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Preface

To paraphrase a comment in the introduction to a classic point-set topology text, this book might have been titled *What Every Young Topologist Should Know.* It grew from lecture notes we wrote while teaching second–year algebraic topology at Indiana University.

The amount of algebraic topology a student of topology must learn can be intimidating. Moreover, by their second year of graduate studies students must make the transition from understanding simple proofs line-by-line to understanding the overall structure of proofs of difficult theorems.

To help our students make this transition, the material in these notes is presented in an increasingly sophisticated manner. Moreover, we found success with the approach of having the students meet an extra session per week during which they took turns presenting proofs of substantial theorems and writing lecture notes to accompany their explanations. The responsibility of preparing and giving these lectures forced them to grapple with "the big picture" and also gave them the opportunity to learn how to give mathematical lectures, preparing for their participation in research seminars. We have collated a number of topics for the students to explore in these sessions; they are listed as projects in the table of contents and are enumerated below.

Our perspective in writing this book was to provide the topology graduate students at Indiana University (who tend to write theses in geometric topology) with the tools of algebraic topology they will need in their work, to give them a sufficient background to be able to interact with and appreciate the work of their homotopy theory cousins, and also to make sure that they are exposed to the critical advances in mathematics which came about with the development of topology in the years 1950-1980. The topics discussed in varying detail include homological algebra, differential topology, algebraic K-theory, and homotopy theory. Familiarity with these topics is important not just for a topology student but any student of pure mathematics, including the student moving towards research in geometry, algebra, or analysis.

The prerequisites for a course based on this book include a working knowledge of basic point-set topology, the definition of CW-complexes, fundamental group/covering space theory, and the construction of singular homology including the Eilenberg-Steenrod axioms. In Chapter 9, familiarity with the basic results of differential topology is helpful. In addition, a command of basic algebra is required. The student should be familiar with the notions of R-modules for a commutative ring R (in particular the definition of tensor products of two R-modules) as well as the structure theorem for modules over a principal ideal domain. Furthermore, in studying non-simply connected spaces it is necessary to work with tensor products over (in general noncommutative) group rings, so the student should know the definition of a right or left module over such a ring and their tensor products. Basic terminology from category theory is used (sometimes casually), such as category, functor, and natural transformation. For example, if a theorem asserts that some map is natural, the student should express this statement in categorical language.

In a standard first-year course in topology, students might also learn some basic homological algebra, including the universal coefficient theorem, the cellular chain complex of a CW-complex, and perhaps the ring structure on cohomology. We have included some of this material in Chapters 2, 3, and 4 to make the book more self-contained and because we will often have to refer to the results. Depending on the pace of a first-year course, a course based on this book could start with the material of Chapter 3 (Homological Algebra), Chapter 4 (Products), or Chapter 5 (Fiber Bundles).

Chapter 7 (Fibrations, Cofibrations and Homotopy Groups) and Chapter 10 (Spectral Sequences) form the core of the material; any second-year course should cover this material. Geometric topologists must understand how to work with non-simply-connected spaces, and so Chapter 6 (Homology with Local Coefficients) is fundamental in this regard. The material in Chapters 8 (Obstruction Theory and Eilenberg-MacLane Spaces) and 9 (Bordism, Spectra, and Generalized Homology) introduces the student to the modern perspective in algebraic topology. In Chapter 11 (Further Applications of Spectral Sequences) many of the fruits of the hard labor that preceded this chapter are harvested. Chapter 12 (Simple-Homotopy theory) introduces the ideas which lead to the subject of algebraic K-theory and to the s-cobordism theorem. This material has taken a prominent role in research in topology, and although we cover only a few of the topics in this area (K_1, the Whitehead group, and Reidemeister torsion), it serves as good preparation for more advanced courses.

These notes are meant to be used in the classroom, freeing the student from copying everything from the chalkboard and hopefully leaving more time to think about the material. There are a number of exercises in the text; these are usually routine and are meant to be worked out when the student studies. In many cases, the exercises fill in a detail of a proof or provide a useful generalization of some result. Of course, this subject, like any subject in mathematics, cannot be learned without thinking through some exercises. Working out these exercises as the course progresses is one way to keep up with the material. The student should keep in mind that, perhaps in contrast to some areas in mathematics, topology is an example driven subject, and so working through examples is the best way to appreciate the value of a theorem.

We will omit giving a diagram of the interdependence of various chapters, or suggestions on which topics could be skipped, on the grounds that teachers of topology will have their own opinion based on their experience and the interests of the students. (In any case, every topic covered in this book is related in some way to every other topic.) We have attempted (and possibly even succeeded) to organize the material in such a way as to avoid the use of technical facts from one chapter to another, and hence to minimize the need to shuffle pages back and forth when reading the book. This is to maximize its usefulness as a textbook, as well as to ensure that the student with a command of the concepts presented can learn new material smoothly and the teacher can present the material in a less technical manner. Moreover, we have not taken the view of trying to present the most elementary approach to any topic, but rather we feel that the student is best served by learning the high-tech approach, since this ultimately is faster and more useful in research. For example, we do not shrink from using spectral sequences to prove basic theorems in algebraic topology.

Some standard references on much of the material covered in this course include the books [**19**], [**17**], [**45**], [**54**], [**12**], [**22**] [**37**], and [**10**]. A large part of the material in these notes was distilled from these books. Moreover, one can find some of the material covered in much greater generality and detail in these tomes. Our intention is not to try to replace these wonderful books, but rather to offer a textbook to accompany a course in which this material is taught.

We recommend that students look at the article "Fifty years of homotopy theory" by G. Whitehead [**55**] for an overview of algebraic topology, and look back over this article every few weeks as they are reading this book. The books a student should read after finishing this course (or in conjunction with this course) are Milnor and Stasheff, *Characteristic Classes* [**36**] (every mathematician should read this book), and Adams, *Algebraic Topology: A Student's Guide* [**1**].

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Projects

The following is a list of topics to be covered in the extra meetings and lectured on by the students. They do not always match the material of the corresponding chapter but are usually either related to the chapter material or preliminary to the next chapter. Sometimes they form interesting subjects which could reasonably be skipped. Some projects are quite involved (e.g. "state and prove the Hurewicz theorem"), and the students and instructor should confer to decide how deeply to cover each topic. In some cases (e.g. the Hopf degree theorem, the Hurewicz theorem, and the Freudenthal suspension theorem) proofs are given in later chapters using more advanced methods.

- **Chapter 1.**
  1. The cellular approximation theorem.
- **Chapter 2.**
  1. Singular homology theory.
  2. De Rham cohomology.
- **Chapter 3.**
  1. The acyclic models theorem and the Eilenberg-Zilber map.
- **Chapter 4.**

1. Algebraic limits and the Poincaré duality theorem.
2. Exercises on intersection forms.

- **Chapter 5.**
  1. Fiber bundles over paracompact bases are fibrations.
  2. Classifying spaces.
- **Chapter 6.**
  1. The Hopf degree theorem.
  2. Colimits and limits.
- **Chapter 7.**
  1. The Hurewicz theorem.
  2. The Freudenthal suspension theorem.
- **Chapter 8.**
  1. Postnikov systems.
- **Chapter 9.**
  1. Basic notions from differential topology.
  2. Definition of topological K-theory.
  3. Spanier-Whitehead duality.
- **Chapter 10.**
  1. Construction of the Leray-Serre-Atiyah-Hirzebruch spectral sequence.
- **Chapter 11.**
  1. Unstable homotopy theory.
- **Chapter 12.**
  1. Handlebody theory and torsion for manifolds.

Chapter 1

Review of Singular Homology

This chapter gives a quick review of the definition of singular homology and the axioms it satisfies. There is also an extended discussion of CW-complexes and cellular homology, as well as many of the categorical constructions which we will be using throughout this book. Our intention is that this chapter is a review, but perhaps it could be useful as an overview to a student being exposed to homology for the first time.

1.1. Some category theory

The language and concepts of category theory are indispensable in modern topology and algebra. Category theory is not only a unifying principle which applies to disparate areas of mathematics, but it is also a guide to the most important aspects of a mathematical object, which may not be the construction or the definition, but rather its properties, especially if the properties characterize the object up to isomorphism.

A *category* $\mathcal{C}$ consists of

- a collection of objects $\mathrm{Ob}(\mathcal{C})$
- for any two objects X and Y, a collection of morphisms $\mathcal{C}(X,Y)$
- for any three objects X, Y, and Z, a composition law

$$\begin{aligned}\mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) &\to \mathcal{C}(X,Z)\\ (g,f) &\mapsto f \circ g\end{aligned}$$

- for every object an identity morphism $\mathrm{Id}_X \in \mathcal{C}(X,X)$

These satisfy the associativity property $(f \circ g) \circ h = f \circ (g \circ h)$ and the identity property $f \circ \mathrm{Id}_X = f = \mathrm{Id}_Y \circ f$.

Instead of $f \in \mathcal{C}(X, Y)$ one also writes $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$ or $f : X \to Y$ or $X \xrightarrow{f} Y$. Sometimes morphisms are referred to as arrows in a psychological attempt to convince the reader that they don't have to be functions. Objects X and Y are *isomorphic*, written $X \cong Y$, if there are morphisms $f : X \to Y$ and $g : Y \to X$ so that the composites $f \circ g$ and $g \circ f$ are identity morphisms.

Our standard categories, listing objects and morphisms, are

- Set (sets and functions),
- Top (topological spaces and continuous functions),
- Ab (abelian groups and homomorphisms),
- R-Mod (R-modules and homomorphisms), and
- Grp (groups and homomorphism).

We also introduce notation for some categories which will be discussed below:

- Gr_R ($\mathbf{Z}$-graded R-modules and graded R-module homomorphisms),
- Gr_R^+ ($\mathbf{Z}_{\geq 0}$-graded R-modules and graded R-module homomorphisms),
- Ch_R (Chain complexes over R-modules and chain maps),
- Ch_R^+ (Chain complexes over R-modules and chain maps which vanish in negative degrees),
- CW (CW-complexes and cellular maps), and
- CGH (Compactly generated Hausdorff spaces and continuous functions).

(Unless otherwise mentioned, all rings will have a unit and all modules will be left modules.)

For a more abstract but very useful example, note that any category with a single object and all of whose morphisms are isomorphisms defines a group, and conversely, any group defines such a category.

Categories can be compared via a *functor* $F : \mathcal{C} \to \mathcal{D}$ which is given by functions $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ so that $F(f \circ g) = F(f) \circ F(g)$ and $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$. A trivial exercise is to show that $X \cong Y$ implies $F(X) \cong F(Y)$. Furthermore functors $F, G : \mathcal{C} \to \mathcal{D}$ can be compared via a *natural transformation* $T : F \Rightarrow G$ which assigns to every object X of $\mathcal{C}$ a morphism $T_X : F(X) \to G(X)$ in $\mathcal{D}$ that $T_Y \circ F(f) =$

$G(f) \circ T_X$ for any $f : X \to Y$. In other words, the diagram below commutes.

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "T_X"] & F(Y) \arrow[d, "T_Y"] \\ G(X) \arrow[r, "G(f)"] & G(Y) \end{tikzcd}$$

A natural transformation T is a *natural isomorphism* if T_X is an isomorphism for all X. Our focus here is not on functors and natural transformations, so we will move on, but the reader may wish to ponder what functors and natural transformations are in the special case where the categories have one object and only isomorphisms, i.e. the categories are groups.

An object X of a category $\mathcal{C}$ is *initial* if for any object Y, there exists a unique morphism $X \to Y$. For example, the categories Set, Top, Ab, R-Mod, Grp, and Ring have initial objects $\emptyset$, $\emptyset$, 0, 0, 1, and $\mathbf{Z}$. (Note that a point is not an initial object in Set or Top.) The category associated to a nontrivial group has no initial object. Any two initial objects in a category are isomorphic. Indeed suppose X and X' are initial objects. Since X is initial, there is a morphism $f : X \to X'$ and since X' is initial, there is a morphism $g : X' \to X$. Since there is a unique morphism from an initial object $g \circ f = \mathrm{Id}_X : X \to X$ and $f \circ g = \mathrm{Id}_{X'} : X' \to X'$.

Such an argument, using existence and uniqueness of a property to produce an isomorphism, will be called an *universal property argument*. For example, one can define a final object in a category $\mathcal{C}$, and the uniqueness of a final object can be proved directly or by saying that a final object in $\mathcal{C}$ is the same as an initial object in the opposite category $\mathcal{C}^{\mathrm{op}}$ whose objects are the same as those of $\mathcal{C}$, but $\mathcal{C}^{\mathrm{op}}(X, Y) := \mathcal{C}(Y, X)$ and $f \circ_{\mathcal{C}^{\mathrm{op}}} g := g \circ_{\mathcal{C}} f$.

A *contravariant functor* F is a functor from $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. It is given by functions $F : \mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(Y), F(X))$ so that $F(f \circ g) = F(g) \circ F(f)$ and $F(\mathrm{Id}_X) = \mathrm{Id}_{F(X)}$.

A functor is sometimes called a a covariant functor to contrast it with a contravariant functor.

The most basic operations in arithmetic are products and sums, likewise in category theory.

Definition 1.1. A *product* of objects X_1 and X_2 in $\mathcal{C}$ is a triple

$$(X, \pi_1 : X \to X_1, \pi_2 : X \to X_2)$$

satisfying the property that given any other triple $(Y, f_1 : Y \to X_1, f_2 : Y \to X_2)$ there exists a unique morphism $f : Y \to X$ so that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

In other words we have the commuting diagram

$$\begin{tikzcd}
 & & X_1 \\
Y \arrow[urr, bend left, "f_1"] \arrow[r, dotted, "\exists! f"] \arrow[drr, bend right, "f_2"'] & X \arrow[ur, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & & X_2
\end{tikzcd}$$

Here the solid arrows are given and the dotted arrow is uniquely determined.

Remark. In general, whenever a commutative diagram is given with one dotted arrow, we will consider it as a problem whose solution is a morphism which can be substituted for the dotted arrow to give a commutative diagram. This makes sense in any category; we will use it mostly in the categories R-Mod and the category of topological spaces.

One can ask about existence and uniqueness of products. A product may not exist; consider the category given by a nontrivial group. In our six standard examples of categories above, a product is given by the cartesian product $X_1 \times X_2$ and the coordinate projection maps. Motivated by this, one often informally calls all products $X_1 \times X_2$, ignoring the uniqueness issue and not referring to the projection maps explicitly as part of the structure.

Clearly products are not unique, if (X, π_1, π_2) is a product of X_1 and X_2 and $\alpha : X' \to X$ is an isomorphism, than $(X', \pi_1 \circ \alpha, \pi_2 \circ \alpha)$ is also a product. However, the following exercise says that this is the only thing that can happen.

Exercise 1. Let (X, π_1, π_2) and (X', π_1', π_2') be products of X_1 and X_2. Use a universal property argument to define an isomorphism $\alpha : X' \to X$ so that $\pi_1' = \pi_1 \circ \alpha$ and $\pi_2' = \pi_2 \circ \alpha$. (One can do this directly or define a category whose objects consist of a pair of morphisms from an object to X_1 and to X_2, then a product is a final object.)

Although the notion of a product is quite easy, we have already learned two interesting things. First, the key properties of products is that they have projections onto the factors and that maps into a product are given by coordinate functions. This is, perhaps, more interesting than the actual definition of the product topology in the category Top. Second, Exercise 1 illustrates the key concept that an object can be characterized up to isomorphism by its properties.

By reversing all the arrows in the definition of a product one obtains the definition of a sum (or coproduct).

Definition 1.2. A *coproduct* of objects X_1 and X_2 in $\mathcal{C}$ is a triple

$$(X, i_1 : X_1 \to X, i_2 : X_2 \to X)$$

satisfying the property indicated by the diagram below

$$\begin{tikzcd}
& & X_1 \arrow[dl, "i_1"] \arrow[dll, bend right, "f_1"'] \\
Y & X \arrow[l, dotted, "\exists! f"'] & \\
& & X_2 \arrow[ul, "i_2"'] \arrow[ull, bend left, "f_2"]
\end{tikzcd}$$

Sometimes a coproduct is denoted by $X = X_1 + X_2$ and $f = f_1 + f_2$. The symbols $\oplus$, $\amalg$ and $*$ are also used.

As above, coproducts exist and are unique up to isomorphism for our six categories, but unlike for products, their construction varies from category to category. In Ab and R-Mod the coproduct X is in fact given by the cartesian product, with $i_1(x_1) = (x_1, 0)$ and $i_2(x_2) = (0, x_2)$. In the category of groups Grp, the coproduct of G_1 and G_2 is given by the free product $G_1 * G_2$. In the category Set a coproduct is called the *disjoint union*, denoted $X_1 \amalg X_2$. If X_2 is empty define $X_1 \amalg X_2 = X_1$ and similarly if X_1 is empty. Otherwise, fix points $p \in X_1$ and $q \in X_2$ and let

$$X_1 \amalg X_2 = \{(x_1, q, 1) \mid x_1 \in X_1\} \cup \{(p, x_2, 2) \mid x_2 \in X_2\} \subset X_1 \times X_2 \times \{1, 2\}.$$

Then define $i_1(x_1) = (x_1, q, 1)$ and $i_2(x_2) = (p, x_2, 2)$. Of course if you were fortunate enough to have X_1 and X_2 given to you as disjoint open subsets of a set U, you could just take $X_1 \cup X_2$ for the coproduct. One sees that the maps i_1 and i_2 are injections in the specific construction above, and hence by uniqueness are injections for any coproduct in Set. One usually abuses notation and omits reference to the maps i_1 and i_2 and considers X_1 and X_2 as subsets of $X_1 \amalg X_2$.

In Top, we start with the disjoint union $(X_1 \amalg X_2, i_1, i_2)$ in the category Set and declare a set $O \subset X_1 \amalg X_2$ to be open if and only if $i_j^{-1}(O)$ is open in X_j for $j = 1, 2$. Thus a disjoint union is always disconnected provided both spaces are nonempty. We won't discuss coproducts in the category of rings, other than to say that it is given by a free product type construction.

Exercise 2. Define the product and coproduct of a collection of objects $\{X_i\}_{i \in I}$. Note that when $\mathcal{C} = $ Top, the product $\prod X_i$ is given the standard product topology, not the pathological box topology. When $\mathcal{C} = $ Ab, denote the coproduct by $\bigoplus X_i$ and the product by $\prod A_i$. Use the definition of the coproduct and the product to define a map $\bigoplus X_i \to \prod X_i$. Identify the image.

Pushouts and pullbacks are slightly more complicated and yet incredibly useful categorical constructions.

Definition 1.3. A *pullback* of $X_1 \xrightarrow{g_1} X \xleftarrow{g_2} X_2$ is a triple

$$(P, \pi_1 : P \to X_1, \pi_2 : P \to X_2)$$

satisfying the properties that $g_1 \circ \pi_1 = g_2 \circ \pi_2$ and that given any other triple $(Q, f_1 : Q \to X_1, f_2 : Q \to X_2)$ with $f_1 \circ \pi_1 = f_2 \circ \pi_2$, there exists a unique morphism $f : Q \to P$ satisfying $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

$$\begin{tikzcd}
Q \arrow[drr, bend left, "f_1"] \arrow[dr, dotted, "\exists! f"] \arrow[ddr, bend right, "f_2"'] & & \\
& P \arrow[r, "\pi_1"] \arrow[d, "\pi_2"] & X_1 \arrow[d, "g_1"] \\
& X_2 \arrow[r, "g_2"] & X
\end{tikzcd}$$

The square in the diagram above is called a *pullback diagram.* A universal property argument shows that if pullbacks exist, they are unique up to isomorphism. In our five example categories, Set, Top, Ab, R-Mod, and Grp, pullbacks are constructed by

$$P = \{(x_1, x_2) \in X_1 \times X_2 \mid g_1(x_1) = g_2(x_2)\}$$

and letting π_i be the projection maps onto the factors. The object P in a pullback is often denoted by $P = X_1 \times_X X_2$. If X is a final object, then a pullback of $X_1 \xrightarrow{g_1} X \xleftarrow{g_2} X_2$ is the same thing as a product of X_1 and X_2.

Definition 1.4. The dual notion, a *pushout* of $X_1 \xleftarrow{g_1} X \xrightarrow{g_2} X_2$ is defined by reversing all the arrow in the above diagram. Thus a pushout is a triple $(P, i_1 : X_1 \to P, i_2 : X_2 \to P)$ satisfying the property indicated by the diagram

$$\begin{tikzcd}
X \arrow[r, "g_1"] \arrow[d, "g_2"] & X_1 \arrow[d, "i_1"] \arrow[ddr, bend left, "f_1"] & \\
X_2 \arrow[r, "i_2"] \arrow[drr, bend right, "f_2"'] & P \arrow[dr, dotted, "\exists! f"] & \\
& & Q
\end{tikzcd}$$

If pushouts exist they are unique up to isomorphism. Often they are denoted by $P = X_1 +_X X_2$, perhaps with $+$ replaced by whatever symbol represents the categorical coproduct. For example, in Grp, the pushout is given by the amalgamated product of groups $X_1 *_X X_2$ which comes up in the

Seifert-van Kampen Theorem (Theorem 1.14) for computing fundamental groups. For the categories Ab and R-mod, pushouts are given by

$$P = \frac{X_1 \oplus X_2}{\langle g_1(x) - g_2(x) \mid x \in X \rangle}$$

In the categories Set and Top, the pushout is given by $X_1 \amalg X_2/\sim$; the equivalence relation is generated by $g_1(x) \sim g_2(x)$ for $x \in X$. (Informally, one says X_1 and X_2 are glued together along X.) In this situation one often writes $P = X_1 \cup_X X_2$. If X is a topological space expressed as the union of two open sets U and V, then X itself is the pushout of $U \leftarrow U \cap V \to V$, a situation familiar from the Mayer-Vietoris Theorem. (Why open?) A consequence is the Gluing Lemma: If $X = U \cup V$ with U and V open in X, then a function $X \to Y$ is continuous if and only if the restricted functions $U \to Y$ and V to Y are continuous.

Exercise 3. Let B and C be topological spaces, let A be a closed subspace of C, and let $h : A \to B$ be a map. Express $B \cup_h C = (B \coprod C)/h(a) \sim a$ as a pushout. Show that B and $C - A$ embed in $B \cup_h C$, but show, by example, that C does not always embed.

One calls $B \cup_h C$ an *adjunction space* and says that one glues C to B using the attaching map h.

A special case of the above is where $C = D^n$ and $A = S^{n-1}$. In this case we say $B \cup_h D^n$ *is obtained by adding an n-cell to B* and write $B \cup e^n$ for the resulting space.

You probably have guessed that products, coproducts, pullbacks, and pushouts are all special cases of a more general construction, that of limits and colimits indexed by a category. That is, in fact, the case, but we postpone the discussion until the second project of Chapter 6.

1.2. Definition of singular homology

Let R be a ring (always with unit). Usually the ring R will be the integers in which case an R-module is simply an abelian group. A $\mathbf{Z}$-*graded R-module* C_* is a sequence of R-modules $\{C_n\}_{n\in\mathbf{Z}}$. A *morphism $f_* : C_* \to D_*$ of degree n* between graded R-modules is a sequence of R-module homomorphisms $f_k : C_k \to D_{k+n}$, $k \in \mathbf{Z}$. When not specified, the degree of a morphism should be assumed to be zero. We denote by Gr_R the category of $\mathbf{Z}$-graded R-modules and Gr_R^+ the category of $\mathbf{Z}_{\geq 0}$-graded R-modules with morphisms of degree zero.

A *chain complex $C_\bullet = (C_*, \partial_*)$ over R* is a $\mathbf{Z}$-graded R-module C_* equipped with an endomorphism $\partial_* : C_* \to C_*$ of degree -1 which satisfies $\partial_*^2 = 0$. More explicitly, each double composite in the sequence of homomorphisms

$$\cdots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \cdots$$

satisfies $\partial_n \circ \partial_{n+1} = 0$. The endomorphism ∂_* is called the *differential* or *boundary operator* of the chain complex.

A *chain map* $f_\bullet : (C_*, \partial_*) \to (C'_*, \partial'_*)$ is a sequence of homomorphisms $\{f_n : C_n \to C'_n\}_{n\in\mathbf{Z}}$ so that $f_{n-1} \circ \partial_n = \partial'_n \circ f_n$ for all n.

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

The category of chain complexes over R is denoted Ch_R. The subcategory of chain complexes which vanish in negative degrees is denoted Ch^+_R.

The *homology $H_*(C_\bullet)$ of a chain complex* (also written as $H_*(C_*, \partial_*)$, $H_*(C_*, \partial)$, $H_*(C_*)$, or H_*C) is the graded R-module with

$$H_n(C_\bullet) = \frac{\ker \partial_n}{\operatorname{im} \partial_{n+1}}$$

Elements in $\ker \partial_n$ are called *n-cycles* and elements of $\operatorname{im} \partial_{n+1}$ are called *n-boundaries.*

Exercise 4. Show that a chain map $f_\bullet : (C_*, \partial_*) \to (C'_*, \partial'_*)$ induces a map of graded R-modules $f_* : H_*(C_\bullet) \to H_*(C'_\bullet)$.

Thus homology defines a functor $H_* : \mathrm{Ch}_R \to \mathrm{Gr}_R$.

Here are 3 examples of chain complexes of abelian groups associated to a space:

1. The singular chain complex $S_\bullet(X) = (S_*(X), \partial)$ of a topological space X.
2. The cellular chain complex $C_\bullet(X) = (C_*(X), \partial)$ of a CW-complex X.
3. The simplicial chain complex $\Delta_\bullet(X) = (\Delta_*(K), \partial)$ of a simplicial complex K.

1.2.1. Construction of the singular chain complex. The *(geometric) n-simplex* Δ^n is defined by

$$\Delta^n = \left\{(t_0, t_1, \ldots, t_n) \in \mathbf{R}^{n+1} \mid \Sigma t_i = 1,\ t_i \geq 0 \ \text{ for all } i\right\}.$$

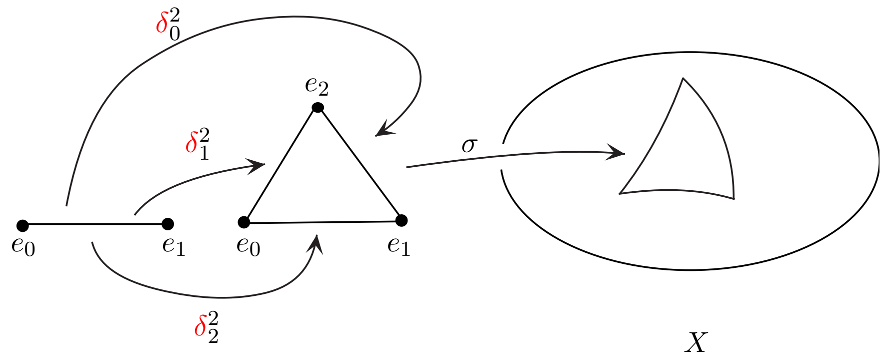
The *face maps* are the functions

$$\delta_j^n : \Delta^{n-1} \to \Delta^n$$

defined by

$$(t_0, t_1, \ldots, t_{n-1}) \mapsto \underset{\uparrow\, j^{\text{th}}\text{coordinate}}{(t_0, \ldots, t_{j-1}, 0, t_j, \ldots, t_{n-1})}$$

A *singular n-simplex in a space X* is a continuous map $\sigma : \Delta^n \to X$.



