} and $f_{23} \circ f_{02}$; a path γ_2 joining f_{03} and $f_{13} \circ f_{01}$; and a path γ_3 joining f_{02} and $f_{12} \circ f_{01}$.
- A (continuous) homotopy (with fixed endpoints) between the two resulting paths joining f_{03} and $f_{23} \circ f_{12} \circ f_{01}$, namely,
$$\gamma_1 \bullet (\mathrm{const}_{f_{23}} \circ \gamma_3) \qquad \text{and} \qquad \gamma_2 \bullet (\gamma_0 \circ \mathrm{const}_{f_{01}}).$$
(Here, $\bullet$ denotes the concatenation of paths; $\circ$ denotes the composition law in the category $\mathcal{C}$; and const_f denotes the constant path at a given element f of some Hom space of $\mathcal{C}$.)

In principle, one could define $\mathfrak{N}(\mathcal{C})_n$ for all $n \in \mathbb{N}$ along these lines, but it is not convenient to work with the increasingly long lists of data with which one will have to deal. Fortunately, Lurie explains a much more concise definition of $\mathfrak{N}(\mathcal{C})$ in §1.1.5 of [T]; he calls it the *topological nerve* of $\mathcal{C}$. An important fact, which follows from Proposition 1.1.5.9 in *loc. cit.*, is that $\mathfrak{N}(\mathcal{C})$ is always an ∞-category.

Moreover, it is obvious from the description of $\mathfrak{N}(\mathcal{C})$ sketched above that the notion of a topological nerve does generalize the classical notion of a nerve, in the sense that if $\mathcal{C}$ is an ordinary category, viewed as a topological category where all the Hom spaces in $\mathcal{C}$ are equipped with the discrete topology, then the topological nerve $\mathfrak{N}(\mathcal{C})$ coincides with the ordinary nerve of $\mathcal{C}$, by definition.

A remark on our notation. In [T], Lurie uses the notation $N(\mathcal{C})$ to denote the ordinary nerve, the topological nerve, or the simplicial nerve, of a given category $\mathcal{C}$, depending on whether $\mathcal{C}$ is an ordinary category, or is enriched over topological spaces or over simplicial sets, respectively. As he points out, this choice of notation could lead to some confusion, since the enriched nerve is often quite different from the ordinary one. Moreover, he denotes the functor that is left adjoint to the simplicial nerve functor by the Fraktur letter $\mathfrak{C}$. Thus we find it reasonable in these notes to

[7]Continuous paths between morphisms play the role of 2-morphisms in the present situation.

denote the simplicial nerve and the topological nerve functors by the Fraktur letter $\mathfrak{N}$, and reserve the letter N for the ordinary nerve functor.

1.5. **Simplicial categories.** It is clear from the description of the topological nerve $\mathfrak{N}(\mathcal{C})$ of a topological category $\mathcal{C}$ given above that $\mathfrak{N}(\mathcal{C})$ only depends on the category $\operatorname{Sing}(\mathcal{C})$ obtained from $\mathcal{C}$ by replacing all of the Hom spaces with their total singular complexes. More generally, simplicial categories are a useful tool for understanding the relationship between topological categories (which provide a very intuitive approach to higher category theory) and ∞-categories (which yield the most technically convenient setting for higher category theory).

2. General notation and conventions

2.1. **Universes.** To avoid set-theoretical difficulties that might occur at a very basic level, we will use Grothendieck's approach. Namely, we fix two universes, $\mathscr{U}$ and $\mathscr{V}$, with the property that $\mathscr{U} \in \mathscr{V}$. Elements of $\mathscr{U}$ will be called *sets*. For more details on this we refer the reader to §1.2.15 of [T].

2.2. **Categories.** In these notes we will exclusively deal with *locally small* categories $\mathcal{C}$, i.e., those that have the property that the class of objects of $\mathcal{C}$ is an element of $\mathscr{V}$, and for any two objects X, Y of $\mathcal{C}$, morphisms from X to Y in $\mathcal{C}$ form an element of $\mathscr{U}$. The set of morphisms from X to Y in $\mathcal{C}$ will always be denoted by $\operatorname{Hom}_{\mathcal{C}}(X, Y)$. This notation is consistent with that used in Lurie's book [T].

The classes of objects and morphisms of a category $\mathcal{C}$ will be denoted by $\mathcal{O}b(\mathcal{C})$ and $\mathcal{A}r(\mathcal{C})$, respectively (and morphisms may sometimes be called "arrows"). By the standard abuse of notation, we will often write $X \in \mathcal{C}$ in place of $X \in \mathcal{O}b(\mathcal{C})$.

The *opposite category* (or *dual category*) of a category $\mathcal{C}$ will be denoted by $\mathcal{C}^{op}$.

2.3. **Small categories.** A category is said to be *small* if its class of objects is a set (i.e., lies in $\mathscr{U}$). Unless explicitly stated otherwise, most of the categories that we consider will be small. Two notable exceptions[8] are the category $\mathcal{S}et$ of all sets and the category $\mathcal{C}at$ of all small categories. In accordance with the conventions of §2.2, if $\mathcal{C}, \mathcal{D} \in \mathcal{C}at$, we write $\operatorname{Hom}_{\mathcal{C}at}(\mathcal{C}, \mathcal{D})$ for the *set* of all functors $\mathcal{C} \longrightarrow \mathcal{D}$.

2.4. **Yoneda embedding.** If $\mathcal{C}$ is a category and $X \in \mathcal{C}$, we define the functor $h_X^{\mathcal{C}} : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$ by $h_X^{\mathcal{C}}(Y) = \operatorname{Hom}_{\mathcal{C}}(Y, X)$. It is well known that for any functor $F : \mathcal{C}^{op} \longrightarrow \mathcal{S}et$, there is a natural bijection between the set of natural transformations $\varphi : h_X^{\mathcal{C}} \longrightarrow F$ and the set $F(X)$, given by $\varphi \longmapsto \varphi_X(\mathrm{id}_X)$. In particular, for all $X, Y \in \mathcal{C}$, we obtain a natural bijection between the set of natural transformations $h_X^{\mathcal{C}} \longrightarrow h_Y^{\mathcal{C}}$ and $\operatorname{Hom}_{\mathcal{C}}(X, Y)$, which is compatible with compositions.

[8]Other exceptions, such as the category $\mathcal{T}op$ of topological spaces, the category $\mathcal{S}et_{\Delta}$ of simplicial sets, and the category $\mathcal{C}at_{\Delta}$ of small simplicial categories, are introduced later.

If $\mathcal{C}$ is small, we obtain a fully faithful embedding $h^{\mathcal{C}}$ of $\mathcal{C}$ into the category of all functors $\mathcal{C}^{op} \longrightarrow \mathcal{S}et$, called the *Yoneda embedding*. The only reason we require $\mathcal{C}$ to be small is that otherwise we cannot consider *all* functors $\mathcal{C}^{op} \longrightarrow \mathcal{S}et$ at once.

2.5. **Enriched categories.** If $\mathcal{H}$ is any monoidal category, one knows how to define the notion of a category *enriched over* $\mathcal{H}$. If $\mathcal{C}$ is such a category and $X, Y \in \mathcal{C}$ are objects, we write $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{H}$ for the corresponding object of morphisms between X and Y. This is again consistent with [T].

If the category $\mathcal{H}$ does not have a specified monoidal structure, then whenever we speak about $\mathcal{H}$-enriched categories, we always assume that $\mathcal{H}$ has finite products and is equipped with the Cartesian monoidal structure[9]. For instance, the category $\mathcal{C}at$ is naturally enriched over itself. Thus, for $\mathcal{C}, \mathcal{D} \in \mathcal{C}at$, we denote by $\mathrm{Map}_{\mathcal{C}at}(\mathcal{C}, \mathcal{D})$ the *category* of functors from $\mathcal{C}$ to $\mathcal{D}$.

2.6. **Lifting properties.** Let $\mathcal{C}$ be a category (not necessarily small), and let $\mathscr{L}$ and $\mathscr{R}$ be two classes of morphisms in $\mathcal{C}$. Suppose that the ordered pair $(\mathscr{L}, \mathscr{R})$ has the following property: given any commutative square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle \ell}\downarrow & & \downarrow{\scriptstyle r} \\ B & \longrightarrow & Y \end{array}$$

in $\mathcal{C}$ with $\ell \in \mathscr{L}$ and $r \in \mathscr{R}$, there exists a (possibly non-unique) morphism $x : B \longrightarrow X$ in $\mathcal{C}$ making the diagram above commute. In this case we say that:

- morphisms in $\mathscr{L}$ have the *left lifting property* (L.L.P.) with respect to $\mathscr{R}$ (or "with respect to morphisms in $\mathscr{R}$"); and
- morphisms in $\mathscr{R}$ have the *right lifting property* (R.L.P.) with respect to $\mathscr{L}$.

If $\mathscr{M}$ is any class of morphisms in $\mathcal{C}$, we denote by $\lambda(\mathscr{M})$ and $\rho(\mathscr{M})$ the classes of all morphisms in $\mathcal{C}$ having the L.L.P. and the R.L.P. with respect to $\mathscr{M}$, respectively.

3. Small categories and diagram schemes

3.1. **Diagram schemes.** In these notes, in order to prove certain general results from category theory, we will employ a useful formalism that we learned from the book [GZ]. A more familiar special case of this story is recalled in §3.2.

Definition 3.1. A *diagram scheme* is an ordered quadruple $\mathcal{D} = (O, A, s, t)$, where O, A are sets and $s, t : A \longrightarrow O$ are maps. Elements of O are called the "objects" or "vertices" of the diagram scheme $\mathcal{D}$, and elements of A are called the "arrows" of the diagram scheme $\mathcal{D}$. If $a \in A$, then $s(a) \in O$ and $t(a) \in O$ are called the "source" and "target" of the arrow a, respectively. Arrows in $\mathcal{D}$ will usually be

[9]That is, the monoidal bifunctor is the direct (or Cartesian) product bifunctor, with the obvious associativity and unit constraints arising from the universal properties of direct products.

drawn as follows: $s(a) \xrightarrow{a} t(a)$. Sometimes, by abuse of notation, we will write $x \in \mathcal{D}$ in place of $x \in O$. Thus "$x \in \mathcal{D}$" means "x is an object of $\mathcal{D}$".

Thus, what we call a diagram scheme is also sometimes referred to as an oriented graph, and sometimes as a quiver. We chose our terminology to be compatible with that of [GZ], and also so as to remind the reader of the relation of the notion we introduced to category theory.

The notion of a morphism of diagram schemes is defined in the obvious way. We will denote the category of diagram schemes by $\mathcal{D}ia$. Note that it is *not* small.

Remark 3.2. Consider the category $\mathcal{K}$ which has two objects, α and ω, and two nonidentity arrows $\omega \longrightarrow \alpha$. It is obvious that $\mathcal{D}ia$ is equivalent to the category of presheaves of sets on $\mathcal{K}$, i.e., functors $\mathcal{K}^{op} \longrightarrow \mathcal{S}et$.

3.2. **Free monoids.** In the rest of this section we will explore the relationship between small categories and diagram schemes. Roughly speaking, the notion of a diagram scheme is convenient for defining what it means for a category to be "freely generated" by a given set of morphisms. Let us recall that a category with one object is "the same thing as" a monoid. On the other hand, a diagram scheme whose set of objects (or vertices) has one element is "the same thing as" a set, and it is well known what it means for a monoid to be freely generated by a given set.

Let us formulate these comments in a category theory language. Let $\mathcal{M}on$ denote the category of monoids. If $* = \{0\}$ denotes a fixed set with one element, we have a natural fully faithful functor $\mathcal{S}et \hookrightarrow \mathcal{D}ia$ defined by

$$S \longmapsto (*, S, p_S, p_S), \quad \text{where } p_S : S \longrightarrow * \text{ is the unique map,}$$

and a natural fully faithful functor $\mathcal{M}on \hookrightarrow \mathcal{C}at$ defined by

$$M \longmapsto \text{the category with set of objects } * \text{ and } \mathrm{End}(0) = M \text{ as monoids.}$$

On the other hand, if $S \in \mathcal{S}et$, we define $\mathcal{FM}(S)$ to be the monoid whose elements are finite, possibly empty, strings $(s_1, s_2, \ldots, s_n)$ of elements of S (so $n \geq 0$), and where composition is defined by concatenation of strings (so the empty string is the unit element). We call $\mathcal{FM}(S)$ the *free monoid* on S. It is clear that $\mathcal{FM}$ can be extended to a functor $\mathcal{S}et \longrightarrow \mathcal{M}on$. The following fact is essentially obvious.

Lemma 3.3. *The functor $\mathcal{FM}$ is left adjoint to the forgetful functor $\mathcal{M}on \longrightarrow \mathcal{S}et$.*

Recall that there is an analogous construction, which is perhaps even more familiar, in group theory: if S is a set, one can define the free group $FG(S)$ on S. However, there is an important difference between groups and monoids: whereas a free group can be free on many[10] different sets of generators, the set of free generators of a free monoid is uniquely determined. Namely, let us define an element x of a monoid M to be *indecomposable* if $x \neq 1$ and $x \neq yz$ for any $y, z \in M \setminus \{1\}$.

[10] In fact, infinitely many if the group is not cyclic.

Lemma 3.4. (*a*) *If S is any set, the indecomposable elements of the monoid $\mathcal{FM}(S)$ are precisely the strings of length* 1*. Thus the set of indecomposable elements of $\mathcal{FM}(S)$ can be naturally identified with S.*

(*b*) *Let $\mathcal{S}et^\times$ and $\mathcal{M}on^\times$ denote the groupoids obtained from the categories $\mathcal{S}et$ and $\mathcal{M}on$ by discarding all noninvertible morphisms. The functor $\mathcal{S}et^\times \longrightarrow \mathcal{M}on^\times$ obtained by restricting $\mathcal{FM}$ to $\mathcal{S}et^\times$ is fully faithful.*

It turns out that the forgetful functor $\mathcal{M}on \longrightarrow \mathcal{S}et$ can be extended to a functor $U : \mathcal{C}at \longrightarrow \mathcal{D}ia$ (see the beginning of §3.3), and the functor $\mathcal{FM} : \mathcal{S}et \longrightarrow \mathcal{M}on$ can be extended to a functor $\mathcal{P}a : \mathcal{D}ia \longrightarrow \mathcal{C}at$ (see Definition 3.7). The functor $\mathcal{P}a$ is left adjoint to U, and an analogue of Lemma 3.4 holds for $\mathcal{P}a$ (Exercise 9.9).

3.3. **Categories and diagram schemes.** Recall that $\mathcal{C}at$ denotes the category of small categories. If $\mathcal{C} \in \mathcal{C}at$, we write $\mathcal{O}b(\mathcal{C})$ for the set of objects of $\mathcal{C}$, and $\mathcal{A}r(\mathcal{C})$ for the set of morphisms in $\mathcal{C}$. The two maps $\mathcal{A}r(\mathcal{C}) \longrightarrow \mathcal{O}b(\mathcal{C})$, taking a morphism $X \xrightarrow{f} Y$ in $\mathcal{C}$ to X and Y, respectively, define a diagram scheme $U(\mathcal{C})$, which we will call the *underlying diagram scheme* of the category $\mathcal{C}$. We obtain a functor

$$U : \mathcal{C}at \longrightarrow \mathcal{D}ia,$$

which is faithful, but certainly neither full nor essentially surjective[11].

Lemma 3.5. *The functor U has a left adjoint.*

In Definition 3.7 below, we will explicitly describe a functor $\mathcal{P}a : \mathcal{D}ia \longrightarrow \mathcal{C}at$. We leave it as a trivial exercise for the reader to verify that $\mathcal{P}a$ is left adjoint to U.

Definition 3.6. Let $\mathcal{D} = (O, A, s, t)$ be a diagram scheme and $n \in \mathbb{N}$. A *path of length n* in $\mathcal{D}$ is a sequence of arrows $a_1, \dots, a_n \in A$ such that $t(a_j) = s(a_{j+1})$ for all $1 \leq j \leq n-1$. Pictorially, we represent such a path as follows:

$$x \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} \bullet \xrightarrow{a_n} y,$$

where $x = s(a_1)$ and $y = t(a_n)$. We also say that this path *joins x to y*. It is convenient to define a *path of length* 0 in $\mathcal{D}$ to be an object $x \in O$; we say that such a path *joins x to x*.

Definition 3.7. Let $\mathcal{D} = (O, A, s, t)$ be a diagram scheme. The *path category*, $\mathcal{P}a(\mathcal{D})$, of $\mathcal{D}$, is defined as the category whose objects are elements of O, and where, for any $x, y \in O$, the set of morphisms from x to y equals to the set of all paths in $\mathcal{D}$ of arbitrary (finite) length ≥ 0 joining x to y. Composition of morphisms in $\mathcal{P}a(\mathcal{D})$ is defined as concatenation of paths. In particular, the paths of length 0 are the unit morphisms in $\mathcal{P}a(\mathcal{D})$. This obviously yields a functor $\mathcal{P}a : \mathcal{D}ia \longrightarrow \mathcal{C}at$.

[11]For instance, if $\mathcal{D}$ is any diagram scheme which is in the essential image of the functor U, then for any object x of $\mathcal{D}$, there must exist at least one arrow $x \longrightarrow x$.

3.4. **Small limits and colimits.** In the standard treatment of limits and colimits, one usually defines (co)limits of functors $\mathscr{I} \longrightarrow \mathcal{C}$, where $\mathcal{C}$ is an arbitrary category and $\mathscr{I}$ is a small category of "indices". Here we recall an equivalent approach, via (co)limits of diagrams, which we also borrowed from [GZ].

Definition 3.8. Let $\mathcal{D}$ be a diagram scheme. If $\mathcal{D}'$ is another diagram scheme, a *diagram of type* $\mathcal{D}$ *in* $\mathcal{D}'$ is a morphism of diagram schemes $\mathcal{D} \longrightarrow \mathcal{D}'$. If $\mathcal{C}$ is a (not necessarily small) category, a *diagram of type* $\mathcal{D}$ *in* $\mathcal{C}$ is a functor $\mathcal{P}a(\mathcal{D}) \longrightarrow \mathcal{C}$.

Remark 3.9. If $\mathcal{C}$ is a small category, then, of course, a diagram of type $\mathcal{D}$ in $\mathcal{C}$ is a morphism of diagram schemes $\mathcal{D} \longrightarrow U(\mathcal{C})$. The only reason we cannot repeat the same definition in general is that we only defined "small" diagram schemes, and hence $U(\mathcal{C})$ also makes sense only for small categories.

If $\mathcal{C}$ is any category, we can also describe diagrams of type $\mathcal{D} = (O, A, s, t)$ in $\mathcal{C}$ more concretely. Namely, giving such a diagram is equivalent to giving a rule that to every $x \in O$ assigns an object $F(x) \in \mathcal{O}b(\mathcal{C})$, and to every $a \in A$ assigns a morphism $F(a) : F(s(a)) \longrightarrow F(t(a))$ in $\mathcal{C}$.

