



# Early work of N.G. (Dick) de Bruijn in analysis and some of my own

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## Abstract

There are parallels between de Bruijn's early work in analysis and that of the author. However, Dick's work soon became much broader and deeper. While the present paper reviews several topics of common interest, its main content is a short version of Dick's important article related to Riemann's Hypothesis, entitled 'The roots of trigonometric integrals' [N.G. de Bruijn, The roots of trigonometric integrals, Duke Math. J. 17 (1950) 197–226]. The associated 'de Bruijn–Newman constant' is also discussed.

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## 1. Introduction

Through the years Dick's work became very broad. He made important contributions to many areas: analysis, number theory, combinatorics, group theory, asymptotics, lambda calculus, generalized functions, quasi-crystals, Penrose patterns, and computer verification of proofs: his project Automath. I have known and admired Dick for almost 70 years. Much to his regret, he had to retire from the 'TH' (Technical University) Eindhoven, in 1984, at the age of 66. In his farewell address he spoke of 'Looking back in admiration', referring to mathematicians whose work had affected his own; cf. [21]. For me, in relation to Dick, it has always been: 'Looking *up* in admiration'.

We were both stimulated by a great teacher (and excellent mathematician), H.D. Kloosterman at the University of Leiden. For several years, we solved many of the diverse problems in the

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‘Wiskundige Opgaven’ of the Dutch mathematical society (of which Dick later became the editor). We were also inspired by the problem books ‘Aufgaben und Lehrsätze’ by Pólya and Szegő [71].

Some of our early papers in *Duke Mathematical Journal* were based on the articles by Pólya mentioned in Section 4. Visits by Paul Erdős were also important to both of us.

In the following I write about some parallels in our work in analysis. Sections 4–6, however, contain a short account (with proofs) of Dick’s deep results on the zeros of trigonometric integrals related to Riemann’s zeta function. Section 6 ends with a discussion of the related de Bruijn–Newman constant.

## 2. Early experiences

There are several parallels between our early mathematical experiences. In high school we were both attracted to mathematics. We both started our university study at Leiden, Dick in 1936, then already in possession of two high-school teaching certificates, I in 1940. We were both impressed by the style of Kloosterman, and stimulated by the beauty of complex analysis, through the inexpensive books by Knopp. Because of the war, neither of us could continue our study at Leiden, and we did much work on our own. We served successively as assistants at the TH Delft. After that Dick came in contact with more applied areas at the Philips Physics Laboratory (the ‘Nat Lab’) in Eindhoven. Shortly after the war he became a professor at the TH Delft, while I learned some applied mathematics at the Mathematical Centre (‘Mathematisch Centrum’) in Amsterdam. (A little later, I would briefly return to Delft as a professor, but then Dick had already moved on to the University of Amsterdam.)

In the beginning there were also similarities in our publication history. We both published several articles before the Ph.D. thesis. At that time this was easier than nowadays. One could start in not-so-demanding journals: *Mathematica B*, *Simon Stevin*, the ‘Handelingen van het Natuur- en Geneeskundig Congres’ or the ‘Nieuw Archief voor Wiskunde’. Next came the Proceedings of the KNAW, the Royal Netherlands Academy of Arts and Sciences, where your paper might be presented by a sympathetic Academy member. (Note that in the bibliography, I have only listed the corresponding articles in *Indagationes Mathematicae*.)

In the early 1940s we both started on difficult problems. Dick treated special cases [7] of the well-known Bieberbach conjecture of 1916, which asserts that the coefficients of a normalized univalent function on the unit disc,

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad (2.1)$$

satisfy the inequalities  $|a_n| \leq n$ . I tried to find a simple proof for a Tauberian theorem that would imply the prime number theorem [41,42]. (Tauberian theorems are conditioned converses of continuity theorems such as Abel’s theorem for power series. Converses by Hardy and Littlewood were mentioned (without proof) in student lecture notes of Kloosterman’s introductory analysis course — and presented a real challenge to me!)