Define $S_n(X)$ to be the free abelian group with basis the singular n-simplices $\{\sigma : \Delta^n \to X\}$. This is called the *group of singular n-chains in X*. Define $\partial_n : S_n(X) \to S_{n-1}(X)$ to be the $\mathbf{Z}$-linear map given on a singular n-simplex σ by

$$\partial_n(\sigma) = \sum_{j=0}^{n} (-1)^j \sigma \circ \delta_j^n.$$

Thus, on a chain $\sum_{i=1}^{\ell} a_i \sigma_i$, ∂_n has the formula

$$\partial_n \left(\sum_{i=1}^{\ell} a_i \sigma_i \right) = \sum_{i=1}^{\ell} a_i \left(\sum_{j=0}^{n} (-1)^j \sigma_i \circ \delta_j^n \right).$$

One calculates that $\partial_{n-1} \circ \partial_n = 0$, so that $\partial = \{\partial_n\}$ is a differential, and hence $S_\bullet(X) = (S_*(X), \partial) = (\{S_n(X)\}, \{\partial_n\})$ is a chain complex.

Definition 1.5. The chain complex $S_\bullet(X) = (S_*(X), \partial)$ is called the *singular chain complex of X*. Its homology

$$H_n(X) = \frac{\ker \partial : S_n(X) \to S_{n-1}(X)}{\operatorname{im} \partial : S_{n+1}(X) \to S_n(X)}$$

is called the *singular homology of X*.

To distinguish this from the homolgy with more general coefficients defined below, we sometimes call this the singular homology with coefficients in $\mathbf{Z}$ and write $H_*(X;\mathbf{Z})$ rather than $H_*(X)$. At times we streamline the notation and write H_*X rather than $H_*(X)$.

Exercise 5. Show that the homology of a point is $\mathbf{Z}$ in degree zero and is 0 otherwise.

Note that

$$\begin{aligned} S_n(-) &: \mathrm{Top} \to \mathrm{Ab} \\ S_\bullet(-) &: \mathrm{Top} \to \mathrm{Ch}^+_{\mathbf{Z}} \\ H_*(-) &: \mathrm{Top} \to \mathrm{Gr}^+_{\mathbf{Z}} \end{aligned}$$

are all functors. For example, a continuous map $f : X \to Y$ induces a homomorphism of free modules $S_n(f) = f_* : S_n(X) \to S_n(Y)$. One defines f_* on a singular n-simplex $\sigma : \Delta^n \to X$ by $f_*(\sigma) = f \circ \sigma$ and extends by linearity. Since functors take isomorphisms to isomorphisms, homeomorphic spaces have isomorphic singular homology groups.

To say that (X, A) is a *pair of spaces* means that A is a subspace of a topological space X.

We next recall the definition of the relative singular chain complex of a pair of spaces. If $A \subset X$, define

$$S_n(X,A) = \frac{S_nX}{S_nA}.$$

Thus $S_n(X, A)$ is a free $\mathbf{Z}$-module; its basis is the set of all singular n-simplices in X whose image is not contained in A. One obtains a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & S_nA \arrow[r] \arrow[d, "\partial"'] & S_nX \arrow[r] \arrow[d, "\partial"'] & S_n(X,A) \arrow[r] \arrow[d, "\partial"'] & 0 \\ 0 \arrow[r] & S_{n-1}A \arrow[r] & S_{n-1}X \arrow[r] & S_{n-1}(X,A) \arrow[r] & 0 \end{tikzcd}$$

by defining $\partial[\alpha] = [\partial\alpha]$ for $\alpha \in S_nX$ and $[\alpha] \in S_n(X, A)$.

Exercise 6. Show that $\partial : S_n(X, A) \to S_{n-1}(X, A)$ is well-defined and $\partial^2 = 0$.

The complex $S_\bullet(X, A) = (S_n(X, A), \partial)$ is called the *singular chain complex for the pair* (X, A).

Its homology is defined by

$$H_n(X,A) = \frac{\ker \partial : S_n(X,A) \to S_{n-1}(X,A)}{\operatorname{im} \partial : S_{n+1}(X,A) \to S_n(X,A)}.$$

A relative homology class is represented by a relative cycle, which is in turn represented by a singular chain in X with boundary in A.

Let Top^2 be the category where an object is a pair (X, A) of topological spaces and a morphism $f : (X, A) \to (Y, B)$ is a continuous map $f : X \to Y$ so that $f(A) \subset B$.

Exercise 7. Show that singular homology gives functors $H_* : \mathrm{Top} \to \mathrm{Gr}^+_{\mathbf{Z}}$ and $H_* : \mathrm{Top}^2 \to \mathrm{Gr}^+_{\mathbf{Z}}$. Note that Top is a subcategory of Top^2 via $X \mapsto (X, \emptyset)$ and that $H_* X \cong H_*(X, \emptyset)$.

For a map $f : (X, A) \to (Y, B)$, one can write $H_n(f)$ or just f_* for the induced map of abelian groups $H_n(X, A) \to H_n(Y, B)$.

A *short exact sequence of chain complexes*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

is a pair of chain maps $A_\bullet \to B_\bullet$ and $B_\bullet \to C_\bullet$ so that for each n,

$$0 \to A_n \to B_n \to C_n \to 0$$

is a short exact sequence of R-modules.

Lemma 1.6 (zig-zag lemma)**.** *Let* $0 \to A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \to 0$ *be a short exact sequence of chain complexes. Then there is a long exact sequence*

$$\cdots \to H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{\partial} H_{n-1}(A_\bullet) \to \dots$$

Exercise 8. Define the *connecting homomorphism* (also called the boundary map) $\partial : H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ and prove the zig-zag lemma. (If one writes the chain complexes vertically, then the boundary map is defined using a "zig-zag.")

For a pair (X, A), there is a short exact sequence of chain complexes

$$0 \to S_\bullet A \to S_\bullet X \to S_\bullet(X, A) \to 0,$$

hence the zig-zag lemma implies that there is an exact sequence

$$\cdots \to H_n A \to H_n X \to H_n(X, A) \xrightarrow{\partial} H_{n-1} A \to \cdots$$
$$\cdots \to H_1(X, A) \xrightarrow{\partial} H_0 A \to H_0 X \to H_0(X, A) \to 0.$$

1.3. Homotopy

A *homotopy* is a continuous map $H : X \times I \to Y$. One thinks of a homotopy as a path of maps $H_t : X \to Y$, where $H_t(-) = H(-, t)$. If H is a homotopy, then one says that H_0 and H_1 are *homotopic* and writes $H_0 \simeq H_1$ for the corresponding equivalence relation. A *homotopy equivalence* is a map f :

$X \to Y$ with a "homotopy inverse" $g : Y \to X$, that is, $g \circ f \simeq \mathrm{Id}_X$ and $f \circ g \simeq \mathrm{Id}_Y$. One then says that X and Y are *homotopy equivalent* or that X and Y have the *same homotopy type.* One never, never, never says that spaces are homotopic; only maps are homotopic.

These notions generalize to pairs. A *homotopy of pairs* is a map $H : (X, A) \times I \to (Y, B)$; in other words $H : X \times I \to Y$ is a homotopy so that $H_t(A) \subset B$ for all t.

The notation $[X, Y]$ is used to denote the set of homotopy classes of continuous maps from X to Y. Similarly, if (X, A) and (Y, B) are pairs of spaces, $[(X, A), (Y, B)]$ is used to denote the set of homotopy classes of the pairs.

There are parallel notions for chain complexes. A *chain homotopy* between chain maps $f, g : C_\bullet \to C'_\bullet$ is a sequence of maps $h_n : C_n \to C'_{n+1}$ so that $f_n - g_n = \partial'_{n+1} \circ h_n + h_{n-1} \circ \partial_n$ for all n. We say the maps f and g are chain homotopic and write $f \simeq g$. Here is a diagram indicating the above

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\partial"] \arrow[d, "{f,g}"] & C_n \arrow[r, "\partial"] \arrow[d, "{f,g}"] \arrow[dl, dotted, "h"] & C_{n-1} \arrow[r] \arrow[d, "{f,g}"] \arrow[dl, dotted, "h"] & \cdots \\
\cdots \arrow[r] & C'_{n+1} \arrow[r, "\partial'"] & C'_n \arrow[r, "\partial'"] & C'_{n-1} \arrow[r] & \cdots
\end{tikzcd}$$

Exercise 9. Show that chain homotopic maps $f, g : C_\bullet \to C'_\bullet$ induce the same map $f_* = g_* : H_*(C_\bullet) \to H_*(C'_\bullet)$.

A *chain equivalence* is a chain map $f_\bullet : C_\bullet \to C'_\bullet$ so that there exists a chain map $g : C'_\bullet \to C_\bullet$ so that $f \circ g$ and $g \circ f$ are both chain homotopic to the identity maps. One writes $C_\bullet \simeq C'_\bullet$. A chain equivalence is also called a chain homotopy equivalence.

A chain map is a *quasi-isomorphism* if it induces an isomorphism on homology. A key lemma gives a converse (see [**45**, pp. 192] or Exercise 246).

Lemma 1.7. *Let $f_\bullet : C_\bullet \to C'_\bullet$ be a chain map between chain complexes of free R-modules where the chain complexes vanish in negative degrees. Then $f_\bullet$ is a quasi-isomorphism if and only if $f_\bullet$ is a chain equivalence.*

A basic, but difficult theorem is that singular homology is homotopy invariant. (See Project 3.8.)

Theorem 1.8. *Homotopic maps $f \simeq g : (X, A) \to (Y, B)$ induce chain homotopic maps $S_\bullet(X, A) \to S_\bullet(Y, B)$ and hence the same map on homology: $f_* = g_* : H_n(X, A) \to H_n(Y, B)$.*

Exercise 10. The *homotopy category* hTop is the category whose objects are topological spaces and whose morphisms are homotopy classes of maps. Thus $\mathrm{hTop}(X, Y) = [X, Y]$. Show that hTop is a category. Define the homotopy category of pairs and define functors $H_* : \mathrm{hTop} \to \mathrm{Gr}^+_{\mathbf{Z}}$ and $H_* : \mathrm{hTop}^2 \to \mathrm{Gr}^+_{\mathbf{Z}}$.

There is an obvious functor $h : \mathrm{Top} \to \mathrm{hTop}$ sending a space to itself and a map to its homotopy class. We can use this to express standard vocabulary. Two maps $f, g \in \mathrm{Top}(X, Y)$ are *homotopic* if $h(f) = h(g)$. A *homotopy equivalence* is an $f \in \mathrm{Top}(X, Y)$ so that $h(f)$ is an isomorphism. A *homotopy inverse* of $f \in \mathrm{Top}(X, Y)$ is a $g \in \mathrm{Top}(Y, X)$ so that $h(f)$ and $h(g)$ are inverses. Spaces X and Y are *homotopy equivalent* if $h(X) \cong h(Y)$.

A 1-cycle in the circle S^1 is given by $\exp : \Delta^1 \to S^1$ with $\exp(t_0, t_1) = e^{2\pi i t_0}$. Using Mayer-Vietoris (or excision or cellular homology) one can show that $H_1(S^1; \mathbf{Z})$ is infinite cyclic with $[\exp]$ a generator. We can use homotopy invariance to define the *Hurewicz map*

$$\rho : \pi_1(X, x_0) \to H_1(X; \mathbf{Z})$$

by

$$\rho[\alpha] = [\alpha \circ \exp] = \alpha_*[\exp].$$

Here $\pi_1(X, x_0)$ denotes the *fundamental group of X based at $x_0 \in X$*

$$\pi_1(X, x_0) = [(S^1, 1), (X, x_0)].$$

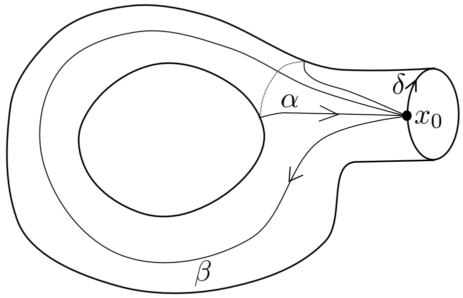
Recall that the fundamental group of a based space is a nonabelian group in general. The Hurewicz theorem for the fundamental group is the following.

Theorem 1.9. *Suppose that X is path-connected. Then the Hurewicz map $\rho : \pi_1(X, x_0) \to H_1(X; \mathbf{Z})$ is a surjection with kernel the commutator subgroup of $\pi_1(X, x_0)$. Hence $H_1(X; \mathbf{Z})$ is isomorphic to the abelianization of $\pi_1(X, x_0)$.*

An illustration of Theorem 1.9 is given in the following figure. Take X to be a genus 1 surface with one boundary component.

Exercise 11. Show that the fundamental group $\pi_1(X, x_0)$ is a free group generated by the loops α and β and that the curve δ represents the nontrivial commutator $\alpha\beta\alpha^{-1}\beta^{-1}$ in $\pi_1(X, x_0)$. Show that $\rho(\delta) = 0$ by finding a singular 2-chain c with $\partial c = \delta$.

This picture illustrates the difference between the first homotopy group and the first homology group. A loop is trivial in homotopy if it bounds a disk and is trivial in homology if it bounds a surface.



1.4. Excision and Mayer-Vietoris

The key computational tool for homology is the excision theorem.

Theorem 1.10 (Excision Theorem)**.** *If $A \subset X$, $\overline{A} \subset \text{Int } U$, then $H_n(X - A, U - A) \to H_n(X, U)$ is an isomorphism for all n.*

To "excise" means to "to cut out." So the excision theorem implies that one can excise closed sets from open sets without changing the relative homology.

Exercise 12. Use excision, homotopy invariance, and induction to compute the homology of S^n.

Definition 1.11. Let A and B be subspaces of a topological space X. $(X; A, B)$ is a *excisive triad* if for all n,

$$H_n(A, A \cap B) \to H_n(X, B)$$

is an isomorphism. $\{A, B\}$ is an *excisive couple* if $(A \cup B; A, B)$ is an excisive triad.

The excision theorem says that if $\{A, B\}$ is an open cover of X (or more generally $\{\text{Int } A, \text{Int } B\}$ is an open cover of X), then $(X; A, B)$ is an excisive triad, but this is not the end of the story. For example, by using homotopy invariance, it is easy to see that (sphere; upper hemisphere, lower hemisphere) is an excisive triad. We will discuss CW-complexes in the next section. If X is a CW-complex and A and B are subcomplexes whose union is X, then $(X; A, B)$ is an excisive triad.

Let $S_\bullet A + S_\bullet B$ be the subchain complex of $S_\bullet X$ generated by the singular simplicies whose image lies entirely in A or entirely in B. The following characterization of excisive triads shows, for example, that $(X; A, B)$ is an excisive triad if and only if $(X; B, A)$ is an excisive triad.

Proposition 1.12. *Let $A, B \subset X$. The following are equivalent.*

1. *$(X; A, B)$ is an excisive triad.*

2. *$H_*(S_\bullet A + S_\bullet B) \to H_* X$ is an isomorphism.*
3. *The inclusion map $S_\bullet A + S_\bullet B \subset S_\bullet X$ is a chain equivalence.*

Proof. 1) is equivalent to 2) by the following short exact sequence of chain complexes

$$0 \to \frac{S_\bullet A}{S_\bullet(A \cap B)} \to \frac{S_\bullet X}{S_\bullet B} \to \frac{S_\bullet X}{S_\bullet A + S_\bullet B} \to 0$$

and the zig-zag lemma. 2) is equivalent to 3) by Lemma 1.7 which gives a criterion for a quasi-isomorphism to be a chain equivalence. $\square$

Theorem 1.13 (Mayer-Vietoris exact sequence)**.** *If $(X; A, B)$ is an excisive triad, there is a long exact sequence*

$$\cdots \to H_n(A \cap B) \xrightarrow{(i_*, j_*)} H_n A \oplus H_n B \xrightarrow{k_* - l_*} H_n X \xrightarrow{\partial} H_{n-1}(A \cap B) \to \ldots$$

Here the maps are induced by the various inclusions $i : A \cap B \to A$, $j : A \cap B \to B$, $k : A \to X$, and $l : B \to X$, and, if $z \in H_n X$ is represented by a chain $a + b$ with $a \in S_n A$ and $b \in S_n B$, then $\partial z = [a] \in H_{n-1}(A \cap B)$.

Proof. In fact if A, B are any subspaces of X, there is a short exact sequence of chain complexes

$$0 \to S_\bullet(A \cap B) \xrightarrow{(i_\bullet, j_\bullet)} S_\bullet A \oplus S_\bullet B \xrightarrow{k_\bullet - l_\bullet} S_\bullet A + S_\bullet B \to 0.$$

The zig-zag lemma and the definition of excisive triad gives the desired result. $\square$

Exercise 13. Give an example where $X = A \cup B$, but where $(X; A, B)$ is not an excisive triad.

Exercise 14. Compute the homology of the Klein bottle using excision or Mayer-Vietoris. Compute the homology of $\frac{S^1 \times I}{(z,0) \sim (z^3,0)}$.

For completeness, we give the counterpart of the Mayer-Vietoris sequence for fundamental groups.

Theorem 1.14 (Seifert-van Kampen Theorem)**.** *Let X be a topological space and suppose that $A, B \subset X$ are open subsets satisfying A, B and $A \cap B$ are path connected and $X = A \cup B$. Choose $x_0 \in A \cap B$. Then $\pi_1(X, x_0)$ is the pushout of $\pi_1(A, x_0) \leftarrow \pi_1(A \cap B, x_0) \to \pi_1(B, x_0)$ in the category* Grp.

1.5. Reduced homology

For a nonempty space X, the *augmentation map* $\varepsilon : S_0X \to \mathbf{Z}$ is the homomorphism $\varepsilon(\sum a_i\sigma_i) = \sum a_i$. A simple calculation shows that $\varepsilon \circ \partial_1 = 0$. The *augmented singular chain complex* is

$$\cdots \to S_2X \to S_1X \to S_0X \xrightarrow{\varepsilon} \mathbf{Z} \to 0.$$

Its homology, denoted $\widetilde{H}_*X$, is called the *reduced homology of* X. Thus $\widetilde{H}_nX = H_nX$ for $n > 0$, and there is a short exact sequence

$$0 \to \widetilde{H}_0X \hookrightarrow H_0X \xrightarrow{\varepsilon} \mathbf{Z} \to 0.$$

Equivalently, if $X \to \mathrm{pt}$ is the constant map to a one-point space then there is a short exact sequence

$$0 \to \widetilde{H}_0X \hookrightarrow H_0X \to H_0(\mathrm{pt}) \to 0.$$

Of course the short exact sequences above are split since $\mathbf{Z}$ is free, but to get a splitting we choose a point $x_0 \in X$. Since x_0 is a retract of X ($i : x_0 \to X$ has a one-sided inverse), $H_0X = \widetilde{H}_0X \oplus i_*H_0(x_0) \cong H_0(X, x_0) \oplus H_0(x_0)$.

Why bother with reduced homology? One reason is that the Mayer-Vietoris exact sequence works with reduced homology and the long exact sequence of a pair is

$$\cdots \to \widetilde{H}_nA \to \widetilde{H}_nX \to H_n(X, A) \to \widetilde{H}_{n-1}A \to \dots$$

This is a minor convenience when computing because one can safely ignore certain $\mathbf{Z}$'s. Reduced homology also simplifies the formulas for the homology of a wedge or of a join.

A better justification for reduced homology is the suspension isomorphism. The suspension of a space X is the space

$$SX = \frac{X \times I}{(x, 0) \sim (x', 0), (x, 1) \sim (x', 1)}.$$

Exercise 15. Define a natural isomorphism $\widetilde{H}_{n+1}(SX) \to \widetilde{H}_nX$ for all n. Deduce the homology of a sphere as a consequence.

The suspension isomorphism also works for a generalized reduced homology theory, and, in fact, is one of the axioms – see Section 9.10.

Definition 1.15. A pair of spaces (X, A) is a *good pair* if the quotient map

$$H_*(X, A) \to H_*(X/A, A/A)$$

is an isomorphism.

Since relative homology relative to a point is canonically isomorphic to reduced homology, an alternate definition of a good pair is that $H_*(X, A) \cong \widetilde{H}_*(X/A)$. Thus for a good pair, there is a long exact sequence

$$\cdots \to \widetilde{H}_n A \to \widetilde{H}_n X \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1} A \to \dots$$

A subspace A of X is called a *strong deformation retract* if there is a homotopy $H : X \times I \to X$ so that $H_0 = \mathrm{Id}_X$, $H_t|_A = \mathrm{Id}_A$ for all t, and $H_1(X) = A$. In particular, A is a retract of X and A has the homotopy type of X, but this is a tad stronger.

The key observation for recognizing a good pair is the following lemma.

Lemma 1.16. *A pair (X, A) is good if A is a strong deformation retract of a neighborhood of A in X.*

Exercise 16. Prove this lemma. You may assume that $U \times I \to U/A \times I$ is a quotient map (see Theorem 7.6).

Examples of good pairs will be given in the next section and in Chapter 7.

1.6. Definition of a CW-complex

Given a space A and a map $\phi : S^{n-1} \to A$, define the space $X = A \cup_\phi D^n$ to be the pushout of $D^n \hookleftarrow S^{n-1} \xrightarrow{\phi} A$. One says that X is *obtained from A by attaching an n-cell*, where an n-cell is a subspace homeomorphic to the interior of the n-disk. Informally speaking, a CW-complex is a space obtained from a discrete set of points by successively attaching cells. We give two formal definitions. The first is essentially the original one given by J.H.C. Whitehead in 1949 and reprinted in [**1**].

Definition 1.17. A CW-*complex* is a pair $(X, \{e_i^n\})$ consisting of a Hausdorff space X and a collection of subspaces indexed by $n = 0, 1, 2, \dots$ and $i \in I_n$. This pair must satisfy the following properties.

1. Every point in X is contained in a unique e_i^n.
2. For each e_i^n, there exists a map $\chi_i^n : D^n \to X$ whose restriction to the interior $\chi_i^n| : \mathrm{int}\, D^n \to e_i^n$, is a homeomorphism (in particular, e_i^n is an n-cell).
3. (Closure finite) For each e_i^n, $\chi_i^n(S^{n-1})$ is contained in a finite union of cells of dimension less than n.
4. (Weak topology) A set $B \subset X$ is closed if and only if $B \cap \overline{e_i^n}$ is closed in $\overline{e_i^n}$ for all n, i.

Often one refers to the space X as a CW-complex; the cell structure is understood. The *n-skeleton* X^n of a CW-complex is the union of the cells of dimension less than or equal to n. The map $\chi_i^n : D^n \to X^n$ is called the *characteristic map* for the cell e_i^n and the restriction to the sphere $\phi_i^n = \chi_i^n| : S^{n-1} \to X^{n-1}$ is called the *attaching map*. Note that X^0 is a discrete set of points. If $X = X^n$ we say X is an *n-dimensional* CW*-complex*. The set I_n is called the *indexing set for the n-cells.*

A one-dimensional CW-complex is called a *graph*. A *finite* CW*-complex* is a CW-complex with a finite number of cells. Conditions (3) and (4) give the initials CW, but are extraneous in the case of a finite CW-complex.

A *subcomplex* of a CW-complex $(X, \{e_i^n\})$ is a CW-complex $(A, \{f_j^n\})$ with $A \subset X$ and $\{f_j^n\} \subset \{e_i^n\}$. In other words, a subcomplex of a CW-complex is a subspace which is also a CW-complex. Thus the cells of A are the cells of X contained in A. An example of a subcomplex is the n-skeleton. If A is subcomplex of X, then A is a closed subset of X. If A is a subcomplex of X, we say that (X, A) is a CW*-pair*.

For a CW-complex X, a subspace $K \subset X$ is compact if and only if K is closed and is contained in a finite union of cells. (The key point is that a set consisting of a point from each cell is discrete.) In particular, a CW-complex is compact if and only if it is finite. CW-complexes are reasonably well-behaved in terms of point set topology. They are normal, paracompact, locally contractible, locally path-connected, and compactly generated. However, infinite CW-complexes need not be locally compact or first countable. But CW-complexes form a nice collection of topological spaces which include most spaces of interest in geometric and algebraic topology.

A *cellular map* $f : X \to Y$ is a continuous function so that $f(X^n) \subset Y^n$ for all n. The collection of all CW-complexes together with all cellular maps defines the category CW of CW-complexes. The cellular approximation theorem (Project 1.8.1) says that any continuous map of CW-complexes is homotopic to a cellular map. One can show that if $f : X \to Y$ is both cellular and an embedding, then $(Y, f(X))$ is a CW-pair. An *embedding* is a continuous map $f : X \to Y$ which induces a homeomorphism from $X \to f(X)$. The symbol $f : X \hookrightarrow Y$ indicates that f is an embedding.

Two examples of CW-complexes are $S^n = e^0 \cup e^n$ and $S^n = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup \cdots \cup e^n_+ \cup e^n_-$. For $0 \leq i < n$, the skeleta satisfy $(S^n)^i = e^0$ and $(S^n)^i = S^i$ in the respective cases. Here are more examples: the disk $D^n = e^0 \cup e^{n-1} \cup e^n$, the torus $T^2 = e^0 \cup e^1_1 \cup e^1_2 \cup e^2$, real projective space $\mathbf{R}P^n = e^0 \cup e^1 \cup \cdots \cup e^n$, and complex projective space $\mathbf{C}P^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$. Here $\mathbf{R}P^n$ is the space of lines through the origin in $\mathbf{R}^{n+1}$. More precisely

$$\mathbf{R}P^n = \mathbf{R}^{n+1} - \{0\}/\mathbf{x} \sim \lambda\mathbf{x} = S^n/\mathbf{x} \sim -\mathbf{x}.$$

In the above CW-decomposition, $(\mathbf{R}P^n)^i = \mathbf{R}P^i \subset \mathbf{R}P^n$ and $(\mathbf{C}P^n)^i = \mathbf{C}P^{[i/2]} \subset \mathbf{C}P^n$.

We now give a more modern, yet equivalent definition of a CW-complex. This definition can be looked at in two ways. First, it makes precise the notion that a CW-complex is constructed inductively by successively adding n-cells, $(n+1)$-cells, etc. Second, it is more categorical, giving the definition in terms of properties satisfied.

Definition 1.18. A CW-*complex* is a pair $(X, \{X^n\}_{n=-1,0,1,2,\ldots})$ where X is a topological space and the X^n are subspaces so that

$$X^{-1} = \emptyset \subset X^0 \subset X^1 \subset X^2 \subset \ldots$$

with $X = \cup X^n$. This pair must satisfy the following properties.

1. For every $n \geq 0$ there is a pushout diagram

$$\begin{tikzcd} \amalg S^{n-1} \arrow[r] \arrow[d] & X^{n-1} \arrow[d] \\ \amalg D^n \arrow[r] & X^n \end{tikzcd}$$

2. A set B in X is closed if and only if $B \cap X^n$ is closed in X^n for all n.

In Condition (1), the disjoint union is indexed by a set I_n. Condition (2) could be expressed in a more categorical (and elegant) way by requiring $X = \operatorname{colim} X^n$. Project 6.5.2 covers colimits.