Definition 3.10. Let $\mathcal{D} = (O, A, s, t)$ be a diagram scheme, let $\mathcal{C}$ be a category, and let F be a diagram of type $\mathcal{D}$ in $\mathcal{C}$, as described in Remark 3.9. A *colimit*[12] *of* F is the datum of an object Y of $\mathcal{C}$, and of a morphism $i_x : F(x) \longrightarrow Y$ in $\mathcal{C}$ for every $x \in O$, so that $i_y \circ F(a) = i_x$ for every arrow $x \xrightarrow{a} y$ of $\mathcal{D}$; and such that this datum is universal(ly repelling, i.e., "initial") among all such data.

The universal property mentioned in this definition is as follows. Let $Z \in \mathcal{C}$, and suppose that for every $x \in O$ we are given a morphism $j_x : F(x) \longrightarrow Z$ in $\mathcal{C}$, so that $j_y \circ F(a) = j_x$ for every arrow $x \xrightarrow{a} y$ of $\mathcal{D}$. Then there exists a unique morphism $f : Y \longrightarrow Z$ in $\mathcal{C}$ such that $f \circ i_x = j_x$ for all $x \in O$.

Often, by abuse of notation, we will denote a colimit of F by the single symbol $\varinjlim F$, or by $\varinjlim_{x \in \mathcal{D}} F(x)$; this will also denote the object Y from Definition 3.10.

Remark 3.11. It is easy to see that this language is really equivalent to the language of colimits of functors from small categories to the category $\mathcal{C}$. On the one hand, with the setup above, $\mathcal{P}a(\mathcal{D})$ is a small category, and the definition of the colimit of the diagram F is equivalent to the definition of the colimit of the corresponding functor $F : \mathcal{P}a(\mathcal{D}) \longrightarrow \mathcal{C}$. On the other hand, if $\mathscr{I}$ is any small category and $G : \mathscr{I} \longrightarrow \mathcal{C}$ is any functor, let $U(G)$ denote the diagram of type $U(\mathscr{I})$ in $\mathcal{C}$ obtained from G in the tautological manner. It is easy to see that a colimit of the diagram $U(G)$ is also a colimit of the functor G, and vice versa.

[12]The term "direct limit" (or "inductive limit") is used in [GZ] in place of "colimit".

Definition 3.12. We say that a (not necessarily small) category $\mathcal{C}$ *has all small colimits*, or *is cocomplete*, if for every diagram scheme $\mathcal{D}$, every diagram of type $\mathcal{D}$ in $\mathcal{C}$ has a colimit in the sense of Definition 3.10.

Remark 3.13. Informally speaking, the notion of a *limit* of a diagram in a category can be obtained from the notion of a colimit by inverting all the arrows. To make this statement more precise, given a diagram scheme $\mathcal{D}$, let us define the "opposite diagram scheme", $\mathcal{D}^{op}$, to be the diagram scheme obtained from $\mathcal{D}$ by switching the source map with the target map. If $\mathcal{C}$ is any category, a diagram of type $\mathcal{D}$ in $\mathcal{C}$ can be regarded as a diagram of type $\mathcal{D}^{op}$ in $\mathcal{C}^{op}$. By definition, a limit of the former diagram in $\mathcal{C}$ is the same as a colimit of the latter diagram in $\mathcal{C}^{op}$.

For the reader's convenience, let us write out the definition of a limit explicitly in the format identical to that of Definition 3.10.

Definition 3.14. Let $\mathcal{D} = (O, A, s, t)$ be a diagram scheme, let $\mathcal{C}$ be a category, and let G be a diagram of type $\mathcal{D}$ in $\mathcal{C}$, as described in Remark 3.9. A *limit*[13] *of* G is the datum of an object Y of $\mathcal{C}$, and of a morphism $p_x : Y \longrightarrow G(x)$ in $\mathcal{C}$ for every $x \in O$, so that $G(a) \circ p_x = p_y$ for every arrow $x \xrightarrow{a} y$ of $\mathcal{D}$; and such that this datum is universal(ly attracting, i.e., "final") among all such data.

The universal property mentioned in this definition is as follows. Let $Z \in \mathcal{C}$, and suppose that for every $x \in O$ we are given a morphism $q_x : Z \longrightarrow G(x)$ in $\mathcal{C}$, so that $G(a) \circ q_x = q_y$ for every arrow $x \xrightarrow{a} y$ of $\mathcal{D}$. Then there exists a unique morphism $g : Z \longrightarrow Y$ in $\mathcal{C}$ such that $p_x \circ g = q_x$ for all $x \in O$.

3.5. **Small colimits of small categories.** It is easy to check that the category of diagram schemes has all small colimits, which are computed, so to speak, "entry-wise". In other words, for a fixed diagram scheme $\mathcal{D}$ and a diagram F of type $\mathcal{D}$ in $\mathcal{D}ia$, if we write $F(x) = (O_x, A_x, s_x, t_x)$ for every vertex x of $\mathcal{D}$, then

$$\varinjlim_{x\in\mathcal{D}} \big(O_x, A_x, s_x, t_x\big) = \Big(\varinjlim_{x\in\mathcal{D}} A_x, \varinjlim_{x\in\mathcal{D}} O_x, \varinjlim_{x\in\mathcal{D}} s_x, \varinjlim_{x\in\mathcal{D}} t_x\Big).$$

For instance, this claim follows immediately from Remark 3.2.

One can use this observation to prove the following

Proposition 3.15. *The category* $\mathcal{C}at$ *of small categories has all small colimits. Moreover, the functor* $\mathcal{P}a : \mathcal{D}ia \longrightarrow \mathcal{C}at$ *preserves small colimits.*

Proof. The second statement is formal, because $\mathcal{P}a$ is left adjoint to the functor $U : \mathcal{C}at \longrightarrow \mathcal{D}ia$. To prove the first statement, let us fix a diagram scheme $\mathcal{D}$ and a diagram F of type $\mathcal{D}$ in $\mathcal{C}at$. Then $U \circ F$ is a diagram of type $\mathcal{D}$ in $\mathcal{D}ia$, so by the previous observation, it has a colimit. Let us call it $\mathcal{L}$, and let us denote the "structure morphisms" by $I_x : U(F(x)) \longrightarrow \mathcal{L}$.

[13]The term "inverse limit" (or "projective limit") is used in [GZ] in place of "limit".

Now $\mathcal{L}$ is a diagram scheme, so we can form the path category $\mathcal{P}a(\mathcal{L}) \in \mathcal{C}at$, introduced in Definition 3.7. Moreover, we have a natural morphism $\mathcal{L} \longrightarrow U(\mathcal{P}a(\mathcal{L}))$, and we obtain the induced compositions $I'_x : U(F(x)) \longrightarrow U(\mathcal{P}a(\mathcal{L}))$ for all $x \in \mathcal{D}$.

If these compositions came from functors $F(x) \longrightarrow \mathcal{P}a(\mathcal{L})$, then $\mathcal{P}a(\mathcal{L})$ would be the colimit of the diagram F in $\mathcal{C}at$, and the proof would be complete. In general, there is no reason for this to be the case (remember that the functor U is faithful, but not full). Instead, we must replace $\mathcal{P}a(\mathcal{L})$ by a suitable quotient which, so to speak, forces all the I'_x to become functors.

Let us consider the smallest equivalence relation $\sim$ on the set of morphisms in $\mathcal{P}a(\mathcal{L})$ with the following properties:

- if $f, g \in \mathcal{A}r(\mathcal{P}a(\mathcal{L}))$ and $f \sim g$, then $h \circ f \circ h' \sim h \circ g \circ h'$ for all morphisms $h, h' \in \mathcal{A}r(\mathcal{P}a(\mathcal{L}))$ such that the compositions are defined;
- if $x \in \mathcal{D}$ and $\alpha, \beta \in \mathcal{A}r(F(x))$ are such that $\alpha \circ \beta$ is defined in $F(x)$, then
$$I'_x(\alpha \circ \beta) \sim I'_x(\alpha) \circ I'_x(\beta) \qquad \text{in } \mathcal{A}r(\mathcal{P}a(\mathcal{L}));$$
- if $x \in \mathcal{D}$ and $Y \in F(x)$, then $I'_x(\mathrm{id}_Y) \sim \mathrm{id}_{I'_x(Y)}$ in $\mathcal{A}r(\mathcal{P}a(\mathcal{L}))$.

If $\mathcal{C}$ is the category with $\mathcal{O}b(\mathcal{C}) = \mathcal{O}b(\mathcal{P}a(\mathcal{L}))$ and $\mathcal{A}r(\mathcal{C}) = \mathcal{A}r(\mathcal{P}a(\mathcal{L}))/\sim$, with the composition of morphisms induced by that in $\mathcal{P}a(\mathcal{L})$, it is clear that the diagram morphisms $I'_x : U(F(x)) \longrightarrow U(\mathcal{P}a(\mathcal{L}))$ induce functors $i_x : F(x) \longrightarrow \mathcal{C}$ for every $x \in \mathcal{D}$, and it is straightforward to verify that the datum $(\mathcal{C}, i_x)$ satisfies the universal property defining the colimit of the diagram F in $\mathcal{C}at$. $\square$

4. Homotopy theory of simplicial sets

The goal of this section is to recall some terminology and basic facts from the homotopy theory of simplicial sets. The main reference is Chapter V of [GM].

4.1. **Notation and terminology.** The category of simplicial sets will be denoted by $\mathcal{S}et_\Delta$. By definition, it is the category of functors from Δ^{op} to the category of sets, where Δ is the category with objects $[n] := \{0, 1, \ldots, n\}$, where $[n]$ is viewed as a totally ordered set in the usual way and $n = 0, 1, 2, \ldots$, and with morphism the nondecreasing maps.

More generally, if $\mathcal{C}$ is any category, a *simplicial object* in $\mathcal{C}$ is a functor $\Delta^{op} \longrightarrow \mathcal{C}$. Simplicial objects in categories other than $\mathcal{S}et$ will not be used in these notes.

By abstract nonsense, the category $\mathcal{S}et_\Delta$ has (small) limits and colimits; in particular, it has an initial object, $\varnothing$, and a final object, $*$. By definition, $\varnothing([n])$ is the empty set, and $*([n])$ has exactly one element, for every $n \geq 0$. Given $X \in \mathcal{S}et_\Delta$, the canonical maps $\varnothing \longrightarrow X$ and $X \longrightarrow *$ will be denoted by i_X and p_X, respectively.

4.1.1. *Simplices and related notions.* Let us introduce some standard and classical terms that are used in working with simplicial sets.

(a) If $X \in \mathcal{S}et_{\triangle}$, we will usually write X_n in place of $X([n])$; elements of the set X_n are called the *n-simplices* of X. If $f : [m] \longrightarrow [n]$ is a morphism in the category $\triangle$, the induced map $X_n \longrightarrow X_m$ will be denoted by f^*.

(b) If $X \in \mathcal{S}et_{\triangle}$, elements of X_0 are usually called the *vertices* of X, while elements of X_1 are called the *edges* of X. Note that the edges of X are oriented, in the sense that every $e \in X_1$ defines an ordered pair $(s(e), t(e))$, where $s(e) = j_0^*(e) \in X_0$ is the *source* of e, and $t(e) = j_1^*(e) \in X_0$ is the *target* of e. Here, $j_0, j_1 : [0] \hookrightarrow [1]$ are the maps defined by $0 \mapsto 0$ and $0 \mapsto 1$, respectively.

(c) One generalization of the notions of the source and target of an edge is provided by the notion of a vertex of simplex. Namely, if $X \in \mathcal{S}et_{\triangle}$ and $0 \leq i \leq n$, we define, for each $z \in X_n$, the *i-th vertex*, $v_i(z) \in X_0$, of z, as the image of z under the map $X_n \longrightarrow X_0$ induced by the inclusion $[0] \hookrightarrow [n]$, $0 \mapsto i$.

(d) Two special kinds of morphisms in the category $\triangle$ play a prominent role. Given $0 \leq i \leq n$ with $n \geq 1$, we define $\partial_n^i : [n-1] \longrightarrow [n]$ as the unique strictly increasing map whose image does not contain i. For instance, the maps j_0 and j_1 introduced above are the same as ∂_1^1 and ∂_1^0, respectively.

(e) With the same notation, if $X \in \mathcal{S}et_{\triangle}$ and $z \in X_n$, the $(n-1)$-simplex $\partial_i(z) := (\partial_n^i)^*(z)$ is called the *i-th face of z*. Informally speaking, $\partial_i(z)$ is "the face of z opposite to the i-th vertex of z". Note that, by definition, vertices of X do not have faces. The map $\partial_i : X_n \longrightarrow X_{n-1}$ is called the *i-th face map*.

(f) If $0 \leq i \leq n$, we let $\sigma_n^i : [n+1] \longrightarrow [n]$ be the unique nondecreasing surjective map of sets that takes on the value i twice. It is easy to see that every morphism in the category $\triangle$ can be factored uniquely as a surjective map followed by an injective map, and that, in turn, every surjective (respectively, injective) morphism in $\triangle$ can be written as a composition of the maps σ_n^i (respectively, ∂_n^i). We refer the reader, for instance, to §II.2 of [GZ] for a proof of the fact that all relations among these maps in $\triangle$ are generated by the following relations:

$$\partial_{n+1}^j \circ \partial_n^i = \partial_{n+1}^i \circ \partial_n^{j-1} \qquad \text{if} \quad i < j;$$

$$\sigma_n^j \circ \sigma_{n+1}^i = \sigma_n^i \circ \sigma_{n+1}^{j+1} \qquad \text{if} \quad i \leq j;$$

$$\sigma_{n-1}^j \circ \partial_n^i = \begin{cases} \partial_{n-1}^i \circ \sigma_{n-2}^{j-1} & \text{if } i < j, \\ \mathrm{id}_{[n-1]} & \text{if } i = j \text{ or } i = j+1, \\ \partial_{n-1}^{i-1} \circ \sigma_{n-2}^j & \text{if } i > j+1. \end{cases}$$

(g) If $X \in \mathcal{S}et_{\triangle}$ and $0 \leq i \leq n$, we write $s_i = (\sigma_n^i)^* : X_n \longrightarrow X_{n+1}$, and call it the *$i$-th degeneracy map* for X. A simplex (of any dimension) in X is said to be *nondegenerate* if it is *not* in the image of any of the degeneracy maps. Note that, by definition, all 0-simplices of X are nondegenerate.

4.1.2. *First examples of simplicial sets.* The following special classes of simplicial sets are ubiquitous in homotopy theory, and will appear often in these notes.

(1) For each $n = 0, 1, 2, \ldots$, we write Δ^n for the object of $\mathcal{S}et_{\Delta}$ represented by $[n]$, i.e., $(\Delta^n)_m = \mathrm{Hom}_{\Delta}([m], [n])$ for all $m \geq 0$. This object is called the *standard* n-simplex.

(2) The *simplicial* $(n-1)$-*sphere*[14] is defined as the subobject $\dot{\Delta}^n \subset \Delta^n$ given by

$$\big(\dot{\Delta}^n\big)_m = \big\{ f \in \mathrm{Hom}_{\Delta}([m], [n]) \,\big|\, f \text{ is not surjective} \big\}.$$

(3) If $0 \leq i \leq n$, we define the (n, i)-*horn* $\Lambda^n_i \subset \Delta^n$ as the subobject given by

$$\big(\Lambda^n_i\big)_m = \big\{ f \in \mathrm{Hom}_{\Delta}([m], [n]) \,\big|\, \mathrm{Im}(f) \neq [n],\ \mathrm{Im}(f) \neq [n] \setminus \{k\} \big\}.$$

This horn is said to be *inner* if $0 < i < n$.

4.2. **General comment.** The main goal of this section is to recall the definitions and basic properties of several classes of morphisms in the category of $\mathcal{S}et_{\Delta}$, namely:

- cofibrations, defined simply as the monomorphisms in $\mathcal{S}et_{\Delta}$;
- Kan fibrations;
- weak equivalences;
- acyclic Kan fibrations (i.e., Kan fibrations that are also weak equivalences);
- acyclic cofibrations, also known as the *anodyne morphisms.*

The notion of a Kan fibration is always defined in terms of a certain lifting property. The notion of a cofibration is of course the most understandable one. The notion of a weak equivalence is probably easiest to grasp when it is defined using the geometric realization functor (see §4.3). However, this notion also has a purely combinatorial definition: see, for instance, §V.1 of [GM], and also §4.10 below. A summary of all the relevant definitions is given in §4.11 at the end of this section.

Remark 4.1. It is proved in *loc. cit.* that there exists a model category structure on $\mathcal{S}et_{\Delta}$ for which the cofibrations are the monomorphisms, the fibrations are the Kan fibrations, and the weak equivalences are the weak equivalences between simplicial sets. The notion of a model category[15] will not appear in this section. Nevertheless, the classes of the morphisms mentioned above will be very important for us.

4.3. **Geometric realization.** Let $\mathcal{T}op$ denote the category of topological spaces and continuous maps. We have an adjoint pair of functors,

$$|\cdot| : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{T}op \qquad \text{and} \qquad \mathrm{Sing} : \mathcal{T}op \longrightarrow \mathcal{S}et_{\Delta},$$

where $|\cdot|$ is called the *geometric realization* functor and Sing is called the *total singular complex* functor; Sing is right adjoint to $|\cdot|$.

[14] Many authors use the notation $\partial\Delta^n$ in place of $\dot{\Delta}^n$.

[15] Introduced by D.G. Quillen in 1967.

Let us briefly recall how these functors are constructed. For each $n = 0, 1, 2, \dots$, we define the *standard n-dimensional simplex* as the following subset of $\mathbb{R}^n$:

$$|\Delta^n| = \big\{(x_1, \dots, x_n) \in \mathbb{R}^n \,\big|\, 0 \le x_1 \le \cdots \le x_n \le 1\big\}.$$

We equip $|\Delta^n|$ with the topology induced by the standard topology on $\mathbb{R}^n$. By convention, both $\mathbb{R}^0$ and $|\Delta^0|$ consist of one point.

We would like to turn the assignment $[n] \longmapsto |\Delta^n|$ into a functor[16] $\mathbb{\Delta} \longrightarrow \mathcal{T}op$. Given a nondecreasing map $f : [m] \longrightarrow [n]$, let us explain a geometric construction of the corresponding continuous map $|f| : |\Delta^m| \longrightarrow |\Delta^n|$. A different description of the functor $[n] \longmapsto |\Delta^n|$ can be found in §4.4 below.

Let us first note that the set of all points of $|\Delta^n|$ has a natural partial order, given by $x \le y \iff x_j \le y_j$ for all $1 \le j \le n$. This partial ordering induces a total ordering on the set of all vertices of the simplex $|\Delta^n|$ (the vertices are the points $(0, \dots, 0, 1)$, $(0, \dots, 0, 1, 1)$, $\dots$, $(1, 1, \dots, 1)$). Thus there is a unique order-preserving bijection between $[n]$ and the set of vertices of $|\Delta^n|$. With this in mind, we define $|f| : |\Delta^m| \longrightarrow |\Delta^n|$ to be the unique affine map that takes vertices of $|\Delta^m|$ to vertices of $|\Delta^n|$ in the way determined by the map $f : [m] \longrightarrow [n]$.