Without much background knowledge we did not get very far. (The Bieberbach conjecture was finally proved by Louis de Branges in 1984; see [6] and the review in *Math. Reviews* by Peter Duren; cf. also [51,52].)

Shortly after the war Paul Erdős visited the Netherlands. Invited to the Mathematical Centre in Amsterdam by its director van der Corput, he came to lecture on his version of the elementary proof of the prime number theorem by Selberg and himself. Here ‘elementary’ means: free of complex analysis; see the resulting early publication [25], as well as [75,31]. Erdős’s visit would

affect the work of both of us. Dick wrote several joint papers with him, one of them [22] related to the work of Mrs van Aardenne-Ehrenfest on ‘just distributions’; cf. Section 3. After listening and talking to Erdős, I realized how naïve I had been in my early Tauberian work! However, one of his suggestions later led to good remainder estimates in Tauberian theorems for power series; cf. [45] and, more importantly, [46,48]. (Analogous results were independently obtained by Geza Freud and Tord Ganelius; cf. [53, Chapter VII].)

When one is guided by the experience of a master, one more readily arrives at high-level results, as Dick did in his Ph.D. thesis [8] on modular forms under the direction of Kloosterman.

### 3. Slowly varying functions

In the spring of 1948 Dick and I participated in one of the stimulating seminars that Mrs van Aardenne-Ehrenfest occasionally organized at her home in Dordrecht. This was one of her ways, as a housewife, to stay in contact with mathematics. (Among mathematicians she became well-known for her proof of van der Corput’s conjecture that there is no ‘just distribution’ of an infinite sequence of points on an interval; cf. [1,2].)

In connection with Tauberian theorems, Fred van der Blij and I had become interested in the ‘slowly oscillating functions’ of Karamata [37,38]. Such functions  $L(x)$ , defined on a half-line  $x > x_0$ , continuous (or just measurable) and eventually positive, are now called slowly varying. They are defined by the condition

$$\frac{L(\lambda x)}{L(x)} \rightarrow 1 \quad \text{as } x \rightarrow \infty \text{ for every number } \lambda > 0. \quad (3.1)$$

In the seminar I asked for a derivation of a general Karamata-type integral representation:

$$L(x) = c(x) \exp \left\{ \int_{x_1}^x \varepsilon(y) \frac{dy}{y} \right\}. \quad (3.2)$$

Here  $c(x)$  tends to a limit  $c > 0$  as  $x \rightarrow \infty$ , while  $\varepsilon(y)$  is bounded and tends to 0 at  $\infty$ . Mrs van Aardenne-Ehrenfest, Dick and I arrived at a nice proof that is often cited; see [54]. (For a detailed discussion, see Korevaar [53, Chapter IV].) An extension of the result is in de Bruijn [19]. Slowly varying functions now play an important role in probability theory and other areas; cf. Bingham et al. [3].

### 4. Dick’s work [16] related to Riemann’s Hypothesis (RH)

Dick and I were both inspired by work of Pólya on the zeros of entire functions. Dick was fascinated by Pólya’s results [65–69] on the zeros of functions given by trigonometric integrals. I was attracted by Pólya’s papers on the approximation of entire functions by polynomials whose zeros satisfy certain conditions: [62,63,55]. All these articles by Pólya have been reproduced with commentary in his Collected papers, vol. II [70]. Dick and I both published extensions of Pólya’s work in *Duke Mathematical Journal*.

Since I will be brief about my own work I mention it first. My Ph.D. thesis [43] and the papers [44,47] involved characterizations of entire functions that can be approximated by polynomials whose zeros lie in a prescribed region (the case of an infinite strip would show up in Dick’s work). Prototypes were the old theorems of Laguerre and Pólya for the case where the polynomials have only positive or real zeros. In particular, an entire function  $f(z)$  is the limit,

uniformly in every bounded domain, of a sequence of polynomials with only real zeros if and only if it has the form

$$f(z) = Ae^{c_1z+c_2z^2}z^m \prod_k (1 - z/z_k)e^{z/z_k} \tag{4.1}$$

with  $c_1, c_2$  real,  $c_2 \leq 0$ ,  $z_k$  real, and  $\sum_k z_k^{-2} < \infty$ ; cf. Pólya [62,64].