In a CW-complex, $X^n - X^{n-1}$ is homeomorphic to a disjoint union of n-cells; this gives a transition from this definition to the previous one.

1.6.1. Definition of the cellular chain complex of a CW-complex. If X is a CW-complex and $X^n \subset X$ is the n-skeleton of X, the *group of cellular n-chains* is defined as the relative homology group

$$C_n(X) = H_n(X^n, X^{n-1}).$$

An excision argument shows that

$$(\amalg \chi_i^n)_* : H_n(\amalg D^n, \amalg S^{n-1}) \xrightarrow{\cong} H_n(X^n, X^{n-1}).$$

Thus $C_n(X)$ is a free $\mathbf{Z}$-module with a basis in one-to-one correspondence with the n-cells of X.

Exercise 17. What is the excision argument? (See Exercise 16.) How can one use characteristic maps $\chi_i : (D^n, S^{n-1}) \to (X^n, X^{n-1})$ for cells to give basis elements? To what extent do the basis elements depend on the choice of characteristic maps? Despite the ambiguity in sign it is traditional to denote the basis element by e_i^n.

Following standard convention, we will use the symbol e_i^n to denote two things, first a subset of X^n homeomorphic to an open n-disk and second an element of $C_n(X)$ given by the image of a generator of $H_n(D^n, S^{n-1})$ under the map induced by a characteristic map. To make things worse, the latter is only determined up to sign. To mitigate this notational sin, we will call $e_i^n \subset X^n$ a cell and $e_i^n \in C_n(X)$ an oriented n-cell.

The differential $\partial : C_n(X) \to C_{n-1}(X)$ is defined as the composite

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}(X^{n-1}) \xrightarrow{\pi} H_{n-1}(X^{n-1}, X^{n-2})$$

where the first map comes from the long exact sequence of the homology of the pair (X^n, X^{n-1}) and the second map comes from the long exact sequence of the homology of the pair (X^{n-1}, X^{n-2}). The *cellular chain complex* is $C_\bullet(X) = (C_*(X), \partial)$.

The differential $\partial : C_n(X) \to C_{n-1}(X)$ can be interpreted in a geometric way which involves the notion of the *degree* of a map $f : S^n \to S^n$. If you don't know what the degree of such a map is, look it up. If you know the algebraic topology definition of degree, then look up the differential topology definition of degree for a smooth map $f : S^n \to S^n$.

Geometric interpretation of ∂. The quotient X^{n-1}/X^{n-2} is homeomorphic to a one-point union of $(n-1)$-spheres, one for each $(n-1)$-cell of X, since the boundary of each $(n-1)$-cell has been collapsed to a point, and a $(n-1)$-cell with its boundary collapsed to a point is a $(n-1)$-sphere. For each oriented n-cell e_i^n consider the attaching map $S^{n-1} \to X^{n-1}$. The composite of this map with the quotient map to X^{n-1}/X^{n-2} defines a map from a $(n-1)$-sphere to a one-point union of $(n-1)$-spheres. Taking the degree of this map in the summand corresponding to an oriented $(n-1)$-cell e_j^{n-1} gives an integer denoted by $[e_i^n : e_j^{n-1}]$.

The *cellular boundary formula* for $\partial : C_n(X) \to C_{n-1}(X)$ is

$$\partial e_i^n = \sum [e_i^n : e_j^{n-1}] e_j^{n-1}$$

(the sum is over all oriented $(n-1)$-cells e_j^{n-1}).

Exercise 18.

1. Use the definition of ∂ to show that $\partial^2 = 0$.
2. Prove that the cellular boundary formula. (Hint: $H_n(D^n, S^{n-1}) \xrightarrow{\partial} H_{n-1}(S^{n-1})$ is an isomorphism.)

A cellular map induces a chain map $f_\bullet : C_\bullet(X) \to C_\bullet(Y)$, since f restricts to a map of pairs $f : (X^n, X^{n-1}) \to (Y^n, Y^{n-1})$.

The proof of the following theorem can be found in any standard first-year algebraic topology textbook. See Exercise 195 for a spectral sequence proof.

Theorem 1.19. *The cellular and singular homology of a* CW*-complex are naturally isomorphic.* □

This can all be generalized to the category of CW-pairs. For a CW-pair (X, A), define

$$C_n(X, A) = H_n(X^n, X^{n-1} \cup A^n).$$

This is a free abelian group with basis the oriented n-cells in $X - A$. The cellular chain complex $C_\bullet(X, A) = (C_*(X, A), \partial)$ has differential defined as above. There is a short exact sequence of chain complexes

$$0 \to C_\bullet(A) \to C_\bullet(X) \to C_\bullet(X, A) \to 0$$

whose homology long exact sequence is isomorphic to the long exact sequence of the pair defined using singular homology.

Let CW^2 be the category whose objects are CW-pairs and whose morphisms are cellular maps. Then cellular homology is the composite functor

$$\mathrm{CW}^2 \to \mathrm{Ch}_R^+ \to \mathrm{Gr}_R^+.$$

Exercise 19. Use cellular homology to compute the homology of the following spaces: S^1, T^2, $\mathbf{R}P^2$, the Klein bottle, and $\mathbf{C}P^n$. Use relative cellular homology to compute $H_*(M, \partial M)$ where M is the Möbius strip.

Finally we can generalize further to CW-complexes relative to a space A. Taking A to be the empty set we recover the classical notion of a CW-complex.

Definition 1.20. Given a topological space A, a CW-*complex relative to* A is a pair $(X, \{(X, A)^n\}_{n=-1,0,1,2,\dots})$ where the $(X, A)^n$ are subspaces of a topological space X so that

$$A = (X, A)^{-1} \subset (X, A)^0 \subset (X, A)^1 \subset (X, A)^2 \subset \dots$$

with $(X, A) = \cup (X, A)^n$. This pair must satisfy the following properties.

1. For every $n \geq 0$ there is a pushout diagram

$$\begin{tikzcd} \amalg S^{n-1} \arrow[r] \arrow[d] & (X, A)^{n-1} \arrow[d] \\ \amalg D^n \arrow[r] & (X, A)^n \end{tikzcd}$$

2. A set B in X is closed if and only if $B \cap (X,A)^n$ is closed in $(X,A)^n$ for all n.

For example, a CW-pair (X, A) determines a CW-complex relative to A whose n-skeleton is $X^n \cup A$.

One often refers to a CW-complex relative to A as a relative CW-complex (X, A). Note that $(X,A)^n - (X,A)^{n-1}$ is a disjoint union of n-cells and that X/A is a CW-complex. The cellular n-chains are defined to be

$$C_n(X,A) = H_n((X,A)^n, (X,A)^{n-1})$$

Thus $C_n(X, A)$ is a free $\mathbf{Z}$-module with basis the free abelian group on the oriented n-cells. One defines the chain complex $C_\bullet(X,A) = (C_*(X,A), \partial)$. The definition of the boundary map is left to the reader.

By Corollaries 7.28 and 7.32, (X, A) is a good pair, so that $H_*(X,A) \cong H_*(X/A, A/A)$. Thus $H_*(X,A) \cong \widetilde{H}_*(X/A) \cong \widetilde{H}_*(C_\bullet(X/A)) \cong H_*(C_\bullet(X,A))$. Hence the homology of the cellular chain complex computes the singular homology of the pair.

1.6.2. Construction of the simplicial chain complex of a simplicial complex. For both historical reasons and to keep current with simplicial techniques in algebraic topology, it is worthwhile to have some familiarity with simplicial homology.

Definition 1.21. An *(abstract) simplicial complex* K is a pair (V, S) where V is a set and S is a collection of nonempty finite subsets of V satisfying:

1. If $v \in V$, then $\{v\} \in S$.
2. If $\tau \subset \sigma \in S$ and τ is nonempty, then $\tau \in S$.

Elements of V are called *vertices*. Elements of S are called *simplices*. A n-*simplex* is an element of S with $n+1$ vertices. If $\sigma \in S$ is a n-simplex, we say $\dim(\sigma) = n$.

Put a (total) ordering on the vertices V.

Define the *simplicial* n-*chains* $\Delta_n(K)$ to be the free R-module with basis the n-simplices of K. Denote a n-simplex by $\langle\sigma\rangle = \langle v_0, v_1, \ldots, v_n\rangle$ where the vertices are listed in increasing order. Define the differential $\partial : \Delta_n(K) \to \Delta_{n-1}(K)$ on a n-simplex by

$$\partial\langle v_0, v_1, \ldots, v_n\rangle = \sum_{m=0}^{n} (-1)^m \langle v_0, v_1, \ldots, \widehat{v_m}, \ldots, v_n\rangle,$$

where $\widehat{v_m}$ means omit the m-th vertex, and then extend by linearity, i.e.

$$\partial\left(\sum_{i=1}^{\ell} a_i\langle\sigma_i\rangle\right) = \sum_{i=1}^{\ell} a_i \partial\langle\sigma_i\rangle.$$

The homology of this chain complex is denoted $H_*(K)$. Notice that these definitions are purely combinatorial; the notions of topological space and continuity are not used. The connection with topology is given by the next definition, which takes a geometric n-simplex for every abstract n-simplex of K and glues the geometric simplices together.

Let K_n be the set of n-simplices of K. Think of this as a topological space with the discrete topology.

Definition 1.22. The *geometric realization* of a simplicial complex K is the quotient space

$$|K| = \frac{\amalg\, K_n \times \Delta^n}{(\delta_m^*(\sigma), x) \sim (\sigma, \delta_m(x))}.$$

where the face maps are given by

$$\begin{aligned} \delta_m^* : K_{n+1} &\to K_n \\ \langle v_0, \cdots, v_{n+1} \rangle &\mapsto \langle v_0, \ldots, v_{m-1}, v_{m+1}, \ldots v_{n+1} \rangle \\ \delta_m : \Delta^{n-1} &\to \Delta^n \\ (t_0, \ldots, t_{n-1}) &\mapsto (t_0, \ldots, t_{m-1}, 0, t_m, \ldots, t_{q-1}) \end{aligned}$$

If one chooses another ordering of the vertices, then one can, with some fuss about orientation, define a canonical isomorphism between the simplicial chain complex (or geometric realization) defined using one ordering to the simplicial chain complex (or geometric realization) defined using the other ordering.

A *triangulation* of a topological space X is a homeomorphism from the geometric realization of a simplicial complex to X.

Exercise 20. Find a triangulation of $\mathbf{R}P^2$ and compute its simplicial homology.

The homology $H_*(K)$ of an abstract simplicial complex K is isomorphic to $H_*(|K|)$, the singular homology of its geometric realization. This can be seen by noting that $|K|$ is naturally a CW-complex, the n-skeleton is the union of simplices of dimension $\leq n$, and the n-cells are the the n-simplices. The cellular chain complex of $|K|$ is isomorphic to the simplicial chain complex of K.

Another construction of homology uses the cubical singular complex (this is the point of view taken in Massey's book [**28**]). This gives yet another chain complex associated to a topological space. It is not hard, using the acyclic models theorem, to show that the simplicial and cubical singular homology functors are naturally isomorphic.

1.7. Homology's greatest hits

1.7.1. Euler characteristic. The *rank* of an abelian group is the cardinality of a maximally linearly independent subset. The n-th *Betti number* of a space X is the rank of $H_n X$ and is the key numerical invariant derived from homology.

A graded abelian group A_* has *finite rank* if $\sum \operatorname{rank} A_n < \infty$. We say a space or chain complex has (homological) finite rank if its homology has finite rank.

Definition 1.23. The *Euler characteristic* of a space X of finite rank is

$$\chi(X) = \sum (-1)^n \operatorname{rank} H_n X.$$

The Euler characteristic of a finite rank chain complex $C_\bullet$ of abelian groups is

$$\chi(C_\bullet) = \sum (-1)^n \operatorname{rank} H_n C_\bullet.$$

You should compute the Euler characteristic of familiar examples: spheres, projective spaces, tori, and products thereof.

The key algebraic property of rank is its additivity: if $0 \to A \to B \to C \to 0$ is a short exact sequence, then $\operatorname{rank} B = \operatorname{rank} A + \operatorname{rank} C$. A consequence is invariance of the Euler characteristic under homology:

Theorem 1.24. *Let $C_\bullet$ be a chain complex so that the underlying graded abelian group C_* has finite rank. Then*

$$\sum (-1)^n \operatorname{rank} H_n C_\bullet = \sum (-1)^n \operatorname{rank} C_n$$

Proof. We will use the two fundamental exact sequences of homology: if $C_\bullet = (C_n, \partial_n)$ is a chain complex with cycles $Z_n = \ker \partial_n$, boundaries $B_n = \operatorname{im} \partial_{n+1}$, and homology $H_n = H_n(C_\bullet) = Z_n/B_n$, then there are the short exact sequences

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$
$$0 \to B_n \to Z_n \to H_n \to 0.$$

Thus

$$\begin{aligned}
\sum (-1)^n \operatorname{rank} H_n &= \sum (-1)^n \operatorname{rank} Z_n - \sum (-1)^n \operatorname{rank} B_n \\
&= \sum (-1)^n \operatorname{rank} Z_n + \sum (-1)^n \operatorname{rank} B_{n-1} \\
&= \sum (-1)^n \operatorname{rank} C_n.
\end{aligned}$$

□

The most famous application is Euler's formula: for any triangulation of the 2-sphere, $2 = v - e + f$, where v, e, and f are the number of vertices, edges, and faces respectively. More generally, we can apply the theorem to the cellular chain complex of a finite CW-complex (or simplical complex) and deduce that the Euler characteristic is the alternating sum of the number of n-cells.

Exercise 21.

1. Show that if $0 \to C_k \to C_{k-1} \to \cdots \to C_0 \to 0$ is an exact sequence and if C_* has finite rank, then $\sum(-1)^n \operatorname{rank} C_n = 0$.
2. Show that if $A \subset X$ and both have finite rank, then $\chi(X) = \chi(A) + \chi(X, A)$.
3. If $E \to B$ is an n-fold cover, how are their Euler characteristics related?
4. Suppose $(X; A, B)$ is an excisive triad of spaces of finite rank. Show that $\chi(X) = \chi(A) + \chi(B) - \chi(A \cap B)$.
5. Find a formula for the Euler characteristic of a connected sum $M \# N$ of manifolds (you might have to look up some definitions).

The most famous theorem of differential geometry is the Gauss-Bonnet Theorem: If M is a compact Riemannian two-dimensional manifold and K is its curvature, then $2\pi\chi(M) = \int_M K dA$. The Euler characteristic also features prominently in the Poincaré-Hopf Theorem, which implies in particular that a vector field on a closed manifold with nonzero Euler characteristic must have a zero. Thus one can't comb the hairy ball!

1.7.2. Classical applications of homology.

Theorem 1.25.

1. *Brouwer No Retraction Theorem: S^{n-1} is not a retract of D^n.*
2. *Brouwer Fixed Point Theorem: Every continuous map $f : D^n \to D^n$ has a fixed point; there is an $x \in D^n$ so that $f(x) = x$.*

Proof. 1. Suppose there was a map $r : D^n \to S^{n-1}$ which restricted to the identity on the boundary sphere. Since the composite $S^{n-1} \hookrightarrow D^n \xrightarrow{r} S^{n-1}$ is the identity, so is the composite $\widetilde{H}_{n-1}S^{n-1} \to \widetilde{H}_{n-1}D^n \to \widetilde{H}_{n-1}S^{n-1}$. Hence the composite of $\mathbf{Z} \to 0 \to \mathbf{Z}$ is the identity. Contradiction. (The proof shows that if A is a retract of X, then the homology H_*A is a direct summand of the homology H_*X.)

2. Suppose $f : D^n \to D^n$ has no fixed point. Then define a retraction $r : D^n \to S^{n-1}$ by setting $r(x)$ to be the unique point on the sphere S^{n-1} so that x is on the line segment connecting $r(x)$ with $f(x)$. Contradiction. $\square$

The following theorem has several interesting consequences.

Theorem 1.26 (Alexander duality; special case).

1. *For any embedding* $h : D^k \hookrightarrow S^n$, $\widetilde{H}_*(S^n - h(D^k)) = 0$.
2. *For any embedding* $h : S^k \hookrightarrow S^n$, $H_*(S^n - h(S^k)) \cong H_*(S^{n-k-1})$.

Proof. 1. Instead we replace the disk D^k by its homeomorph the cube I^k and show that $\widetilde{H}_*(S^n - h(I^k)) = 0$ by induction on k. Clearly this is true for $k = 0$ since $S^n - h(I^0) \cong \mathbf{R}^n$. Assume it is true for any embedding of I^{k-1}. Suppose, by the way of contradiction, there is an $[a] \in \widetilde{H}_i(S^n - h(I^k))$ with $[a] \neq 0$. We will find a sequence of nested intervals

$$[0,1] = I_0 \supset I_1 \supset I_2 \supset \cdots$$

with $0 \neq [a] \in \widetilde{H}_i(S^n - h(I^{k-1} \times I_j))$ and length $I_j = 2^{-j}$. Note that $\cap I_j = \{p\}$ for some point $p \in [0,1]$.

To define I_1, consider the open cover $\{S^n - h(I^{k-1} \times [0,1/2]), S^n - h(I^{k-1} \times [1/2,1])\}$ of $S^n - h(I^{k-1} \times \{1/2\})$. By induction on k, the reduced homology of the space vanishes, so by the Mayer-Vietoris exact sequence, $0 \neq [a] \in S^n - h(I^{k-1} \times I_1)$ where I_1 equals $[0,1/2]$ or $[1/2,1]$. One then defines I_j inductively by chopping I_{j-1} in half and letting I_j be the left half or right half as needed.

Suppose $X_0 \subset X_1 \subset X_2 \subset \cdots$ is a increasing sequence of open subsets of a topological space $X = \cup X_j$. If $[a] \in H_i X_0$ and $0 = [a] \in H_i X$, we claim that $[a] = 0 \in H_i X_j$, some j. To prove this claim, choose an $(i+1)$-chain b in X so that $\partial b = a$. Since b has compact support (if $b = \sum_k a_k \sigma_k$, the *support of* b is the union of the images of σ_k's), b has support in some X_j.

We have finally reached our contradiction. We assumed that $0 \neq [a] \in \widetilde{H}_i(S^n - h(I^k))$. Let $X_j = S^n - h(I^{k-1} \times I_j)$. We have shown both that $0 \neq [a] \in \widetilde{H}_i X_j$ for all j and that there exists a j so that $0 = [a] \in \widetilde{H}_i X_j$. Contradiction.

2. We show this by induction on k. Since $S^n - h(S^0) \cong \mathbf{R}^n - \text{pt}$, it is true for $k = 0$.

Write $S^k = D^k_+ \cup D^k_-$ as a union of the upper and lower hemispheres. The Mayer-Vietoris exact sequence associated to the open cover $\{S^n - h(D^k_-), S^n - h(D^k_+)\}$ of $S^n - h(S^{k-1})$ and part 1 shows that $\widetilde{H}_i(S^n - h(S^{k-1})) \cong \widetilde{H}_{i-1}(S^n - h(S^k))$, so the result follows by induction. □

Exercise 22. Formulate and prove Alexander duality for complements of embeddings $D^k \hookrightarrow \mathbf{R}^n$ and $S^k \hookrightarrow \mathbf{R}^n$.

The following exercise, motivated by the proof of Theorem 1.26, should be only attempted by those readers who know about colimits (see Project 6.5.2).

Exercise 23. Suppose $X_0 \subset X_1 \subset X_2 \subset \cdots$ is a increasing sequence of open subsets of a topological space $X = \cup X_j$. Then the map $\operatorname{colim}_{i \to \infty} H_* X_i \to H_* X$ is an isomorphism.

Applying the Alexander duality theorem in codimension one ($n - k = 1$) gives the following important corollary.

Theorem 1.27 (Jordan-Brouwer Separation Theorem)**.** *Let $h : S^{n-1} \hookrightarrow S^n$ be an embedding. Then the complement $S^n - h(S^{n-1})$ has two path components. These path components are also connected components and are open subsets of S^n.*

Proof. Since $H^0(S^n - h(S^{n-1})) \cong H^0 S^0 \cong \mathbf{Z}^2$, the complement has two path components. Since $S^n - h(S^{n-1})$ is an open subset of S^n, it is locally path connected, hence the path components are connected components. Since connected components are closed and there are a finite number of them, the components are open in $S^n - h(S^{n-1})$ and hence in S^n. $\square$

Informally speaking, an embedding $h : S^{n-1} \hookrightarrow \mathbf{R}^n$ cuts Euclidean space into two parts, the inside and the outside. The inside has compact closure.

Theorem 1.28 (Schoenflies Theorem)**.**

1. *For any embedding $h : S^1 \hookrightarrow S^2$, there is a homeomorphism $H : S^2 \to S^2$ so that $H \circ h : S^1 \hookrightarrow S^2$ is the standard equatorial embedding.*
2. *If $h : S^{n-1} \hookrightarrow S^n$ is a locally flat embedding (i.e. h extends to an embedding $h : S^{n-1} \times (-1, 1) \hookrightarrow S^n$), then there is a homeomorphism $H : S^n \to S^n$ so that $H \circ h : S^{n-1} \hookrightarrow S^n$ is the standard equatorial embedding.*

We will not give a proof of the Schoenflies Theorem. The locally flat hypothesis is necessary because of the beautiful (or ugly, depending on taste) example of the Alexander horned sphere $S^2 \hookrightarrow S^3$ where one of the components of the complement is not even simply connected!

The history of the above theorems is quite convoluted. Camile Jordan first proved the Jordan-Brouwer Separation Theorem when $n = 2$. This special case is usually called the Jordan Curve Theorem. L. E. J. Brouwer proved the Separation Theorem in all dimensions. Arthur Schoenflies showed the Jordan-Schoenflies Theorem in the special case $n = 2$. The Jordan-Schoenflies Theorem in dimensions greater than two is usually called the

Generalized Scheonflies Theorem and was proved independently by Morton Brown and Barry Mazur.

Alexander duality in codimension 2 ($n - k = 2$) is foundational for knot theory. A *knot* K is the image of an embedding $S^{n-2} \hookrightarrow S^n$ and a *classical knot* is the image of $S^1 \hookrightarrow S^3$. Theorem 1.26 shows that the knot complement $S^n - K$ has the homology of a circle. This, together with the Hurewicz Theorem, shows that the abelianization of $\pi_1(S^n - K, x_0)$ is infinite cyclic.

We next discuss Brouwer's theorem on invariance of domain. We already know that $\mathbf{R}^m$ is not homeomorphic to $\mathbf{R}^n$ if $m \neq n$ (since $\mathbf{R}^m - \text{pt}$ and $\mathbf{R}^n - \text{pt}$ have different homology) and in fact a nonempty open set $U \subset \mathbf{R}^m$ is not homeomorphic to a nonempty open set $V \subset \mathbf{R}^n$ (since $(U, U - \text{pt})$ and $(V, V - \text{pt})$ have different relative homology). But Brouwer's theorem is stronger yet.

Classically, the term for an open subset of Euclidean space is a domain. Recall that an open map is a continuous function which takes open sets to open sets.

Theorem 1.29 (Invariance of domain). *Let $U \subset \mathbf{R}^n$ be an open set and let $h : U \to \mathbf{R}^n$ be an injective continuous map. Then $h : U \to \mathbf{R}^n$ is an open map. Thus h is an embedding and $h(U)$ is an open set in $\mathbf{R}^n$.*

Proof. It suffices to prove that if $U \subset S^n$ is open and if $h : U \to S^n$ is continuous and injective, then h is an open map. For any point $x \in U$, choose D' homeomorphic to an n-disk, with $x \in D' \subset U$. Then $h : D' \hookrightarrow S^n$ is an embedding, since h is injective and continuous, D' is compact, and S^n is Hausdorff. Let S' be the boundary of D' and let $D = h(D')$ and $S = h(S')$. Our goal is to show that $D - S$ is open. But $S^n - S$ has two components by the Jordan-Brouwer Separation Theorem, and $S^n - S = (S^n - D) \cup (D - S)$. Alexander duality shows that $S^n - D$ is connected and clearly $D - S$ is connected. It follows that $D - S$ is one of the components of $S^n - S$ and hence open. □

In particular, a subset of $\mathbf{R}^n$ which is homeomorphic to an open subset of $\mathbf{R}^n$ is open in $\mathbf{R}^n$.

An topological n-manifold is a second countable, Hausdorff topological space where every point has a neighborhood homeomorphic to $\mathbf{R}^n$. The following corollary implies, for example, that S^2 does not embed in $\mathbf{R}^2$.

Corollary 1.30. *Let $h : M^n \to N^n$ be a continuous injective map from a nonempty, compact n-manifold to a connected n-manifold. Then f is a homeomorphism.*

Proof. The image of h is closed, since M is compact and N is Hausdorff. The image of h is open by applying invariance of domain to sufficiently small open sets in M. By connectedness of N, we see that h is open. We now have a continuous, bijective map from a compact space to a Hausdorff space. This must be a closed map, and hence a homeomorphism. □

Another greatest hit is the classification of compact surfaces, which includes the result that if two compact surfaces have isomorphic homology then they are homeomorphic. But we will not pursue this.

1.8. Projects: Cellular approximation theorem

1.8.1. Cellular approximation theorem. Recall that a cellular map $f : X \to Y$ is a map between CW-complexes which satisfies $f(X^n) \subset Y^n$ for all n. The cellular approximation theorem says that any map between CW-complexes is homotopic to a cellular map. Prove the cellular approximation theorem. State the relative version (see Theorem 7.52). Give applications to homotopy groups. A good reference is [**43**]. See also [**19**].

Chapter 2

Chain Complexes, Homology, and Cohomology

This chapter develops the necessary homological algebra to define homology and cohomology with coefficients. The Eilenberg-Steenrod axioms characterize these theories.

2.1. Tensor products, adjoint functors, and Hom

2.1.1. Tensor products. Let A and B be modules over a commutative ring R.

Definition 2.1. The *tensor product of A and B* is the R-module $A \otimes_R B$ defined as the quotient

$$\frac{F(A \times B)}{R(A \times B)}$$

where $F(A \times B)$ is the free R-module with basis $A \times B$ and $R(A \times B)$ the submodule generated by

1. $(a_1 + a_2, b) - (a_1, b) - (a_2, b)$
2. $(a, b_1 + b_2) - (a, b_1) - (a, b_1)$
3. $r(a, b) - (ra, b)$
4. $r(a, b) - (a, rb)$.

One denotes the image of a basis element (a, b) in $A \otimes_R B$ by $a \otimes b$. Note that one has the relations

1. $(a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b$
2. $a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2$
3. $(ra \otimes b) = r(a \otimes b) = (a \otimes rb)$.

For example, these formulas can be used to show that $\mathbf{Z}/2 \otimes_{\mathbf{Z}} \mathbf{Z}/3 = 0$.

Informally, $A \otimes_R B$ is the largest R-module generated by the set of symbols $\{a \otimes b\}_{a \in A, b \in B}$ satisfying the above "product type relations". Any element of $A \otimes_R B$ can be expressed as a finite sum $\sum_{i=1}^n a_i \otimes b_i$, but it may not be possible to take $n = 1$, nor is the representation as a sum unique.