We have thus described the geometric realization functor $|\cdot|$ on the full subcategory of $\mathcal{S}et_{\mathbb{\Delta}}$ formed by the representable functors. This is enough to recover the functor $|\cdot|$ on all of $\mathcal{S}et_{\mathbb{\Delta}}$, using the fact that this functor must commute with small colimits. We do not wish to recall the precise description of $|\cdot|$, since it is quite standard[17].

The total singular complex functor $\operatorname{Sing} : \mathcal{T}op \longrightarrow \mathcal{S}et_{\mathbb{\Delta}}$ is defined as follows: given any $X \in \mathcal{T}op$, the functor $\operatorname{Sing}(X) : \mathbb{\Delta}^{op} \longrightarrow \mathcal{S}et$ is the composition

$$\mathbb{\Delta}^{op} \xrightarrow{\text{Yoneda}} \mathcal{S}et_{\mathbb{\Delta}}^{op} \xrightarrow{|\cdot|} \mathcal{T}op^{op} \xrightarrow{\operatorname{Hom}(\cdot, X)} \mathcal{S}et.$$

4.4. **An alternate description of $|\Delta^n|$.** There exists a different description of the topological space $|\Delta^n|$, explained in [Dri], which makes the functorial nature of the assignment $[n] \longmapsto |\Delta^n|$ especially obvious. Let us briefly explain this description.

Consider all nondecreasing maps $\gamma : [0, 1] \longrightarrow [n]$, where $[0, 1]$ denotes the closed unit interval with the standard ordering. Any such map is necessarily piecewise constant. Let us introduce an equivalence relation on the set $\mathcal{S}^n$ of all such maps, defined as follows: $\gamma_1 \sim \gamma_2$ if γ_1 and γ_2 agree outside of a finite subset of $[0, 1]$.

There is a natural bijection between $|\Delta^n|$ and the set $\mathcal{S}^n/{\sim}$ of the equivalence classes for this relation, given by sending a point $(x_1, \dots, x_n) \in |\Delta^n|$ to the equivalence class formed by the maps $[0, 1] \longrightarrow [n]$ that take $(0, x_1)$ to 0, (x_1, x_2) to 1,

[16]By definition, such a functor is called a "cosimplicial topological space".

[17]For the references, we especially recommend §I.2.2 in [GM] and [Dri].

(x_2, x_3) to 2, etc., and $(x_n, 1)$ to n. It is not hard to check that this bijection becomes a homeomorphism if we equip $\mathcal{S}^n/\sim$ with the topology induced by the metric

$$d\big([\gamma_1], [\gamma_2]\big) = \mu\big(\{x \in [0,1] \,\big|\, \gamma_1(x) \neq \gamma_2(x)\}\big),$$

where $[\gamma_1], [\gamma_2]$ are the equivalence classes of $\gamma_1, \gamma_2 \in \mathcal{S}^n$, and μ is the standard Lebesgue measure on $[0, 1]$.

Now, given any morphism $f : [m] \longrightarrow [n]$ in $\triangle$, composition with f defines a continuous map $\mathcal{S}^m/\sim \longrightarrow \mathcal{S}^n/\sim$, and using the identification above, we obtain a continuous map $|f| : |\Delta^m| \longrightarrow |\Delta^n|$.

4.5. **A category-theoretical remark.** The construction of the functors $|\cdot|$ and Sing recalled above can be thought of as a particular instance a much more general pattern. Namely, it can be obtained by applying Proposition 4.2 below to the functor $\triangle \longrightarrow \mathcal{T}op$, $[n] \longmapsto |\Delta^n|$, which was described in §4.3 and in §4.4.

To state the next result, we introduce the following notation. If $\mathcal{D}$ is a small category, we write $\mathrm{PreSh}(\mathcal{D})$ for the category of functors $\mathcal{D}^{op} \longrightarrow \mathcal{S}et$, and call it the category of *presheaves of sets* on $\mathcal{D}$. (We would not have been able to define $\mathrm{PreSh}(\mathcal{D})$ without assuming that $\mathcal{D}$ is small.) Thus, for example, $\mathcal{S}et_{\triangle} = \mathrm{PreSh}(\triangle)$.

Let us recall from §2.4 that we have the Yoneda embedding $h^{\mathcal{D}} : \mathcal{D} \longrightarrow \mathrm{PreSh}(\mathcal{D})$, which is a fully faithful functor given by $h^{\mathcal{D}}(X) \equiv h^{\mathcal{D}}_X = \mathrm{Hom}_{\mathcal{D}}(\cdot, X)$.

Proposition 4.2 ([GZ], Proposition II.1.3)**.** *Let $\mathcal{D}$ be a small category, and let $\mathcal{C}$ be an arbitrary category which has all small colimits.*

(*a*) *For an arbitrary functor*

$$L : \mathrm{PreSh}(\mathcal{D}) \longrightarrow \mathcal{C},$$

the following two statements are equivalent:
  (*i*) *the functor L commutes with all small colimits;*
  (*ii*) *the functor L has a right adjoint, $R : \mathcal{C} \longrightarrow \mathrm{PreSh}(\mathcal{D})$.*

(*b*) *The functor $L \longmapsto L \circ h^{\mathcal{D}}$ defines an equivalence between the full subcategory of the category of functors $\mathrm{PreSh}(\mathcal{D}) \longrightarrow \mathcal{C}$ formed by those functors that commute with direct limits, and the category of all functors $\mathcal{D} \longrightarrow \mathcal{C}$.*

(*c*) *Thus, we see that every functor $F : \mathcal{D} \longrightarrow \mathcal{C}$ determines a unique* (*up to canonical isomorphism*) *adjoint pair of functors, $L : \mathrm{PreSh}(\mathcal{D}) \longrightarrow \mathcal{C}$ and $R : \mathcal{C} \longrightarrow \mathrm{PreSh}(\mathcal{D})$, so that $L \circ h^{\mathcal{D}} = F$. Furthermore, we have $R(Z) = h^{\mathcal{C}}_Z \circ F$, where F is viewed as a functor $\mathcal{D}^{op} \longrightarrow \mathcal{C}^{op}$.*

To the reader who is seeing this material for the first time, we recommend proving this proposition as an instructive exercise.

Remark 4.3. Part (b) of Proposition 4.2 can be interpreted (somewhat imprecisely) as follows: one gets the category of presheaves of sets on $\mathcal{D}$ from the category $\mathcal{D}$ itself by formally adding small colimits of objects of $\mathcal{D}$ in a universal way.

4.6. **Compactly generated spaces.** Sometimes it is more convenient to view the geometric realization functor as taking values in a certain full subcategory of $\mathcal{T}op$, namely, the category $\mathcal{CG}$ of "compactly generated spaces", introduced below. For instance, the functor $|\cdot| : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{T}op$ does not commute with finite limits (see Exercise 9.1), while working with $\mathcal{CG}$ resolves this issue. On the other hand, it is well known that in "abstract" point-set topology and algebraic topology, one should work with the category $\mathcal{CG}$ to avoid various pathologies that occur in $\mathcal{T}op$.

Definition 4.4. (a) A topological space X is said to be *weak Hausdorff* if for every compact Hausdorff space K, the image of any continuous map $K \longrightarrow X$ is closed in X (thus Hausdorff $\equiv T_2 \Longrightarrow$ weak Hausdorff $\Longrightarrow T_1$).

(b) A subset A of a topological space X is said to be *compactly closed* if for every compact Hausdorff space K and every continuous map $f : K \longrightarrow X$, the preimage $f^{-1}(A)$ is closed in K.

(c) A topological space X is said to be a *k-space* if every compactly closed subset of X is closed.

(d) A *compactly generated*[18] space is a weak Hausdorff k-space.

We let $\mathcal{CG}$ denote the category of all compactly generated topological spaces.

Proposition 4.5. *For any simplicial set X, the geometric realization $|X|$ is a compactly generated space. Viewed as a functor $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{CG}$, the geometric realization functor $|\cdot|$ commutes with finite limits and arbitrary (small) colimits.*

For a discussion of this result, we refer the reader to [Dri]. (Of course, the result itself is much older; it goes back to the work of J. Milnor in the 1950s).

4.7. **Kan fibrations.** A morphism of simplicial sets is said to be a *Kan fibration* if it satisfies the right lifting property (see §2.6) with respect to all horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$, for all $0 \leq i \leq n$. A simplicial set E is said to be *Kan* (or a Kan complex) if the canonical morphism $p_E : E \longrightarrow *$ is a Kan fibration.

For instance, if X is any topological space, then the total singular complex $\mathrm{Sing}(X)$ is a Kan complex (Exercise 9.19). For another class of examples of Kan complexes, see Exercise 9.20.

Remark 4.6. A simplicial set E is Kan if and only if the canonical (adjunction) morphism $\eta_E : E \longrightarrow \mathrm{Sing}|E|$ admits a retraction, i.e., if and only if there exists a morphism $f : \mathrm{Sing}|E| \longrightarrow E$ with $f \circ \eta_E = \mathrm{id}_E$. Indeed, the "if" direction follows from the definition of Kan fibrations and the easy exercise that if $\mathcal{C}$ is any category and $\mathscr{M}$ is any class of morphisms in $\mathcal{C}$, then the class of morphisms $\rho(\mathscr{M})$, defined in §2.6, is stable under retractions. The "only if" direction follows from the fact

[18] In [GZ], the term "Kelley space" is used in place of "compactly generated topological space". Also, in [GZ], Kelley spaces are required to be Hausdorff. However, the weak Hausdorff requirement is both more natural (see, e.g., Exercise 9.3) and technically more convenient.

that η_E is an acyclic cofibration (see §§4.9–4.10 below) and that Kan fibrations have the R.L.P. with respect to all acyclic cofibrations; thus, the commutative square

$$\begin{array}{ccc} E & \xrightarrow{\mathrm{id}_E} & E \\ {\scriptstyle \eta_E}\downarrow & & \downarrow \\ \mathrm{Sing}|E| & \longrightarrow & * \end{array}$$

can be completed to a commutative diagram via a morphism $f : \mathrm{Sing}|E| \longrightarrow E$.

4.8. **The inner Hom in $\mathcal{S}et_\Delta$.** The category $\mathcal{S}et_\Delta$ is *Cartesian closed*, meaning that for any pair of objects $X, Y \in \mathcal{S}et_\Delta$, there exists an object $X^Y \in \mathcal{S}et_\Delta$ together with a functorial collection of bijections

$$\mathrm{Hom}_{\mathcal{S}et_\Delta}(Z \times Y, X) \cong \mathrm{Hom}_{\mathcal{S}et_\Delta}(Z, X^Y) \qquad \text{for all } Z \in \mathcal{S}et_\Delta.$$

Explicitly, X^Y is the simplicial set given by $(X^Y)_n = \mathrm{Hom}_{\mathcal{S}et_\Delta}(\Delta^n \times Y, X)$ (one could also define X^Y abstractly via a suitable application of Proposition 4.2).

In fact, we see that $\mathcal{S}et_\Delta$ becomes enriched over itself, i.e., becomes a simplicial category. In order to emphasize this fact, we will sometimes write $\mathrm{Map}(Y, X)$ or $\mathrm{Map}_{\mathcal{S}et_\Delta}(X, Y)$ in place of X^Y, for a pair of simplicial sets X, Y.

Definition 4.7 (Homotopy classes of morphisms). For any simplicial set Z, we write $\pi_0(Z)$ for the quotient of the set $Z_0 \equiv Z([0])$ by the equivalence relation generated by the relation $z \approx w$ if there exists $\sigma \in Z_1$ with $\partial_0(\sigma) = z$ and $\partial_1(\sigma) = w$.

If X and Y are simplicial set, we define a set $[X, Y]$ by $[X, Y] = \pi_0(\mathrm{Map}(X, Y))$; its elements are called the (combinatorial) *homotopy classes of morphisms* from X to Y. We warn the reader that usually, $[X, Y]$ is only the right object to consider when Y is a Kan complex.

4.9. **Cofibrations and anodyne morphisms.** We define a morphism $f : X \longrightarrow Y$ of simplicial sets to be a *cofibration* if f is a monomorphism in $\mathcal{S}et_\Delta$; equivalently, if the induced map of sets $f_n : X_n \longrightarrow Y_n$ is injective for all $n \geq 0$ (Exercise 9.6).

Cofibrations in $\mathcal{S}et_\Delta$ admit an alternate characterization. Namely, let us define $\mathscr{T}$ to be the class of all morphisms in $\mathcal{S}et_\Delta$ that have the right lifting property (§2.6) with respect to all sphere inclusions $\overset{\bullet}{\Delta}{}^n \hookrightarrow \Delta^n$ $(n \geq 0)$.

Lemma 4.8 (cf. Lemma V.2.5 in [GM])**.** *The monomorphisms in $\mathcal{S}et_\Delta$ are precisely the morphisms having the left lifting property with respect to all morphisms in $\mathscr{T}$.*

Definition 4.9. A morphism in $\mathcal{S}et_\Delta$ is said to be *anodyne* if it has the left lifting property with respect to all Kan fibrations (see §4.7).

It is perhaps not immediately obvious, but true, that anodyne morphisms in $\mathcal{S}et_\Delta$ are cofibrations. Moreover, anodyne morphisms can be characterized as those cofibrations in $\mathcal{S}et_{\triangle}$ that are also weak equivalences (see §4.10 and Lemma 4.13). A typical example of an anodyne morphism is a horn inclusion $\Lambda^n_i \hookrightarrow \Delta^n$, where $0 \leq i \leq n$; this follows tautologically from Definition 4.9.

The next result is useful. It is left as an instructive exercise for the reader.

Lemma 4.10 (cf. Lemma V.2.8 in [GM])**.** *Let $f : X \hookrightarrow Y$ be any cofibration, and let $g : Z \longrightarrow W$ be an anodyne morphism, in $\mathcal{S}et_{\triangle}$. The induced morphism*

$$(X \times W) \coprod_{X\times Z} (Y \times Z) \longrightarrow Y \times W$$

is also anodyne, where the left hand side is the pushout of the morphisms $(\mathrm{id}_X \times g) : X \times Z \longrightarrow X \times W$ *and* $(f \times \mathrm{id}_Z) : X \times Z \longrightarrow Y \times Z$.

4.10. **Weak equivalences between simplicial sets.** If one is allowed to use the geometric realization functor (§4.3), the easiest way to define weak equivalences of simplicial sets is as follows. Let us recall that a continuous map $f : X \longrightarrow Y$ of topological spaces is said to be a *weak homotopy equivalence* if for every $x \in X$, the map $\pi_i(X, x) \longrightarrow \pi_i(Y, f(x))$ induced by f is bijective for all $i \geq 0$.

Definition 4.11. A morphism $f : X \longrightarrow Y$ of simplicial sets is a *weak equivalence* if the induced continuous map $|f| : |X| \longrightarrow |Y|$ is a weak homotopy equivalence of topological spaces.

Remark 4.12. In fact, it is known that $|X|$ and $|Y|$ are CW complexes, so a weak homotopy equivalence between them is the same as a homotopy equivalence.

The following results are well known (see §V.2 of [GM], and [May]).

Lemma 4.13. *A morphism in $\mathcal{S}et_{\triangle}$ is anodyne if and only if it is both a cofibration and a weak equivalence.*

Proposition 4.14 (Characterizations of weak equivalences)**.** *For a morphism $f : X \longrightarrow Y$ of simplicial sets, the following are equivalent:*

(*i*) *f is a weak equivalence;*

(*ii*) *f can be factored as $g \circ h$, where g satisfies the R.L.P. (§2.6) with respect to all sphere inclusions $\dot{\Delta}^n \hookrightarrow \Delta^n$ (see §4.1), and h satisfies the L.L.P. with respect to all Kan fibrations (§4.7);*

(*iii*) *for every Kan complex E, the map $[Y, E] \longrightarrow [X, E]$ induced by f is a bijection.*

We remark that the equivalence between properties (ii) and (iii) is proved in Theorem V.2.3 of [GM].

4.11. **Summary.** Let $\mathscr{F}$, $\mathscr{C}$ and $\mathscr{W}$ denote the classes of Kan fibrations, cofibrations, and weak equivalences in the category $\mathcal{S}et_{\Delta}$, introduced in §4.7, §4.9 and §4.10, respectively. Let $\mathscr{H}$ and $\mathscr{S}$ denote the classes of *horns* and *spheres* in $\mathcal{S}et_{\Delta}$, defined as the arbitrary disjoint unions of the horn inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ and the sphere inclusions $\overset{\bullet}{\Delta}{}^n \hookrightarrow \Delta^n$, respectively, where $0 \leq i \leq n$. With the notation of §2.6, the relations between these classes of morphisms can be summarized as follows:

- $\mathscr{F} = \rho(\mathscr{H}) = \rho\lambda\rho(\mathscr{H}) = \rho(\mathscr{C} \cap \mathscr{W})$ (the last two equalities are formal);
- $\mathscr{C} = \lambda\rho(\mathscr{S}) = \lambda(\mathscr{F} \cap \mathscr{W}) = \{\text{all monomorphisms in } \mathcal{S}et_{\Delta}\}$;
- $\mathscr{W}$ is the class of morphisms $f : X \longrightarrow Y$ in $\mathcal{S}et_{\Delta}$ whose geometric realization $|f| : |X| \longrightarrow |Y|$ is a weak homotopy equivalence (§4.10); combinatorially,
$$\mathscr{W} = \rho(\mathscr{S}) \circ \lambda\rho(\mathscr{H}) = (\mathscr{F} \cap \mathscr{W}) \circ (\mathscr{C} \cap \mathscr{W}).$$
- $\mathscr{F} \cap \mathscr{W} = \rho(\mathscr{S}) = \rho(\mathscr{C})$ (the last equality is formal);
- $\mathscr{C} \cap \mathscr{W} = \lambda(\mathscr{F}) \equiv \lambda\rho(\mathscr{H}) \overset{\text{def}}{=} \{\text{anodyne morphisms in } \mathcal{S}et_{\Delta}\}$.

4.12. **An application of Quillen's formalism.** In this subsection we give an illustration of how the various relations between Kan fibrations, cofibrations and weak equivalences in $\mathcal{S}et_{\Delta}$ can be used formally to prove non-obvious results. Namely, recall that a Kan complex is a simplicial set E characterized by the property that for all $0 \leq k \leq n$, every morphism $\Lambda^n_k \xrightarrow{f} E$ extends to a morphism $\Delta^n \longrightarrow E$. However, this extension is often non-unique[19]. Therefore, it is natural to ask if something can be said about the set of all possible extensions of f. From the naive point of view, these extensions form a discrete set, so of course, we cannot say anything interesting about it. Instead, we observe that this set can be naturally viewed as the set of 0-simplices of a *simplicial* set, namely, the fiber over f of the morphism $\mathrm{Map}(\Delta^n, E) \longrightarrow \mathrm{Map}(\Lambda^n_k, E)$ induced by the inclusion $\Lambda^n_k \hookrightarrow \Delta^n$ (see §4.8).