Dick’s paper [16] was much deeper. His aim was to obtain precise information on the zeros of trigonometric integrals related to Riemann’s zeta function  $\zeta(s)$ , or more directly, to Riemann’s Xi function  $\Xi(z)$ . Setting  $s = \frac{1}{2} + iz$  one has

$$\Xi(z) = \xi(s) = (1/2)s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s). \tag{4.2}$$

The functional equation for  $\zeta(s)$  is equivalent to the statement that  $\Xi(z)$  is even. The zeros of  $\Xi(z)$  lie in the strip  $|\text{Im } z| < \frac{1}{2}$ , and RH asserts that they are all real.

Although he referred to (the 1930 predecessor of) Titchmarsh’s zeta-function book [80], Dick used a non-standard integral representation

$$\Xi(2z) = \int_{\mathbb{R}} \phi(t)e^{izt} dt, \tag{4.3}$$

where

$$\phi(t) = \sum_{n=1}^{\infty} (2n^4\pi^2e^{9t/4} - 3n^2\pi e^{5t/4})e^{-n^2\pi e^t}. \tag{4.4}$$

The function  $\phi(t)$  is even, positive,  $t$  and has an analytic continuation to the strip  $|\text{Im } t| < \frac{1}{2}\pi$ . For  $t \rightarrow \pm\infty$  one has

$$\phi(t) \sim \phi_1(t) = 4\pi^2 \cosh(9t/4) e^{-2\pi \cosh t}, \tag{4.5}$$

and  $\int_{\mathbb{R}} \phi_1(t)e^{izt} dt$  has only real zeros; see Section 5 below or Pólya [66,67]. Dick also gave a second approximation  $\phi_2(t)$  to  $\phi(t)$  (sharper than Pólya’s), and showed that its transform has only real zeros.

**Remarks 4.1.** The integrand  $\phi(t)$  in Dick’s representation (4.3) for the Xi function is different from the integrand of Riemann and Pólya (which would later be used also by Csordas, Norfolk and Varga [29]). The connection is as follows: Riemann et al. wrote  $\Xi(z/2)$  as the Fourier transform of  $\Phi(t) = 4\phi(4t)$ . Following Titchmarsh [80], Newman [60] (cf. Section 6 below) would write  $\Xi(z)$  as the Fourier transform of  $F(u) = 2\phi(2u)$ .

**5. Dick’s extension in [16] of Pólya’s work**

Following Pólya [66–69], Dick considered Fourier integrals

$$\hat{F}(z) = \int_{\mathbb{R}} F(t)e^{izt} dt, \tag{5.1}$$

where  $F(t)$  is integrable over  $\mathbb{R}$  and such that

$$F(-t) = F(t)^* = \mathcal{O}(e^{-|t|^b}) \quad \text{as } t \rightarrow \infty \text{ for some } b > 2. \tag{5.2}$$

(Here  $*$  denotes the complex conjugate.) Pólya had proved reality of the zeros in various cases, for example,  $F(t) = e^{-t^{2n}}$  and  $F(t) = e^{-\lambda \cosh t}$  with  $\lambda > 0$ . He also introduced *universal factors*: functions  $U(t)$  (my notation) with the following property. If  $\hat{F}(z)$ , with  $F(t)$  as in (5.2), has only real zeros, the same holds for

$$\widehat{FU}(z) = \int_{\mathbb{R}} F(t)U(t)e^{izt} dt. \tag{5.3}$$

In his papers, Pólya had identified a good many universal factors. In particular, every function  $U(t)$  such that  $U(it)$  is a limit (uniformly in bounded domains) of polynomials with only real zeros, is a universal factor. Simple examples are  $\cosh \lambda t$  with  $\lambda > 0$  and  $e^{\gamma t^2}$  with  $\gamma > 0$ . The above results confirm that the Fourier transform of  $\phi_1(t)$  in (4.5) has only real zeros.