Recall that a function $\phi : A \times B \to M$ is *R-bilinear* if A, B, and M are R-modules and

1. $\phi(a_1 + a_2, b) = \phi(a_1, b) + \phi(a_2, b)$
2. $\phi(a, b_1 + b_2) = \phi(a, b_1) + \phi(a, b_2)$
3. $\phi(ra, b) = r\phi(a, b) = \phi(a, rb)$.

For example, the map $\pi : A \times B \to A \otimes_R B$, $(a, b) \mapsto a \otimes b$ is R-bilinear. The universal property of the tensor product is that this map π is initial in the category of bilinear maps with domain $A \times B$.

Proposition 2.2. *Given an R-bilinear map $\phi : A \times B \to M$, there is a unique R-module map $\bar{\phi} : A \otimes_R B \to M$ so that $\bar{\phi} \circ \pi = \phi$.*

$$\begin{tikzcd}
A \times B \arrow[r, "\phi"] \arrow[d, "\pi"'] & M \\
A \otimes B \arrow[ur, dotted, "\exists! \bar{\phi}"'] &
\end{tikzcd}$$

Proof. If $\overline{\phi}$ exists, then $\overline{\phi}(\sum a_i \otimes b_i) = \sum \overline{\phi}(a_i \otimes b_i) = \sum \overline{\phi} \circ \pi(a_i, b_i) = \sum \phi(a_i, b_i)$. Thus uniqueness is clear. For existence, define $\hat{\phi} : F(A \times B) \to M$ on basis elements by $(a, b) \mapsto \phi(a, b)$ and extend by R-linearity. The bilinearity of ϕ implies $\hat{\phi}(R(A \times B)) = 0$, so $\hat{\phi}$ induces $\overline{\phi} : A \otimes_R B \to M$ by the universal property of quotients. $\square$

Proposition 2.2 is useful for defining maps *out* of tensor products, and the following exercise indicates that this is the defining property of tensor products.

Exercise 24. Suppose $p : A \times B \to T$ is an R-bilinear map so that for any R-bilinear map $\psi : A \times B \to M$, there is a unique R-module map $\overline{\psi} : T \to M$ so that $\overline{\psi} \circ p = \psi$. Then $T \cong A \otimes_R B$.

$$\begin{tikzcd} A \times B \arrow[r, "\psi"] \arrow[d, "p"'] & M \\ T \arrow[ur, dotted, "\exists ! \bar{\psi}"'] & \end{tikzcd}$$

Whenever the ring R is understood from context, we will omit the subscript R from the tensor product. The basic properties of the tensor product are given by the next theorem.

Theorem 2.3.

1. *$A \otimes B \cong B \otimes A$.*
2. *$R \otimes B \cong B$.*
3. *$(A \otimes B) \otimes C \cong A \otimes (B \otimes C)$.*
4. *$(\oplus_\alpha A_\alpha) \otimes B \cong \oplus_\alpha (A_\alpha \otimes B)$.*
5. *Given R-module maps $f : A \to C$ and $g : B \to D$, there is an R-module map $f \otimes g : A \otimes B \to C \otimes D$ so that $a \otimes b \mapsto f(a) \otimes g(b)$.*
6. *The functor $- \otimes M$ is right exact. That is, given an R-module M, and an exact sequence*

$$A \xrightarrow{f} B \xrightarrow{g} C \to 0,$$

the sequence

$$A \otimes M \xrightarrow{f \otimes \mathrm{Id}} B \otimes M \xrightarrow{g \otimes \mathrm{Id}} C \otimes M \to 0$$

is exact.

Proof.

1. There is a map $A \otimes B \to B \otimes A$ with $a \otimes b \mapsto b \otimes a$. More formally, the map $A \times B \to B \otimes A$, $(a, b) \mapsto b \otimes a$ is bilinear; for example, one sees $(a_1 + a_1, b) \mapsto b \otimes (a_1 + a_2) = b \otimes a_1 + b \otimes a_2$. By the universal property there is a map $A \otimes B \to B \otimes A$ with $\sum a_i \otimes b_i \mapsto \sum b_i \otimes a_i$. The inverse map is clear.
2. Define $R \otimes B \to B$ by $r \otimes b \mapsto rb$ and $B \to R \otimes B$ by $b \mapsto 1 \otimes b$.
3. $(a \otimes b) \otimes c \leftrightarrow a \otimes (b \otimes c)$.
4. $(\oplus a_\alpha) \otimes b \leftrightarrow \oplus\ (a_\alpha \otimes b)$.
5. $A \times B \to C \otimes D$, $(a, b) \mapsto f(a) \otimes g(b)$ is R-bilinear.

6. First note that $(g \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id}) = 0$. Thus it suffices to explicitly define an isomorphism
$$\overline{g \otimes \mathrm{Id}} : \frac{B \otimes M}{(f \otimes \mathrm{Id})(A \otimes M)} \to C \otimes M.$$
Since $(g \otimes \mathrm{Id}) \circ (f \otimes \mathrm{Id}) = (g \circ f) \otimes \mathrm{Id} = 0$, the map $g \otimes \mathrm{Id}$ descends to the map $\overline{g \otimes \mathrm{Id}}$ by the universal property of quotients. The inverse map is given by defining an R-bilinear map $C \times M \to \frac{B \otimes M}{(f \otimes \mathrm{Id})(A \otimes M)}$ by $(c, m) \mapsto [\hat{c} \otimes m]$ where $g(\hat{c}) = c$. Note that the map is independent of the choice of lift $\hat{c}$, indeed if $\hat{c}'$ is another lift, then $\hat{c} - \hat{c}' \in \ker g = \operatorname{im} f$, so $[\hat{c} \otimes m] - [\hat{c}' \otimes m] = 0$.

□

Example 2.4. Let M be an abelian group. Applying Properties 5 and 2 of Theorem 2.3 we see that if we tensor the short exact sequence
$$0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to \mathbf{Z}/n \to 0$$
by M, we obtain the exact sequence
$$M \xrightarrow{\times n} M \to \mathbf{Z}/n \otimes_{\mathbf{Z}} M \to 0.$$
Notice that $\mathbf{Z}/n \otimes_{\mathbf{Z}} M \cong M/nM$ and that the sequence is not short exact if M has torsion whose order is not relatively prime to n. Thus $- \otimes M$ is not left exact (see Definition 3.1).

Example 2.5. If V and W are vector spaces over $\mathbf{R}$ with bases $\{e_i\}$ and $\{f_j\}$ respectively, then $V \otimes_{\mathbf{R}} W$ has basis $\{e_i \otimes f_j\}$; thus $\dim(V \otimes_{\mathbf{R}} W) = (\dim V)(\dim W)$.

Exercise 25. Compute $A \otimes_{\mathbf{Z}} B$ for any finitely generated abelian groups A and B.

Exercise 26. Show that $- \otimes_R - : R\text{-}\mathrm{Mod} \times R\text{-}\mathrm{Mod} \to R\text{-}\mathrm{Mod}$ is a functor.

Exercise 27. Recall that an abelian group A is torsion if every element has finite order. Show that an abelian group A is torsion if and only if $A \otimes_{\mathbf{Z}} \mathbf{Q} = 0$.

2.1.2. Adjoint functors. Note that an R-bilinear map $\beta : A \times B \to C$ is the same as an element of $\mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C))$. The universal property of the tensor product can be rephrased as follows.

Proposition 2.6 (Adjoint Property of Tensor Products). *There is an isomorphism of R-modules*

$$\mathrm{Hom}_R(A \otimes_R B, C) \cong \mathrm{Hom}_R(A, \mathrm{Hom}_R(B, C)),$$

natural in A, B, C given by $\phi \leftrightarrow (a \mapsto (b \mapsto \phi(a \otimes b)))$. □

This is more elegant than the universal property for three reasons: It is a statement in terms of the category of R-modules, it gives a reason for the duality between tensor product and Hom, and it leads us to the notion of adjoint functor.

Henceforth in this book we will assume that any category $\mathcal{C}$ we discuss has the property $\mathcal{C}(X, Y)$ is a set for any objects X and Y. Such categories are called *locally small* or *enriched in the category* Set.

Definition 2.7. An *adjunction* is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ and a bijection $\mathcal{D}(Fc, d) \longleftrightarrow \mathcal{C}(c, Gd)$, for all $c \in \mathrm{Ob}\ \mathcal{C}$ and $d \in \mathrm{Ob}\ \mathcal{D}$, natural in c and d. The functor F is said to be the left adjoint of G and G is the right adjoint of F.

The adjoint property of tensor products says that for any R-module B, the functors

$$- \otimes_R B : R\text{-Mod} \to R\text{-Mod}$$

and

$$\mathrm{Hom}_R(B, -) : R\text{-Mod} \to R\text{-Mod}$$

form an adjoint pair.

It turns out that the right exactness of $- \otimes_R B$ and the left exactness of $\mathrm{Hom}_R(B, -)$ are formal consequences of being an adjoint pair, but we won't pursue this. A random functor may not have a left (or right) adjoint, but if it does, the adjoint is unique up to natural isomorphism.

Exercise 28. The *forgetful functor* R-Mod $\to$ Set takes a module to its underlying set. Find an adjoint for the forgetful functor. Find another adjoint pair of your own. "Adjoints are everywhere."

2.1.3. Hom.

Exercise 29. For any finitely generated abelian groups A and B, compute $\mathrm{Hom}_{\mathbf{Z}}(A, B)$, the group of all homomorphisms from A to B.

For an R-module A, define $A^* = \mathrm{Hom}_R(A, R)$. The module A^* is often called the *dual* of A. For an R-module map $f : A \to B$, the dual map $f^* : B^* \to A^*$ is defined by $f^*(\varphi) = \varphi \circ f$. Hence taking duals defines a contravariant functor from the category of R-modules to itself.

More generally, for R-modules A and M, $\mathrm{Hom}_R(A, M)$ is the R-module of homomorphisms from A to M. It is contravariant in its first variable and covariant in its second variable. For an R-map $f : A \to B$, we have $\mathrm{Hom}_R(f, M) : \mathrm{Hom}_R(B, M) \to \mathrm{Hom}_R(A, M)$, defined by $\varphi \mapsto \varphi \circ f$. Usually we write f^* for $\mathrm{Hom}_R(f, M)$. The following computational facts may help with Exercise 29.

1. $\mathrm{Hom}_R(R, M) \cong M$.
2. $\mathrm{Hom}_R(\oplus_\alpha A_\alpha, M) \cong \prod_\alpha \mathrm{Hom}_R(A_\alpha, M)$.
3. $\mathrm{Hom}_R(A, \prod_\alpha M_\alpha) \cong \prod_\alpha \mathrm{Hom}_R(A, M_\alpha)$.

The distinction between direct sum and direct product in the category of modules is relevant only when the indexing set is infinite, in which case the direct sum allows only a finite number of nonzero coordinates.

2.2. Homology with coefficients

We will turn our attention now to the algebraic study of (abstract) chain complexes $C_\bullet = (C_*, \partial)$ of R-modules. We do not assume that the chain groups are free R-modules, although they are for the three geometric examples of Section 1.2.

A useful strategy is to use functors out of chain complexes to construct new co/homology invariants of spaces. For example, we could form the tensor product of a R-module chain complex and an R-module M.

This is the functor $- \otimes_R M : \mathrm{Ch}_R \to \mathrm{Ch}_R$ defined by

$$C_\bullet = (C_*, \partial) \mapsto C_\bullet \otimes_R M = (C_* \otimes_R M,\ \partial \otimes \mathrm{Id})$$

with

$$(\partial \otimes \mathrm{Id}) \left(\sum c_i \otimes m_i \right) = \sum (\partial c_i) \otimes m_i.$$

Since $(\partial \otimes \mathrm{Id})^2 = 0$, $C_\bullet \otimes_R M$ is a chain complex. You should show that this is a *functor;* i.e. write down the formula for the map $C_* \otimes_R M \to C'_* \otimes_R M$ induced by a chain map $C_\bullet \to C'_\bullet$ and check that it is a chain map.

One can then take homology:

$$H_n(C_\bullet \otimes_R M) = \frac{\ker \partial \otimes \mathrm{Id} : C_n \otimes M \to C_{n-1} \otimes M}{\mathrm{im}\, \partial \otimes \mathrm{Id} : C_{n+1} \otimes M \to C_n \otimes M}.$$

Applying this to the singular complex of a space leads to the following definition of homology with coefficients.

Definition 2.8. The *singular chain complex of a space X with coefficients in an abelian group M* is $S_\bullet(X; M) = S_\bullet X \otimes_{\mathbf{Z}} M$. The *homology of a space X with coefficients in an abelian group M* is $H_*(X; M) = H_*(S_\bullet(X; M))$.

In the same way one can define $H_*(X, A; M)$, the relative singular homology with coefficients in M. A similar construction applies to the cellular complex to give cellular homology with coefficients. With our notation, $H_*X = H_*(X; \mathbf{Z})$, homology with integer coefficients.

In fact, we have the following theorem.

Theorem 2.9. *Let M be an abelian group. There are functors*

$$S_\bullet(-,-;M) : \mathrm{Top}^2 \to \mathrm{Ch}^+_{\mathbf{Z}}$$
$$H_*(-,- : M) : \mathrm{Top}^2 \to \mathrm{Gr}^+_{\mathbf{Z}}$$

Homology with coefficients is homotopy invariant. Indeed, if $f \simeq g : (X, A) \to (Y, B)$, then the induced maps on the singular complex $S_\bullet(X, A) \to S_\bullet(Y, B)$ are chain homotopic, so the same applies to $S_\bullet(X, A) \otimes_{\mathbf{Z}} M \to S_\bullet(Y, B) \otimes_{\mathbf{Z}} M$. Thus $f_* = g_* : H_*(X, A; M) \to H_*(Y, B; M)$.

Let R be a commutative ring and M be an R-module. Note that by forgetting the module structure, M can be considered as an abelian group. We make two important remarks (which also apply to the relative case):

1. $S_\bullet(X; M)$ is a chain complex of R-modules and, hence $H_*(X; M)$ is a graded R-module.
2. $S_\bullet(X; M) = S_\bullet(X; R) \otimes_R M$.

The reason for the first point is that $S_n(X; M) = S_nX \otimes_{\mathbf{Z}} M$ is an R-module with

$$r\left(\sum_i c_i \otimes m_i\right) = \sum_i c_i \otimes rm_i.$$

Then the differentials are R-module homomorphisms.

The reason for the second point is the associativity isomorphism

$$A \otimes_{\mathbf{Z}} (B \otimes_R C) \cong (A \otimes_{\mathbf{Z}} B) \otimes_R C$$

applied to $A = S_nX$, $B = R$, and $C = M$.

These two remarks will be important when we apply the homological algebra in Chapters 3 and 4 to the simple case where the ring R is a field.

Exercise 30. Let X be a finite CW-complex and F be a field. Show that the Euler characteristic satisfies

$$\chi(X) = \sum (-1)^n \dim_F H_n(X; F)$$

2.3. Cohomology

Cohomology arises from cochain complexes. A *cochain complex* $C^\bullet = (C^*, \delta^*)$ *over* R is a sequence of R-module homomorphisms

$$\cdots \to C^{n-1} \xrightarrow{\delta^{n-1}} C^n \xrightarrow{\delta^n} C^{n+1} \to \cdots$$

so that all the double composites satisfy $\delta \circ \delta = 0$. There is not much difference between a chain complex and a cochain complex other than that the differential has degree $+1$. In fact if (C^*, δ^*) is a cochain complex, then (C^{-*}, δ^{-*}) is a chain complex. The category of cochain complexes over R is denoted Coch_R.

The *cohomology* of a cochain complex $H^*(C^\bullet)$ is the graded R-module with

$$H^n(C^\bullet) = \frac{\ker \delta^n}{\operatorname{im} \delta^{n-1}} = \frac{\text{cocycles}}{\text{coboundaries}}$$

There is a functor $\mathrm{Hom}_R(-, M) : (\mathrm{Ch}_R)^{\mathrm{op}} \to \mathrm{Coch}_R$ with

$$\cdots \to \mathrm{Hom}_R(C_{n-1}, M) \xrightarrow{\delta^{n-1}} \mathrm{Hom}_R(C_n, M) \xrightarrow{\delta^n} \mathrm{Hom}_R(C_{n+1}, M) \to \cdots$$

where the differential δ is the dual to ∂; i.e. $\delta = \mathrm{Hom}_R(\partial, M)$ (sometimes denoted by ∂^*). Explicitly $\delta : \mathrm{Hom}_R(C_n, M) \to \mathrm{Hom}_R(C_{n+1}, M)$ is defined by $(\delta f)(c) = f(\partial c)$. Then $\delta^2 = 0$ since $(\delta^2 f)(c) = (\delta f)(\partial c) = f(\partial^2 c) = 0$.

Applying this to the singular complex of a space leads to the following definition.

Definition 2.10. The *singular cochain complex of a space X with coefficients in an abelian group M* is $S^\bullet(X; M) = \mathrm{Hom}_{\mathbf{Z}}(S_\bullet(X), M)$. The *cohomology of a space X with coefficients in an abelian group M* is $H^*(X; M) = H^*(S^\bullet(X; M))$.

We define $H^*(X)$ to be $H^*(X; \mathbf{Z})$, dropping the coefficients.

In the same way one can define $H^*(X, A; M)$, the relative singular cohomology with coefficients in M. A similar construction applies to the cellular complex to give cellular cohomology with coefficients.

For a useful way to think about singular cochains, extending by linearity shows that

$$S^n(X; M) = \text{functions}(\{\text{singular } n\text{-simplexes}\}, M).$$

Note that if $f : (X, A) \to (Y, B)$ is continuous, then there is a cochain map $S^\bullet(f; M) : S^\bullet(Y, B; M) \to S^\bullet(X, A; M)$ given by $\varphi \mapsto (\sum a_i \sigma_i \mapsto \varphi(\sum a_i (f \circ \sigma_i))$.

Theorem 2.11. *Let M be an abelian group. There are contravariant functors*

$$S^\bullet(-,-;M) : (\mathrm{Top}^2)^{\mathrm{op}} \to \mathrm{Ch}^+_{\mathbf{Z}}$$
$$H^*(-,-;M); (\mathrm{Top}^2)^{\mathrm{op}} \to \mathrm{Gr}^+_{\mathbf{Z}}$$

Cohomology with coefficients is homotopy invariant. Indeed, if $f \simeq g : (X,A) \to (Y,B)$, then the induced maps on the singular complex $S_\bullet(X,A) \to S_\bullet(Y,B)$ are chain homotopic, so the same applies to $\mathrm{Hom}_{\mathbf{Z}}(S_\bullet(Y,B),M) \to \mathrm{Hom}_{\mathbf{Z}}(S_\bullet(X,A),M)$. Thus $f^* = g^* : H^*(Y,B;M) \to H^*(X,A;M)$.

Let R be a commutative ring and M be an R-module. Note that by forgetting the module structure, M can be considered as an abelian group. We make two important remarks (which also apply to the relative case):

1. $S^\bullet(X;M)$ is a cochain complex of R-modules and, hence $H^*(X;M)$ is a graded R-module.
2. $S^\bullet(X;M) = \mathrm{Hom}_R(S_\bullet(X;R),M)$.

The reason for the first point is that $S^n(X;M) = \mathrm{Hom}_{\mathbf{Z}}(S_nX,M)$ is an R-module with $(rf)c := rf(c)$. Then the differentials are R-module homomorphisms.

The reason for the second point is the adjoint isomorphism

$$\mathrm{Hom}_Z(A, \mathrm{Hom}_R(B,C)) \cong \mathrm{Hom}_R(A \otimes_{\mathbf{Z}} B, C)$$

applied to $A = S_nX$, $B = R$, and $C = M$.

These two remarks will be important when we apply the homological algebra in Chapters 3 and 4 to the simple case where the ring R is a field.

Let $1 \in S^0(X;\mathbf{Z}) = \mathrm{Hom}_{\mathbf{Z}}(S_0(X),\mathbf{Z})$ be the homomorphism defined by $1(\sum a_i\sigma_i) = \sum a_i$.

Exercise 31. Show that 1 is a cocycle, and hence represents an element (also called 1) of H^0X. Give a computation of H^0X in terms of path components. Show that $H_0(\mathbf{Q};\mathbf{Z}) \ncong H^0(\mathbf{Q};\mathbf{Z})$, where we consider the rational numbers $\mathbf{Q}$ as a subspace of the real numbers $\mathbf{R}$. Hint: First show that if a space X has path components $\{X_\alpha\}$, then $S_\bullet(X;M) = \oplus_\alpha\, S_\bullet(X_\alpha;M)$.

The primary motivation for introducing cohomology comes from the fact that $H^*(X;R)$ admits a ring structure, while homology does not. This will be discussed in Chapter 4.

Exercise 32. Let $C_\bullet$ be a chain complex over a ring R. Let $C^\bullet$ be the associated dual cochain complex $\mathrm{Hom}_R(C_\bullet, R)$. Show that a cocycle applied to a boundary is zero and a coboundary applied to a cycle is zero. Deduce that there is a bilinear pairing (the *Kronecker pairing*)

$$H^n(C^\bullet) \times H_n(C_\bullet) \to R$$

given by the formula

$$\langle [f], [c] \rangle = f(c).$$

Deduce by taking adjoints that the Kronecker pairing defines a map

$$H^n(C^\bullet) \to H_n(C_\bullet)^* = \operatorname{Hom}_R(H_n(C_\bullet), R).$$

The Kronecker pairing on the homology and cohomology of a space

$$H^n(X; R) \times H_n(X; R) \to R$$

should be thought of as an analogue (in fact a generalization) of integrating a differential n-form along an n-dimensional submanifold. (See the paragraph on the de Rham complex on page 41.)

We will study the Kronecker pairing in detail for R a principal ideal domain (PID) in Section 3.6. It is important to note that cohomology is not the dual of homology in general. The map $H^n(C^\bullet) \to H_n(C_\bullet)^*$ need not be injective nor surjective in general, although we will show that it is surjective when R is a PID and bijective when R is a field. A precise relationship between cohomology and the dual of homology provided by the universal coefficient theorem (Theorem 3.29) when R is a PID.

The following example illustrates the failure of injectivity.

The cellular chain complex of $\mathbf{R}P^2$ is

$$\begin{tikzcd}
\cdots \arrow[r, "\partial"] & C_3 \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"] & C_2 \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"] & C_1 \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"] & C_0 \arrow[r, "\partial"] \arrow[d, leftrightarrow, "\cong"] & 0 \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbf{Z} \arrow[r, "\times 2"'] & \mathbf{Z} \arrow[r, "\times 0"'] & \mathbf{Z} \arrow[r] & 0
\end{tikzcd}$$

so

$$H_0(\mathbf{R}P^2; \mathbf{Z}) = \mathbf{Z},\ H_1(\mathbf{R}P^2; \mathbf{Z}) = \mathbf{Z}/2,\ \text{and } H_2(\mathbf{R}P^2; \mathbf{Z}) = 0.$$

The corresponding cochain complex is

$$\begin{tikzcd}
\cdots & C^3 \arrow[l, "\delta"'] \arrow[d, leftrightarrow, "\cong"] & C^2 \arrow[l, "\delta"'] \arrow[d, leftrightarrow, "\cong"] & C^1 \arrow[l, "\delta"'] \arrow[d, leftrightarrow, "\cong"] & C^0 \arrow[l, "\delta"'] \arrow[d, leftrightarrow, "\cong"] & 0 \arrow[l] \\
\cdots & \operatorname{Hom}(C_3, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(C_2, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(C_1, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(C_0, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & 0 \arrow[l] \\
\cdots & 0 \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & \operatorname{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[l] \arrow[d, leftrightarrow, "\cong"] & 0 \arrow[l] \\
\cdots & 0 \arrow[l] & \mathbf{Z} \arrow[l] & \mathbf{Z} \arrow[l, "\times 2"] & \mathbf{Z} \arrow[l, "\times 0"] & 0 \arrow[l]
\end{tikzcd}$$

Thus

$$H^0(\mathbf{R}P^2; \mathbf{Z}) = \mathbf{Z}$$
$$H^1(\mathbf{R}P^2; \mathbf{Z}) = 0$$

$$H^2(\mathbf{R}P^2;\mathbf{Z}) = \mathbf{Z}/2.$$

In particular $H^2(\mathbf{R}P^2;\mathbf{Z}) \neq \operatorname{Hom}_{\mathbf{Z}}(H_2(\mathbf{R}P^2;\mathbf{Z}),\mathbf{Z})$. Hence the Kronecker pairing is singular.

Exercise 33. We will show that if R is a field, then homology and cohomology are dual. Verify this for $\mathbf{R}P^2$ and $R = \mathbf{Z}/2$.

Remark for those readers who know about differential forms. Suppose X is a smooth manifold, for example, an open subset of Euclidean space. Let $\Omega^n X$ be the vector space of differential n-forms on a manifold. Let $d : \Omega^n X \to \Omega^{n+1} X$ be the exterior derivative. Then $\Omega^\bullet X =$

$$\cdots \to \Omega^{n-1} X \xrightarrow{d} \Omega^n X \xrightarrow{d} \Omega^{n+1} \to \cdots$$

is an $\mathbf{R}$-cochain complex whose cohomology is denoted by $H^*_{\mathrm{DR}}(X)$ and is called the *de Rham cohomology* of X. This gives geometric analogues: n-form and n-cochain, d and δ, closed form and cocycle, exact form and coboundary. For more details on this topic, see the project for this chapter, and [**6**] and [**52**].

De Rham's theorem states that the de Rham cohomology of a manifold X is isomorphic to the singular cohomology $H^*(X;\mathbf{R})$. More precisely, let $S_n^{\mathrm{smooth}}(X;\mathbf{R})$ be the free $\mathbf{R}$-module generated by smooth singular simplices $\sigma : \Delta^n \to X$. There is the chain map

$$S_\bullet^{\mathrm{smooth}}(X;\mathbf{R}) \to S_\bullet(X;\mathbf{R})$$

given by inclusion and the cochain map

$$\Omega^\bullet(X) \to S_\bullet^{\mathrm{smooth}}(X;\mathbf{R})^*$$

given by integrating a n-form along a n-chain. De Rham's theorem follows from the fact that both maps are chain equivalences; i.e. they have inverses up to chain homotopy.

Consider the following 1-form on $\mathbf{R}^2 - \{0\}$.

$$\omega = -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

Exercise 34. Show that $H^1_{\mathrm{DR}}(\mathbf{R}^2 - \{0\}) \cong \mathbf{R}$. Show that ω is an closed 1-form (i.e. $d\omega) = 0$, and hence is a 1-cocycle. Show that ω is not an exact form (i.e. $\omega \neq df$), by integrating ω on the circle. Thus ω generates the first de Rham cohomology.

Note that ω is "locally exact" and is often denoted as $d\theta$ or $d\arctan(y/x)$. For a curve $\gamma : S^1 \to \mathbf{R}^2 - \{0\}$, the integral $\int_\gamma \omega$ is a integral multiple of 2π; one calls this integer *the winding number of* γ *about* 0.