It turns out that this simplicial set is a *contractible Kan complex* (a Kan complex Z is said to be contractible if the morphism $p_Z : Z \longrightarrow *$ is a weak equivalence). This statement is a special case of its relative version:

Proposition 4.15. *Let $f : X \longrightarrow Y$ be a Kan fibration of simplicial sets, and consider a commutative square*

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{\widetilde{\alpha}} & X \\ {\scriptstyle\iota}\downarrow & & \downarrow{\scriptstyle f} \\ \Delta^n & \xrightarrow[\alpha]{} & Y \end{array}$$

where ι is the natural inclusion. Consider the simplicial set of all lifts $\Delta^n \longrightarrow X$ preserving commutativity, i.e., the fiber, Z, of the natural morphism

$$(f_*, \iota^*) : \mathrm{Map}(\Delta^n, X) \longrightarrow \mathrm{Map}(\Delta^n, Y) \times \mathrm{Map}(\Lambda^n_k, X)$$

[19] It is *always* unique if and only if E is isomorphic to the nerve of a small groupoid.

over the 0*-simplex* $(\alpha, \widetilde{\alpha})$*. This fiber is a contractible Kan complex.*

Proof. We must show that the morphism $p_Z : Z \longrightarrow *$ is an acyclic Kan fibration. Acyclic Kan fibrations are characterized as the morphisms having the R.L.P. with respect to all cofibrations, i.e., monomorphisms of simplicial sets. Thus let $j : S \hookrightarrow S'$ be any monomorphism of simplicial sets, and consider a morphism $\beta : S \longrightarrow Z$. We must show that there is a morphism $\beta' : S' \longrightarrow Z$ with $\beta = \beta' \circ j$.

By definition, giving β [respectively, β'] amounts to giving a morphism $S \longrightarrow \mathrm{Map}(\Delta^n, X)$ [respectively, $S' \longrightarrow \mathrm{Map}(\Delta^n, X)$] whose composition with (f_*, ι^*) equals the constant morphism given by $(\alpha, \widetilde{\alpha})$. In turn, this is equivalent to giving a morphism $\gamma : S \times \Delta^n \longrightarrow X$ [respectively, $\gamma' : S' \times \Delta^n \longrightarrow X$] such that:

- the restriction of γ to $S \times \Lambda^n_k$ equals $\widetilde{\alpha} \circ \mathrm{pr}_2$, where $\mathrm{pr}_2 : S \times \Lambda^n_k \longrightarrow \Lambda^n_k$ is the second projection [respectively, the restriction of γ' to $S' \times \Lambda^n_k$ equals $\widetilde{\alpha} \circ \mathrm{pr}'_2$, where $\mathrm{pr}'_2 : S' \times \Lambda^n_k \longrightarrow \Lambda^n_k$ is the second projection]; and
- the composition $f \circ \gamma$ equals $\alpha \circ \mathrm{pr}_2$, where $\mathrm{pr}_2 : S \times \Delta^n \longrightarrow \Delta^n$ is the second projection [respectively, the composition $f \circ \gamma'$ equals $\alpha \circ \mathrm{pr}'_2$, where $\mathrm{pr}'_2 : S' \times \Delta^n \longrightarrow \Delta^n$ is the second projection].

Thus we see that the existence of the desired extension β' of β is equivalent to the existence of an extension γ' of γ as specified above. We can rephrase this extension problem in a more compact way as follows. Consider the pair of morphisms $\gamma : S \times \Delta^n \longrightarrow X$ and $\widetilde{\alpha} \circ \mathrm{pr}'_2 : S' \times \Lambda^n_k \longrightarrow X$, where the notation is as above. The restrictions of these morphisms to $S \times \Lambda^n_k$ coincide by construction. Thus we obtain a morphism from the corresponding pushout to X that fits into a commutative square

$$\begin{array}{ccc} \big(S \times \Delta^n\big) \coprod_{S \times \Lambda^n_k} \big(S' \times \Lambda^n_k\big) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ S' \times \Delta^n & \xrightarrow{\alpha \circ \mathrm{pr}'_2} & Y \end{array}$$

Moreover, finding γ' amounts to finding a lift $S' \times \Delta^n \longrightarrow X$ in this diagram that preserves commutativity. However, the left vertical arrow in this diagram is an anodyne morphism by Lemma 4.10. Therefore the desired lift exists because Kan fibrations have the R.L.P. with respect to all anodyne morphisms. □

5. Small categories and simplicial sets

5.1. **Categories and posets.** We use the standard abbreviation: "poset" means "partially ordered set". If P is any poset, we form a small category whose objects are elements of P, and where, given $x, y \in P$, the set $\mathrm{Hom}(x, y)$ has precisely one element if $x \leq y$, and is empty otherwise (this determines composition of morphisms uniquely). By abuse of notation, we will denote this category by P as well.

Observe that if P_1 and P_2 are posets, then defining a functor $P_1 \longrightarrow P_2$ between the corresponding categories is the same as defining a nondecreasing map $P_1 \longrightarrow P_2$. Thus the category of posets and nondecreasing maps is a *full* subcategory of $\mathcal{C}at$.

5.2. **The nerve of a small category.** Let $\mathcal{C}$ be a small category. The *nerve* of $\mathcal{C}$ is the simplicial set $N(\mathcal{C}) : \triangle^{op} \longrightarrow \mathcal{S}et$ defined by[20]

$$N(\mathcal{C})([n]) \equiv N(\mathcal{C})_n = \mathrm{Hom}_{\mathcal{C}at}([n], \mathcal{C}).$$

We recall that $\mathrm{Hom}_{\mathcal{C}at}(\mathcal{C}_1, \mathcal{C}_2)$ denotes the *set* of all functors between two small categories, $\mathcal{C}_1$ and $\mathcal{C}_2$; and that in the formula above, the (totally) ordered set $[n] = \{0, 1, \dots, n\}$ is viewed as a category in the way described in §5.1.

5.3. **The Poincaré category of a simplicial set.** We recall the following

Proposition 5.1. *The nerve functor $N : \mathcal{C}at \longrightarrow \mathcal{S}et_\triangle$ has a left adjoint.*

We will explicitly describe a functor $\mathcal{P} : \mathcal{S}et_\triangle \longrightarrow \mathcal{C}at$ in Definition 5.3. It is a simple exercise to check that this functor is left adjoint to N. The construction of the functor $\mathcal{P}$ is taken from the proposition in §II.4.2 of [GZ].

Definition 5.2. Let X be a simplicial set. We denote by $\mathcal{D}(X)$ the diagram scheme $(X_1, X_0, \partial_0, \partial_1)$, where $X_0 = X([0])$, $X_1 = X([1])$, and $\partial_0, \partial_1 : X_1 \longrightarrow X_0$ are the maps induced by the inclusions $[0] \hookrightarrow [1]$ given by $0 \mapsto 0$ and $0 \mapsto 1$, respectively.

Definition 5.3. Let X be a simplicial set. The *Poincaré category*[21] of X, written $\mathcal{P}(X)$, is the quotient of the path category $\mathcal{P}a(\mathcal{D}(X))$ by the equivalence relation on morphisms generated by the "elementary relation" $\approx$ defined as follows.

If $\sigma \in X_2 = X([2])$, we declare $\partial_1(\sigma) \approx \partial_2(\sigma) \circ \partial_0(\sigma)$, where $\partial_0, \partial_1, \partial_2 : X_2 \longrightarrow X_1$ are induced by the three strictly increasing maps $[1] \hookrightarrow [2]$ whose images omit 0, 1 and 2, respectively.

Remark 5.4. Observe that if X is any simplicial set, the Poincaré category $\mathcal{P}(X)$ depends only on the 2-skeleton[22] of X.

Remark 5.5. The nerve functor $N : \mathcal{C}at \longrightarrow \mathcal{S}et_\triangle$ and the Poincaré category functor $\mathcal{P} : \mathcal{S}et_\triangle \longrightarrow \mathcal{C}at$ could be formally defined by applying Proposition 4.2 to the functor $\triangle \longrightarrow \mathcal{C}at$ that takes $[n]$ to the ordered set $[n]$ viewed as a category.

[20]The action of $N(\mathcal{C})$ on morphisms in $\triangle$ is defined in the obvious way.

[21]In the literature, $\mathcal{P}(X)$ is sometimes called the "fundamental category" of X, or the category "generated by", or "presented by", X. We thank Michael Shulman for explaining its definition to us. Lurie calls $\mathcal{P}(X)$ the "homotopy category" of X; it is studied in §1.2.3 of [T].

[22]That is, the minimal simplicial subset $\mathrm{sk}_2 X \subset X$ such that $(\mathrm{sk}_2 X)_n = X_n$ for $n = 0, 1, 2$.

5.4. **Characterization of nerves of small categories.** It turns out that a small category can be recovered from its nerve up to an isomorphism (not merely an equivalence), and that the class of all simplicial sets that are isomorphic to nerves of small categories admits a simple characterization. These facts lie at the heart of the approach to ∞-categories that we will explain below.

The proofs of the following results are left as exercises.

Lemma 5.6. *The nerve functor $N : \mathcal{C}at \longrightarrow \mathcal{S}et_{\triangle}$ is fully faithful. If $\mathcal{C}$ is any small category, the adjunction morphism $\mathcal{P}(N(\mathcal{C})) \longrightarrow \mathcal{C}$ (induced by the identity morphism $N(\mathcal{C}) \longrightarrow N(\mathcal{C})$) is an isomorphism of categories.*

Lemma 5.7. *If $X \in \mathcal{S}et_{\triangle}$, then $X \cong N(\mathcal{C})$ for some small category $\mathcal{C}$ if and only if for all $0 < i < n$, every morphism $f : \Lambda_i^n \longrightarrow X$ admits a unique extension to a morphism $\Delta^n \longrightarrow X$.*

6. Topological and simplicial categories

6.1. **Definitions of topological and simplicial categories.** This section is devoted to the analogue for simplicial categories of the story explained in Section 5. For us, the original motivation comes from trying to understand the relationship between topological categories and ∞-categories (the latter will be discussed in Section 7). We defined and discussed topological categories in §1.2 and §1.4, but for consistency with [T], let us slightly change the definition:

Definition 6.1. A *topological category* is a category enriched over the category $\mathcal{CG}$ of compactly generated topological spaces (Definition 4.4). Thus, to give a topological category, we must give an ordinary category $\mathcal{C}$ and a compactly generated topology on each of the sets $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ such that the composition laws

$$\mathrm{Hom}_{\mathcal{C}}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X,Z)$$

are continuous for all objects $X, Y, Z \in \mathcal{C}$. Note that the product on the left hand side is computed in $\mathcal{CG}$, so its topology may be finer than the product topology.

As mentioned in §2.5, if $\mathcal{C}$ is a topological category, we will write $\mathrm{Map}_{\mathcal{C}}(X,Y)$, rather than $\mathrm{Hom}_{\mathcal{C}}(X,Y)$, for the topological space of morphisms $X \longrightarrow Y$ in $\mathcal{C}$.

Definition 6.2. A *simplicial category* is a category enriched over the category $\mathcal{S}et_{\triangle}$ of simplicial sets. Thus, to give a simplicial category $\mathcal{C}$, we must give a class $\mathcal{O}b(\mathcal{C})$ of objects of $\mathcal{C}$, and, for every pair $X, Y \in \mathcal{O}b(\mathcal{C})$, a simplicial set $\mathrm{Map}_{\mathcal{C}}(X,Y)$, together with composition laws

$$\mathrm{Map}_{\mathcal{C}}(Y,Z) \times \mathrm{Map}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Map}_{\mathcal{C}}(X,Z)$$

for all triples $X, Y, Z \in \mathcal{O}b(\mathcal{C})$, satisfying the usual associativity and unit[23] axioms.

[23]The identity endomorphism of any $X \in \mathcal{O}b(\mathcal{C})$ is necessarily a 0-simplex in $\mathrm{Map}_{\mathcal{C}}(X,X)$.

It is useful to keep the following comments in mind.

Remarks 6.3. (1) A simplicial category $\mathcal{C}$ gives rise to a countable collection of ordinary categories. Namely, for each $n \geq 0$, we can consider the category $\mathcal{C}_n$ which has the same objects as $\mathcal{C}$, and with $\mathrm{Hom}_{\mathcal{C}_n}(X,Y) = \mathrm{Map}_{\mathcal{C}}(X,Y)_n$ for all X, Y.

(2) The sequence $\mathcal{C}_n$ mentioned in the previous remark is a simplicial object in "the category of categories". Conversely, an arbitrary simplicial object in "the category of categories" arises from a simplicial category in the sense of Definition 6.2 if and only if its underlying simplicial set of objects is constant. (We are using quotation marks because one has to formulate this remark more carefully if one wishes to allow large categories. If we only work with small simplicial categories, there are no technical problems.)

(3) It is also easy to check that a category object in the category of simplicial sets is "the same thing as" a simplicial object in the category of small categories.

6.2. **Relation between three types of higher categories.** As we mentioned in Section 1, the notion of a topological category provides one of the most intuitive approaches to higher category theory. On the other hand, the notion of a quasi-category [Joy], a.k.a. ∞-category [T], is technically the most convenient one. In order to compare these two notions, one employs an appropriate version of the nerve construction. In §1.4, we briefly explained why the ordinary nerve construction is not suitable in the setting of topological categories, and outlined an approach for obtaining the "correct" notion of a nerve, called the topological nerve. It is possible to write down a complete definition of the topological nerve $\mathfrak{N}(\mathcal{C})$ of a (small) topological category $\mathcal{C}$ along the lines of §1.4. However, this approach has the disadvantage of producing long lists of data that are rather difficult to manage.

Fortunately, Lurie explains in [T, §1.1.5] a much more concise approach to the definition of $\mathfrak{N}(\mathcal{C})$. His approach uses simplicial categories, which, more generally, are a very convenient tool for comparing ∞-categories with other models of higher category theory.

The relationship between simplicial categories and topological categories is easy to describe. Let $\mathcal{C}at_\Delta$ and $\mathcal{C}at_{top}$ denote the categories of all small simplicial (respectively, topological) categories. The geometric realization and the total singular complex functors,

$$|\cdot| : \mathcal{S}et_\Delta \longrightarrow \mathcal{CG} \qquad \text{and} \qquad \mathrm{Sing} : \mathcal{CG} \longrightarrow \mathcal{S}et_\Delta,$$

are adjoint, and they both preserve finite products (for Sing, this follows from abstract nonsense; for $|\cdot|$, see Proposition 4.5). It follows that by applying these two functors to the spaces of maps in a simplicial (respectively, topological) category, we obtain an adjoint pair of functors

$$|\cdot| : \mathcal{C}at_\Delta \longrightarrow \mathcal{C}at_{top} \qquad \text{and} \qquad \mathrm{Sing} : \mathcal{C}at_{top} \longrightarrow \mathcal{C}at_\Delta.$$

(The functor $|\cdot|$ is again left adjoint to Sing.)

In order to compare simplicial categories and ∞-categories (more generally, arbitrary simplicial sets), one also uses an adjoint pair of functors, $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$ and $\mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta}$, which we study below.

6.3. **Simplicial Poincaré category.** The rest of this section is devoted to a study of the functor

$$\mathfrak{C} : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$$

introduced in §1.1.5 of [T]. Its definition is recalled in §6.4 below. Lurie does not give this functor a name. Following a suggestion of A. Beilinson, given $X \in \mathcal{S}et_{\Delta}$, we will call $\mathfrak{C}(X)$ the *simplicial Poincaré category* of X. The terminology is motivated by §5.3 and the fact that $\mathfrak{C}$ is left adjoint to the *simplicial nerve functor*

$$\mathfrak{N} : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta},$$

which is also introduced in §1.1.5 of [T].

Remark 6.4. The simplicial nerve functor is denoted by N in [T]. However, for the purposes of these notes, it seems reasonable to use the letter $\mathfrak{N}$, so as to avoid confusion with the usual nerve functor N. The definition of $\mathfrak{N}$ is recalled in §6.4.

6.4. **An abstract definition of $\mathfrak{C}$ and $\mathfrak{N}$.** In order to define the adjoint pair of functors $\mathfrak{C} : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$ and $\mathfrak{N} : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta}$ (where $\mathfrak{C}$ is left adjoint to $\mathfrak{N}$), Lurie defines a functor $\mathfrak{C} : \Delta \longrightarrow \mathcal{C}at_{\Delta}$, $[n] \longmapsto \mathfrak{C}[\Delta^n]$, and applies the construction of Proposition 4.2 to this functor.

Let us recall Lurie's definition of $\mathfrak{C}[\Delta^n]$.

Definition 6.5 (auxiliary). Fix integers $i, j \geq 0$. We define a poset $P_{i,j}$ as follows. The elements of $P_{i,j}$ are sets of integers I such that $i, j \in I$, and if $k \in I$, then $i \leq k \leq j$. The partial ordering on the set $P_{i,j}$ is by inclusion.

Thus, for example, $P_{i,j} = \varnothing$ whenever $i > j$. If $j = i$ or $j = i + 1$, then $P_{i,j}$ has exactly one element. If $j > i$, then $P_{i,j}$ has 2^{j-i-1} elements.

Definition 6.6. For $n = 0, 1, 2, \dots$, we define $\mathfrak{C}[\Delta^n]$ as the simplicial category with the set of morphisms $\{0, 1, \dots, n\}$, and with the morphism spaces given by

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(i, j) = N(P_{i,j}),$$

where $N(P_{i,j})$ is the nerve (§5.2) of the category defined by the poset $P_{i,j}$ (see §5.1). Note that $P_{i,j} = \varnothing$ if $i > j$, in which case we also have $N(P_{i,j}) = \varnothing$. Composition in the simplicial category $\mathfrak{C}[\Delta^n]$ is induced by the union maps $\cup : P_{j,k} \times P_{i,j} \longrightarrow P_{i,k}$.

Remark 6.7. If S is any set, the poset of all subsets of S can be identified with the poset of functions $f : S \longrightarrow \{0, 1\}$, equipped with the partial ordering defined by

$$f \leq g \qquad \Longleftrightarrow \qquad f(s) \leq g(s) \quad \forall\, s \in S.$$

It follows that the nerve of the category defined by this poset is isomorphic to an S-cube, i.e., the product of copies of the standard 1-simplex Δ^1 indexed by S.

Now if $j \geq i$, then the poset $P_{i,j}$ can be identified with the poset of all subsets of the set $\{i+1, i+2, \ldots, j-1\}$ (the latter set is empty unless $j \geq i+2$). In particular, for $j > i$, we see that $N(P_{i,j})$ is a $(j-i-1)$-dimensional cube.