Dick found *special* universal factors  $S(t)$  (satisfying  $S(-t) = S(t)^*$ ) with stronger properties. If  $F(t)$  as in (5.2) is such that all zeros of  $\hat{F}(z)$  lie in a strip  $|\text{Im } z| \leq \Delta$  ( $\Delta > 0$ ), then the zeros of

$$\widehat{FS}(z) = \int_{\mathbb{R}} F(t)S(t)e^{izt} dt \tag{5.4}$$

lie in a narrower ‘strip’  $|\text{Im } z| \leq \Delta_1$ , with  $0 \leq \Delta_1 < \Delta$  independent of  $F(t)$ . (There is a further requirement for what Dick called a strong universal factor, which we will ignore.) Special universal factors  $S(t)$  are described in Dick’s Theorems 11–13. We quote Theorem 12:

**Theorem 5.1.** *Let  $S(t) = \prod_{k=1}^N (\xi_k e^{\lambda_k t} + \xi_k^* e^{-\lambda_k t})$ , where  $|\xi_k| = 1$  and  $\lambda_k > 0$ , and let  $F(t)$  be as in (5.2). If all zeros of  $\hat{F}(z)$  lie in the strip  $|\text{Im } z| \leq \Delta$ , then the zeros of  $\widehat{FS}(z)$  lie in the strip*

$$|\text{Im } z| \leq \Delta_N = \left( \max \left\{ \Delta^2 - \sum_{k=1}^N \lambda_k^2, 0 \right\} \right)^{\frac{1}{2}}.$$

For the proof it is enough to consider the special case  $S(t) = \xi e^{\lambda t} + \xi^* e^{-\lambda t}$ , following which one may use iteration. To complete the proof one takes a limit in a related result for polynomials; see Section 6.

**Corollary 5.2.** *Let  $\phi(t)$  be the function of (4.4) that is associated with the Xi function. Then for any number  $\lambda \geq \frac{1}{4}$ , the transform*

$$\int_{\mathbb{R}} \phi(t) \cosh(\lambda t) e^{izt} dt \tag{5.5}$$

*has only real zeros.*

Indeed, the zeros of  $\hat{\phi}(z) = \Xi(2z)$  lie in the strip  $|\text{Im } z| \leq \frac{1}{4}$ . Now take  $S(t) = e^{\lambda t} + e^{-\lambda t} = 2 \cosh \lambda t$ . For  $\lambda \geq \frac{1}{4}$  one has  $\Delta_1 = 0$ .

By a further limit process one obtains Dick’s Theorem 13:

**Theorem 5.3.** *Let  $F(t)$  be as in (5.2). If all zeros of  $\hat{F}(z)$  lie in the strip  $|\text{Im } z| \leq \Delta$ , then the zeros of*

$$g(z) = g(z, \lambda) = \int_{\mathbb{R}} F(t)e^{\frac{1}{2}\lambda^2 t^2} e^{izt} dt \tag{5.6}$$

lie in the strip

$$|\operatorname{Im} z| \leq \Delta_1 = (\max\{\Delta^2 - \lambda^2, 0\})^{\frac{1}{2}}. \tag{5.7}$$

For the proof Dick observed that by [Theorem 5.1](#), the zeros of

$$g_N(z) = \int_{\mathbb{R}} F(t)(\cosh \lambda t/N)^{N^2} e^{izt} dt$$

lie in the strip (5.7). Hence by Hurwitz’s theorem on the zeros of limit functions, it is sufficient to show that  $g_N(z) \rightarrow g(z)$  uniformly in every bounded domain. The latter can be derived from (5.2) and the fact that for  $\mu^2 > \lambda^2$  and  $N \rightarrow \infty$ , one has

$$e^{-\frac{1}{2}\mu^2 t^2} (\cosh \lambda t/N)^{N^2} \rightarrow e^{-\frac{1}{2}\mu^2 t^2 + \frac{1}{2}\lambda^2 t^2}$$

uniformly for  $t \in \mathbb{R}$ . (Use the inequality  $\cosh y < e^{\frac{1}{2}y^2}$ , which implies that  $(\cosh \lambda t/N)^{N^2} \leq e^{\frac{1}{2}\lambda^2 t^2}$ .)