The wedge product of differential forms induces a ring structure on $\Omega^\bullet X$. We will see in Chapter 4 that $S^\bullet(X;\mathbf{R})$ also has a ring structure. These ring structures pass to ring structures on cohomology and the isomorphism of de Rham's theorem is an isomorphism of rings.

The reader may find the dictionary below helpful.

topological space	smooth manifold
n-chain $c \in S_n X$	smooth n-chain $c \in S_n^{\text{smooth}}(X; \mathbf{R})$
n-cochain $f \in S^n X$	differential n-form $\omega \in \Omega^n X$
evaluation $\langle f, c\rangle \in \mathbf{Z}$	integration $\langle \omega, c\rangle = \int_c \omega \in \mathbf{R}$
differential $\delta : S^n X \to S^{n+1} X$	exterior derivative $d : \Omega^n X \to \Omega^{n+1} X$
definition of δ $\langle \delta f, c\rangle = \langle f, \partial c\rangle$	Stokes' Theorem $\langle d\omega, c\rangle = \langle \omega, \partial c\rangle$
cocycle $\delta f = 0$	closed form $d\omega = 0$
coboundary $f = \delta g$, some g	exact form $\omega = d\alpha$, some α
Kronecker pairing	integration of a closed form over a cycle
cup product $f \cup g$	wedge product $\alpha \wedge \beta$.

2.3.1. The long exact sequence of a pair in cohomology . Recall that the relative singular chain complex of a pair (X, A) is defined by taking the chain groups $S_n(X, A) = S_n X / S_n A$. Similarly, let M be an abelian group and define the relative singular cochain complex by

$$S^n(X, A; M) = \operatorname{Hom}_{\mathbf{Z}}(S_n(X, A), M)$$

$$\delta = \operatorname{Hom}_R(\partial, M),\ \ \delta(f) = f \circ \partial.$$

Lemma 2.12. *The diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & S^n(X, A; M) & \longrightarrow & S^n(X; M) & \longrightarrow & S^n(A; M) & \longrightarrow & 0 \\
 & & \downarrow \delta & & \downarrow \delta & & \downarrow \delta & & \\
0 & \longrightarrow & S^{n+1}(X, A; M) & \longrightarrow & S^{n+1}(X; M) & \longrightarrow & S^n(A; M) & \longrightarrow & 0
\end{array}$$

commutes and the horizontal rows are exact.

The proof will depend on a few exercises.

Exercise 35. The diagram commutes, in other words the horizontal maps are cochain maps.

We recall what it means for homomorphisms to split.

Definition 2.13.

1. An injection $0 \longrightarrow A \xrightarrow{\alpha} B$ is said to *split* if there is a map $\delta : B \to A$ so that $\delta \circ \alpha = \operatorname{Id}_A$. The map δ is called a *splitting*.

2. A surjection $B \xrightarrow{\beta} C \longrightarrow 0$ *splits* if there is a map $\gamma : C \to B$ so that $\beta \circ \gamma = \operatorname{Id}_C$.

A surjection $B \to C \to 0$ splits if C is free (prove this basic fact). In general, for an injection $0 \longrightarrow A \xrightarrow{\alpha} B$ the dual $\operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(A, M)$

need not be a surjection (find an example!), but if α is split by δ, then the dual map is a split surjection with splitting map $\mathrm{Hom}_R(\delta, M)$.

Lemma 2.14. *Given a short exact sequence of R-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

show that α splits if and only if β splits. (If either of these possibilities occur, we say the short exact sequence splits.) Show that in this case $B \cong A \oplus C$.

Exercise 36. Prove this lemma.

Corollary 2.15. *If $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ is a short exact sequence of R-modules which splits, then*

$$0 \to \mathrm{Hom}(C, M) \to \mathrm{Hom}(B, M) \to \mathrm{Hom}(A, M) \to 0$$

is exact and splits.

Exercise 37. For a commutative ring R, $S_n(X, A; R)$ is a free R-module with basis

$$\{\sigma : \Delta^n \to X \mid \sigma(\Delta^n) \not\subset A\}.$$

Lemma 2.12 now follows from Corollary 2.15 and Exercise 37. □

Applying the zig-zag lemma immediately implies the following corollary.

Theorem 2.16. *To a pair (X, A) of spaces there corresponds a long exact sequence in singular cohomology*

$$0 \to H^0(X, A; M) \to H^0(X; M) \to H^0(A; M) \xrightarrow{\delta} H^1(X, A; M) \to \\ \cdots \to H^{n-1}(A; M) \xrightarrow{\delta} H^n(X, A; M) \to H^n(X; M) \to \cdots .$$

□

Note that the connecting homomorphism δ has degree $+1$, in contrast to the homology connecting homomorphism ∂ in homology which has degree -1.

Exercise 38. Compute all the groups and homomorphisms in the long exact sequence in cohomology for the pair $(X, A) = (M, \partial M)$ with M the Möbius strip and ∂M its boundary circle. Give a CW-decomposition of $(M, \partial M)$. Find a cellular 2-cocycle $[f]$ representing a generator of $H^2(M, \partial M) \cong \mathbf{Z}/2$ and find a cellular 1-cochain g so that $2f = \delta g$.

Exercise 39. Using the facts that $S_\bullet(X)$ and $S_\bullet(X, A)$ are *free* chain complexes with bases consisting of singular simplices (see Exercise 37), show that for an abelian group M

1. $S_n(X;M) = S_n(X) \otimes_{\mathbf{Z}} M$ can be expressed as the set of all sums $\left\{\sum_{i=1}^{\ell} \sigma_i \otimes m_i \mid m_i \in M, \sigma_i \text{ a } n\text{-simplex}\right\}$ and that $\partial(\sigma \otimes m) = \partial(\sigma) \otimes m$. What is the corresponding statement for $S_n(X, A; M)$?
2. $S^n(X;M) = \mathrm{Hom}_{\mathbf{Z}}(S_n(X), M)$ is in 1-1 correspondence with the set of functions from the set of singular n-simplices to M. Under this identification, given a cochain
$$f : \{\text{Singular } n\text{-simplices}\} \to M,$$
its differential $\delta f : \{\text{Singular } (n+1)\text{-simplices}\} \to M$ corresponds to the map
$$\delta f(\sigma) = \Sigma(-1)^i f(\sigma \circ \delta_i^{n+1})$$
and $S^n(X, A; M) \subset S^n(X;M)$ corresponds to those functions which vanish on the n-simplices entirely contained in A.

Exercise 40. Define and identify the cellular cochain complex in two different ways: as the dual of the cellular chain complex and in terms of relative cohomology of the skeleta. (This will be easier after you have learned the universal coefficient theorem.)

2.4. The Eilenberg-Steenrod axioms

An important conceptual advance took place in algebraic topology when Eilenberg and Steenrod [16] "axiomatized" homology and cohomology.

Definition 2.17. An *(ordinary) homology theory* is a sequence of functors
$$H_n : \mathrm{Top}^2 \to \mathrm{Ab},$$
(where Top^2 is the category of pairs of topological spaces) together with a sequence of natural transformations
$$\partial_n : H_n(X, A) \to H_{n-1}A := H_{n-1}(A, \emptyset)$$
so that

1. A pair of spaces (X, A) induces a long exact sequence
$$\cdots \to H_nA \to H_nX \to H_n(X, A) \xrightarrow{\partial_n} H_{n-1}A \to \cdots$$
(*Long exact sequence of a pair*)
2. If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then the induced maps on homology are equal, $H_n(f) = H_n(g) : H_n(X, A) \to H_n(Y, B)$.
(*Homotopy invariance*)
3. If $A \subset X$, $\overline{A} \subset \mathrm{Int}\ U$, then $H_n(X - A, U - A) \to H_n(X, U)$ is an isomorphism for all n.
(*Excision*)

4. If "pt" denotes a one-point space, then $H_n(\mathrm{pt}) = 0$ when $n \neq 0$.
   (*Dimension Axiom*)
5. If $X = \coprod_\alpha X_\alpha$, then the sum of the inclusions induces an isomorphism $\bigoplus_\alpha H_n(X_\alpha) \cong H_n X$.
   (*Additivity*)

Theorem 2.18 (Existence)**.** *For any abelian group M, there is a homology theory H_* with $H_0(\mathrm{pt}) \cong M$.* □

In fact, existence is shown by proving that singular homology with coefficients in M satisfies the axioms.

Definition 2.19. An *(ordinary) cohomology theory* is a sequence of contravariant functors

$$H^n : (\mathrm{Top}^2)^{\mathrm{op}} \to \mathrm{Ab},$$

together with a sequence of natural transformations

$$\delta^n : H^n A \to H^{n+1}(X, A)$$

so that

1. A pair of spaces (X, A) induces a long exact sequence
   $$\cdots \to H^{n-1}A \xrightarrow{\delta^{n-1}} H^n(X, A) \to H^n X \to H^n A \to \cdots$$
   (*Long exact sequence of a pair*)
2. If $f, g : (X, A) \to (Y, B)$ are homotopic maps, then $H^n(f) = H^n(g) : H^n(Y, B) \to H^n(X, A)$.
   (*Homotopy invariance*)
3. If $A \subset X$, $\overline{A} \subset \mathrm{Int}\ U$, then $H^n(X, U) \to H^n(X - A, U - A)$ is an isomorphism.
   (*Excision*)
4. $H^n(\mathrm{pt}) = 0$ when $n \neq 0$.
   (*Dimension Axiom*)
5. If $X = \coprod_\alpha X_\alpha$, then the product of the maps induced by the inclusions give an isomorphism $H^n X \cong \prod_\alpha H^n(X_\alpha)$.
   (*Additivity*)

Theorem 2.20 (Existence)**.** *For any abelian group M, there is a cohomology theory with $H^0(\mathrm{pt}) \cong M$.* □

Once again, existence is shown by showing that singular cohomology $H^*(X, A; M)$ satisfies the axioms.

There are several reasons for wanting a set of axioms. First and foremost, the axioms isolate the key features of (co)homology. Any proof or computation that uses only the axioms is likely to be elegant. Second, once we have a uniqueness theorem, it can be useful for saying that two (co)homology theories coincide, for example, singular and cellular homology when restricted to CW-pairs. And, as we will see below, we use the axioms as a flexible guide to the features of the theory we have in mind.

There are many different approaches to constructing homology and cohomology theories; the choice of method is often dictated by the kind of problem one is attacking. Singular homology and cohomology are defined for all spaces. The abstract definition simplifies the proofs of many theorems and is makes it easy to see that (co)homology is a homeomorphism invariant, but the singular complex is too large to be effective for computations. It is often useful to consider (co)homology theories defined on subcategories of Top^2. De Rham cohomology is defined for smooth manifolds and has many nice properties, including direct relationships to solutions of differential equations on manifolds. There exist some extensions of de Rham theory to more general spaces; these tend to be technical. Cellular homology is often the most useful for computing, but of course applies only to CW-complexes.

Čech (co)homology theory is another theory that satisfies the axioms (at least for the subcategory of pairs of compact spaces), but the Čech (co)homology of the topologist's sine curve is not isomorphic to the singular cohomology. Thus the axioms do not determine the (co)homology of all spaces. They *do* determine the (co)homology for CW-complexes. However, the excision axiom must be modfied to an axiom internal to the category CW^2.

(*CW-excision*) If X is a CW-complex and A and B are subcomplexes, then $H_n(A, A \cap B) \to H_n(X, B)$ is an isomorphism for all n.

CW-excision holds for both singular and cellular homology. The uniqueness theorem below implies that cellular homology is isomorphic to singular homology, and, furthermore, that the proof of this fact only uses the axioms. We state the uniqueness theorem for for homology, but the dual result holds for cohomology also.

Theorem 2.21 (Uniqueness). *Let* (H_*, ∂_*) *and* $(\hat{H}_*, \hat{\partial}_*)$ *satisfy the modified Eilenberg-Steenrod Axioms on the category* CW^2 *of CW-pairs.*

1. *Given a homomorphism $H_0(\mathrm{pt}) \to \hat{H}_0(\mathrm{pt})$, there is a natural transformation $H_* \to \hat{H}_*$ compatible with the boundary maps inducing the given homomorphism.*
2. *Any natural transformation $H_* \to \hat{H}_*$ compatible with the boundary maps inducing an isomorphism for a point is an isomorphism for all CW-complexes.*

Exercise 41. Show that cellular and singular homology are isomorphic for CW-pairs.

In light of this theorem, one can do all computations of homology groups of CW-complexes using the axioms, i.e. without resorting to the definition of the singular or cellular chain complex. This is not always the best way to proceed, but usually in doing homology computations one makes repeated use of the axioms and a few basic computations.

There are also many functors from spaces to abelian groups which satisfy all the Eilenberg-Steenrod axioms except the Dimension axiom. These are called generalized (co)homology theories, and are introduced in Chapter 9.

2.5. Projects: Singular homology; De Rham cohomology

2.5.1. Singular homology theory. Give an outline of the proof that singular homology theory satisfies the Eilenberg-Steenrod axioms, concentrating on the excision axiom. State the Mayer-Vietoris exact sequence, give a computational example of its use, and show how it follows from the Eilenberg-Steenrod axioms.

2.5.2. De Rham cohomology. Construct the vector space $\Omega^n(X)$ of differential n-forms on a smooth manifold X, the exterior derivative $d : \Omega^n(X) \to \Omega^{n+1}(X)$, and prove $d^2 = 0$, yielding the de Rham cochain complex $(\Omega^*(X), d)$ and its cohomology, the de Rham cohomology $H^*_{\mathrm{DR}}(X)$. Show that a smooth map $f : X \to Y$ between smooth manifolds induces a chain map $f^\bullet : \Omega^\bullet(Y) \to \Omega^\bullet(X)$. Compute $H^0_{\mathrm{DR}}(X)$ for any X, and compute $H^1_{\mathrm{DR}}(S^1)$.

Define the wedge product $\wedge : \Omega^m(X) \times \Omega^n(X) \to \Omega^{m+n}(X)$ and verify that it satisfies $a \wedge b = (-1)^{mn} b \wedge a$. Prove that $d(a \wedge b) = da \wedge b + (-1)^m a \wedge db$. Conclude that the wedge product descends to a ring structure on $H^*_{\mathrm{DR}}(X)$.

Outline the proof, using Stokes' theorem, that integrating differential forms defines a chain map from $(\Omega^*(X), d)$ to the smooth singular cochain complex $S^\bullet(X; \mathbf{R})_{\mathrm{smooth}}$, where

$$S^n(X; \mathbf{R})_{\mathrm{smooth}} := \mathrm{Funct}(C^\infty(\Delta^n, X), \mathbf{R}).$$

The *de Rham Theorem* asserts that this chain map is a chain homotopy equivalence and that the restriction map

$$S^{\bullet}(X;\mathbf{R}) \to S^{\bullet}(X;\mathbf{R})_{\text{smooth}}$$

(induced by the inclusion $C^{\infty}(\Delta^n, X) \subset C^0(\Delta^n, X)$) is a chain homotopy equivalence. Hence

$$H^*_{\text{DR}}(X) \cong H^*(S^{\bullet}(X;\mathbf{R})_{\text{smooth}}) \cong H^*(S^{\bullet}(X;\mathbf{R})) = H^*(X;\mathbf{R}).$$

Good references include [**6**] and [**52**].

Chapter 3

Homological Algebra

In this chapter R denotes a commutative ring.

3.1. Axioms for Tor and Ext; projective resolutions

Definition 3.1. An *exact functor* R-Mod $\to$ R-Mod is a functor which takes short exact sequences to short exact sequences.

More generally, a covariant functor $F : R\text{-Mod} \to R\text{-Mod}$ is called *right exact* (resp. *left exact*) if $F(A) \to F(B) \to F(C) \to 0$ is exact (resp. $0 \to F(A) \to F(B) \to F(C)$ is exact) whenever $0 \to A \to B \to C \to 0$ is a short exact sequence. Similarly a contravariant functor is called *right exact* (resp. *left exact*) if $F(C) \to F(B) \to F(A) \to 0$ is exact (resp. $0 \to F(C) \to F(B) \to F(A)$ is exact) whenever $0 \to A \to B \to C \to 0$ is a short exact sequence.

We have already seen that the functors $- \otimes_R M$, $\mathrm{Hom}_R(M, -)$, and $\mathrm{Hom}_R(-, M)$ are not exact in general. For example, taking $R = \mathbf{Z}$, $M = \mathbf{Z}/2$, and the short exact sequence

$$0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2 \to 0,$$

we obtain

$$\begin{tikzcd}
\mathbf{Z} \otimes \mathbf{Z}/2 \arrow[r] \arrow[d, "\cong"'] & \mathbf{Z} \otimes \mathbf{Z}/2 \arrow[r] \arrow[d, "\cong"'] & \mathbf{Z}/2 \otimes \mathbf{Z}/2 \arrow[r] \arrow[d, "\cong"'] & 0 \\
\mathbf{Z}/2 \arrow[r, "\times 2"] & \mathbf{Z}/2 \arrow[r, "\mathrm{Id}"] & \mathbf{Z}/2 \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2) \arrow[d, "\cong"'] \\
0 \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \mathbf{Z}/2
\end{tikzcd}$$

and

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(\mathbf{Z}/2,\mathbf{Z}/2) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{Z},\mathbf{Z}/2) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{Z},\mathbf{Z}/2) \arrow[d, "\cong"'] \\
0 \arrow[r] & \mathbf{Z}/2 \arrow[r] & \mathbf{Z}/2 \arrow[r, "\times 2"] & \mathbf{Z}/2
\end{tikzcd}$$

However, we have seen in Theorem 2.3 that $-\otimes_R M$ is right exact and in Exercise 42 that $\mathrm{Hom}_R(M,-)$ and $\mathrm{Hom}_R(-,M)$ are left exact.

Exercise 42. Given any short exact sequence of R-modules

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

show that

$$0 \to \mathrm{Hom}_R(C,M) \to \mathrm{Hom}_R(B,M) \to \mathrm{Hom}_R(A,M)$$

and

$$0 \to \mathrm{Hom}_R(M,A) \to \mathrm{Hom}_R(M,B) \to \mathrm{Hom}_R(M,C)$$

are exact.

Exercise 43. If F is a free module, show that $-\otimes_R F$ and $\mathrm{Hom}_R(F,-)$ are exact functors. Show by example that $\mathrm{Hom}_R(-,F)$ need not be exact.

The idea of homological algebra is to find *natural* functors which measure the failure of a functor to preserve short exact sequences. (A first stab at this for $-\otimes_R M$ might be to take the kernel of $A\otimes_R M \to B\otimes_R M$ as the value of this functor. Unfortunately, this does not behave nicely with respect to morphisms.) To construct these functors the only things we will use are the left/right exactness properties, the above exercise and the observation that for any module M there is a surjective map from a free module to M.

Theorem 3.2 (existence)**.**

1. *There exist functors*

   $\mathrm{Tor}_n^R : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ *for all* $n = 0,1,2,\ldots$

   $(M_1,M_2) \mapsto \mathrm{Tor}_n^R(M_1,M_2)$ *covariant in* M_1 *and* M_2 *satisfying the following axioms:*
   T1) $\mathrm{Tor}_0^R(M_1,M_2) = M_1 \otimes_R M_2$.

T2) *If* $0 \to A \to B \to C \to 0$ *is any short exact sequence of* R*-modules and* M *is any* R*-module, then there is a natural long exact sequence*

$$\cdots \to \operatorname{Tor}_n^R(A, M) \to \operatorname{Tor}_n^R(B, M) \to \operatorname{Tor}_n^R(C, M) \to \operatorname{Tor}_{n-1}^R(A, M) \to \cdots$$
$$\cdots \to \operatorname{Tor}_1^R(C, M) \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0.$$

T3) $\operatorname{Tor}_n^R(F, M) = 0$ *if* F *is a free module and* $n > 0$.

The functor $\operatorname{Tor}_n^R(-, M)$ *is called the* n^{th} derived functor *of the functor* $- \otimes_R M$.

2. *There exist functors*

$$\operatorname{Ext}_R^n : R\text{-Mod}^{\mathrm{op}} \times R\text{-Mod} \to R\text{-Mod} \quad \textit{for all} \quad n = 0, 1, 2, \ldots$$

$(M_1, M_2) \mapsto \operatorname{Ext}_R^n(M_1, M_2)$ *contravariant in* M_1 *and covariant in* M_2 *satisfying the following axioms:*

E1) $\operatorname{Ext}_R^0(M_1, M_2) = \operatorname{Hom}_R(M_1, M_2)$.

E2) *If* $0 \to A \to B \to C \to 0$ *is any short exact sequence of* R*-modules and* M *is any* R*-module, then there is a natural long exact sequence*

$$0 \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(B, M) \to \operatorname{Hom}_R(A, M) \to \operatorname{Ext}_R^1(C, M) \to \cdots$$
$$\cdots \to \operatorname{Ext}_R^n(B, M) \to \operatorname{Ext}_R^n(A, M) \to \operatorname{Ext}_R^{n+1}(C, M) \to \cdots$$

E3) $\operatorname{Ext}_R^n(F, M) = 0$ *if* F *is a free module and* $n > 0$.

The functor $\operatorname{Ext}_R^n(-, M)$ *is called the* n^{th} derived functor *of the functor* $\operatorname{Hom}_R(-, M)$.

Before we embark on the proof of this theorem, we prove that these axioms characterize the functors Tor and Ext.

Theorem 3.3 (uniqueness)**.** *Any two functors satisfying T1), T2), and T3) are naturally isomorphic. Any two functors satisfying E1), E2), and E3) are naturally isomorphic.*

Proof. We will show that values of $\operatorname{Tor}_n^R(M_1, M_2)$ are determined by the axioms by induction on n. This is true for $n = 0$ by T1). Next note that for any module M_1, there is a surjection $F \xrightarrow{\phi} M_1 \to 0$ where F is a free module. For example, let $S \subset M_1$ be a set which generates M_1 as an R-module (e.g. $S = M_1$), and let $F = F(S)$ be the free module with basis S. There is an obvious surjection ϕ. Let $K = \ker \phi$. Apply T2) to the short exact sequence

$$0 \to K \to F \to M_1 \to 0.$$

Then by T2) and T3), one has

$$\operatorname{Tor}_1^R(M_1, M_2) \cong \ker(K \otimes_R M_2 \to F \otimes_R M_2)$$

and

$$\mathrm{Tor}_n^R(M_1, M_2) \cong \mathrm{Tor}_{n-1}^R(K, M_2) \quad \text{for} \quad n > 1.$$

The values of Tor_{n-1}^R are known by induction. The proof for Ext is similar. □

The technique of the above proof is called dimension shifting, and it can be useful for computations. For example, if F is a free module and

$$0 \to K \to F' \to M \to 0$$

is a short exact sequence with F' free, then

$$\mathrm{Tor}_1^R(M, F) \cong \ker(K \otimes F \to F' \otimes F),$$

but this is zero by Exercise 43. Thus $\mathrm{Tor}_1^R(-, F)$ is identically zero. But $\mathrm{Tor}_n^R(M, F) \cong \mathrm{Tor}_{n-1}^R(K, F)$ for $n > 1$, so inductively we see $\mathrm{Tor}_n^R(-, F)$ is zero for $n > 0$. To compute $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/2, \mathbf{Z})$, we apply E2) to the exact sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2 \to 0$ to get the exact sequence

$$\begin{tikzcd}
\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[r, "(\times 2)^*"] \arrow[d, "\cong"'] & \mathrm{Hom}(\mathbf{Z}, \mathbf{Z}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^1(\mathbf{Z}/2, \mathbf{Z}) \arrow[r] \arrow[d, "\cong"'] & \mathrm{Ext}^1(\mathbf{Z}, \mathbf{Z}) \arrow[d, "\cong"'] \\
\mathbf{Z} \arrow[r, "\times 2"] & \mathbf{Z} \arrow[r] & \mathrm{Ext}^1(\mathbf{Z}/2, \mathbf{Z}) \arrow[r] & 0
\end{tikzcd}$$

so $\mathrm{Ext}^1_{\mathbf{Z}}(\mathbf{Z}/2, \mathbf{Z}) \cong \mathbf{Z}/2$.

The following proposition gives some simple but useful computations. This result should be memorized. (The subscript or superscript R is often omitted when the choice of the ring R is clear from context.)

Proposition 3.4. *Let R be a commutative ring and $a \in R$ a nonzero divisor (i.e. $ab = 0$ implies $b = 0$). Let M be an R-module. Let $M/a = M/aM$ and ${}_aM = \{m \in M \mid am = 0\}$. Then*

1. $R/a \otimes M \cong M/a$,
2. $\mathrm{Tor}_1(R/a, M) \cong {}_aM$,
3. $\mathrm{Hom}(R/a, M) \cong {}_aM$,
4. $\mathrm{Ext}^1(R/a, M) \cong M/a$.

Proof. Since a is not a divisor of zero, there is a short exact sequence

$$0 \to R \xrightarrow{\times a} R \to R/a \to 0.$$

Apply the functors $- \otimes M$ and $\mathrm{Hom}(-, M)$ to the above short exact sequence. By the axioms we have exact sequences

$$0 \to \mathrm{Tor}_1(R/a, M) \to R \otimes M \to R \otimes M \to R/a \otimes M \to 0$$

and

$$0 \to \mathrm{Hom}(R/a, M) \to \mathrm{Hom}(R, M) \to \mathrm{Hom}(R, M) \to \mathrm{Ext}^1(R/a, M) \to 0.$$

The middle maps in the exact sequences above can be identified with

$$M \xrightarrow{\times a} M,$$

which has kernel ${}_aM$ and cokernel M/a. □

In particular if n is a nonzero integer and $R = \mathbf{Z}$, the four functors $\mathrm{Tor}_1, \otimes, \mathrm{Hom}$, and Ext^1 applied to the pair $(\mathbf{Z}/n, \mathbf{Z}/n)$ are all isomorphic to $\mathbf{Z}/n$. If m and n are relatively prime integers, then applied to the pair $(\mathbf{Z}/m, \mathbf{Z}/n)$ they are all zero.

Proposition 3.5.

1. *If R is a field, then $\mathrm{Tor}_n^R(-,-)$ and $\mathrm{Ext}_R^n(-,-)$ are zero for $n > 0$.*
2. *If R is a PID, then $\mathrm{Tor}_n^R(-,-)$ and $\mathrm{Ext}_R^n(-,-)$ are zero for $n > 1$.*

Proof. 1. All modules over a field are free so this follows from axioms T3) and E3).

2. A submodule of a free module over a PID is free, so for any module M there is a short exact sequence

$$0 \to F_1 \to F_0 \to M \to 0$$

with F_1 and F_0 free. Then by T2), T3), E2), and E3), for $n > 1$, $\mathrm{Tor}_n^R(M,-)$ and $\mathrm{Ext}_n^R(M,-)$ sit in long exact sequences flanked by zero, and hence must vanish. □

The functors $\mathrm{Tor}_1^{\mathbf{Z}}$ and $\mathrm{Ext}_{\mathbf{Z}}^1$ are typically abbreviated Tor and Ext.

Exercise 44. Using the axioms, compute $\mathrm{Tor}(A, B)$ and $\mathrm{Ext}(A, B)$ for all finitely generated abelian groups.