Definition 6.8. The *simplicial nerve* functor $\mathfrak{N} : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta}$ is defined by

$$\mathfrak{N}(\mathcal{C})([n]) \equiv \mathfrak{N}(\mathcal{C})_n := \mathrm{Hom}_{\mathcal{C}at_{\Delta}}(\mathfrak{C}[\Delta^n], \mathcal{C}).$$

The *simplicial Poincaré category* functor $\mathfrak{C} : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$ is defined as its left adjoint, which exists by Proposition 4.2. By the same result, $\mathfrak{C}$ is also the unique colimit-preserving extension of the functor $\Delta^n \longmapsto \mathfrak{C}[\Delta^n]$ introduced in Definition 6.6, defined on the full subcategory of $\mathcal{S}et_{\Delta}$ formed by the representable presheaves.

Following [T], given any $X \in \mathcal{S}et_{\Delta}$, we will write $\mathfrak{C}[X]$ in place of $\mathfrak{C}(X)$. To the best of our knowledge, [T] does not contain an explicit description of the functor $\mathfrak{C}$. The main goal of this section is to present such a description.

6.5. **Digression: properties of the functor $\mathfrak{N}$.** Before studying the functor $\mathfrak{C}$ in more detail, we make a small digression and mention an important property of the functor $\mathfrak{N}$. First, let us agree that if $\mathcal{C}$ is a small *topological* category, then $\mathfrak{N}(\mathcal{C})$ denotes the simplicial set $\mathfrak{N}(\mathrm{Sing}(\mathcal{C}))$, where $\mathrm{Sing}(\mathcal{C})$ is the simplicial category defined by $\mathcal{C}$ (see §6.2). Further, in this situation, we call $\mathfrak{N}(\mathcal{C})$ the *topological nerve* of $\mathcal{C}$. This should cause no confusion with the simplicial nerve functor (§6.4).

We leave it to the reader to check (cf. Exercise 9.42) that the *ad hoc* construction of the topological nerve of a topological category outlined in §1.4 agrees with the definition that we just explained, at least at the level of n-simplices, where $0 \leq n \leq 3$.

Proposition 6.9 ([T], Proposition 1.1.5.9)**.** *Let $\mathcal{C}$ be a* fibrant *simplicial category, i.e., having the property that* $\mathrm{Map}_{\mathcal{C}}(X, Y)$ *is a Kan complex for all $X, Y \in \mathcal{C}$. Then* $\mathfrak{N}(\mathcal{C})$ *is an ∞-category* (*see* §*1.3 for a definition*).

Corollary 6.10. *If $\mathcal{C}$ is a topological category, then $\mathfrak{N}(\mathcal{C})$ is an ∞-category.*

6.6. **Explicit description of the functor $\mathfrak{C}$.** In this subsection we state a sequence of results that may help the reader understand the structure of the simplicial category $\mathfrak{C}[X]$, for an arbitrary simplicial set X. The proofs of these results are left as exercises, since it is much more instructive to find them on your own.

We do not explain *why* the category $\mathfrak{C}[\Delta^n]$ is defined the way it is defined; such an explanation is beyond the scope of these notes. However, taking the definition of $\mathfrak{C}[\Delta^n]$ as given, we hope that our approach will help the reader visualize $\mathfrak{C}[X]$ for an arbitrary simplicial set X. It also has a concrete application (Corollary 6.15).

Step 1. Let us recall (Remark 6.3(a)) that a simplicial category $\mathcal{C}$ gives rise to a sequence of ordinary categories $\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \ldots$. We begin with a statement that can be verified directly using the definition of the simplicial category $\mathfrak{C}[\Delta^d]$. We believe that it would be instructive for the reader to prove this lemma before moving on.

Lemma 6.11. *For all integers $d, n \geq 0$, the ordinary category $\mathfrak{C}[\Delta^d]_n$ is free, i.e., isomorphic to the path category of a certain diagram scheme with vertices $0, 1, \ldots, d$.*

We refer the reader to §3.3 for the terminology used in this lemma. If $\mathcal{C}$ is a small category isomorphic to $\mathcal{P}a(\mathcal{D})$ for a diagram scheme $\mathcal{D}$, then, by a slight abuse of notation, the morphisms in $\mathcal{C}$ corresponding to arrows of $\mathcal{D}$ will be called the *free generators*, or just *generators*, of $\mathcal{C}$ (we recall from Exercise 9.9 that for a free category, just as for a free monoid, the set of free generators is uniquely determined).

Step 2. It turns out that the statement of Lemma 6.11 remains true with Δ^n replaced by an arbitrary simplicial set X:

Proposition 6.12. *If $X \in \mathcal{S}et_{\Delta}$, then for every integer $n \geq 0$, the ordinary category $\mathfrak{C}[X]_n$ is free.*

However, unlike Lemma 6.11, the last proposition seems to be difficult to prove directly without first having a guess about what the free generators of $\mathfrak{C}[X]_n$ are. They are exhibited very explicitly in part (d) of the next result.

Theorem 6.13. *Let X be an arbitrary simplicial set.*

(*a*) *For every integer $n \geq 0$, all the degeneracy maps $\mathfrak{C}[X]_n \longrightarrow \mathfrak{C}[X]_{n+1}$ take free generators of $\mathfrak{C}[X]_n$ to free generators of $\mathfrak{C}[X]_{n+1}$.*

(*b*) *For every integer $n \geq 1$, all the face maps $\mathfrak{C}[X]_n \longrightarrow \mathfrak{C}[X]_{n-1}$, perhaps except for ∂_0 and ∂_n, take free generators of $\mathfrak{C}[X]_n$ to free generators of $\mathfrak{C}[X]_{n-1}$.*

(*c*) *The nondegenerate*[24] *free generators of $\mathfrak{C}[X]_n$ are in one-to-one correspondence with pairs $(\sigma, \mathcal{F})$ consisting of a nondegenerate simplex $\sigma \in X_k$ for some $k \geq 1$ and a strictly ascending chain of subsets*

$$\{0, k\} = \mathcal{F}_0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_{n-1} \subsetneq \mathcal{F}_n = \{0, 1, \ldots, k\}.$$

(*d*) *More generally, for each $n \geq 0$, consider the set A_n of pairs $(\sigma, \mathcal{F})$, where $\sigma \in X_k$ is a nondegenerate simplex for some $k \geq 1$, and $\mathcal{F}$ is a (possibly non-strictly) ascending chain of subsets*

$$\{0, k\} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_{n-1} \subseteq \mathcal{F}_n = \{0, 1, \ldots, k\}.$$

Consider the map $\gamma_n : A_n \longrightarrow Ar\big(\mathfrak{C}[X]_n\big)$, defined as follows. If $(\sigma, \mathcal{F}) \in A_n$, we can view $\mathcal{F}$ as an n-simplex of the simplicial set $\mathrm{Map}_{\mathfrak{C}[\Delta^k]}(0, k)$, while σ defines a simplicial functor $\mathfrak{C}[\sigma] : \mathfrak{C}[\Delta^k] \longrightarrow \mathfrak{C}[X]$. We put $\gamma_n(\sigma, \mathcal{F}) = \mathfrak{C}[\sigma](\mathcal{F})$.

[24]It follows formally from Proposition 6.12 that every generator of $\mathfrak{C}[X]_n$ can be obtained by applying a sequence of degeneracy maps to a nondegenerate generator of $\mathfrak{C}[X]_m$ for some $m \leq n$.

The map γ_n is a bijection between A_n and the set of all free generators of the category $\mathfrak{C}[X]_n$.

(e) *Suppose $n \geq m \geq 0$ and $f : [n] \longrightarrow [m]$ is a surjective nondecreasing map. With the notation of* (d)*, there is a commutative diagram*

$$\begin{array}{ccc} A_m & \xrightarrow{\gamma_m} & \mathcal{A}r\big(\mathfrak{C}[X]_m\big) \\ {\scriptstyle f^*}\big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ A_n & \xrightarrow{\gamma_n} & \mathcal{A}r\big(\mathfrak{C}[X]_n\big) \end{array}$$

where the vertical arrow on the right is the map coming from the simplicial category structure on $\mathfrak{C}[X]$, while the vertical arrow on the left is defined by $f^*(\sigma, \mathcal{F}) = (\sigma, f^*\mathcal{F})$, with $f^*\mathcal{F}$ being the chain of subsets

$$\{0, k\} = \mathcal{F}_0 = \mathcal{F}_{f(0)} \subseteq \mathcal{F}_{f(1)} \subseteq \cdots \subseteq \mathcal{F}_{f(n-1)} \subseteq \mathcal{F}_{f(n)} = \mathcal{F}_m = \{0, 1, \ldots, k\}.$$

Remarks 6.14. (1) In parts (c) and (d) of the theorem, it is clear that if $n = 0$, then the integer k is forced to be equal to 1, whereas if $n \geq 1$, then, in principle, k is allowed to be arbitrary (so long as X has a nondegenerate k-simplex).

(2) There is obviously some redundancy in the statements above; Proposition 6.12 and Theorem 6.13 could be reformulated together in a more compact way (for instance, parts (a) and (c) of the theorem follow formally from parts (d) and (e)) . However, we find the sequence of the statements above, in the order in which we presented them, easier to digest.

(3) In turn, part (e) of the theorem follows trivially from the statement (d) and the fact that, with the notation of (d), $\mathfrak{C}[\sigma]$ is a simplicial functor. Moreover, statement (b) is not hard to deduce from (d) as well.

(4) Thus, the heart of our description of the functor $\mathfrak{C}$ lies in Proposition 6.12 combined with Theorem 6.13(d). The proof is left as an exercise (see Exercise 9.44, where we provide some hints).

An application. As far as we know, the next result is useful, but not obvious from the definition of the functor $\mathfrak{C}$ given in [T]. It follows easily from the results stated above, and, once again, is left as an exercise for the reader (see Exercise 9.45).

Corollary 6.15. *If $f : S \longrightarrow S'$ is a monomorphism of simplicial sets, the induced functor $\mathfrak{C}[f] : \mathfrak{C}[S] \longrightarrow \mathfrak{C}[S']$ is faithful in the sense that for any pair of objects $x, y \in \mathfrak{C}[S]$, the corresponding morphism of simplicial sets $\mathrm{Map}_{\mathfrak{C}[S]}(x, y) \longrightarrow \mathrm{Map}_{\mathfrak{C}[S']}(x', y')$ is a monomorphism, where $x' = \mathfrak{C}[f](x)$ and $y' = \mathfrak{C}[f](y)$.*

7. Basic notions of ∞-category theory

7.1. **Objects and morphisms.** First we recall the definition of an ∞-category. For all integers $n \geq i \geq 0$, we have the standard n-simplex $\Delta^n \in \mathcal{S}et_{\Delta}$ and the "horn" $\Lambda^n_i \hookrightarrow \Delta^n$, introduced in §4.1.2. A simplicial set X is said to be an *∞-category* [T] (or a *quasi-category* [Joy], or a *weak Kan complex*) if for all $n > i > 0$, every morphism $\Lambda^n_i \longrightarrow X$ can be extended, possibly non-uniquely, to a morphism $\Delta^n \longrightarrow X$. Thus the notion of an ∞-category is a simultaneous weakening of the notion of a Kan complex (§4.7) and of the property that characterizes the nerves of small categories among all simplicial sets (Lemma 5.7).

The main goal of this section is to explain the analogues of the most basic notions of classical category theory (notably, that of an equivalence of categories) in the ∞-categorical setting, and to give a few elementary illustrations of what makes ∞-categories special by comparison with more general simplicial sets.

If one analyzes the process by which an ordinary (small) category can be recovered from its nerve, one immediately arrives at the following

Definition 7.1. If X is an ∞-category, the *objects* of X are the elements of the set X_0 (i.e., the vertices of X), while the *morphisms* (or *arrows*) of X are the elements of the set X_1 (i.e., the edges of X). Given $f \in X_1$, we already introduced (§4.1.1(b)) the *source* and *target* of f, which are objects $s(f), t(f) \in X_0$. If $x \in X_0$, the *identity* morphism $x \xrightarrow{\mathrm{id}_x} x$ is the constant 1-simplex at x.

As usual, instead of writing "let f be a morphism in X with source x and target y," we will often simply write "let $x \xrightarrow{f} y$ be a morphism in X."

The first difference between ∞-categories and ordinary categories reveals itself when we try to define the composition of two morphisms in an ∞-category.

7.2. **A naive approach to compositions.** Let X be an ∞-category, and consider a composable pair of morphisms $x \xrightarrow{f} y \xrightarrow{g} z$ in X. This pair obviously defines a morphism $\Lambda^2_1 \longrightarrow X$. If X were (isomorphic to) the nerve of an ordinary (small) category, this map could be extended to a unique morphism $\Delta^2 \longrightarrow X$, and the image of the edge $\partial_1(\Delta^2)$ under this extension would give the composition $x \xrightarrow{g\circ f} z$.

In general, however, we no longer have the uniqueness property for the extension, so we are forced to introduce the following definition.

Definition 7.2. A *composition* of f and g in X is a morphism $x \xrightarrow{h} z$ that is the image of $\partial_1(\Delta^2)$ under *some* extension of the morphism $\Lambda^2_1 \longrightarrow X$ defined by the pair f, g to a morphism $\Delta^2 \longrightarrow X$.

Remark 7.3. Implicit in this definition is the important fact that not only can f and g have many different compositions, but also, a given morphism $x \xrightarrow{h} z$ can be realized as a composition of f and g in many different ways.

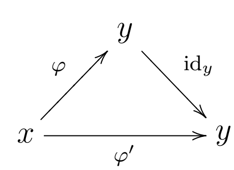
Fortunately, the non-uniqueness of compositions is not very serious from the viewpoint of homotopy theory, in the sense that, informally speaking, "the space of all possible choices is contractible." More generally (and more precisely), given any ∞-category X, any pair of integers $0 < i < n$ and any morphism $\Lambda_i^n \xrightarrow{\alpha} X$, the set of all possible extensions of α to a morphism $\Delta^n \longrightarrow X$ can be naturally identified with the set of vertices of a certain contractible Kan complex. We refer the reader to Exercise 9.33 for the details.

7.3. **An explicit description of $\mathcal{P}(X)$ for an ∞-category X.** As a first illustration of how to work with ∞-categories, we present an alternate construction of the Poincaré category $\mathcal{P}(X)$ (which can be defined for any simplicial set X, cf. Definition 5.3) in the case where X is an ∞-category.

Let us first recall that the objects of $\mathcal{P}(X)$ are by definition the vertices of X, and the morphisms in $\mathcal{P}(X)$ are certain equivalence classes of oriented paths. Suppose now that X is an ∞-category. If $x \xrightarrow{f} y \xrightarrow{g} z$ are morphisms in X and $x \xrightarrow{h} z$ is a composition of f and g in the sense of Definition 7.2, it is clear that $g \circ f$ and h are equivalent as paths, and hence define the same morphism in $\mathcal{P}(X)$. By induction, it follows that every morphism in $\mathcal{P}(X)$ can be represented by a path of length ≤ 1.

We see that if X is an ∞-category, the morphisms in $\mathcal{P}(X)$ can be represented as certain equivalence classes of morphisms in X in the sense of Definition 7.1. It turns out that the corresponding equivalence relation has a very simple description. We explain it in the next result, following [T, §1.2.3].

Let us say that two morphisms, $x \xrightarrow{\varphi} y$ and $x \xrightarrow{\varphi'} y$, in X, are *homotopic*, written $\varphi \sim \varphi'$, if there exists a 2-simplex $\sigma \in X$ such that the edges $\partial_2(\sigma)$, $\partial_1(\sigma)$ and $\partial_0(\sigma)$ are the morphisms φ, φ' and id_y, respectively. Pictorially, we think of σ as a triangle[25]



Proposition 7.4. (*a*) *The homotopy relation $\sim$ is an equivalence relation on X_1.*

(*b*) *If $x \xrightarrow{f} y \xrightarrow{g} z$ are morphisms in X and $x \xrightarrow{h} z$ is a composition of f and g, the equivalence class of h with respect to $\sim$ depends only on the equivalence classes of f and g.*

(*c*) *In view of the above, we get a well defined composition law on $X_1/\sim$. It is associative, and allows us to define an ordinary category πX whose set of objects is X_0, and whose morphisms are the homotopy classes of morphisms in X.*

[25]It is very important to remember that σ also determines the "interior" of this triangle.

(*d*) *The natural functor* $\mathcal{P}(X) \longrightarrow \pi X$ *is an isomorphism of categories.*

The proof of this result is left as an exercise (Exercise 9.32).

Remarks 7.5. (1) The reader may object that the definition of the homotopy relation $\sim$ is not self-dual, i.e., it is not obviously preserved if we replace X by the opposite ∞-category [T, §1.2.1]. Of course, it is a consequence of the proposition that $\sim$ is in fact self-dual, but this fact depends on X being an ∞-category.

(2) The analogue of πX can be easily defined for other sorts of higher categories. Namely, if $\mathcal{C}$ is a simplicial category or a topological category, let us define $\pi\mathcal{C}$ to be the ordinary category whose objects are the objects of $\mathcal{C}$, and whose morphisms are defined by $\operatorname{Hom}_{\pi\mathcal{C}}(X,Y) = \pi_0 \operatorname{Map}_{\mathcal{C}}(X,Y)$.

(3) If X is an arbitrary simplicial set, the Poincaré category of X can be recovered from the simplicial Poincaré category of X: namely, we have a canonical isomorphism $\mathcal{P}(X) \cong \pi\mathfrak{C}[X]$. This follows from abstract nonsense (Exer. 9.41).

7.4. **Equivalences in higher categories.** The correct analogue of the notion of an isomorphism between two objects of an ordinary category is provided by

Definition 7.6. A morphism in an ∞-category (respectively, a simplicial or a topological category) $\mathcal{C}$ is an *equivalence* if it becomes an isomorphism in $\pi\mathcal{C}$.

First, let us note that this is indeed a *generalization* of the classical notion of an isomorphism. Namely, if $\mathcal{C}$ is an ordinary small category, a morphism in $\mathcal{C}$ is an equivalence in the corresponding ∞-category $N(\mathcal{C})$ if and only if it is an isomorphism in $\mathcal{C}$ in the usual sense. This statement is obvious, since we know that $\pi N(\mathcal{C}) \cong \mathcal{P}(N(\mathcal{C}))$ by Proposition 7.4 and $\mathcal{P}(N(\mathcal{C})) \cong \mathcal{C}$ by Lemma 5.6.

Joyal found convenient characterizations of equivalences in ∞-categories; and of Kan complexes among all ∞-categories. We state his results without proofs.

Proposition 7.7 (Joyal; see Proposition 1.2.4.3 in [T])**.** *Let X be an ∞-category. A morphism $\varphi \in X_1$ is an equivalence in X if and only if for each $n \geq 2$, every morphism $f_0 : \Lambda^n_0 \longrightarrow X$ such that $f_0\big|_{\Delta^{\{0,1\}}} = \varphi$ extends to Δ^n.*

Here, we use the standard notation, where $\Delta^{\{0,1\}}$ denotes the edge of Δ^n joining the vertices 0 and 1 (this edge is contained in Λ^n_0 since $n \geq 2$).