**Corollary 5.4.** *If  $g(z, \lambda)$  has only real zeros, the same holds for  $g(z, \mu)$  whenever  $\mu^2 > \lambda^2$ .*

This would also follow from the fact that  $e^{\gamma t^2}$  is a universal factor when  $\gamma > 0$ .

For the case of  $F(t) = \phi(t)$  as in (4.4), one can formulate a corollary similar to [Corollary 5.2](#); see [Section 6](#).

## 6. Auxiliary results in [16] and later developments

Dick derived his basic [Theorem 12](#) (our [Theorem 5.1](#)) from an auxiliary result for polynomials, his [Theorem 3](#):

**Theorem 6.1.** *Let  $f(z)$  be a polynomial with real coefficients whose zeros lie in the strip  $|\operatorname{Im} z| \leq \Delta$ . Then for  $|\xi| = 1$  and  $\lambda > 0$ , the zeros of the polynomial*

$$Tf(z) \stackrel{\text{def}}{=} \xi f(z + i\lambda) + \xi^* f(z - i\lambda) \tag{6.1}$$

lie in the strip  $|\operatorname{Im} z| \leq \Delta_1$  given by (5.7).

More precisely Dick showed that the zeros of  $Tf(z)$  lie in the set  $\Omega(f)$  defined as follows. Let

$$f(z) = A \prod_{i=1}^m \{(z - a_i)^2 + \Delta_i^2\} \prod_{j=1}^n (z - b_j), \tag{6.2}$$

where  $a_i, b_j$  are real,  $\Delta_i > 0, A \neq 0$ . With any  $\Delta_i > \lambda$  one associates the closed circular disc  $D_i$  given by  $(x - a_i)^2 + y^2 \leq \Delta_i^2 - \lambda^2$ ; if  $\Delta_i \leq \lambda$  one takes  $D_i$  empty. The set  $\Omega(f)$  is the union of the real axis and the discs  $D_i$ .

For the proof one takes  $z$  outside  $\Omega(f)$  and  $\operatorname{Im} z > 0$ , say. Comparing  $f(z + i\lambda)$  and  $f(z - i\lambda)$  factor by factor, one then finds that  $|f(z + i\lambda)| > |f(z - i\lambda)|$ . Thus the zeros of  $Tf(z)$  must lie in  $\Omega(f)$ , which is a subset of the strip (5.7).

**Proof of Theorem 5.1.** Let  $F(t)$  be as in (5.2) and suppose that the zeros of the Fourier transform  $\hat{F}(z)$  (5.1) lie in the strip  $|\text{Im } z| \leq \Delta$ . By (5.2)  $\hat{F}(z)$  will be a real entire function (it is real-valued for real  $z$ ). By a straightforward estimate for  $|\hat{F}(x + iy)|$ , this function will be of order  $\leq b/(b - 1) < 2$ . Hence by Hadamard’s theorem,  $\hat{F}(z)$  has a product representation of the form (4.1), with  $A, c_1$  real and  $c_2 = 0$ . The zeros  $z_k$  are either real, or occur in complex conjugate pairs  $x_k \pm iy_k$  with  $|y_k| \leq \Delta$ , and  $\sum_k |z_k|^{-2} < \infty$ . It readily follows that  $\hat{F}(z)$  can be written as the limit, uniformly in every bounded domain, of a sequence of real polynomials  $f_n(z)$  whose zeros lie in the strip  $|\text{Im } z| \leq \Delta$ . Setting  $S(t) = \xi^* e^{\lambda t} + \xi e^{-\lambda t}$  as we may, one has

$$\widehat{FS}(z) = \xi \hat{F}(z + i\lambda) + \xi^* \hat{F}(z - i\lambda) = T\hat{F}(z) = \lim_{n \rightarrow \infty} T f_n(z), \tag{6.3}$$

uniformly in every bounded domain. Now the zeros of the functions  $T f_n(z)$  lie in the strip (5.7), hence by Hurwitz’s theorem, the same is true for the limit function  $T\hat{F}(z)$ .  $\square$

Dick’s 1950 article [16] contains more general results for (real) polynomials than [Theorem 6.1](#), with applications to universal factors and Ramanujan’s zeta function. Earlier he had written on the theme that for a real polynomial, the zeros of the derivative tend to lie closer to the real axis than those of the original polynomial; see [9] and the joint papers with his assistant Springer [23,24]. Still other papers by Dick [13,14] involved more general composition polynomials. (One may remark here that a recent paper by Malamud [58] includes a proof of a general conjecture in [23].)