A couple of natural questions must have occurred to you. What is the behavior of these functors with respect to exact sequences in the second variable? Is $\mathrm{Tor}_n(A, B) \cong \mathrm{Tor}_n(B, A)$? This seems likely since $A \otimes B \cong B \otimes A$. (Since $\mathrm{Hom}(A, B) \ncong \mathrm{Hom}(B, A)$ the corresponding question for Ext could not have possibly occurred to you!) Your questions are answered by the following theorem.

Theorem 3.6 (existence′)**.**

1. *The functors*

   $\mathrm{Tor}_n^R : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod}$ *for all* $n = 0, 1, 2, \ldots$

   satisfy the following axioms.

T1′) $\mathrm{Tor}_0^R(M_1, M_2) = M_1 \otimes_R M_2$.

T2′) *If* $0 \to A \to B \to C \to 0$ *is any short exact sequence of R-modules and M is any R-module, then there is a natural long exact sequence*

$$\cdots \to \mathrm{Tor}_n^R(M, A) \to \mathrm{Tor}_n^R(M, B) \to \mathrm{Tor}_n^R(M, C) \to \mathrm{Tor}_{n-1}^R(M, A) \to \cdots$$
$$\cdots \to \mathrm{Tor}_1^R(M, C) \to M \otimes_R A \to M \otimes_R B \to M \otimes_R C \to 0.$$

T3′) $\mathrm{Tor}_n^R(M, F) = 0$ *if F is a free module and* $n > 0$.

2. *The functors*

$$\mathrm{Ext}_R^n : R\text{-Mod} \times R\text{-Mod} \to R\text{-Mod} \quad \textit{for all } n = 0, 1, 2, \ldots$$

satisfy the following axioms:

E1′) $\mathrm{Ext}_R^0(M_1, M_2) = \mathrm{Hom}_R(M_1, M_2)$.

E2′) *If* $0 \to A \to B \to C \to 0$ *is any short exact sequence of R-modules and M is any R-module, then there is a natural long exact sequence*

$$0 \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(M, B) \to \mathrm{Hom}_R(M, C) \to \mathrm{Ext}_R^1(M, A) \to \cdots$$
$$\cdots \to \mathrm{Ext}_R^n(M, B) \to \mathrm{Ext}_R^n(M, C) \to \mathrm{Ext}_R^{+1}(M, A) \to \cdots$$

E3′) $\mathrm{Ext}_R^n(M, I) = 0$ *if I is an injective module (see Definition 3.12) and* $n > 0$.

We postpone the proof of Theorem 3.6 until Section 3.5.

Corollary 3.7. *The functors* $\mathrm{Tor}_n^R(A, B)$ *and* $\mathrm{Tor}_n^R(B, A)$ *are naturally isomorphic.*

Proof. By Theorem 3.6, the functor $(A, B) \mapsto \mathrm{Tor}_n^R(B, A)$ satisfies the axioms T1), T2), and T3) and thus by the uniqueness theorem, Theorem 3.3, it must be naturally isomorphic to $(A, B) \mapsto \mathrm{Tor}_n^R(A, B)$. □

Tor and Ext are derived versions of $\otimes_R$ and Hom, so they have analogous properties. For example we offer without proof:

1. $\mathrm{Tor}_n^R(\oplus_\alpha A_\alpha, B) \cong \oplus_\alpha \mathrm{Tor}_n^R(A_\alpha, B)$,
2. $\mathrm{Ext}_R^n(\oplus_\alpha A_\alpha, B) \cong \prod_\alpha \mathrm{Ext}_R^n(A_\alpha, B)$, and
3. $\mathrm{Ext}_R^n(A, \prod_\alpha B_\alpha) \cong \prod_\alpha \mathrm{Ext}_R^n(A, B_\alpha)$.

The proofs of Theorems 3.2 and 3.6 are carried out using *projective modules* and *projective resolutions*. The functors Ext_R^n can also be defined using *injective resolutions*. We will carry out the details in the projective case over the next few sections and sketch the approach to Ext using injective resolutions.

Tor, Ext, Hom, and tensor product also make sense for modules over noncommutative rings. The issue here is that one has to distinguish between left and right modules. In Definition 6.2 we define the tensor product $M \otimes_R N$ of a right R-module M and a left R-module N. Armed with this definition, it is straightforward to define $\mathrm{Tor}_n^R(M, N)$. Similarily, if M and N are both left R-modules, one can define $\mathrm{Hom}_R(M, N)$ and $\mathrm{Ext}_R^n(M, N)$.

Much of what we say can be done in the more general setting of *abelian categories*; these are categories where the concept of exact sequence makes sense (for example the category of sheaves or the category of representations of a Lie algebra) provided there are "enough projectives" or "enough injectives" in the category.

3.2. Projective and injective modules

Recall that if F is a free R-module, A, B are R-modules, and

$$\begin{tikzcd} & F \arrow[d, "\beta"] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$$

is a diagram with α *onto*, then there exists a $\gamma : F \to A$ so that

$$\begin{tikzcd} & F \arrow[d, "\beta"] \arrow[dl, "\gamma"'] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$$

commutes. We say

"The problem $\begin{tikzcd} & F \arrow[d, "\beta"] \arrow[dl, dotted, "\gamma"'] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$ has a solution."

We make the following definition which encapsulates this basic property of free modules.

Definition 3.8. An R-module P is called *projective* if for any A, B, α, β with α onto, the problem

$$\begin{tikzcd} & P \arrow[d, "\beta"] \arrow[dl, dotted, "\gamma"'] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$$

has a solution γ.

Lemma 3.9. *An R-module P is projective if and only if there exists an R-module Q so that $P \oplus Q$ is a free R-module.*

Proof. If P is projective, choose an epimorphism $F \to P$ with F free. Let $Q = \ker(F \to P)$, so

$$0 \to Q \to F \to P \to 0$$

is exact. Since P is projective, the sequence splits, as one sees by considering the problem

$$\begin{tikzcd} & P \arrow[dl, dotted] \arrow[d, "Id"] & \\ F \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

so $P \oplus Q \cong F$.

Conversely, if there exists an R-module Q so that $P \oplus Q$ is free, extend

$$\begin{tikzcd} & P \arrow[d, "\beta"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd} \quad \text{to} \quad \begin{tikzcd} & P \oplus Q \arrow[d, "\beta\oplus 0"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

Since $P \oplus Q$ is free, there exists a solution $f : P \oplus Q \to A$ to

$$\begin{tikzcd} & P \oplus Q \arrow[dl, dotted, "f"'] \arrow[d, "\beta\oplus 0"] & \\ A \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

But then let $\overline{f} = f \circ i$ where $i : P \to P \oplus Q$ is given by $p \mapsto (p, 0)$. Then $\overline{f}$ solves the problem

$$\begin{tikzcd} & P \arrow[dl, dotted, "\bar{f}"'] \arrow[d, "\beta"] & \\ A \arrow[r, "\alpha"'] & B \arrow[r] & 0 \end{tikzcd}$$

Hence P is projective. □

Thus projective modules generalize free modules by isolating one of their main properties. Furthermore the definition of a projective module is purely in terms of arrows in R-Mod, and hence is more elegant than the definition of a free module. On the other hand projective modules are less familiar.

Exercise 45. An R-module P is projective if and only if it is the image of a *projection*, that is, a homomorphism $\pi : F \to F$ with F free and $\pi \circ \pi = \pi$.

Exercise 46. Let P be a projective module.

1. Any short exact sequence $0 \to A \to B \to P \to 0$ is split.
2. If P is finitely generated, there is a Q so that $P \oplus Q$ is finitely generated free.

Proposition 3.10.

1. *Any module over a field is projective.*
2. *Any projective module over a PID is free.*

Proof. All modules over a field are free, hence projective. A projective module P is a submodule of the free module $P \oplus Q$, and for PIDs submodules of free modules are free. $\square$

There are many examples of *nonfree* projective modules over rings R. Note that R must be complicated, i.e. not a field nor a PID. For example, if $R = \mathbf{Z}/6$, then $P = \mathbf{Z}/2$ is a projective module. (To see this, use the Chinese remainder theorem $\mathbf{Z}/6 = \mathbf{Z}/2 \times \mathbf{Z}/3$.)

Here is a more interesting example, related to K-theory. Let R be the ring of continuous functions on the circle, $R = C^0(S^1, \mathbf{R})$. Let $E \to S^1$ be the twisted real line bundle over S^1 (so $E =$ open Möbius band). Then as vector bundles $E \ncong S^1 \times \mathbf{R}$, but $E \oplus E \cong S^1 \times \mathbf{R}^2$. So, if $M = C^0E$ (continuous sections of E), M is *not a free R-module* (why?), but $M \oplus M \cong C^0(S^1, \mathbf{R}) \oplus C^0(S^1, \mathbf{R}) = R \oplus R$. Thus M is projective.

Exercise 47. Show that the following are examples of projectives which are not free.

1. Let R be the ring of 2-by-2 matrices with real entries. Let $P = \mathbf{R}^2$ where the action of R on P is by multiplying a matrix by a vector. (Hint: Think of P as 2-by-2 matrices with the second column all zeroes.)
2. Let $R = \mathbf{R} \times \mathbf{R}$ (addition and multiplication are component-wise) and $P = \mathbf{R} \times \{0\}$.

One of the quantities measured by the functor K_0 of *algebraic K-theory* is the difference between projective and free modules over a ring. See Chapter 12 for another aspect of algebraic K-theory, namely the geometric meaning of the functor K_1.

As far as Tor and Ext are concerned, observe that

$$\operatorname{Tor}_n^R(A \oplus B, M) \cong \operatorname{Tor}_n^R(A, M) \oplus \operatorname{Tor}_n^R(B, M).$$

This is because $A \oplus B$ fits into the split exact sequence

$$0 \to A \to A \oplus B \to B \to 0.$$

Functoriality and Axiom T2) put $\mathrm{Tor}_n^R(A \oplus B, M)$ in a corresponding split exact sequence. Applying this to $P \oplus Q \cong$ free (and applying a similar argument to Ext), one obtains the following result.

Corollary 3.11. *For a projective module P, for $n > 0$, and for any module M, both $\mathrm{Tor}_n^R(P, M)$ and $\mathrm{Ext}_R^n(P, M)$ vanish.* □

Thus for purposes of computing Tor and Ext (e.g. dimension shifting), projective modules work just as well as free modules.

In the categorical framework in which we find ourselves, something interesting usually happens if one reverses all arrows. Reversing the arrows in the definition of projective modules leads to the definition of *injective* modules.

Definition 3.12. An R-module I is called *injective* if

$$\begin{tikzcd} & I & \\ A \arrow[ur, dotted] & B \arrow[l, "\beta"] \arrow[u, "\alpha"'] & 0 \arrow[l] \end{tikzcd}$$

has a solution for all A, B, α, β (with β injective).

Exercise 48. Let P be a projective R-module and I be an injective R-module. Show that the following functors are exact: $P \otimes_R -$, $\mathrm{Hom}_R(P, -)$, and $\mathrm{Hom}_R(-, I)$.

We will define Ext using projective modules instead of injective modules, so we omit most details about injective modules. See Rotman [**41**] or MacLane [**27**] for more. We list here a few results.

Theorem 3.13. *An abelian group A is injective if and only if A is divisible (i.e. the equation $nx = a$ has a solution $x \in A$ for each $n \in \mathbf{Z} - \{0\}$, $a \in A$).* □

Thus some examples of injective abelian groups are $\mathbf{Q}$ and $\mathbf{Q}/\mathbf{Z}$. (Note that a quotient of a divisible group is divisible, hence injective.)

Theorem 3.14.

1. *Given any R-module M, there exists a projective R-module P and an epimorphism $P \to M \to 0$.*
2. *Given any R-module M, there exists an injective R-module I and a monomorphism $0 \to M \to I$.*

Proof. We have already proved 1, by taking P to be a free module on a set of generators of M. The proof of 2 is more involved. One proves it first for an abelian group. Here is one way. Express $M = (\oplus\mathbf{Z})/K$. This injects to $D = (\oplus\mathbf{Q})/K$ which is divisible and hence injective.

Now suppose M is an R-module. Then, considered as an *abelian group*, there is an injection $\varphi : M \to D$ where D is divisible. One can show the map $M \to \mathrm{Hom}_{\mathbf{Z}}(R, D)$, $m \mapsto (r \mapsto \varphi(rm))$ is an R-module monomorphism and that $\mathrm{Hom}_{\mathbf{Z}}(R, D)$ is an injective R-module when D is divisible. □

Here is an application of injective modules to a computation of Ext. Let R be a PID and let K be its quotient field. The *torsion dual* (or Pontryagin dual) of an R-module M is the R-module $M^\wedge = \mathrm{Hom}_R(M, K/R)$. Note for an nonzero $a \in R$, that $R/a \cong (R/a)^\wedge$ with $[1] \mapsto ([r] \mapsto [r/a])$. It follows that if T is a finitely generated torsion R-module, then $T^\wedge \cong T$, but noncanonically.

Now suppose that T is a torsion R-module. Then by Theorem 3.6 applied to the short exact sequence $0 \to R \to K \to K/R \to 0$, one obtains an isomorphism

$$\mathrm{Ext}(T, R) \cong T^\wedge,$$

natural in T.

If M is a finitely generated R-module, then $M \cong T \oplus F$ with T torsion and F free, and $\mathrm{Ext}(M, R) \cong \mathrm{Ext}(T, R) \cong T$, but *none* of these isomorphisms need be valid when M is not finitely generated.

3.3. Resolutions

We begin with the definition of projective and injective resolutions of an R-module.

Definition 3.15.

1. A *projective resolution* of an R-module M is a sequence (possibly infinitely long)

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

   where
   (a) the sequence is exact, and
   (b) each P_i is a projective R-module.

2. An *injective resolution* of M is a sequence

$$0 \to M \to I^0 \to I^1 \to I^2 \to \cdots \to I^n \to \cdots$$

   where
   (a) the sequence is exact, and

(b) each I^n is an injective R-module.

Definition 3.16. Given a projective resolution, define the *deleted resolution* to be

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_0 \to 0.$$

We will use the notation $\mathbf{P}_\bullet$ or $\mathbf{P}_M$. Note that $H_q(\mathbf{P}_M)$ is zero for $q \neq 0$ and is isomorphic to M for $q = 0$.

Theorem 3.17. *Every R-module M has (many) projective and injective resolutions.*

Proof. Choose a surjection $P_0 \to M$ with P_0 projective. Assume by induction that you have an exact sequence

$$P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

Let $K_n = \ker d_n$. Using the previous theorem, choose a projective module P_{n+1} which surjects to K_n. Then splice

$$P_{n+1} \to K_n \to 0 \qquad \text{to} \qquad 0 \to K_n \to P_n \to \cdots \to P_0 \to M$$

to get an exact sequence

$$P_{n+1} \to P_n \to \cdots \to P_0 \to M.$$

The proof for injective resolutions is obtained by rewriting the proof for projective resolutions but turning the arrows around.

To see that projective resolutions are not unique, notice that if

$$\to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution and Q is projective, then

$$\to P_{n+1} \to P_n \oplus Q \xrightarrow{d_n \oplus Id} P_{n-1} \oplus Q \to P_{n-2} \to \cdots \to P_0 \to M$$

is also a projective resolution. $\square$

If at any stage in the above construction the kernel K_n is projective, then one may stop there since

$$\cdots \to 0 \to 0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is a projective resolution. We omit typing the 0's.

We also record the following lemma which we used in constructing resolutions.

Lemma 3.18 (splicing lemma)**.** *If the sequences $A \to B \xrightarrow{\alpha} C \to 0$ and $0 \to C \xrightarrow{\beta} D \to E$ are exact, then $A \to B \xrightarrow{\beta\circ\alpha} D \to E$ is exact.* $\square$

Exercise 49. Prove the splicing lemma.

Theorem 3.19.

1. *If R is a field and M is any R-module, then*

$$0 \to M \xrightarrow{\mathrm{Id}} M \to 0$$

is a projective resolution. In other words, every module over a field has a length 0 *projective resolution. (It stops at P_0.)*

2. *Every module over a PID has a length* 1 *projective resolution*

$$0 \to P_1 \to P_0 \to M \to 0.$$

3. *Every abelian group ($R = \mathbf{Z}$) has a length* 1 *injective resolution*

$$0 \to M \to I^0 \to I^1 \to 0.$$

Proof. 1. This is clear.

2. Every submodule of a free module over a PID is free. Thus if P_0 is a free module surjecting to M, and P_1 is its kernel,

$$0 \to P_1 \to P_0 \to M \to 0$$

is a projective (in fact free) resolution of M.

3. If $0 \to M \to D_0$ is an injection with D_0 divisible, then D_0/M is divisible, since the quotient of any divisible group is divisible. Thus $0 \to M \to D_0 \to D_0/M \to 0$ is an injective resolution. □

Comment about Commutative Algebra. A *Dedekind Domain* is a commutative domain (no zero divisors) in which every module has a projective resolution of length 1. Equivalently submodules of projective modules are projective. A PID is a Dedekind domain. From the point of view of category theory, they are perhaps more natural than PIDs. If $\zeta_n = e^{2\pi i/n}$ is a primitive n-th root of unity, then $\mathbf{Z}[\zeta_n]$ is a Dedekind domain. Projective modules (in fact ideals) which are not free first arise at $n = 23$. Nonfree ideals are what made Fermat's Last Theorem so hard to prove.

A commutative Noetherian ring R has *height* equal to n ($\mathrm{ht}(R) = n$) if the longest chain of nontrivial prime ideals in R has length n:

$$0 \subset P_1 \subset \cdots \subset P_n \subset R.$$

The *homological dimension of* R, $\mathrm{hdim}(R)$, is the least upper bound on the length of projective resolutions for all finitely generated modules over R. The homological dimension of a field is 0 and a Dedekind domain is 1. If a ring has homological dimension n, then any module M has a projective resolution with $P_k = 0$ for $k > n$. The numbers $\mathrm{ht}(R)$ and $\mathrm{hdim}(R)$ are related. For a large class of rings (regular rings) they are *equal*.

3.4. Definition of Tor and Ext - existence

In this section we will complete the proof of Theorem 3.2.

Let M, N be R-modules. Let $\cdots \to P_n \to \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} M \to 0$ be a projective resolution of M. Applying $- \otimes_R N$ to a deleted resolution $\mathbf{P}_M$, one obtains the sequence $\mathbf{P}_M \otimes_R N$

$$\mathbf{P}_M \otimes_R N = \{\cdots \to P_n \otimes_R N \to \cdots \xrightarrow{d_2 \otimes \mathrm{Id}} P_1 \otimes_R N \xrightarrow{d_1 \otimes \mathrm{Id}} P_0 \otimes_R N \to 0\}.$$

Note that $\mathbf{P}_M \otimes_R N$ is a chain complex (since $(d_{n-1} \otimes \mathrm{Id}) \circ (d_n \otimes \mathrm{Id}) = d_{n-1} \circ d_n \otimes \mathrm{Id} = 0$), and by right exactness of $- \otimes_R N$, the 0th homology is $M \otimes_R N$. However, since $- \otimes_R N$ need not be an exact functor in general, $\mathbf{P}_M \otimes_R N$ might not be exact.

Similarly, by applying the functor $\mathrm{Hom}_R(-, N)$ to the deleted projective resolution $\mathbf{P}_M$, one obtains the cochain complex

$$\mathrm{Hom}_R(\mathbf{P}_M, N) = \{0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{d_1^*} \mathrm{Hom}_R(P_1, N) \xrightarrow{d_2^*} \mathrm{Hom}_R(P_2, N) \to \cdots\}.$$

We will eventually define $\mathrm{Tor}_n^R(M, N)$ as $H_n(\mathbf{P}_M \otimes_R N)$ and we will define $\mathrm{Ext}_R^n(M, N)$ as $H^n(\mathrm{Hom}_R(\mathbf{P}_M, N))$. (We could also define Tor and Ext as $H_*(M \otimes_R \mathbf{P}_N)$ and $H^*(\mathrm{Hom}_R(M, \mathbf{I}_N))$.) For now we record some obvious facts. What is not obvious is the fact that two different resolutions will give isomorphic results.

Theorem 3.20. *Let M and N be R-modules and let $\mathbf{P}_M$ be a deleted projective resolution of M.*

1. *For $n = 0, 1, 2, \ldots$ the assignment $(M, N) \mapsto H_n(\mathbf{P}_M \otimes_R N)$ is a covariant functor in N and satisfies Axioms T1′), T2′), and T3′) of Theorem 3.6. Furthermore, if M is free (or just projective), one can choose the resolution so that axiom T3) of Theorem 3.2 is satisfied.*
2. *For $n = 0, 1, 2, \ldots$ the assignment $(M, N) \mapsto H^n(\mathrm{Hom}_R(\mathbf{P}_M, N))$ is a covariant functor in N and satisfies Axioms E1′), E2′), and E3′) of Theorem 3.6. Furthermore, if M is free (or just projective), one can choose the resolution so that axiom E3) of Theorem 3.2 is satisfied.*

Exercise 50. Prove this theorem.

3.5. The fundamental lemma of homological algebra

Taking inventory, we still need to show that our candidates for Tor and Ext are functorial in the first variable, and that short exact sequences in the first variable give long exact sequences in Tor and Ext. Functoriality will follow from the fundamental lemma of homological algebra; the long exact sequences will follow from the horseshoe lemma.

Definition 3.21. A *projective chain complex*

$$P_\bullet = \{\cdots \to P_2 \to P_1 \to P_0\}$$

is a chain complex where all the modules P_i are projective. An *acyclic chain complex*

$$C_\bullet = \{\cdots \to C_2 \to C_1 \to C_0\}$$

is a chain complex where $H_i(C_\bullet) = 0$ for all $i > 0$ (i.e. $C_\bullet$ is an exact sequence).

Theorem 3.22 (fundamental lemma of homological algebra)**.** *Let $P_\bullet$ be a projective chain complex and $C_\bullet$ be an acyclic chain complex over a ring R. Then given a homomorphism $\varphi : H_0P_\bullet \to H_0C_\bullet$, there is a chain map $f_\bullet : P_\bullet \to C_\bullet$ inducing φ on H_0. Furthermore, any two such chain maps are chain homotopic.*

We derive a few corollaries before turning to the proof.

Corollary 3.23. *Any two deleted projective resolutions of M are chain homotopy equivalent.*

Proof. Let $\mathbf{P}_M$ and $\mathbf{P}'_M$ be deleted projective resolutions of M. They are both projective and acyclic and they have $H_0 = M$. The existence part of the fundamental lemma gives chain maps $f_\bullet : \mathbf{P}_M \to \mathbf{P}'_M$ and $g_\bullet : \mathbf{P}'_M \to \mathbf{P}_M$ inducing the identity on H_0. The uniqueness part of the fundamental lemma gives a chain homotopy equivalence between $g_\bullet \circ f_\bullet$ and Id since they are both chain maps $\mathbf{P}_M \to \mathbf{P}_M$ inducing the identity map on H_0. Likewise $f_\bullet \circ g_\bullet$ is chain homotopy to Id. □

Corollary 3.24. *For every module M, choose a projective resolution P_M. The assignments $(M, N) \mapsto H_n(\mathbf{P}_M \otimes_R N)$ and $(M, N) \mapsto H^n(\mathrm{Hom}_R(\mathbf{P}_M, N))$ are functorial in both variables and satisfy the $'$ axioms of Theorem 3.6.* □

Proof. By the fundamental lemma, a module map $M \to M'$ induces a chain map $\mathbf{P}_M \to \mathbf{P}_{M'}$, unique up to chain homotopy, and hence a well-defined map $H_n(\mathbf{P}_M \otimes_R N) \to H_n(\mathbf{P}_{M'} \otimes_R N)$. It is easy to check that this is a functor, respecting composition and identities. □

Corollary 3.25. *Let M and N be modules and let $\mathbf{P}_M$ and $\mathbf{P}'_M$ be deleted projective resolutions of M. Then $H_n(\mathbf{P}_M \otimes_R N) \cong H_n(\mathbf{P}'_M \otimes_R N)$.*

Proof. There are two proofs of this fundamental result. It follows from Corollary 3.23 (uniqueness of projective resolutions up to chain homotopy equivalence) and from the axiomatic characterization of Tor and Ext (see Theorem 3.3). □

Proof of the fundamental lemma, Theorem 3.22. Let $M = H_0 P_\bullet$ and $M' = H_0 C_*$. We wish to solve the following problem (i.e. fill in the dotted arrows so that the diagram commutes).

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, dotted, "f_{n+1}"] & P_n \arrow[r, "\partial_n"] \arrow[d, dotted, "f_n"] & P_{n-1} \arrow[r] \arrow[d, dotted, "f_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, dotted, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, dotted, "f_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
\cdots \arrow[r] & C_{n+1} \arrow[r, "d_{n+1}"'] & C_n \arrow[r, "d_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "\varepsilon'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

Here the P_i are projective and the horizontal sequences are exact. We construct f_i by induction.

Step 0. The map f_0 exists since P_0 is projective:

$$\begin{tikzcd}
 & P_0 \arrow[dl, dotted, "f_0"'] \arrow[d, "\varphi\circ\varepsilon"] & \\
C_0 \arrow[r, "\varepsilon'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

Step n. Suppose we have constructed $f_0, f_1, \cdots, f_{n-1}$. The problem

$$\begin{tikzcd}
 & P_n \arrow[dl, dotted, "f_n"'] \arrow[d, "f_{n-1}\circ\partial_n"] & \\
C_n \arrow[r, "d_n"'] & \ker d_{n-1} \arrow[r] & 0
\end{tikzcd}$$

makes sense since $d_{n-1} \circ f_{n-1} \circ \partial_n = f_{n-2} \circ \partial_{n-1} \circ \partial_n = 0$. Furthermore $\ker d_{n-1} = \operatorname{im} d_n$ since $C_\bullet$ is acyclic, so the bottom map is onto. Then f_n exists since P_n is projective.

This completes the existence part of the fundamental lemma; we switch now to uniqueness up to chain homotopy. Suppose $f_\bullet$, $g_\bullet$ are two choices of chain maps which induce φ on H_0.

$$\begin{tikzcd}
{} \arrow[r] & P_n \arrow[r, "\partial_n"] \arrow[d, "f_n-g_n"'] & P_{n-1} \arrow[r] \arrow[dl, dotted, "s_{n-1}"'] \arrow[d, "f_{n-1}-g_{n-1}"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1-g_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[dl, dotted, "s_0"'] \arrow[d, "f_0-g_0"] & M \arrow[r] \arrow[d, "\varphi"] & 0 \\
{} \arrow[r] & C_n \arrow[r, "d_n"'] & C_{n-1} \arrow[r] & \cdots \arrow[r] & C_1 \arrow[r, "d_1"'] & C_0 \arrow[r, "\varepsilon'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

Here we want to define maps $s_n : P_n \to C_{n+1}$, but contrary to our usual convention, we don't want the diagram to commute, but instead we want s

to be a chain homotopy, i.e. $f_0 - g_0 = d_1 \circ s_0$ and $f_n - g_n = d_{n+1} \circ s_n + s_{n-1} \circ \partial_n$ for $n > 0$.

We will construct a *chain homotopy* by induction on n.