Proposition 7.8 (Joyal [Joy])**.** *For a simplicial set X, the following are equivalent:*

(*i*) *X is an ∞-category and πX is a groupoid;*

(*ii*) *X is a Kan complex.*

Remark 7.9. There are two natural candidates for the notion of an ∞-groupoid. One is to define an ∞-groupoid to be the same things as a Kan complex, as we already discussed. However, by analogy with classical category theory, one could also define an ∞-groupoid to be an ∞-category where every morphism is an equivalence. Proposition 7.8 implies that these two definitions are in fact equivalent.

7.5. **Equivalences between higher categories.** In this subsection we consider the question of what is the appropriate notion of an equivalence between two ∞-categories, or simplicial categories, or topological categories. One motivation for asking this question comes from the observation that every "reasonable" construction in the world of higher categories should be invariant under equivalences.

In the setting of topological or simplicial categories, we have the notion of an equivalence in the sense of enriched category theory. For instance, we can define a topological functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ between topological categories to be a *strong equivalence* if there exists a topological functor $G : \mathcal{C}_2 \longrightarrow \mathcal{C}_1$ such that $F \circ G$ and $G \circ F$ are isomorphic to the identity functors $\mathrm{Id}_{\mathcal{C}_2}$ and $\mathrm{Id}_{\mathcal{C}_1}$, respectively. However, this notion is too restrictive to be useful. As an example, consider a contractible topological monoid M, and let $\mathcal{C}$ be the topological category with one object defined by M. From the point of view of homotopy theory, $\mathcal{C}$ should be equivalent to the trivial category (with one object and one morphism); however, this is not so in the sense of enriched category theory.

The correct notion of an equivalence is defined as follows.

Definition 7.10. (a) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are topological categories and $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a topological functor, we say that F is an *equivalence* if F induces a weak homotopy equivalence $\mathrm{Map}_{\mathcal{C}_1}(X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}_2}(F(X), F(Y))$ for every pair $X, Y \in \mathcal{C}_1$, and the functor $\pi F : \pi\mathcal{C}_1 \longrightarrow \pi\mathcal{C}_2$ is essentially surjective[26].

(b) If $\mathcal{C}_1$ and $\mathcal{C}_2$ are simplicial categories and $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ is a simplicial functor, we say that F is an *equivalence* if F induces a weak equivalence of simplicial sets $\mathrm{Map}_{\mathcal{C}_1}(X, Y) \xrightarrow{\sim} \mathrm{Map}_{\mathcal{C}_2}(F(X), F(Y))$ for every pair $X, Y \in \mathcal{C}_1$, and the induced functor $\pi F : \pi\mathcal{C}_1 \longrightarrow \pi\mathcal{C}_2$ is essentially surjective.

(c) A morphism $f : S_1 \longrightarrow S_2$ between simplicial sets is a *categorical equivalence* if $\mathfrak{C}[f] : \mathfrak{C}[S_1] \longrightarrow \mathfrak{C}[S_2]$ is an equivalence of simplicial categories, in the sense we just defined.

Remark 7.11. It turns out [T] that the correct definition of a functor between ∞-categories is simply as a morphism of simplicial sets. If X, Y are ∞-categories, we write $\mathrm{Fun}(X, Y) = \mathrm{Map}_{\mathcal{S}et_\Delta}(X, Y)$; this is a simplicial set whose vertices are functors $X \longrightarrow Y$. This simplicial set is automatically an ∞-category as well. More generally, if K is any simplicial set and $\mathcal{C}$ is an ∞-category, then $\mathrm{Map}_{\mathcal{S}et_\Delta}(K, \mathcal{C})$ is also an ∞-category (Exercise 9.34).

7.6. **The Joyal model category structure on $\mathcal{S}et_\Delta$.** Categorical equivalences between simplicial sets turn out to be a part of a certain model category structure on $\mathcal{S}et_\Delta$, which was discovered by Joyal. We state his result without proof[27].

[26] And thus, in view of the previous requirement, is an equivalence of ordinary categories

[27] As Lurie explains in [T], Joyal's original definition of this model structure is different.

Theorem 7.12 (Joyal; see Theorem 2.2.5.1 in [T])**.** *There exists a model category structure on $\mathcal{S}et_{\Delta}$ for which the cofibrations are the monomorphisms of simplicial sets, and the weak equivalences are the categorical equivalences.*

The fibrations in the Joyal model structure are called the *categorical fibrations* of simplicial sets. From the general properties of model categories, it follows that categorical fibrations can be characterized as those morphisms of simplicial sets, which have the right lifting property with respect to monomorphisms that are also categorical equivalences. We state a few more facts, also without proofs.

Remarks 7.13. (1) Joyal defines a morphism $A \xrightarrow{f} B$ of simplicial sets to be a *weak categorical equivalence* if for every ∞-category $\mathcal{C}$, the induced functor $\pi \operatorname{Fun}(B, \mathcal{C}) \longrightarrow \pi \operatorname{Fun}(A, \mathcal{C})$ is an equivalence of ordinary categories. Lurie proves in [T, Proposition 2.2.5.8] that a morphism in $\mathcal{S}et_{\Delta}$ is a weak categorical equivalence in the sense of Joyal's definition if and only if it is a categorical equivalence in the sense of Definition 7.10(c).

(2) The fibrant objects for the Joyal model structure are precisely the ∞-categories; this already explains why this structure is useful.

(3) In view of the previous remark, one might naively expect that the categorical fibrations are the same as the *inner fibrations*, defined as the morphisms having the right lifting property with respect to inner horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 < i < n$. **This is false.** Every categorical fibration is an inner fibration [T, Remark 2.2.5.5], but not conversely (see Exercise 9.52). Of course, the previous remark implies that if X is any simplicial set, then the natural map $X \longrightarrow *$ is a categorical fibration if and only if it is an inner fibration.

(4) It is also known that a categorical equivalence between simplicial sets is also a weak equivalence, but the converse is again false (Exercise 9.51).

(5) It follows formally from the previous remark that every Kan fibration is also a categorical fibration. The converse, of course, is again false (e.g., not every ∞-category is an ∞-groupoid).

(6) The class of acyclic cofibrations for Quillen's model structure on $\mathcal{S}et_{\Delta}$ has a convenient set of generators (namely, the horn inclusions $\Lambda_i^n \hookrightarrow \Delta^n$, $0 \leq i \leq n$). It would be interesting to describe explicitly a (countable) set of morphisms generating the class of acyclic cofibrations for the Joyal model structure. Such a description is not known to us.

(7) Similarly, one can ask whether the acyclic cofibrations for the Joyal model structure admit a convenient "geometric" (or "visual") characterization. (For comparison, recall that anodyne morphisms can be characterized as monomorphisms of simplicial sets whose geometric realizations are homotopy equivalences, and in practice one can often "see" whether this property holds or not.) Once again, the answer to this question is not known to us.

7.7. **An alternate viewpoint on categorical equivalences.** In §7.5 we explained that the usual notion of an equivalence in the sense of enriched category theory is unsuitable for defining the correct notion of an equivalence between simplicial or topological categories. On the other hand, it is clear that the correct notion can still be approached from the viewpoint of enriched category theory, only the "enrichment" has to be changed.

Indeed, after some preliminaries, we will associate to every simplicial or topological category, or an ∞-category, $\mathcal{C}$, its "homotopy category" $h\mathcal{C}$ (see §7.10), which is a category enriched over "the homotopy category $\mathcal{H}$ of spaces" (defined in §7.8).

In particular, if $\mathcal{C}$ is an ∞-category, we will have four different ways of associating an enriched category to $\mathcal{C}$. Namely, we can consider:

(i) the Poincaré category $\mathcal{P}(\mathcal{C}) \cong \pi\mathcal{C}$, which is an ordinary category; or
(ii) the homotopy category $h\mathcal{C}$, which is enriched over $\mathcal{H}$; or
(iii) the simplicial Poincaré category $\mathfrak{C}[\mathcal{C}]$, which is enriched over $\mathcal{S}et_{\Delta}$; or
(iv) the category $|\mathfrak{C}[\mathcal{C}]|$, which is enriched over $\mathcal{CG}$.

It is clear that $\mathcal{P}(\mathcal{C})$ is the most crude invariant of $\mathcal{C}$. From the viewpoint of homotopy theory, (iii) and (iv) above are more or less equivalent to each other. Finally, (ii) occupies an intermediate position between (i) and (iii)–(iv).

7.8. **The homotopy category of spaces.** In this subsection we recall the definition of the "homotopy category of spaces", denoted $\mathcal{H}$. This category has (at least) four equivalent constructions, two of which are topological, while the other two are simplicial. It is often useful to keep all four approaches in mind. Some remarks about them, and an explanation of why they are equivalent, are given in §7.9.

First definition of $\mathcal{H}$. We define $\mathcal{H}$ as the category whose objects are CW complexes, and whose sets of morphisms are defined by

$$\mathrm{Hom}_{\mathcal{H}}(X,Y) = \{\text{continuous maps } f : X \longrightarrow Y\}/\sim,$$

where $\sim$ is the equivalence relation of homotopy.

Second definition of $\mathcal{H}$. We define $\mathcal{H}$ to be the category obtained from the category $\mathcal{CG}$ of compactly generated topological spaces by formally inverting all weak homotopy equivalences (the latter are defined at the beginning of §4.10).

Third definition of $\mathcal{H}$. We define $\mathcal{H}$ to be the category whose objects are Kan complexes, and whose morphism sets are defined by $\mathrm{Hom}_{\mathcal{H}}(K,L) = [K,L]$ (the set of "combinatorial" homotopy classes of morphisms $K \longrightarrow L$, cf. Definition 4.7).

Fourth definition of $\mathcal{H}$. We define $\mathcal{H}$ to be the category obtained from the category $\mathcal{S}et_{\Delta}$ of simplicial sets by formally inverting all weak equivalences (Definition 4.11).

7.9. **A remark on the four definitions of $\mathcal{H}$.** The first and third definitions of $\mathcal{H}$ that we mentioned clearly pose no set-theoretical problems. The second and fourth definitions are potentially problematic, since $\mathcal{CG}$ and $\mathcal{S}et_{\Delta}$ are "large" categories. Besides, one must also check that all four definitions of $\mathcal{H}$ are in fact equivalent to each other.

In fact, suppose for the moment that we defined $\mathcal{H}$ using the first approach. Then one can construct natural functors $\mathcal{CG} \longrightarrow \mathcal{H}$ and $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{H}$ that satisfy the universal properties of the corresponding localizations.

If $X \in \mathcal{CG}$, it is well known that there exist a CW complex X' and a weak homotopy equivalence $X' \xrightarrow{\sim} X$. Moreover, X' is unique up to canonical homotopy equivalence. Thus we obtain a functor $\mathcal{CG} \longrightarrow \mathcal{H}$, and (essentially by a theorem of Whitehead) this functor satisfies the universal property of the localization of $\mathcal{CG}$ with respect to all weak homotopy equivalences.

The desired functor $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{H}$ is induced by the geometric realization functor $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{CG}$, which automatically takes values in the category of CW complexes. By a theorem of Quillen, the functor $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{H}$ satisfies the universal property of the localization of $\mathcal{S}et_{\Delta}$ with respect to all weak equivalences.

Finally, to establish the equivalence between the first and third approaches, it is enough, in view of Remark 4.6 and the previous comments, to prove the following

Lemma 7.14. *If X and Y are Kan complexes and $f, g : X \longrightarrow Y$ are morphisms of simplicial sets, then f and g are "combinatorially" homotopic (i.e., define the same element of $[X, Y]$) if and only if the geometric realizations $|f|, |g| : |X| \longrightarrow |Y|$ are homotopic in the usual sense of algebraic topology.*

The proof of this lemma is left as an exercise (see Exercise 9.18).

7.10. **The homotopy category of a simplicial set.** It is not hard to check that the homotopy category $\mathcal{H}$ of spaces has finite products, and that the natural functors $\mathcal{CG} \longrightarrow \mathcal{H}$ and $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{H}$ commute with finite products. In particular, if $\mathcal{C}$ is a simplicial category or a topological category, then, by replacing each $\mathrm{Map}_{\mathcal{C}}(X, Y)$, where $X, Y \in \mathcal{C}$, with its image in $\mathcal{H}$, we obtain a category enriched over $\mathcal{H}$. It is called the *homotopy category* of $\mathcal{C}$ and denoted by $h\mathcal{C}$.

The homotopy category of an ∞-category, or, more generally, of an arbitrary simplicial set, is defined by means of the simplicial Poincaré category functor $\mathfrak{C}$ that we studied in Section 6. Namely, if $X \in \mathcal{S}et_{\Delta}$, we put $hX := h\mathfrak{C}[X]$.

In particular, if $\mathcal{C}$ is either an ∞-category, or a simplicial category, or a topological category, then we know how to associate to $\mathcal{C}$ a category $h\mathcal{C}$ enriched over $\mathcal{H}$. It is immediate from the definition that in each of the three settings, a functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ (in the appropriate sense) is an equivalence in the sense of Definition

7.10 if and only if the corresponding functor $hF : h\mathcal{C}_1 \longrightarrow h\mathcal{C}_2$ is an equivalence of categories enriched over $\mathcal{H}$.

Remark 7.15. Let $\mathcal{C}$ be as above. In [T], Lurie often uses the notation $h\mathcal{C}$ and the term "the homotopy category of $\mathcal{C}$" for the ordinary category obtained from $h\mathcal{C}$ by replacing the homotopy types of morphism spaces with their π_0's. In other words, he uses $h\mathcal{C}$ to denote what we called $\pi\mathcal{C}$. While this abuse of terminology is often justified, in these notes we will always denote by $h\mathcal{C}$ the category enriched over $\mathcal{H}$ associated to $\mathcal{C}$, and reserve the notation $\pi\mathcal{C}$ for the corresponding ordinary category.

7.11. **Mapping spaces in hX for an ∞-category X.** If X is an ∞-category, it turns out that the (homotopy types of) the spaces of maps in the homotopy category hX admit concrete descriptions that do not use the functor $\mathfrak{C}$. Namely, in §1.2.2 of [T], Lurie introduces, for a given ∞-category X and a pair of objects $x, y \in X_0$, the *right mapping space* $\mathrm{Hom}^R_X(x, y)$, which is a simplicial set defined as follows.

For each $n \geq 0$, the elements of $\mathrm{Hom}^R_X(x, y)_n$ are the simplices $z \in X_{n+1}$ such that $v_{n+1}(z) = y$ and $\partial_{n+1}(z)$ is the constant n-simplex at the vertex x (here we are using the notation of §4.1.1). It is easy to make the assignment $[n] \longmapsto \mathrm{Hom}^R_X(x, y)_n$ functorial with respect to $[n] \in \triangle^{op}$.

It turns out that $\mathrm{Hom}^R_X(x, y)$ always has the homotopy type of $\mathrm{Map}_{hX}(x, y)$. The advantage of this description of mapping spaces is that $\mathrm{Hom}^R_X(x, y)$ has a very concrete and manageable definition; moreover, it is always a Kan complex (Exercise 9.39), unlike $\mathrm{Map}_{\mathfrak{C}[X]}(x, y)$. The disadvantage is that there are no naturally defined composition laws $\mathrm{Map}^R_X(y, z) \times \mathrm{Map}^R_X(x, y) \longrightarrow \mathrm{Map}^R_X(x, z)$ for $x, y, z \in X_0$.

8. Homotopy colimits in ∞-categories

8.1. **Generalities.** The notion of a (co)limit of a diagram of objects and morphisms in an ordinary category (see §3.4) plays a central role in classical category theory. In this section we discuss the analogous notion in the setting of ∞-categories. It was introduced in [Joy] and explored in [T]. Our presentation follows [T, §1.2.13], except that we simplify the definitions by only considering special types of (co)limits. Namely, Joyal and Lurie work with (co)limits of a morphism $K \longrightarrow \mathcal{C}$, where K is an arbitrary simplicial set and $\mathcal{C}$ is an ∞-category. On the other hand, we only consider simplicial sets K of the form $N(\mathscr{I})$, where $\mathscr{I}$ is an ordinary small category.

This suffices for a first introduction to the notion of a (co)limit, and allows us to reduce the number of preliminary definitions (for instance, we only have to discuss joins of categories and not of simplicial sets). Besides, many of the (co)limits that arise in practice (such as the cartesian and cocartesian squares appearing in the theory of stable ∞-categories) are "indexed by" ordinary categories.

Let us note that we decided to use the term "homotopy (co)limit", rather than simply "limit" (as used by Joyal and Lurie), to emphasize the connection with the notion of a homotopy (co)limit in algebraic topology, and the fact that we are working with a "smart" notion of a (co)limit[28].

From now on, for concreteness, we choose to work with homotopy colimits; of course, all the definitions related to the notion of a homotopy limit can be formally obtained from the ones we present below by "reversing the arrows"[29].

We will motivate the notion of a homotopy colimit in an ∞-category by first rewriting the classical definition of a colimit in such a way that all the ingredients in the definition will have natural ∞-categorical analogues.

8.2. **Homotopy coherent diagrams.** Of course, the first ingredient in the classical definition of a colimit is the notion of a *commutative* diagram in an ordinary category. It turns out that the "correct" version of this notion in the setting of higher category theory is provided by the notion of a *homotopy coherent* diagram.

If $\mathcal{C}$ is an ∞-category (respectively, a topological category) and K is a simplicial set, Lurie defines a *K-shaped homotopy coherent diagram* in $\mathcal{C}$ as a morphism of simplicial sets $K \longrightarrow \mathcal{C}$ (respectively, $K \longrightarrow \mathfrak{N}(\mathcal{C})$). If $K = N(\mathscr{I})$ for an ordinary small category $\mathscr{I}$, a K-shaped homotopy coherent diagram in $\mathcal{C}$ is called a *functor* $\mathscr{I} \longrightarrow \mathcal{C}$ (of course, if $\mathcal{C}$ is an ∞-category, this is a special case of the notion of a functor between ∞-categories mentioned in Remark 7.11).

The notion of a homotopy coherent diagram should be contrasted with the notion of a *homotopy commutative* diagram in $\mathcal{C}$, defined simply as a commutative diagram in the homotopy category $h\mathcal{C}$. We refer the reader to [T, §1.2.6] for a clear and concise discussion of these two notions, and an explanation of why the former is the "correct" one to use, while the latter is quite unsatisfactory.

8.3. **Colimits in ordinary categories, revisited.** Let $\mathscr{I}$ and $\mathcal{C}$ be ordinary categories, where $\mathscr{I}$ is small, and consider a functor $F : \mathscr{I} \longrightarrow \mathcal{C}$ (the reader may prefer to forget about the composition law on $\mathscr{I}$ and merely view F as an $\mathscr{I}$-shaped diagram in $\mathcal{C}$, since this makes no difference from the point of view of colimits).

The definition of a colimit of F presented in §3.4 is unsuitable for ∞-categorical generalizations, since it is somewhat complicated and involves too many equalities between compositions of morphisms. Our first task is to rewrite the definition in a more compact and more abstract way.