**De Bruijn–Newman constant.** As noted by Charles Newman [60], one has the following corollary to [Theorem 5.3](#). Let  $F(u) = 2\phi(2u)$  where  $\phi(t)$  is as in (4.4), so that  $\hat{F}(z) = \Xi(z)$ ; cf. [Remarks 4.1](#). Now for  $c \in \mathbb{R}$ , set

$$\Xi^c(z) = \int_{\mathbb{R}} F(u) e^{cu^2} e^{izu} du. \tag{6.4}$$

(Newman actually used parameter  $b = -c$ .)

**Corollary 6.2.** *For  $c \geq 0$ , the zeros of  $\Xi^c(z)$  lie in the strip*

$$|\text{Im } z| \leq (\max\{(1/4) - 2c, 0\})^{\frac{1}{2}}. \tag{6.5}$$

*They are all real when  $c \geq \frac{1}{8}$ .*

This follows from [Theorem 5.3](#) since now  $\Delta = \frac{1}{2}$  and  $\frac{1}{2}\lambda^2 = c$ .

Newman went on to prove the following *important result*: there exists a number  $c_0 \in (-\infty, \frac{1}{8}]$  such that  $\Xi^c(z)$  has only real zeros when  $c \geq c_0$ , but has some nonreal zeros when  $c < c_0$ . (It is interesting that in the discussion preceding the proof, Newman referred to the final theorem in Dick’s paper.) Note that *Riemann’s Hypothesis* is true if and only if  $c_0 \leq 0$ , and Newman conjectured that  $c_0 = 0$ .

The number  $c_0$  corresponds to the so-called *de Bruijn–Newman constant* which was subsequently introduced by Csordas, Norfolk and Varga [29]. Because the latter authors started out with the Fourier integral for  $\Phi(t) = 4\phi(4t) = 2F(2t)$ , which is equal to  $\Xi(z/2)$ , it turns out that the ‘official’ de Bruijn–Newman constant  $\Lambda$  is equal to  $4c_0$ . Lower bounds for  $\Lambda$  have been computed by several authors, in the first place by Varga and coauthors. For the crucial role of so-called Lehmer pairs of zeta zeros, see Csordas, Smith and Varga [30]. Other contributors include Herman te Riele [72], Odlyzko [61], and the present record holders: Saouter, Gourdon and

Demichel [73]. The latter showed that  $\Lambda > -1.14541 \times 10^{-11}$ ; recall that RH is equivalent to  $\Lambda \leq 0$ !

The inequality  $\Lambda = 4c_0 \leq \frac{1}{2}$  has been improved to  $\Lambda < \frac{1}{2}$  by Ki, Kim and Lee [40]. For additional references to the constant, see Mathematical Reviews.

It would be interesting to study the nonreal zeros of  $\Xi^c(z)$  for negative  $c$  close to zero. If RH is false, the zeros of  $\Xi^c(z)$  with  $c < 0$  cannot lie in the strip  $|\operatorname{Im} z| \leq \sqrt{2|c|}$ . This follows from Theorem 5.3 with  $F(t) = \phi(t)e^{ct^2}$ .

In [34], Hejhal has studied Fourier integrals corresponding to higher-order approximations for the function  $\phi(t)$  of (4.4). He found that almost all their zeros are real. See also Ki [39], Haglund [33], and Stopple [76].