Step 0. Since $\varepsilon' \circ (f_0 - g_0) = (\varphi - \varphi) \circ \varepsilon = 0 : P_0 \to M'$, $\operatorname{Im}(f_0 - g_0) \subset \ker \varepsilon' : C_0 \to M' = \operatorname{im} d_1 : C_1 \to C_0$.

Then s_0 exists since P_0 is projective

$$\begin{tikzcd} & P_0 \arrow[dl, dotted, "s_0"'] \arrow[d, "f_0 - g_0"] & \\ C_1 \arrow[r, "d_1"'] & \ker \varepsilon' \arrow[r] & 0 \end{tikzcd}$$

Step n. Suppose we have defined

$$s_q : P_q \to C_{q+1} \quad \text{for} \quad q = 0, \cdots, n-1$$

satisfying $f_q - g_q = d_{q+1} s_q + s_{q-1} \partial_q$ for each $q = 0, \cdots, n-1 \quad (s_{-1} = 0)$. Then the problem

$$\begin{tikzcd} & P_n \arrow[dl, dotted, "s_n"'] \arrow[d, "f_n - g_n - s_{n-1}\partial_n"] & \\ C_{n+1} \arrow[r, "d_{n+1}"'] & \ker d_n \arrow[r] & 0 \end{tikzcd}$$

makes sense, since

$$\begin{aligned} d_n(f_n - g_n - s_{n-1}\partial_n) &= (f_{n-1} - g_{n-1})\partial_n - d_n s_{n-1} \partial_n \\ &= (d_n s_{n-1} + s_{n-2}\partial_{n-1})\partial_n - d_n s_{n-1} \partial_n \\ &= s_{n-2}\partial_{n-1}\partial_n = 0. \end{aligned}$$

Therefore $\operatorname{im}(f_n - g_n - s_{n-1}\partial_n) \subset \ker d_n = \operatorname{im}(d_{n+1} : C_{n+1} \to C_n)$. Thus $d_{n+1} s_n = f_n - g_n - s_{n-1}\partial_n$, proving the induction step.

This finishes the proof of the fundamental lemma. □

To show that our functors satisfy the remaining axioms, we need the following lemma.

Lemma 3.26 (horseshoe lemma)**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Let $\mathbf{P}_A$ and $\mathbf{P}_C$ be deleted projective resolutions of A and C. Then there exists a deleted projective resolution $\mathbf{P}_B$ of B, fitting into a short exact sequence of chain complexes $0 \to \mathbf{P}_A \to \mathbf{P}_B \to \mathbf{P}_C \to 0$ which induces the original sequence on H_0.*

Proof. We are given the following "horseshoe" diagram

$$\begin{tikzcd}
 & & & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_n \arrow[r, "\partial"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial"] & P_0 \arrow[r, "\varepsilon"] & A \arrow[r] \arrow[d] & 0 \\
 & & & & & B \arrow[d] & \\
\cdots \arrow[r] & R_n \arrow[r, "d"] & \cdots \arrow[r] & R_1 \arrow[r, "d"] & R_0 \arrow[r, "\tau"] & C \arrow[r] \arrow[d] & 0 \\
 & & & & & 0 &
\end{tikzcd}$$

where the horizontal rows are projective resolutions. We want to add a middle row of projective modules to obtain a commutative diagram with exact rows and short exact columns. Since R_n is projective, the columns will split, and so $Q_n = P_n \oplus R_n$ must go in the n-th slot in the middle. Furthermore we may assume that the maps $P_n \to Q_n$ and $Q_n \to R_n$ are the inclusion and projection maps, but the horizontal maps are yet unclear.

Step 0. The problem

$$\begin{tikzcd}
 & B \arrow[d] \\
R_0 \arrow[ur, dotted, "\Phi"] \arrow[r, "\tau"'] & C \arrow[d] \\
 & 0
\end{tikzcd}$$

has a solution Φ since R_0 is projective. Let $\gamma : Q_0 \to B$ be $\gamma(p, r) = i\varepsilon(p) + \Phi(r)$ where $(p, r) \in Q_0 = P_0 \oplus R_0$ and $i : A \to B$. A diagram chase shows γ is *onto*. Thus we have the commutative diagram

$$
\begin{tikzcd}
 & & & & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_n \arrow[r, "\partial"] & \cdots \arrow[r] & P_1 \arrow[r, "\partial"] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & & & & Q_0 \arrow[r, "\gamma"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & R_n \arrow[r, "d"] & \cdots \arrow[r] & R_1 \arrow[r, "d"] & R_0 \arrow[r, "\tau"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & & & & 0 & 0 &
\end{tikzcd}
$$

Step $n+1$*.* Suppose inductively we have constructed the following commutative diagram with exact rows and columns.

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & P_n \arrow[r, "\partial"] \arrow[d] & \cdots \arrow[r] & P_1 \arrow[r, "\partial"] \arrow[d] & P_0 \arrow[r, "\varepsilon"] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & Q_n \arrow[r] \arrow[d] & \cdots \arrow[r] & Q_1 \arrow[r] \arrow[d] & Q_0 \arrow[r, "\gamma"] \arrow[d] & B \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & R_n \arrow[r, "d"] \arrow[d] & \cdots \arrow[r] & R_1 \arrow[r, "d"] \arrow[d] & R_0 \arrow[r, "\tau"] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
 & 0 & & 0 & 0 & 0 &
\end{tikzcd}
$$

Let $K_n = \ker(P_n \to P_{n-1})$, $L_n = \ker(Q_n \to Q_{n-1})$ and $M_n = \ker(R_n \to R_{n-1})$. We then have the diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
\cdots \arrow[r] & P_{n+2} \arrow[r, "\partial"] & P_{n+1} \arrow[r, "\partial"] & K_n \arrow[r] \arrow[d] & 0 \\
 & & & L_n \arrow[d] & \\
\cdots \arrow[r] & R_{n+2} \arrow[r, "d"] & R_{n+1} \arrow[r, "d"] & M_n \arrow[r] \arrow[d] & 0 \\
 & & & 0 &
\end{tikzcd}$$

By Step 0 we can fill in the next column and horizontal arrow. We then splice this diagram with the previous one to obtain the inductive step. $\square$

It is important to notice that the short exact sequence $0 \to \mathbf{P}_A \to \mathbf{P}_B \to \mathbf{P}_C \to 0$ is not (necessarily) a split short exact sequence of chain complexes, even though each chain module is projective. (What might a projective object in the category of chain complexes be?)

Corollary 3.27. *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules and let $0 \to \mathbf{P}_A \to \mathbf{P}_B \to \mathbf{P}_C \to 0$ be a short exact sequence of deleted projective resolutions provided by the horseshoe lemma. Let N be an R-module. Then there are long exact sequences:*

$$\cdots \to H_{n+1}(\mathbf{P}_C \otimes_R N) \to H_n(\mathbf{P}_A \otimes_R N) \to H_n(\mathbf{P}_B \otimes_R N) \to H_n(\mathbf{P}_C \otimes_R N) \to \cdots$$

and

$$\cdots \to H^n(\mathrm{Hom}_R(\mathbf{P}_C, N)) \to H^n(\mathrm{Hom}_R(\mathbf{P}_B, N)) \to$$
$$H^n(\mathrm{Hom}_R(\mathbf{P}_A, N)) \to H^{n+1}(\mathrm{Hom}_R(\mathbf{P}_C, N)) \to \cdots$$

Proof. Since we have a short exact sequence of deleted projective resolutions, in degree n, the short exact sequence

$$0 \to (\mathbf{P}_A)_n \to (\mathbf{P}_B)_n \to (\mathbf{P}_C)_n \to 0$$

is split; hence

$$0 \to (\mathbf{P}_A \otimes_R N)_n \to (\mathbf{P}_B \otimes_R N)_n \to (\mathbf{P}_C \otimes_R N)_n \to 0$$

is split and hence exact. Thus

$$0 \to \mathbf{P}_A \otimes_R N \to \mathbf{P}_B \otimes_R N \to \mathbf{P}_C \otimes_R N \to 0$$

is a short exact sequence of chain complexes; the zig-zag lemma gives the long exact sequence in homology above. We leave the cohomology proof as an exercise. □

We can finally safely make the following definition.

Definition 3.28.

1. $\mathrm{Tor}_n^R(M,N) = H_n(\mathbf{P}_M \otimes_R N)$.
2. $\mathrm{Ext}_R^n(M,N) = H^n(\mathrm{Hom}_R(\mathbf{P}_M, N))$.

With these definitions, the existence theorem, Theorem 3.2, and the primed version, Theorem 3.6, follow from Corollaries 3.24 and 3.27. As a consequence we can now deduce Corollary 3.7 which states that $\mathrm{Tor}_n^R(M,N)$ and $Tor_n^R(N,M)$ are naturally isomorphic.

We have not proven that $\mathrm{Ext}_R^n(M,N) = H^n(\mathrm{Hom}_R(M, \mathbf{I}_N))$. This follows by using injective versions of the fundamental lemma and the horseshoe lemma to show that the axioms are also satisfied here. For these facts, see any book on homological algebra, or, better, prove it yourself. Once we have defined tensor products and Hom for chain complexes, one can show $\mathrm{Tor}_n^R(M,N) = H_n(\mathbf{P}_M \otimes_R \mathbf{P}_N)$ and $\mathrm{Ext}_R^n(M,N) = \mathrm{Hom}_R(\mathbf{P}_M, \mathbf{I}_N))$; this is an intermediate way between resolving on the left and the right.

Earlier in this chapter you were asked in an exercise to compute $\mathrm{Tor}(A,B)$ and $\mathrm{Ext}(A,B)$ for finitely generated abelian groups. Lest you learn all the theory without any examples, we give a way of stating the result. Let $\mathrm{torsion}(A)$ denote the subgroup of A consisting of elements of finite order. Then

$$\mathrm{Tor}(A,B) \cong \mathrm{torsion}(A) \otimes_{\mathbf{Z}} \mathrm{torsion}(B)$$

and

$$\mathrm{Ext}(A,B) \cong \mathrm{torsion}(A) \otimes_{\mathbf{Z}} B,$$

but these isomorphisms are not natural in A and B. These computations are not valid when A or B is infinitely generated, for example one can show that $\mathrm{Ext}(\mathbf{Q},\mathbf{Z})$ is an uncountable abelian group, in fact it is isomorphic to the product over all primes p of the p-adic integers modulo the diagonal embedding of the ordinary integers. But $\mathrm{Ext}(\mathbf{Z},\mathbf{Q})$ is zero since $\mathbf{Z}$ is free.

Exercise 51. For any commutative ring R show that

$$\mathrm{Ext}_R^q(A \oplus B, M) \cong \mathrm{Ext}_R^q(A,M) \oplus \mathrm{Ext}_R^q(B,M)$$

and

$$\mathrm{Tor}_q^R(A \oplus B, M) \cong \mathrm{Tor}_q^R(A,M) \oplus \mathrm{Tor}_q^R(B,M).$$

We end with the famous exercise from Lang's *Algebra*, Chapter IV ([**26**]):

> Take any book on homological algebra, and prove all the theorems without looking at the proofs given in that book.
>
> Homological algebra was invented by Eilenberg-MacLane. General category theory (i.e. the theory of arrow-theoretic results) is generally known as *abstract nonsense* (the terminology is due to Steenrod).

3.6. Universal coefficient theorems

The universal coefficient theorems show that H_*X (homology with integer coefficients) is universal in the sense that $H_*(X;M)$ and $H^*(X;M)$ (homology and cohomology with coefficients in an abelian group M) are given by formulas depending only on H_*X. Then why bother with cohomology and homology with coefficients? We will see later that cohomology is a more powerful invariant than homology since cohomology admits a ring structure. Also we will see that homology and cohomology is simplified with coefficients in a field.

In this section we will omit the R from the symbols Hom_R, $\otimes_R$, Tor^R, and Ext_R.

Let $C_\bullet = (C_*, \partial)$ be a chain complex over a ring R and let M be an R-module. There is an evaluation map

$$\begin{gathered}\mathrm{Hom}(C_n, M) \times C_n \to M\\ (f, z) \mapsto f(z).\end{gathered}$$

You have already come across this pairing in Exercise 32 and have shown that this pairing passes to the *Kronecker* pairing

$$\langle\ ,\ \rangle : H^n(\mathrm{Hom}(C_\bullet, M)) \times H_n(C_\bullet) \to M$$

of cohomology with homology. This pairing is bilinear, and its adjoint is a homomorphism

$$H^n(\mathrm{Hom}(C_\bullet, M)) \to \mathrm{Hom}(H_n(C_\bullet), M).$$

The example following Exercise 32 shows that this adjoint need not be an isomorphism. To understand the kernel and cokernel of this map is a subtle question. Universal coefficient theorems among other things provide a measure of how this adjoint fails to be an isomorphism in terms of the derived functors Ext^n and Tor_n. The answer can be quite difficult for general commutative rings and arbitrary chain complexes.

We will answer the question completely when R is a PID and $C_\bullet$ is a free chain complex. In this case $H^n(\mathrm{Hom}(C_\bullet, M)) \to \mathrm{Hom}(H_n(C_\bullet), M)$ is surjective with kernel $\mathrm{Ext}(H_{n-1}(C_\bullet), M)$. This will cover the topological situation in the most important cases of coefficients in the integers or in a field, since the singular and cellular complexes of a space are free.

Theorem 3.29 (universal coefficient theorem for cohomology)**.** *Let R be a principal ideal domain. Suppose that M is a module over R, and $C_\bullet = (C_*, \partial)$ is a free chain complex over R (i.e. each C_n is a free R-module).*

For each n, there is an exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(C_\bullet), M) \to H^n(\mathrm{Hom}(C_\bullet, M)) \to \mathrm{Hom}(H_n(C_\bullet), M) \to 0,$$

natural with respect to chain maps of free chain complexes. Moreover, the sequence splits, but not naturally.

We will give a proof of this based on the concept of an exact triangle.

Definition 3.30. An *exact triangle of R-modules* is a diagram of R-modules

$$\begin{tikzcd} A \arrow[rr, "\alpha"] & & B \arrow[dl, "\beta"] \\ & C \arrow[ul, "\gamma"] & \end{tikzcd}$$

satisfying $\ker(\beta) = \mathrm{im}(\alpha)$, $\ker(\gamma) = \mathrm{im}(\beta)$, and $\ker(\alpha) = \mathrm{im}(\gamma)$.

Similarly one defines an exact triangle of *graded* R-modules A_*, B_*, C_* (see Definition 4.1). In this case we require the homomorphisms α, β, and γ each to have a *degree*; so for example if α has degree 2, then $\alpha(A_n) \subset B_{n+2}$.

The basic example of an exact triangle of graded R-modules is the long exact sequence in homology

$$\begin{tikzcd} H_*A \arrow[rr, "i_*"] & & H_*X \arrow[dl, "j_*"] \\ & H_*(X,A) \arrow[ul, "\partial"] & \end{tikzcd}$$

For this exact triangle i_* and j_* have degree 0, and ∂ has degree -1.

Exercise 52. Suppose that

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "j"] & A \arrow[rr, "\alpha"] & & B \arrow[dl, "\beta"] \arrow[r, "k"] & F \arrow[r] & 0 \\ & & & C \arrow[ul, "\gamma"] & & & \end{tikzcd}$$

is a diagram with the top row exact and the triangle exact. Prove that there is a short exact sequence

$$0 \longrightarrow F \xrightarrow{\beta \circ k^{-1}} C \xrightarrow{j^{-1} \circ \gamma} E \longrightarrow 0.$$

State and prove the graded version of this exercise.

Proof of Theorem 3.29. There is a short exact sequence of graded, free R-modules

$$0 \to Z_* \xrightarrow{i} C_* \xrightarrow{\partial} B_* \to 0 \tag{3.1}$$

where Z_n denotes the n-cycles and B_n denotes the n-boundaries. The homomorphism i has degree 0 and ∂ has degree -1. This sequence is in fact a short exact sequence of chain complexes where Z_* and B_* are given the *zero* differential.

Since the sequence (3.1) is an exact sequence of free chain complexes, applying the functor $\mathrm{Hom}(-, M)$ gives a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}(B_*, M) \xrightarrow{\partial^*} \mathrm{Hom}(C_\bullet, M) \xrightarrow{i^*} \mathrm{Hom}(Z_*, M) \to 0.$$

Applying the zig-zag lemma we obtain a long exact sequence (i.e. exact triangle) in cohomology, which, since the differentials for the complexes $\mathrm{Hom}(B_*, M)$ and $\mathrm{Hom}(Z_*, M)$ are zero, gives the exact triangle

$$\begin{tikzcd}
\mathrm{Hom}(Z_*, M) \arrow[rr, "\delta"] & & \mathrm{Hom}(B_*, M) \arrow[dl, "\partial^*"] \\
& H^*(\mathrm{Hom}(C_\bullet, M)) \arrow[ul, "i^*"] &
\end{tikzcd} \tag{3.2}$$

There is also a short exact sequence of graded R-modules

$$0 \to B_* \xrightarrow{j} Z_* \to H_* \to 0 \tag{3.3}$$

coming from the definition of homology, that is

$$Z_* = \ker \partial : C_* \to C_*,$$

$$B_* = \operatorname{im} \partial : C_* \to C_*,$$

and

$$H_* = H_*(C_\bullet) = Z_*/B_*.$$

Notice that in the sequence (3.3), B_* and Z_* are free, since R is a PID and these are submodules of the free module C_*. Thus using axiom E2) of Theorem 3.2 and using the fact that $\mathrm{Ext}(Z_*, M) = 0$, we obtain an exact sequence

$$0 \to \mathrm{Hom}(H_*, M) \to \mathrm{Hom}(Z_*, M) \xrightarrow{j^*} \mathrm{Hom}(B_*, M) \to \mathrm{Ext}(H_*, M) \to 0. \tag{3.4}$$

Exercise 53. Complete the proof of the universal coefficient theorem as follows.

1. Show, taking special care with the grading, that the homomorphism δ of the exact triangle (3.2) coincides with the homomorphism j^* of (3.4). Thus there is a commutative diagram

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & \operatorname{Hom}(H_*, M) \arrow[r] & \operatorname{Hom}(Z_*, M) \arrow[rr, "j^*"] & & \operatorname{Hom}(B_*, M) \arrow[r] \arrow[dl, "\partial^*"] & \operatorname{Ext}(H_*, M) \arrow[r] & 0 \\
& & & H^*(\operatorname{Hom}(C_\bullet, M)) \arrow[ul, "i^*"] & & &
\end{tikzcd}$$

obtained by putting together (3.2) and (3.4).

2. Apply Exercise 52 to obtain a split short exact sequence of graded R-modules

$$0 \to \operatorname{Ext}(H_*, M) \to H^*(\operatorname{Hom}(C_\bullet, M)) \to \operatorname{Hom}(H_*, M) \to 0.$$

Verify that the map $H^*(\operatorname{Hom}(C_\bullet, M)) \to \operatorname{Hom}(H_*, M)$ is induced by evaluating a cochain on a cycle. The splitting of this map is obtained by splitting the inclusion $i : Z_* \to C_*$ by $C_* \to Z_*$, passing to the chain map

$$C_\bullet = (C_*, \partial) \to (H_*(C_\bullet), 0),$$

applying $\operatorname{Hom}(-, M)$, and taking cohomology.

3. By taking the grading into account, finish the proof of Theorem 3.29.

Corollary 3.31. *If R is a field, M is a vector space over R, and $C_\bullet$ is a chain complex over R, then*

$$H^n(\operatorname{Hom}(C_\bullet, M)) \cong \operatorname{Hom}(H_n(C_\bullet), M).$$

Moreover the Kronecker pairing is nondegenerate. □

Applying the universal coefficient theorem to the singular or cellular complexes of a space or a pair of spaces, one obtains the following.

Corollary 3.32. *Let (X, A) be a pair of spaces $A \subset X$, R a PID, and M a module over R. For each n, there is a natural exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}(X, A; R), M) \to H^n(X, A; M) \to \operatorname{Hom}(H_n(X, A; R), M) \to 0$$

which splits, but not naturally. □

Exercise 54. Let $f : \mathbf{R}P^2 \to S^2$ be the map pinching the 1-skeleton to a point. Compute the induced map on $\mathbf{Z}$ and $\mathbf{Z}/2$ cohomology to show the splitting is not natural.

The most important special case of the universal coefficient theorem for cohomology is its use in the computation of $H^nX = H^n(X; \mathbf{Z})$ (cohomology with integer coefficients). For an abelian group A, denote the torsion subgroup (i.e. the subgroup of finite order elements) by torsion(A). Let free$(A) = A/\text{torsion}(A)$. Then for a space X whose homology is finitely generated in every dimension (e.g. a finite CW-complex), the universal coefficient theorem shows that

$$H^nX \cong \text{free}(H_nX) \oplus \text{torsion}(H_{n-1}X).$$

Another formulation is to define the dual of an abelian group A by $A^* = \text{Hom}(A, \mathbf{Z})$ and the torsion dual $A^\frown \cong \text{Hom}(A, \mathbf{Q}/\mathbf{Z})$. The universal coefficient theorem then says that

$$H^nX \cong H_n(X)^* \oplus (\text{torsion}(H_{n-1}X))^\frown.$$

The right hand side is then a contravariant functor in X, as it should be, but the isomorphism is still not natural.

There is also a universal coefficient theorem for homology, and we turn to it now. First note that for a chain complex $C_\bullet$ and a module M over a ring R, there is a homomorphism

$$H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M).$$

This is not an isomorphism in general (unless R is a field), but for a R a PID, the theorem below gives the computation of $H_n(C_\bullet \otimes M)$.

Theorem 3.33 (universal coefficient theorem for homology)**.** *Suppose that R is a PID, $C_\bullet$ a free chain complex over R, and M a module over R. Then there is a natural short exact sequence.*

$$0 \to H_n(C_\bullet) \otimes M \to H_n(C_\bullet \otimes M) \to \text{Tor}(H_{n-1}(C_\bullet), M) \to 0$$

which splits, but not naturally.

Sketch of Proof. The proof is similar to the proof given above of Theorem 3.29. As before, there is a short exact sequence of chain complexes

$$0 \to Z_* \to C_\bullet \to B_* \to 0$$

which remains exact after tensoring with M, since B_* is free.

Applying the zig-zag lemma to the tensored sequence, one obtains the exact triangle

$$\begin{tikzcd} B_* \otimes M \arrow[rr] & & Z_* \otimes M \arrow[dl] \\ & H_*(C_\bullet \otimes M) \arrow[ul] & \end{tikzcd} \tag{3.5}$$

The short exact sequence of graded R-modules

$$0 \to B_* \to Z_* \to H_*(C_\bullet) \to 0$$

gives, using axiom T2) of Theorem 3.2, an exact sequence

$$0 \to \operatorname{Tor}(H_*(C_\bullet), M) \to B_* \otimes M \to Z_* \otimes M \to H_*(C_\bullet) \otimes M \to 0. \tag{3.6}$$

Assembling the triangle (3.5) and the sequence (3.6) as in Exercise 52, one obtains the short exact sequence

$$0 \to H_*(C_\bullet) \otimes M \to H_*(C_\bullet \otimes M) \to \operatorname{Tor}(H_*(C_\bullet), M) \to 0.$$

Taking the grading into account finishes the proof. □

Corollary 3.34. *If (X, A) is a pair of spaces $A \subset X$, R a PID, M a module over R, then for each n the sequence*

$$0 \to H_n(X, A; R) \otimes M \to H_n(X, A; M) \to \operatorname{Tor}(H_{n-1}(X, A; R), M) \to 0$$

is short exact, natural, and splits, but not naturally. □

The above universal coefficient theorems computed homology and cohomology with module coefficients in terms of homology with ring coefficients. In that sense homology is universal. However, there may be situations where one starts with cohomology with ring coefficients. We will state some universal coefficient theorems starting with cohomology, but they will require additional hypothesis. The algebraic reason is that for every chain complex $C_\bullet$ and every module M, there are maps of cochain and chain complexes, respectively

$$\operatorname{Hom}(C_\bullet, R) \otimes M \to \operatorname{Hom}(C_\bullet, M)$$
$$C_\bullet \otimes M \to \operatorname{Hom}(\operatorname{Hom}(C_\bullet, R), M)$$

which may not be isomorphisms. Try and figure out what these maps are and their relationship to the maps in two theorems below.

Theorem 3.35. *Suppose R is a PID, $C_\bullet$ is a free chain complex over R, and either M is a finitely generated R-module or $H_n(C_\bullet)$ is finitely generated for all n. Let $C^\bullet$ be the cochain complex* $\operatorname{Hom}(C_\bullet, R)$. *Then there is a natural exact sequence*

$$0 \to H^n(C^\bullet) \otimes M \to H^n(\operatorname{Hom}(C_\bullet, M)) \to \operatorname{Tor}(H^{n+1}(C^\bullet), M) \to 0$$

which splits, but not naturally.

See [**45**, pg. 246] for a proof.

Theorem 3.36. *Let R be a PID, let $C_\bullet$ be a free chain complex over R such that $H_n(C_\bullet)$ is finitely generated for each n, and let M be an R-module. Let $C^\bullet$ be the cochain complex* $\mathrm{Hom}(C_\bullet, R)$. *Then there is a natural exact sequence*

$$0 \to \mathrm{Ext}(H^{n+1}(C^\bullet), M) \to H_n(C_\bullet \otimes M) \to \mathrm{Hom}(H^n(C^\bullet), M) \to 0$$

which splits, but not naturally.

See [**45**, pg. 248] for a proof.

In particular for a finite CW complex X, one has $H_nX \cong H^nX^* \oplus H^{n+1}X^\wedge$.

Example. Since $H_n(\mathbf{R}P^2; \mathbf{Z}) = \mathbf{Z}, \mathbf{Z}/2, 0, \dots$ for $n = 0, 1, 2, \dots$, by the universal coefficient theorem $H_n(\mathbf{R}P^2; \mathbf{Z}/2) = \mathbf{Z}/2, \mathbf{Z}/2, \mathbf{Z}/2, 0, \dots$ for $n = 0, 1, 2, 3, \dots$ and $H^n(\mathbf{R}P^2; \mathbf{Z}) = \mathbf{Z}, 0, \mathbf{Z}/2, 0, \dots$ for $n = 0, 1, 2, 3, \dots$ What is the geometric meaning of the torsion? Let α be a cycle representing the generator of $H_1(\mathbf{R}P^2; \mathbf{Z})$, i.e. α is a "half-equator". Then $2\alpha = \partial\beta$. The generators of $H_1(\mathbf{R}P^2; \mathbf{Z}/2)$ and $H_2(\mathbf{R}P^2; \mathbf{Z}/2)$ are represented by $\alpha \otimes 1$ and $\beta \otimes 1$ respectively. A representative of the generator of $H^2(\mathbf{R}P^2; \mathbf{Z})$ is represented by a cocycle ω where $\omega(\beta) = 1$.

3.7. Flat modules

Flat modules are quite common; for example $\mathbf{Q}$ is a flat $\mathbf{Z}$-module. Nonetheless, feel free to skip this section on a first reading.

Let R be a commutative ring. Throughout this section we will abbreviate and write $\otimes$ instead of $\otimes_R$.