[28]In the setting of *topological* categories, there is also a naive notion of a (co)limit, which is obtained by disregarding the topologies on the mapping spaces and applying the classical definition of a (co)limit. This notion is rarely useful in practice.

[29]In the setting of ∞-categories, this amounts to replacing a simplicial set with the opposite one; see [T, §1.2.1] for the precise definition and a discussion.

The universal property appearing in the definition of a colimit of F can be rephrased by introducing a new category, $\mathcal{C}_{F/}$, called the *undercategory* of the functor F. This definition is a generalization of the more familiar undercategory $X/\mathcal{C}$, where X is a fixed object of $\mathcal{C}$ (this is the case where $\mathscr{I}$ is a category with one object and one morphism). In general, objects of $\mathcal{C}_{F/}$ are pairs $\big(Z, \{j_x\}_{x\in\mathscr{I}}\big)$, where $Z \in \mathcal{C}$ is an object and $j_x : F(x) \longrightarrow Z$ is a morphism for every $x \in \mathscr{I}$, such that $j_y \circ F(a) = j_x$ for every morphism $x \xrightarrow{a} y$ in $\mathscr{I}$. Morphisms in $\mathcal{C}_{F/}$ are defined in the obvious way.

With this notation, a colimit of the functor F is nothing but an initial object in the undercategory $\mathcal{C}_{F/}$. It turns out that both ingredients in this definition of a colimit have ∞-categorical analogues: namely, there is a good notion of an initial object in an ∞-category (§8.4), and a good construction of $\mathcal{C}_{F/}$ in case $\mathcal{C}$ is a general ∞-category (§8.5). We now proceed to discuss these two notions.

8.4. **Initial and final objects in ∞-categories.** The notion of an initial object in an ∞-category was introduced in [Joy]. For the sake of brevity, we will be content with summarizing §1.2.12 of [T].

Definition 8.1. (a) Let $\mathcal{C}$ be a simplicial or topological category, or an arbitrary simplicial set, so that the homotopy category $h\mathcal{C}$ is defined. An object (respectively, vertex) x of $\mathcal{C}$ is said to be *initial* if $\mathrm{Map}_{h\mathcal{C}}(x, y)$ is contractible (i.e., is a final object of $\mathcal{H}$) for every object (respectively, vertex) y of $\mathcal{C}$.

(b) Let X be a simplicial set. A vertex x of X is *strongly initial* if for each $n \geq 0$, every morphism $f_0 : \dot{\Delta}^n \longrightarrow X$ (see §4.1.2) such that $f_0(0) = x$ can be extended to a morphism $f : \Delta^n \longrightarrow X$.

As usual, one can obtain the notions of a final object (or vertex) and of a strongly final vertex formally by reversing the arrows in the last definition.

Proposition 8.2 (see [Joy] and §1.1.12 of [T])**.** (*a*) *If X is a simplicial set, then every strongly initial vertex of X is also initial, and the converse holds in case X is an ∞-category.*

(*b*) (*Joyal*) *Initial objects in ∞-categories, if they exist, are unique in the following sense. Let $\mathcal{C}$ be an ∞-category, and let $\mathcal{C}'$ be the simplicial subset of $\mathcal{C}$ formed by those simplices, all of whose vertices are initial in $\mathcal{C}$. Then $\mathcal{C}'$ is either empty, or a contractible Kan complex.*

8.5. **Over- and undercategories in the ∞-categorical setting.** In order to motivate the definition of undercategories in the ∞-categorical setting, let us again consider a functor $F : \mathscr{I} \longrightarrow \mathcal{C}$ between ordinary small categories and try to compute the nerve of the undercategory $\mathcal{C}_{F/}$.

To this end, let us first observe that objects of $\mathcal{C}_{F/}$ can be regarded as certain functors. Namely, let $\mathscr{I}^{\triangleright}$ denote the "right cone category" over $\mathscr{I}$, obtained from

$\mathscr{I}$ by adding an extra object $*$ so that $\mathrm{Hom}(X,*)$ has exactly one element for each $X \in \mathscr{I}^{\triangleright}$, and $\mathrm{Hom}(*,X) = \varnothing$ for each $X \in \mathscr{I}$ (this determines the composition law in $\mathscr{I}^{\triangleright}$ uniquely). It is clear that objects of $\mathcal{C}_{F/}$ can be viewed as functors $\mathscr{I}^{\triangleright} \longrightarrow \mathcal{C}$ that extend F. With this interpretation, morphisms in $\mathcal{C}_{F/}$ are natural transformations of functors that restrict on $\mathscr{I}$ to the identity transformation of F.

Using this viewpoint on $\mathcal{C}_{F/}$, it is not hard to describe n-simplices in $N(\mathcal{C}_{F/})$ for all $n \geq 0$. To present this description in a concise way, it is convenient to introduce

Definition 8.3. If $\mathcal{D}$ and $\mathcal{D}'$ are (ordinary) categories, the *join* of $\mathcal{D}$ and $\mathcal{D}'$ is defined as the category $\mathcal{D} \star \mathcal{D}'$, where an object of $\mathcal{D} \star \mathcal{D}'$ is either an object of $\mathcal{D}$ or an object of $\mathcal{D}'$ (i.e., $\mathcal{O}b(\mathcal{D} \star \mathcal{D}') = \mathcal{O}b(\mathcal{D}) \coprod \mathcal{O}b(\mathcal{D}')$), and where the morphism sets are given as follows ($*$ denotes a set with one element):

$$\mathrm{Hom}_{\mathcal{D}\star\mathcal{D}'}(X,Y) = \begin{cases} \mathrm{Hom}_{\mathcal{D}}(X,Y) & \text{if } X, Y \in \mathcal{D}; \\ \mathrm{Hom}_{\mathcal{D}'}(X,Y) & \text{if } X, Y \in \mathcal{D}'; \\ \varnothing & \text{if } X \in \mathcal{D}',\ Y \in \mathcal{D}; \\ * & \text{if } X \in \mathcal{D},\ Y \in \mathcal{D}'. \end{cases}$$

The composition of morphisms in $\mathcal{D} \star \mathcal{D}'$ is defined in the obvious way. Observe that both $\mathcal{D}$ and $\mathcal{D}'$ can be canonically realized as full subcategories of $\mathcal{D} \star \mathcal{D}'$.

The next result is immediate from the definitions.

Lemma 8.4. *For each $n \geq 0$, there is a natural bijection between $N(\mathcal{C}_{F/})_n$ and the set of functors $\mathscr{I} \star [n] \longrightarrow \mathcal{C}$ that extend F.*

Here, the poset $[n]$ is viewed as a category in the usual way (§5.1). The lemma motivates the following

Definition 8.5. Let $\mathcal{C}$ be an ∞-category, let $\mathscr{I}$ be a small ordinary category, and let $F : \mathscr{I} \longrightarrow \mathcal{C}$ be a functor (§8.2). We construct simplicial sets $\mathcal{C}_{F/}$ and $\mathcal{C}_{/F}$ as follows. For each $n \geq 0$, we define $(\mathcal{C}_{F/})_n$ (respectively, $(\mathcal{C}_{/F})_n$) to be the set of all functors $\mathscr{I} \star [n] \longrightarrow \mathcal{C}$ (respectively, $[n] \star \mathscr{I} \longrightarrow \mathcal{C}$) that extend F.

According to Exercise 9.57, the simplicial sets $\mathcal{C}_{F/}$ and $\mathcal{C}_{/F}$ are ∞-categories as well. They will be called the *overcategory* and *undercategory* of the functor F (but remember that they are usually *not* ordinary categories!).

8.6. **Homotopy colimits in ∞-categories.** It was observed by Joyal that the definitions of §§8.4–8.5 can be used to give the "correct" definition of colimits in ∞-categories. It concludes our introduction to the theory of ∞-categories.

Definition 8.6. If $\mathcal{C}$ is an ∞-category, $\mathscr{I}$ is a small ordinary category, and $F : \mathscr{I} \longrightarrow \mathcal{C}$ is a functor, a *homotopy colimit* of F is an initial object of the ∞-category $\mathcal{C}_{F/}$.

The classical notion of a colimit of a functor between small ordinary categories turns out to be a special case of this definition (see Exercise 9.58).

9. Exercises

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The exercises have not been proofread very carefully.
Please let me know if any of them appear to be incorrect.
Further hints and references will be added in due course.
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Exercises related to geometric realizations

9.1. Find two simplicial sets, X and Y, such that the natural continuous map $|X \times Y| \longrightarrow |X| \times |Y|$ is *not* a homeomorphism of topological spaces, where $|X| \times |Y|$ is equipped with the usual product topology.

9.2. If $w\mathcal{H}$ is the category of weak Hausdorff topological spaces, prove that the natural inclusion functor $\mathcal{CG} \hookrightarrow w\mathcal{H}$ admits a right adjoint. Describe it explicitly.

9.3. If X is a k-space, prove that X is weak Hausdorff (and hence compactly generated) if and only if the diagonal is compactly closed in $X \times X$.

9.4. Read the first section of [Dri] and prove Proposition 4.5 using the definition of geometric realization introduced therein. [As a first step, you need to understand how to construct limits in the category $\mathcal{CG}$, using Exercise 9.2.]

Exercises on categories of presheaves

9.5. Prove the claim in the first paragraph of §2.4 (it is called *Yoneda's lemma*).

9.6. If $\mathcal{D}$ is a small category, prove that a morphism $\varphi : F \longrightarrow G$ in the category PreSh($\mathcal{D}$) of functors $\mathcal{D}^{op} \longrightarrow \mathcal{S}et$ is a monomorphism (respectively, epimorphism) if and only if the corresponding map of sets $\varphi_X : F(X) \longrightarrow G(X)$ is injective (respectively, surjective) for all $X \in \mathcal{D}$. [*Hint*: for the "only if" direction, it may be helpful to consider the fiber product $F \underset{G}{\times} F$ (respectively, the pushout $G \coprod_F G$).]

9.7. Prove Proposition 4.2. [*Hint.* An important step in the proof is to observe that every functor $F : \mathcal{D}^{op} \longrightarrow \mathcal{S}et$ can be realized as a colimit of a diagram of representable functors in a canonical way. Namely, consider the diagram scheme whose vertices are pairs (X, s), where $X \in \mathcal{D}$ and $s \in F(X)$, and where an arrow between (X, s) and (X', s') is a morphism $f \in \mathrm{Hom}_{\mathcal{D}}(X, X')$ with $F(f)(s') = s$. Let us denote this diagram scheme[30] by $\mathcal{D}/F$. Show that the map $(X, s) \longmapsto h_X^{\mathcal{D}}$, where $h_X^{\mathcal{D}}$ is the presheaf represented by X, can be made into a diagram of type $\mathcal{D}/F$ in the category PreSh($\mathcal{D}$), and that F can be naturally identified with the colimit of this diagram. Yoneda's lemma plays a role in the proof of the last statement.]

Exercises on free monoids and free categories

[30]In fact, this diagram scheme comes from a category, which is usually called the "category of objects of $\mathcal{D}$ over F", explaining our notation.

9.8. Prove that the functor $\mathcal{P}a : \mathcal{D}ia \longrightarrow \mathcal{C}at$ introduced in Definition 3.7 is left adjoint to the functor $U : \mathcal{C}at \longrightarrow \mathcal{D}ia$ defined at the beginning of §3.3.

9.9. Prove Lemma 3.4, then state and prove its analogue for the functor $\mathcal{P}a$.

9.10. (*Drinfeld*) Prove that a retract of a free monoid is free. In other words, if S is a set, M is any monoid, and $M \xrightarrow{i} \mathcal{FM}(S) \xrightarrow{p} M$ are monoid homomorphisms such that $p \circ i = \mathrm{id}_M$, then M is a free monoid. [*Hint.* First reduce to the case where $p(s) \neq 1$ for any $s \in S$. In this situation, if S' is the set of elements $s \in S$ such that $s = i(p(s))$, verify that $p(S')$ generates M freely.]

Exercises on nerves of small categories

9.11. Prove that the functor $\mathcal{P} : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at$ constructed in Definition 5.3 is left adjoint to the nerve functor $N : \mathcal{C}at \longrightarrow \mathcal{S}et_{\Delta}$ introduced in §5.2.

9.12. Prove Lemma 5.6.

9.13. Show that a small category $\mathcal{C}$ is a groupoid (that is, every morphism in $\mathcal{C}$ is invertible) if and only if its nerve, $N(\mathcal{C})$, is a Kan complex (see §4.7).

9.14. (*Lurie*) Prove Lemma 5.7.

9.15. Given small categories $\mathcal{C}$ and $\mathcal{D}$, construct a natural isomorphism between the nerve of the category of functors $\mathcal{C} \longrightarrow \mathcal{D}$ and $\mathrm{Map}_{\mathcal{S}et_{\Delta}}(N(\mathcal{C}), N(\mathcal{D}))$.

9.16. If $\mathcal{C}$ is a small category, the geometric realization $B\mathcal{C} = |N(\mathcal{C})|$ of its nerve is called the *classifying space* of $\mathcal{C}$. If $\mathcal{G}$ is a small groupoid, prove that:

(a) the set $\pi_0(B\mathcal{G})$ of connected components of $B\mathcal{G}$ can be naturally identified with the set of isomorphism classes of objects of $\mathcal{G}$;

(b) if $x \in \mathcal{G}$, and we view x as a 0-simplex of $N(\mathcal{G})$ and thus as a point of $B\mathcal{G}$, then the fundamental group $\pi_1(B\mathcal{G}, x)$ is naturally isomorphic to the group $\mathrm{Aut}_{\mathcal{G}}(x)$;

(c) with the notation of (b), $\pi_n(B\mathcal{G}, x) = 0$ for all $n \geq 2$.

Note that as a consequence of (a) and (b), the Poincaré groupoid[31] of the topological space $B\mathcal{G}$ is naturally equivalent to $\mathcal{G}$ as a category.

Exercises on simplicial sets

9.17. If S is any simplicial set and K is a Kan complex, prove that $\mathrm{Map}_{\mathcal{S}et_{\Delta}}(S, K)$ is a Kan complex as well (the method of Proposition 4.15 can be used here).

[31] Also known as the "fundamental groupoid".

9.18. If S is any simplicial set and K is a Kan complex, prove that two morphisms $S \longrightarrow K$ of simplicial sets define the same element of $[S, K]$ if and only if their geometric realizations $|S| \longrightarrow |K|$ are homotopic in the usual sense. [*Hint*: Use the adjunction between the functors $|\cdot|$ and Sing together with Remark 4.6.]

9.19. If X is any topological space, prove that $\mathrm{Sing}(X)$ is a Kan complex.

9.20. Let G be a simplicial group, i.e., a functor from $\mathbb{\Delta}^{op}$ to the category of groups. By forgetting the group structures on all the G_n's, we obtain a simplicial set. Prove that the latter is a Kan complex. (A solution of this exercise is given in [GM, ch. II, §2.1(b)] and [May, §17].)

9.21. $*$ In this exercise you should not use the geometric realization functor. The goal is to give purely combinatorial proofs in the spirit of Chapter V of [GM].

Read the proof of Lemma V.2.8 in [GM] and fill in all the missing details. Observe that this lemma is a special case of Lemma 4.10. Then prove Lemma 4.10.

9.22. Now, prove Lemma 4.10 via a topological argument. Namely, the map in question is obviously a cofibration, so you only need to check that its geometric realization is a homotopy equivalence, which is rather easy.

The next series of exercises (9.23–9.31) was suggested by Vladimir Drinfeld. It is meant to help the reader become comfortable with the notion of a Kan complex, a.k.a. Kan simplicial set (defined at the beginning of §4.7).

9.23. Prove that if a Kan complex contains a 1-dimensional simplex with first vertex x and second vertex y, then it also contains a 1-dimensional simplex with first vertex y and second vertex x.

9.24. Prove that if $n > 0$, then Δ^n is not a Kan complex.

9.25. Let $A \subset \Delta^n$ be the union of the 1-simplices $\{0,1\}, \{1,2\}, \dots, \{n-1,n\} \subset \Delta^n$. Then the embedding $A \hookrightarrow \Delta^n$ is an acyclic cofibration (a.k.a. anodyne morphism).

Remark 9.1. There are two definitions of an acyclic cofibration: on the one hand, the class of acyclic cofibrations is, in a certain sense, generated by the inclusions of horns into simplices; on the other hand, acyclic cofibrations can be defined as those embeddings of simplicial sets that induce homotopy equivalences between the geometric realizations. For proving that something is an acyclic cofibration, it is better to use the second definition.

9.26. If a Kan complex has a nondegenerate 1-simplex, then it has a nondegenerate n-simplex for each $n \geq 1$. [*Hint*: use the results of Exercises 9.23 and 9.25.]

9.27. Let $0 < k \leq n$. Show that there exists a collection of k-dimensional simplices $A_1, \dots, A_m \subset \Delta^n$ such that their union is contractible and each 1-simplex $\{i, i+1\} \subset \Delta^n$ is an edge of some A_j.

9.28. If a Kan complex has a nondegenerate k-simplex for some $k > 0$, then it has a nondegenerate n-simplex for each $n \geq k$. [*Hint*: take the smallest k satisfying the assumption of this exercise and apply the result of Exercise 9.27.]

9.29. $*$ Let G be an abelian simplicial group. Then the homotopy groups[32] of G (viewed as a simplicial set) are equal to the homology groups of G. (A solution of this exercise can be found in [May, Theorem 22.1].)

Given a simplicial set S let $C(S)$ denote the simplicial abelian group freely generated by S (so that the homology of the simplicial abelian group $C(S)$ equals the homology of the simplicial set S).

9.30. $*$ Let π be an abelian group and n a nonnegative integer. Then the simplicial group (see Exercise 9.20) $(C(\Delta^n)/C(\dot{\Delta}^n)) \otimes \pi$ has the homotopy type of $K(\pi, n)$ (i.e., its n-th homotopy group equals π and the other homotopy groups are zero).

(We recall that $\dot{\Delta}^n$ denotes the simplicial $(n-1)$-sphere, defined as the "boundary" of the standard n-simplex Δ^n; see §4.1 for the precise definition.)

9.31. $*$ Prove that the simplicial group defined in Exercise 9.30 is a *minimal* Kan complex in the sense of Definition 9.2 below.

Definition 9.2. A Kan complex S is said to be *minimal* if it satisfies one of the following two properties, which are known to be equivalent [May, Lemma 9.2]:

(i) for every non-negative integer m, if two morphisms $s, s' : \Delta^m \longrightarrow S$ are homotopic[33], then they are equal;

(ii) for every integer $N \geq 2$, every horn $\Lambda_i^N \subset \Delta^N$, and every pair of simplices $\alpha, \beta : \Delta^N \longrightarrow S$ with $\alpha\big|_{\Lambda_i^N} = \beta\big|_{\Lambda_i^N}$, one has $\alpha\big|_{\dot{\Delta}^N} = \beta\big|_{\dot{\Delta}^N}$.