The referee has drawn attention to an older paper by P.R. Taylor [77]. As a step towards a possible proof of RH he proved the following. Define  $\xi_1(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s) = 2\xi(s)/\{s(s-1)\}$ . Then all complex zeros of the difference  $\xi_1(s+1/2) - \xi_1(s-1/2)$  lie on the line  $\{\operatorname{Re} s = 1/2\}$ .

### 7. Generalized functions

At the ‘International Congress of Mathematicians’ in Cambridge, Mass., in 1950, one could learn about distributions (generalized functions) from Laurent Schwartz himself; cf. his book [74]. After the Congress, some colleagues and I at Purdue University organized a year-long seminar on distributions. The study of Schwartz’s work was difficult because he made heavy use of a theory of locally convex topological vector spaces (which he had developed especially for distribution theory). It turned out later that one could dispense with such spaces in an ‘elementary theory’ of distributions, which could be taught to students of physics and engineering. Cf. my book [50], and for the early history of distributions, see Lützen [56].

At a much deeper level than my work Dick has developed a very general theory for the so-called Wigner distribution [20]. The latter (not a ‘distribution’ in the sense of Schwartz!) is an important tool in signal analysis. It goes back to quantum-theoretical work of Wigner [81] and describes signals in both time and frequency. For square-integrable functions  $f(x)$  on  $\mathbb{R}$  the Wigner distribution may be given by the formula

$$W(t, \omega; f) = (2\pi)^{-\frac{1}{2}} \int_{\mathbb{R}} e^{-i\omega x} f(t+x/2) f^*(t-x/2) dx. \tag{7.1}$$

In his work, Dick obtained a rigorous theory for the case where  $f$  belongs to a large space of generalized functions in the sense of Gel’fand and Shilov [32]. (Curiously enough, there is a relation between Dick’s generalized functions and the author’s formal Hermite expansions, on which the latter had based a general ‘ordinary’ Fourier theory [49]; cf. Zhang [82].)

Dick’s work on the Wigner distribution was carried further and extensively applied by his student Guido Janssen. See the latter’s Ph.D. thesis [35]; for references to a large number of Guido’s subsequent papers, see Math. Reviews. One should also mention Guido’s very readable introduction to time-frequency analysis in [36].

### 8. Asymptotics

Good communication with students and investigators outside mathematics has always been important to both of us. At the ‘Philips Nat Lab’ Dick regularly collaborated with workers in applied areas; cf. the papers [4,5,10]. Early on Dick developed sophisticated special asymptotics in his papers [11,12,15–17]; for comments, see Pieter Moree [59]. The first three, which treat



functional equations for a function  $f$  that involve a difference  $f(x + 1) - f(x)$ , were the result of a question by Mahler on a certain partition problem; cf. [57].

Later Dick wrote a beautiful book ‘Asymptotic methods in analysis’ [18] that was reprinted several times, in 1981 by ‘Dover’. In it, he explained a number of important methods by careful analysis of well-chosen concrete examples. Cf. the article ‘Asymptotiek’ by Nico Temme [78].

It is now necessary to say something about the work of van der Corput. In the 1920s he had done outstanding work in number theory. Later his attention turned to more applied areas, in particular, asymptotics. Together with many of his students he developed and applied the sophisticated method of stationary phase. After the war, he started a large project ‘Asymptotic Methods’ at the Mathematical Centre, in which many mathematicians participated; cf. his Rouse Ball lecture [26]. It is a pity that a comprehensive work on this ‘Dutch asymptotics’ has never appeared. (In a broad survey of asymptotic methods, Nico Temme [79] refers to both de Bruijn’s work and that of van der Corput.)

In the 1950s van der Corput developed a somewhat abstract ‘neutrix-calculus’; cf. [28]. Its purpose was to automate the use of asymptotics by the introduction of a ‘neutrix’, a well-chosen group of ‘negligible functions’. (Distributions could also be included in this framework; cf. [27].) A problem was that in this system, there was no ‘topology’, and therefore, no possibility of good remainder estimates. Many classical analysts were skeptical about the new theory. Dick de Bruijn’s hilarious comment was: ‘One can’t teach an old dog new tricks (neutrix)’.

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