It is known[34] that every Kan complex S contains a simplicial subset $S' \subset S$ such that S' is a minimal Kan complex and the embedding $S' \hookrightarrow S$ is a weak equivalence. It is also known that weakly equivalent Kan complexes are isomorphic, and, moreover, any weak equivalence between minimal Kan complexes is an isomorphism. So the minimality property from Exercise 9.31 uniquely characterizes the simplicial model of $K(\pi, n)$ defined in Exercise 9.30.

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[32]The definition of homotopy groups of a Kan simplicial set S can be found in [May, Definitions 3.6 and 4.1] or [Hov, §3.4]. It is known that the homotopy groups of S are equal to the homotopy groups of the geometric realization of S; if you wish, you can use this to define the group operation on $\pi_n(S)$. Furthermore, if G is a simplicial group, the group operation on $\pi_n(G)$ coincides with that induced by the group operation on G [May, Proposition 17.2].

[33]The word "homotopic" means that the restrictions of s and s' to $\partial\Delta^m$ are equal and that s and s' are homotopic as morphisms with fixed restriction to $\partial\Delta^m$

[34]See [May, §§9-10] and [Hov, §3.5].

Remark 9.3. The simplicial version of Postnikov's theory [May, §8 and Ch.V] says that every Kan complex is, in a certain sense, built from Kan complexes, each of which has the homotopy type of some $K(\pi, n)$.

Exercises on the definition of an ∞-category

9.32. Prove Proposition 7.4; if any difficulties arise, see [T, §1.2.3]. As a hint, we note that only the weak Kan extension property for 2- and 3-dimensional simplices needs to be used in the proof. We strongly recommend drawing all the 3-dimensional pictures accompanying the proof.

Let us outline what is probably the shortest possible path one could choose (following *loc. cit.*). First, it is clear that the homotopy relation $\sim$ is reflexive. One can show simultaneously that it is symmetric and transitive by proving that if φ, φ' and φ'' are morphisms in X such that $\varphi \sim \varphi'$ and $\varphi \sim \varphi''$, then $\varphi' \sim \varphi''$. This can be done using a single 3-dimensional diagram.

Next, one can use a single diagram to show that if $x \xrightarrow{\varphi} y \xrightarrow{\psi} z$ are morphisms in X, if $\psi \sim \psi'$, if η is any composition of φ and ψ, and η' is any composition of φ and ψ', then $\eta \sim \eta'$. We also need to verify the "dual" property, where the roles of φ and ψ are reversed, but this follows formally by looking at the opposite category of X (see [T, §1.2.1] for the definition).

Thus composition of homotopy classes of morphisms in X is well defined. Next, check that it is associative, so that the category πX can be formed, and construct a canonical functor $\mathcal{P}(X) \longrightarrow \pi X$ using the explicit presentation of $\mathcal{P}(X)$ by generators and relations (Definition 5.3).

Finally, observe that this functor is bijective at the level of objects and surjective at the level of morphisms by construction. To check that it is injective at the level of morphisms, use the observation that every morphism in $\mathcal{P}(X)$ can be represented by a path of length ≤ 1.

9.33. Let X be an ∞-category, let $0 < k < n$, and let $f : \Lambda^n_k \longrightarrow X$ be a morphism of simplicial sets. Show that the simplicial set of all possible extensions of f to a morphism $\Delta^n \longrightarrow X$ is a contractible Kan complex.

Background. To formalize this statement, imitate the discussion in §4.12. To prove it, use the strategy described in the proof of Proposition 4.15. At some point you will need an analogue of Lemma 4.10. This analogue can be formulated as follows. Let us define an *inner fibration* to be a morphism of simplicial sets having the R.L.P. with respect to all inner horn inclusions $\Lambda^n_k \hookrightarrow \Delta^n$ $(0 < k < n)$. Thus a simplicial set X is an ∞-category if and only if the morphism $p_X : X \longrightarrow *$ is an inner fibration. Let us define an *inner anodyne*[35] morphism to be a morphism of simplicial sets that has the L.L.P. with respect to all inner fibrations. By a result of Joyal (see Corollary 2.3.2.4 in [T]), if $A \longrightarrow A'$ is an inner anodyne morphism of simplicial sets and $B \longrightarrow B'$ is any cofibration (i.e., monomorphism) of simplicial sets, the induced morphism $(A \times B') \coprod_{A\times B} (A' \times B) \longrightarrow A' \times B'$ is inner anodyne.

9.34. If K is any simplicial set and $\mathcal{C}$ is an ∞-category, then the simplicial set $\mathrm{Fun}(K, \mathcal{C})$ (defined to be $\mathrm{Map}_{\mathcal{S}et_\Delta}(K, \mathcal{C})$) is also an ∞-category. [*Hint*: Joyal's result mentioned in Exercise 9.33 can be used here as well.]

9.35. Let $\mathcal{C}$ be an ordinary small category, let X be any simplicial set, and let $p : X \longrightarrow N(\mathcal{C})$ be a morphism. Then p is an inner fibration (see Exercise 9.33 for the definition of this notion) if and only if X is an ∞-category.

9.36. (*Lurie*) Let $\mathcal{C}$ be an ordinary small category, and let $f : X \longrightarrow N(\mathcal{C})$ be a morphism of simplicial sets. Suppose that X is an ∞-category. Show that each fiber[36] of f is an ∞-category.

9.37. (*Lurie*) Show that, up to isomorphism, there is a unique simplicial set X with the following properties:
(i) X has a single nondegenerate simplex of each dimension;
(ii) every face of a nondegenerate simplex of X is again nondegenerate.
Prove that X is isomorphic to the nerve of an ordinary category.

Remark (Lurie)**.** We can think of X as the receptacle for the "universal idempotent". In other words, for every ∞-category $\mathcal{C}$, we can think of maps $X \longrightarrow \mathcal{C}$ as objects of x in $\mathcal{C}$ together with a (coherently) idempotent map $f : x \longrightarrow x$.

9.38. If X is a Kan complex, then X, viewed as an ∞-category, has an initial object (see [T, §1.2.12] and §8.4 above) if and only if X is contractible.

9.39. $*$ If X is an ∞-category and $x, y \in X_0$, show that the right mapping space $\mathrm{Map}^R_X(x, y)$, defined in [T, §1.2.2] and in §7.11 above, is a Kan complex.

9.40. $*$ Let X be an ∞-category. Check directly (using the definition of $\mathrm{Map}^R_X(x, y)$ and the definition of morphisms in the category πX given in Proposition 7.4) that for all $x, y \in X_0$, the set $\pi_0 \mathrm{Map}^R_X(x, y)$ can be canonically identified with $\mathrm{Hom}_{\pi X}(x, y)$.

Exercises on the functors $\mathfrak{N}$ and $\mathfrak{C}$

[35] We use the terminology of Lurie [T]. Joyal uses the terms "mid-fibration" and "mid-anodyne morphism" in place of "inner fibration" and "inner anodyne morphism".

[36] Fibers of a morphism $g : Y \longrightarrow Z$ of simplicial sets are to be understood in the following sense. If $z \in Z_0 = Z([0])$, we think of z as a morphism $\Delta^0 \longrightarrow Z$ and form the fiber product $g^{-1}(z) := \Delta^0 \times_Z Y$ in the category $\mathcal{S}et_\Delta$. The simplicial sets $g^{-1}(z)$, $z \in Z_0$, are the fibers of g.

9.41. Prove that the functor $\pi : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{C}at$ is left adjoint to the obvious inclusion functor $\mathcal{C}at \longrightarrow \mathcal{C}at_{\Delta}$ (obtained by viewing every set as a discrete simplicial set). Deduce that for every simplicial set X, there is a canonical isomorphism of categories $\mathcal{P}(X) \cong \pi\mathfrak{C}[X]$.

9.42. Let $\mathfrak{N} : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta}$ denote the simplicial nerve functor, introduced in Definition 1.1.5.5 of [T]. Let $\mathcal{C}$ be a topological category (*op. cit.*, Definition 1.1.1.3), and let $\mathrm{Sing}(\mathcal{C})$ denote the simplicial category obtained by applying the functor $\mathrm{Sing} : \mathcal{T}op \longrightarrow \mathcal{S}et_{\Delta}$ to each of the topological spaces of morphisms in $\mathcal{C}$. The simplicial nerve $\mathfrak{N}(\mathrm{Sing}(\mathcal{C}))$ will be denoted simply by $\mathfrak{N}(\mathcal{C})$. (It is denoted by $N(\mathcal{C})$ and is called the *topological nerve* of $\mathcal{C}$ in [T].) An explicit description of the 0-simplices, 1-simplices and 2-simplices of $\mathfrak{N}(\mathcal{C})$ is given after Example 1.1.5.8 in [T]. Give a similar description of the 3-simplices of $\mathfrak{N}(\mathcal{C})$, as well as of the various face and degeneracy maps between the sets $\mathfrak{N}(\mathcal{C})_n$ for $n = 0, 1, 2, 3$.

9.43. Prove Lemma 6.11.

9.44. Prove Proposition 6.12 together with Theorem 6.13. We suggest the following approach (although we do not claim that it is the only one). First, as explained in the text, reduce the problem to proving the proposition combined with part (d) of the theorem. Then, using the solution of Exercise 9.43, verify that the claim holds when $X = \Delta^d$ for some $d \geq 0$.

Now the idea is to use the fact that $\mathfrak{C}$ is the unique colimit-preserving functor $\mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$ that acts as described in Definition 6.6 on the objects Δ^d. Unfortunately, the sets A_n described in Theorem 6.13(d) are not functorial with respect to X in an obvious way[37]. To remedy this, we use an alternate description of A_n.

Fix a simplicial set X and an integer $n \geq 0$. Consider the set $\widetilde{A}_n$ of pairs $(\sigma, \mathcal{F})$, where σ and $\mathcal{F}$ are as in the definition of A_n, except that $\sigma \in X_k$ can now be an arbitrary k-simplex, with the only requirement that σ is not constant (i.e., does not come from a 0-simplex in X). Introduce a relation $\gg$ on $\widetilde{A}_n$ as follows: given $\sigma' \in X_{k'}$, we say that $(\sigma, \mathcal{F}) \gg (\sigma', \mathcal{F}')$ if there exists an (automatically surjective) nondecreasing map $g : [k] \longrightarrow [k']$ such that $\sigma = g^*(\sigma')$ and $\mathcal{F}'_j = g(\mathcal{F}_j)$ for every $0 \leq j \leq n$. Let $\approx$ be the equivalence relation on $\widetilde{A}_n$ generated by $\gg$.

(a) Show that every equivalence class in $\widetilde{A}_n$ with respect to $\approx$ contains a unique representative $(\sigma, \mathcal{F})$ such that σ is a nondegenerate simplex of X. Thus $\widetilde{A}_n / \approx$ can be canonically identified with A_n.
(b) Prove that $\widetilde{A}_n$ and $\widetilde{A}_n / \approx$ can be naturally made to depend functorially on X.

[37]The problem is that morphisms of simplicial sets may take nondegenerate simplices to degenerate ones.

(c) Let D_n denote the diagram scheme with X_0 as the set of vertices, $\widetilde{A}_n/\approx$ as the set of arrows, and the maps taking $(\sigma, \mathcal{F})$ to the 0-th and k-th vertices, $v_0(\sigma)$ and $v_k(\sigma)$, of σ, as the source and target maps, respectively. Thus D_n depends functorially on X as well. Prove that the formation of D_n commutes with colimits of simplicial sets.

(d) Finally, complete the proof of Proposition 6.12 and Theorem 6.13(d). Keep in mind the fact (which comes from abstract nonsense) that colimits in the category of simplicial objects in $\mathcal{C}at$ can be computed "simplex-wise".

9.45. Let $\mathfrak{C}[-] : \mathcal{S}et_{\Delta} \longrightarrow \mathcal{C}at_{\Delta}$ denote the left adjoint to the simplicial nerve functor $\mathfrak{N} : \mathcal{C}at_{\Delta} \longrightarrow \mathcal{S}et_{\Delta}$. (It is introduced in [T] just before Proposition 1.1.5.9.) Prove that if $f : S \longrightarrow S'$ is a monomorphism of simplicial sets, the induced functor $\mathfrak{C}[f] : \mathfrak{C}[S] \longrightarrow \mathfrak{C}[S']$ is faithful in the sense that for any pair of objects $x, y \in \mathfrak{C}[S]$, the corresponding morphism of simplicial sets $\mathrm{Map}_{\mathfrak{C}[S]}(x, y) \longrightarrow \mathrm{Map}_{\mathfrak{C}[S']}(x', y')$ is a monomorphism, where $x' = \mathfrak{C}[f](x)$ and $y' = \mathfrak{C}[f](y)$.

9.46. (*Lurie*) Describe $\mathfrak{C}[\Lambda^n_i]$ as a simplicial subcategory of $\mathfrak{C}[\Delta^n]$. (The solution of this exercise is contained in the proof of Proposition 1.1.5.9 in [T]. We also recommend reading and understanding this proof as part of the exercise.)

9.47. (*Drinfeld*) Let X be an arbitrary simplicial set, and consider the topological category $|\mathfrak{C}[X]|$ obtained by applying the geometric realization functor to each of the spaces of morphisms in the simplicial Poincaré category $\mathfrak{C}[X]$ of X. By *discarding the topology* on the spaces of morphisms, view $\mathfrak{C}[X]$ as an ordinary category. Prove that this category is free and describe its generators explicitly.

9.48. $*$ Let $\mathcal{S}et^{\times}_{\Delta}$ and $\mathcal{C}at^{\times}_{\Delta}$ denote the groupoids obtained from the categories $\mathcal{S}et_{\Delta}$ and $\mathcal{C}at_{\Delta}$, respectively, by discarding all non-invertible arrows. Prove that the functor $\mathcal{S}et^{\times}_{\Delta} \longrightarrow \mathcal{C}at^{\times}_{\Delta}$, obtained by restricting $\mathfrak{C}$, is fully faithful.

Exercises on equivalences in ∞-categories

9.49. (*Lurie*) Taking Joyal's characterization of equivalences (cf. Proposition 1.2.4.3 in [T], which is stated as Proposition 7.7 above) as a definition, prove that if X is an ∞-category, then any composition of equivalences in X is again an equivalence.

9.50. (*Lurie*) Fill in the details of the proof of Proposition 1.2.5.3 in [T]. In other words, suppose that X is an ∞-category, and let X' be the simplicial subset of X spanned by those simplices, all of whose edges are equivalences in X. Show that X' is a Kan complex.

Exercises on the Joyal model structure on $\mathcal{S}et_{\Delta}$

9.51. If $0 \le i \le n$, show that the inclusion $\Lambda^n_i \hookrightarrow \Delta^n$ is a categorical equivalence if and only if $0 < i < n$.

9.52. Give an example of a functor $\mathcal{C} \longrightarrow \mathcal{D}$ between ordinary small categories such that the induced morphism $N(\mathcal{C}) \longrightarrow N(\mathcal{D})$ is *not* a categorical fibration. (Note, however, that it is always an inner fibration, by Exercise 9.35.)

Exercises on joins of categories and of simplicial sets

9.53. Show that the nerve of the join of two small categories is canonically isomorphic to the join of their nerves.

9.54. $*$ Show that the geometric realization functor takes joins of simplicial sets to joins of compactly generated spaces.

9.55. (*Lurie*) Let X be a simplicial set, and suppose that X is isomorphic to a cone $Y \star \Delta^0$. Show that the isomorphism is unique, and that Y is determined by X up to (unique) isomorphism. (The join operation $\star$ is discussed in §1.2.8 of [T].)

9.56. (*Joyal*) If $\mathcal{C}$ and $\mathcal{C}'$ are ∞-categories, prove that so is $\mathcal{C} \star \mathcal{C}'$.

Exercises on homotopy (co)limits in ∞-categories

9.57. If $\mathcal{C}$ is an ∞-category and $\mathscr{I}$ is an ordinary small category, prove that for every functor $F : \mathscr{I} \longrightarrow \mathcal{C}$, the simplicial sets $\mathcal{C}_{F/}$ and $\mathcal{C}_{/F}$, constructed in Definition 8.5 (see also §1.2.9 in [T]), are ∞-categories as well.

9.58. (*Lurie*) Let $f : \mathscr{I} \longrightarrow \mathcal{C}$ be a functor between ordinary small categories. Show that colimits of f (in the sense of ordinary category theory) coincide with homotopy colimits of the induced morphism $N(f) : N(\mathscr{I}) \longrightarrow N(\mathcal{C})$ in the ∞-categorical sense. (Homotopy limits and colimits in ∞-categories are discussed in §1.2.13 of [T] and in Section 8.)

An exercise on model category structures

9.59. (*Lurie*) This exercise uses the language of model categories (the necessary background is provided in Appendix A.2 to [T]). Consider the following properties of a given model category $\mathcal{A}$.

(i) The collection of weak equivalences in $\mathcal{A}$ is stable under Cartesian product.
(ii) If $f : X \hookrightarrow Y$ is a cofibration and Z is a cofibrant object[38] in $\mathcal{A}$, then the induced morphism $X \times Z \longrightarrow Y \times Z$ is a cofibration in $\mathcal{A}$.
(iii) As a category, $\mathcal{A}$ is Cartesian closed[39].

[38] This means that the unique arrow $\varnothing \longrightarrow Z$ is a cofibration, where $\varnothing$ is an initial object of $\mathcal{A}$.

[39] That is, for every pair of objects $X, Y \in \mathcal{A}$, there exists another object X^Y such that the functors $\mathcal{A} \longrightarrow \mathcal{S}et$ given by $Z \longmapsto \mathrm{Hom}_{\mathcal{A}}(Z, X^Y)$ and $Z \longmapsto \mathrm{Hom}_{\mathcal{A}}(Z \times Y, X)$ are isomorphic.

Do the following.

(a) Prove that if a model category $\mathcal{A}$ satisfies properties (i)–(iii) above, then the homotopy category $h\mathcal{A}$ is Cartesian closed.

(b) Show that the Joyal model structure on $\mathcal{S}et_\Delta$ satisfies (i), (ii) and (iii). (The Joyal model structure is defined and studied in §2.2.5 of [T].)

(c) Show that the model category $\mathcal{C}at_\Delta$ satisfies (i) and (iii). (The model category structure on $\mathcal{C}at_\Delta$ referred to in this exercise is defined in §A.3.1 of [T].)

(d) Show that $\mathcal{C}at_\Delta$ does not satisfy (ii). [*Hint.* Let $\mathcal{C} = \mathfrak{C}[\Delta^1]$ denote the simplicial category containing a pair of objects and a single morphism between them. Show that $\mathcal{C}$ is cofibrant, but $\mathcal{C} \times \mathcal{C}$ is not cofibrant.]

(e) Nevertheless, the homotopy category of $\mathcal{C}at_\Delta$ is Cartesian closed: this follows from the equivalence of $\mathcal{C}at_\Delta$ with the Joyal model structure on $\mathcal{S}et_\Delta$.

Try to prove this directly (the goal being not to succeed, but to gain an appreciation of why some constructions are difficult in $\mathcal{C}at_\Delta$).

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