Definition 3.37. An R-module M is *flat* if $- \otimes M : R\text{-Mod} \to R\text{-Mod}$ is an exact functor.

In other words, if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is too. Equivalently, if $A \to B \to C$ is exact at B, then $A \otimes M \to B \otimes M \to C \otimes M$ is exact at $B \otimes M$.

A free module is flat, and clearly a summand of a flat module is flat, so projectives are flat.

For any chain complex $C_\bullet$ over R and any R-module M, there is a homomorphism

$$H_*(C_\bullet) \otimes M \to H_*(C_\bullet \otimes M)$$
$$[z] \otimes m \mapsto [z \otimes m]$$

If M is flat, this is an isomorphism. In other words, $- \otimes M$ is not just exact, it preserves homology.

Lemma 3.38. *If $C_\bullet$ is a chain complex over R and M is a flat R-module, then $H_*(C_\bullet) \otimes M \to H_*(C_\bullet \otimes M)$ is an isomorphism.*

Proof. Let $Z_n = \ker(\partial_n : C_n \to C_{n-1})$ be the n-cycles, $B_n = \operatorname{im}(\partial_{n+1} : C_{n+1} \to C_n)$ be the n-boundaries, and $H_n = H_n(C_\bullet)$ be the homology. Tensor the fundamental exact sequences of homology

$$0 \to Z_n \to C_n \to B_{n-1} \to 0$$
$$0 \to B_n \to Z_n \to H_n \to 0$$

to obtain the short exact sequences

$$0 \to Z_n \otimes M \to C_n \otimes M \to B_{n-1} \otimes M \to 0$$
$$0 \to B_n \otimes M \to Z_n \otimes M \to H_n \otimes M \to 0$$

Then, using these exact sequences one sees that

$$H_n(C_\bullet \otimes M) = \frac{\ker(\partial_n \otimes \mathrm{Id}_M)}{\operatorname{im}(\partial_{n+1} \otimes \mathrm{Id}_M)} \cong \frac{Z_n \otimes M}{B_n \otimes M} \cong H_n \otimes M$$

□

Corollary 3.39. *Let M be an R-module. The following are equivalent*

1. *M is flat.*
2. $\operatorname{Tor}_n^R(A, M) = 0$ *for all $n > 0$ and for all R-modules A.*
3. $\operatorname{Tor}_1^R(A, M) = 0$ *for all R-modules A.*

Proof. Assume M is flat. Then $\operatorname{Tor}_n(A, M) = H_n(\mathbf{P}_A \otimes M) = H_n(\mathbf{P}_A) \otimes M = 0$ for $n > 0$.

Clearly if $\operatorname{Tor}_n^R(A, M) = 0$ for all $n > 0$ then $\operatorname{Tor}_1^R(A, M) = 0$.

If $0 \to A \to B \to C \to 0$ is short exact and if $\operatorname{Tor}_1^R(C, M) = 0$, then the axioms for Tor show that $0 \to A \otimes M \to B \otimes M \to C \otimes M \to 0$ is short exact. □

We wish to show that $\mathbf{Q}$ is a flat $\mathbf{Z}$-module, or, more generally, that a quotient field of a commutative domain is a flat module. An easy route is to use localization.

A *multiplicative subset* S of a commutative ring R is a subset $S \subset R$ with $1 \in S$ and which is closed under multiplication: $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$. Define an equivalence relation on $R \times S$ by $(r_1, s_1) \sim (r_2, s_2)$ if there is an $s \in S$ so that $r_1 s_2 s = r_2 s_1 s$. (If S consists of nonzero divisors, then one can omit the s). Let $S^{-1}R = R \times S/\sim$. Write an equivalence class as r/s instead of $[r, s]$. Then $S^{-1}R$ is a commutative ring using the usual rules for addition and multiplication of fractions. The map $R \to S^{-1}R, \quad r \mapsto r/1$ is a ring homomorphism which maps S to units. In fact this ring homomorphism is initial with respect to this property and hence this characterizes the localization. The ring $S^{-1}R$ (or sometimes the homomorphism $R \to S^{-1}R$) is called the *localization of R with respect to S.*

The most important example of a localization is the quotient field K of a domain R. Here $S = R - 0$ and $S^{-1}R = K$. Another example is given by $S = \{n \in \mathbf{Z} \mid (p,n) = 1\}$ where p is a prime number. Then $S^{-1}\mathbf{Z} = \mathbf{Z}_{(p)} = \{a/n \mid a, n \in \mathbf{Z}, (p,n) = 1\}$. This ring is called "$\mathbf{Z}$ localized at p."

Exercise 55. Characterize all multiplicative subsets S of $\mathbf{Z}$ and the corresponding subrings $S^{-1}\mathbf{Z} \subset \mathbf{Q}$.

If S is a multiplicative subset of a commutative ring R and M is an R-module, then define $S^{-1}M = M \times S/\!\sim$ where $(m_1, s_1) \sim (m_2, s_2)$ if there is an $s \in S$ so that $s_2 s m_1 = s_1 s m_2$. Then $S^{-1}M$ is an $S^{-1}R$-module, in fact, $S^{-1} : R\text{-Mod} \to S^{-1}R\text{-Mod}$ is a functor.

Here is our motivation for introducing localization.

Theorem 3.40. *Let S be a multiplicative subset of a commutative ring R.*

1. *If A is an R-module, then $S^{-1}A$ and $S^{-1}R \otimes A$ are isomorphic $S^{-1}R$-modules.*
2. *Let $A \to B \to C$ be a sequence of R-modules exact at B. Then $S^{-1}A \to S^{-1}B \to S^{-1}C$ is exact at $S^{-1}B$. Thus S^{-1} is an exact functor.*
3. *$S^{-1}R$ is a flat R-module.*

Proof. 1. Define inverse maps

$$\begin{aligned} S^{-1}A &\to S^{-1}R \otimes A \\ a/s &\mapsto 1/s \otimes a \\ S^{-1}R \otimes A &\to S^{-1}A \\ r/s \otimes a &\mapsto ra/s \end{aligned}$$

2. Let $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ be exact. Clearly $S^{-1}\beta \circ S^{-1}\alpha = 0$. To show that $\ker S^{-1}\beta \subset \operatorname{im} S^{-1}\alpha$, note that $(S^{-1}\beta)(b/s) = 0$ implies there exists $s' \in S$ such that $0 = s'\beta(b) = \beta(s'b)$ which implies that $s'b = \alpha(a)$ for some $a \in A$, and so finally $(S^{-1}\alpha)(a/s's) = b/s$.

3. This is a consequence of Parts 1 and 2.

□

Exercise 56. Show that for an abelian group A, $\operatorname{rank} A = \dim_{\mathbf{Q}} A \otimes_{\mathbf{Z}} \mathbf{Q}$.

Thus the Euler characteristic could alternately be defined as $\chi(X) = \sum(-1)^n \dim_{\mathbf{Q}} H_n(X;\mathbf{Q})$.

Note that $\mathbf{R}$ is flat over $\mathbf{Q}$ since it is a free module, and that $\mathbf{R}$ is flat over $\mathbf{Z}$ since $A \otimes_{\mathbf{Z}} \mathbf{R} = (A \otimes_{\mathbf{Z}} \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{R}$. Also $\mathbf{Q}$ and $\mathbf{R}$ are injective $\mathbf{Z}$-modules. We can now identify, using the universal coefficient theorem, the various versions of the Betti numbers of a space.

Corollary 3.41. *If $\operatorname{rank} H_nX$ is finite, then the following numbers are all equal: $\operatorname{rank} H_nX$, $\dim_{\mathbf{Q}} H_n(X;\mathbf{Q})$, $\dim_{\mathbf{R}} H_n(X;\mathbf{R})$, $\dim_{\mathbf{Q}} H^n(X;\mathbf{Q})$, and $\dim_{\mathbf{R}} H^n(X;\mathbf{R})$.*

In particular if X is a compact smooth manifold, by the above corollary and de Rham cohomology, we see the n-th Betti number is the dimension of the real vector space of closed n-forms modulo exact n-forms.

Exercise 57. Let X be a space of finite rank. Let F be a field. Suppose that $\sum_n \dim_F H_n(X;F) < \infty$. Show that the Euler characteristic satisfies

$$\chi(X) = \sum(-1)^n \dim_F H_n(X;F).$$

Find a space of finite rank for which $\dim_{\mathbf{Z}/2} H_1(X;\mathbf{Z}/2) = \infty$.

There is an alternate proof that $\mathbf{Q}$ is flat (or, more generally that a quotient field of a domain is flat) that uses that tensor products commute with filtered colimits and that $\mathbf{Q}$ is a filtered colimit of the free modules $(1/n)\mathbf{Z}$. This proof has the advantage that it shows that a torsion-free abelian group is flat because it is the union of finitely generated torsion-free groups which are flat.

Tor can be computed using a flat resolution rather than a projective one. Assume this and compute $\operatorname{Tor}(\mathbf{Q}/\mathbf{Z}, A) = H_1(P_{\mathbf{Q}/\mathbf{Z}} \otimes A)$ for any abelian group A.

3.8. Projects: Acyclic models and the Eilenberg-Zilber map

3.8.1. The acyclic models theorem and the Eilenberg-Zilber map.
First state the acyclic models theorem very carefully.

Theorem 3.42 (acyclic models theorem)**.** *Suppose that $\mathcal{C}$ is a category with models $\mathcal{M} \subset \operatorname{Ob}\mathcal{C}$. Let $F, F' : \mathcal{C} \to \operatorname{Ch}^+_R$ be functors so that F is free on the models and F' is acyclic on the models. Then any natural transformation $H_0(F) \Rightarrow H_0(F')$ is induced by a natural transformation $T : F \Rightarrow F'$. Furthermore, any two natural transformations $S, T : F \Rightarrow F'$ inducing the same natural transformation on H_0 are naturally chain homotopic.*

In particular, if both F and F' are free and acyclic and if $H_0(F)$ and $H_0(F')$ are naturally isomorphic, then F and F' are naturally chain homotopy equivalent. □

Here Ch_R^+ is the category of chain complexes $C_\bullet$ over R so that $C_n = 0$ for $n < 0$. To say F' is acyclic on the models means that $H_n(F'(M)) = 0$ for $n > 0$ and $M \in \mathcal{M}$. To say that F is free on the models means that for every $n \geq 0$, the functor $F(-)_n : \mathcal{C} \to R$-Mod is free with basis in $\mathcal{M}$, that is, there is an indexed set $\{b_j \in F_n(M_j)\}_{j \in J}$ where $M_j \in \mathcal{M}$ such that for every $X \in \mathrm{Ob}\,\mathcal{C}$, $F_n(X)$ is a free R-module with basis $\{F_n(u)(b_j) \mid u \in \mathcal{C}(M_j, X), j \in J\}$. For example, if $\mathcal{C} = \mathrm{Top}$, then the singular chain functor $S_\bullet(-)$ is both free and acyclic on the models $\{\Delta^n \mid n = 0, 1, 2, \dots\}$, with the basis in degree n consisting of the singular n-simplex in $S_n(\Delta^n)$ given by the identity map.

A natural transformation $T : F \Rightarrow F'$ is a chain map $T_X : F(X) \to F'(X)$ for each object X which is natural in X, that is, for any morphism $f : X \to Y$, $F'(f) \circ T_X = T_Y \circ F(f)$. Two such natural transformations S and T are naturally chain homotopic if there are R-module homomorphisms $H_X : F(X)_* \to F(X)_{*+1}$, natural in X, so that $S_X - T_X = \partial H_X - H_X \partial$. This implies that $H_*(S_X) = H_*(T_X) : F(X) \to F'(X)$. Finally, to say that functors F and F' are naturally chain homotopy equivalent means there are natural transformations $S : F \Rightarrow F'$ and $T : F' \Rightarrow F$ so that $S \circ T$ and $T \circ S$ are both naturally chain homotopy equivalent to the identity. This implies that $H_*(F(X)) \cong H_*(F'(X))$.

The acyclic models theorem is a tool (often called the method of acyclic models) used prove many of the basic theorems of algebraic topology. Here are five examples:

- $\mathcal{C} = \mathrm{Top}$, $\mathcal{M} = \{\Delta^n\}$, $F(X) = S_\bullet(X), F'(X) = S_\bullet(X \times I)$, $S, T : F \Rightarrow F'$ given by $S_X = S_\bullet(i_0)$ and $T_X = S_\bullet(i_1)$ where $i_0, i_1 : X \to X \times I$ are the maps $x \mapsto (x, 0)$ and $x \mapsto (x, 1)$. This is used to prove homotopy invariance of homology.
- $\mathcal{C} = \mathrm{Top} \times \mathrm{Top}$, $\mathcal{M} = \{\Delta^p \times \Delta^q\}$, $F(X \times Y) = S_\bullet(X \times Y), F'(X) = S_\bullet(X) \otimes S_\bullet(Y)$. This is used to prove the Eilenberg-Zilber Theorem, and, with more work, the Künneth Theorem.
- $\mathcal{C} = \mathrm{Top}$, $\mathcal{M} = \{\Delta^n\}$, $F(X) = S_\bullet(X)$, $F'(X) = S_\bullet(X) \otimes S_\bullet(X)$. This is used to define a diagonal approximation, underlying cup and cap products, and the commutativity and associativity of cup products.
- $\mathcal{C} = \mathrm{Top}$, $\mathcal{M} = \{\Delta^n\}$, $F(X) = S_\bullet(X) = F'(X)$, S given by barycentric subdivision, and T given by the identity. This is used to prove excision.
- Here $R = \mathbf{F}_2[\mathbf{Z}/2] = \mathbf{F}_2[t]/\langle t^2 - 1\rangle$, $W_\bullet = \{\cdots \to \mathbf{F}_2[\mathbf{Z}/2] \xrightarrow{1+t} \mathbf{F}_2[\mathbf{Z}/2] \xrightarrow{1+t} \mathbf{F}_2[\mathbf{Z}/2]\}$, $\mathcal{M} = \{\Delta^n\}$, $F(X) = W_\bullet \otimes_{\mathbf{F}_2} S_\bullet(X; \mathbf{F}_2)$ and $F'(X) = S_\bullet(X; \mathbf{F}_2) \otimes_{\mathbf{F}_2} S_\bullet(X; \mathbf{F}_2)$ with $t(a \otimes b) = b \otimes a$. This is used to construct the Steenrod squares.

It is possible to establish all of these using explicit formulas, see, for example, Hatcher [**19**]. But there is a certain utility is noting that one is using the same sort of complicated proof by induction over and over.

For the project, after stating the acyclic models theorem carefully, deduce the Eilenberg–Zilber theorem, Theorem 4.4. If time is left, prove the acyclic models theorem, or prove the homotopy axiom for homology and cohomology: homotopic maps give chain homotopic maps on the singular chains. References include [**17**, pp. 265–270]. Also see [**45**, pp. 164].

Chapter 4

Products

The two main points of this chapter are the cup product and the Künneth theorem. The mathematics is both intricate and meandering; intricate because of the use of the acyclic models theorem and meandering because of the related products such as the cross product and cap product. Hence a preview is called for.

The advantage of cohomology over homology is that cohomology $H^*X = \oplus\, H^nX$ forms a ring. The key geometric idea is, given a singular n-simplex $\sigma : \Delta^n \to X$ and integers $0 \le p, q \le n$ with $p+q = n$, define the *front p-face of* σ to be the singular p-simplex ${}_p\sigma : \Delta^p \to X$

$$ {}_p\sigma(t_0, \ldots, t_p) = \sigma(t_0, \ldots, t_p, 0, \ldots, 0) $$

and the *back q-face of* σ to be the singular q-simplex $\sigma_q : \Delta^q \to X$

$$ \sigma_q(t_0, \ldots, t_q) = \sigma(0, \ldots, 0, t_0, \ldots, t_q). $$

Then the singular cochains $S^*X = \oplus\, S^nX$ form a ring where multiplication is given by the cup product: for cochains $\alpha \in S^pX$ and $\beta \in S^qX$, define $\alpha \cup \beta \in S^{p+q}X$ (pronounced "α cup β") by

$$ (\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma)\beta(\sigma_q) \in \mathbf{Z}. $$

One can check

$$ \delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^p \alpha \cup \delta\beta. $$

It follows that the cup product of two cocycles is a cocycle and the cup product of a cocycle with a coboundary is a coboundary. Hence the cup product gives a product on cohomology by defining $[\alpha] \cup [\beta]$ to be $[\alpha \cup \beta]$. Thus the definition of the cohomology ring is not difficult. But if one wants to compute or to compare $[\alpha] \cup [\beta]$ with $[\beta] \cup [\alpha]$ one needs considerably more theory.

The second theme of this chapter is the Künneth formula which computes the (co)homology of a product space $X \times Y$ in terms of the (co)homology of X and Y. This comes about by combining two basic constructions. The first is purely algebraic; one forms the tensor product of chain complexes and their dual cochain complexes and studies their relationships. The second construction is topological and applies to the singular complex of a space. It is a natural chain homotopy equivalence between the singular chain complex $S_\bullet(X \times Y)$ and the tensor product of $S_\bullet X$ and $S_\bullet Y$. This result is called the Eilenberg-Zilber theorem, and it is a consequence of the acyclic models theorem.

The two themes of the chapter, the cup product and the Künneth Theorem, are related by the diagonal map $\Delta : X \to X \times X$ with $\Delta(x) = (x, x)$. The cup product is then the composite

$$H^p X \otimes H^q X \xrightarrow{\times} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q} X$$

where the first map arises from the Künneth theorem. The existence of the diagonal map Δ for any space X is the geometric reason why there is a product structure in cohomology, but not in homology.

4.1. Tensor products of chain complexes and the algebraic Künneth theorem

We begin with a discussion about graded R-modules and algebras, tensor products, and Hom.

Definition 4.1. Let R be a commutative ring.

1. A *graded R-module* A_* is a collection of R-modules $\{A_k\}_{k \in \mathbf{Z}}$. Equivalently it is an R-module A with a direct sum decomposition $A = \bigoplus_k A_k$. A graded $\mathbf{Z}$-module is called a *graded abelian group*.
2. A *homomorphism of graded R-modules of degree n* is an element of the product $\prod_k \operatorname{Hom}(A_k, B_{k+n})$.
3. Given graded R-modules A_*, B_*, define $\operatorname{Hom}(A_*, B_*)$ to be the graded R-module
$$\operatorname{Hom}(A_*, B_*)_n = \prod_k \operatorname{Hom}(A_k, B_{k+n}).$$
4. The *tensor product $A_* \otimes B_*$ of graded R-modules A_** and B_* is the graded R-module
$$(A_* \otimes B_*)_n = \bigoplus_{p+q=n} (A_p \otimes B_q).$$

The functors $- \otimes B_*$ and $\operatorname{Hom}(B_*, -)$ are adjoint functors from the category of graded R-modules to itself.

5. A *graded R-algebra* is a graded R-module S_* together with a pair of morphisms

$$\mu : S_* \otimes S_* \to S_*, \;\; u : R \to S_0.$$

satisfying:
(a) The map μ has degree 0, that is, $\mu(S_k \otimes S_l) \subset S_{k+l}$.
(b) The map μ is associative in the sense that $(ab)c = a(bc)$, where we write ab for $\mu(a \otimes b)$.
(c) The map u is multiplicative; $u(r)u(s) = u(rs)$.
(d) The element $1_{S_*} = u(1_R)$ is a unit for μ, i.e. $1_{S_*}a = a1_{S_*} = a$ for all $a \in S_*$.
Often u is injective and one identifies R with $u(R)$. A graded $\mathbf{Z}$-algebra is called a *graded ring*. Equivalently, a graded ring is a ring $S = \oplus_k S_k$ with an additive direct sum decomposition so that $S_k S_l \subset S_{k+l}$.

6. A graded R-algebra is called *commutative* if

$$ab = (-1)^{|a||b|} ba,$$

where $a \in S_{|a|}$ and $b \in S_{|b|}$. A commutative graded R-algebra is also called *graded-commutative* R-algebra.

The graded R-modules and rings of Definition 4.1 are sometimes called $\mathbf{Z}$-*graded* to distinguish them from the more general notion of X-*graded* R-*modules* where X is an arbitrary set and X-*graded ring* where X is an arbitrary group (or monoid).

We apply these constructions to chain complexes $C_\bullet = (C_*, \partial)$ and $C'_\bullet = (C'_*, \partial'_*)$. We allow C_n and C'_n to be nonzero for *any* $n \in \mathbf{Z}$.

Definition 4.2. The *tensor product chain complex* $C_\bullet \otimes C'_\bullet = (C_* \otimes C'_*, d)$ is defined by taking the tensor product of the underlying graded modules, i.e.

$$(C_* \otimes C'_*)_n = \bigoplus_{p+q=n} C_p \otimes C'_q$$

and giving it the differential

$$d(z \otimes w) = \partial z \otimes w + (-1)^p z \otimes \partial' w, \quad \text{if } z \in C_p.$$

(The differential d is sort of a “graded derivative”; it satisfies the product rule by definition.)

One computes:

$$\begin{aligned} d^2(z\otimes w) &= d(\partial z\otimes w+(-1)^p z\otimes \partial' w)\\ &= \partial^2 z\otimes w+(-1)^{p-1}\partial z\otimes\partial' w\\ &\qquad +(-1)^p\partial z\otimes \partial' w+(-1)^{2p} z\otimes {\partial'}^2 w\\ &= 0.\end{aligned}$$

Thus $C_\bullet\otimes C'_\bullet$ is indeed a chain complex.

One geometric motivation for this construction is the following. If X and Y are CW-complexes with cells $\{e_i\}$ and $\{f_j\}$ respectively, then $X\times Y$ is a CW-complex with cells $\{e_i\times f_j\}$. The cellular chain complex $C_\bullet(X\times Y)$ can be identified with (i.e. is isomorphic to) the tensor product $C_\bullet X\otimes C_\bullet Y$.

The question we wish to understand is: To what extent and how does the homology of $C_\bullet$ and $C'_\bullet$ determine the homology of $C_\bullet\otimes C'_\bullet$? A connection between the two is provided by the algebraic homology cross product.

Exercise 58. If $C_\bullet, D_\bullet$ are chain complexes, there is a natural map

$$\times_{\text{alg}}: H_pC_\bullet\otimes H_qD_\bullet\to H_{p+q}(C_\bullet\otimes D_\bullet)$$

called the *algebraic homology cross product* defined by

$$[z]\otimes[w]\mapsto[z\otimes w].$$

Write $[z]\times_{\text{alg}}[w]$ (or just $[z]\times[w]$) for $[z\otimes w]$.

The following theorem measures the extent to which this map is an isomorphism, at least if the ground ring R is a PID.

Theorem 4.3 (Künneth exact sequence). *Suppose $C_\bullet, D_\bullet$ are chain complexes over a PID R, and suppose C_n is a free R-module for each n. Then there is a natural exact sequence*

$$0\to\bigoplus_{p+q=n} H_p(C_\bullet)\otimes H_q(D_\bullet)\xrightarrow{\times_{\text{alg}}} H_n(C_\bullet\otimes D_\bullet)\to\bigoplus_{p+q=n}\operatorname{Tor}^R(H_p(C_\bullet),H_{q-1}(D_\bullet))\to 0$$

which splits (nonnaturally).

Proof. The proof is similar to the proof of the universal coefficient theorem (Theorem 3.29), and so we only sketch the argument, leaving details, notably issues about the grading, to the reader.

Setting $Z_n=\ker\partial: C_n\to C_{n-1}$ and $B_n=\operatorname{im}\partial: C_{n+1}\to C_n$, we obtain the short exact sequence

$$0\to Z_*\to C_*\xrightarrow{\partial} B_*\to 0 \tag{4.1}$$

which we view as a short exact sequence of free chain complexes by giving Z_* and B_* the zero differential (the modules Z_n and B_n are free since they are submodules of the free module C_n and R is a PID).

Since B_n is free, tensoring the short exact sequence (4.1) with $D_\bullet$ yields a new short exact sequence of chain complexes.

$$0 \to Z_\bullet \otimes D_\bullet \to C_\bullet \otimes D_\bullet \to B_\bullet \otimes D_\bullet \to 0. \tag{4.2}$$

Since the differential in the chain complex $Z_\bullet$ is zero, the differential $\partial : Z_\bullet \otimes D_\bullet \to Z_\bullet \otimes D_\bullet$ reduces to

$$z \otimes d \mapsto (-1)^{|z|} z \otimes \partial d,$$

and Z_n is free, hence flat, so passing to homology one gets

$$H_*(Z_\bullet \otimes D_\bullet) = Z_* \otimes H_*(D_\bullet).$$

Similarly

$$H_*(B_\bullet \otimes D_\bullet) = B_* \otimes H_*(D_\bullet).$$

Thus, the long exact sequence in homology obtained by applying the zig-zag lemma to the complex (4.2) reduces to the exact triangle

$$\begin{tikzcd} B_* \otimes H_*(D_\bullet) \arrow[rr] & & Z_* \otimes H_*(D_\bullet) \arrow[dl] \\ & H_*(C_\bullet \otimes D_\bullet) \arrow[ul] & \end{tikzcd} \tag{4.3}$$

On the other hand, applying Theorem 3.2 to the tensor product of the short exact sequence

$$0 \to B_* \to Z_* \to H_*(C_\bullet) \to 0,$$

with $H_*(D_\bullet)$ yields an exact sequence

(4.4)
$$0 \to \mathrm{Tor}(H_*(C_\bullet), H_*(D_\bullet)) \to B_* \otimes H_*(D_\bullet) \to Z_* \otimes H_*(D_\bullet) \to H_*(C_\bullet) \otimes H_*(D_\bullet) \to 0.$$

Combining (4.3) and (4.4), applying Exercise 52, taking care with the grading, and chasing down the definitions of the maps induced finishes the proof that the Künneth sequence is exact.

If $C_\bullet$ and $D_\bullet$ are free chain complexes, the splitting in the Künneth exact sequence is obtained just like the splitting in the universal coefficient theorem. If $D_\bullet$ is not free, the reasoning is more complicated and involves finding a chain equivalence $D'_\bullet \to D_\bullet$ where $D'_\bullet$ is a free chain complex. For details, see [**21**].

□

4.2. The Künneth formula

Until further notice, homology and cohomology with coefficients in a ring R are understood, and we omit writing " ; R ". Similarly, we write $\otimes$ instead of $\otimes_R$.

4.2.1. The Eilenberg-Zilber theorem.

Theorem 4.4 (Eilenberg-Zilber theorem). *Let* $\mathsf{Top} \times \mathsf{Top}$ *be the category whose objects are ordered pairs of spaces* (X, Y) *(we do not assume* $Y \subset X$*) and whose morphisms are pairs* $(f : X' \to X, g : Y' \to Y)$ *of continuous maps. Then the two functors*

$$F : (X, Y) \mapsto S_\bullet(X \times Y)$$

and

$$F' : (X, Y) \mapsto S_\bullet X \otimes S_\bullet Y$$

from $\mathsf{Top} \times \mathsf{Top}$ *to the category of chain complexes are naturally equivalent; more precisely, there exist natural transformations* $A : F \to F'$ *and* $B : F' \to F$ *so that* $A(\sigma) = \mathrm{pr}_X\sigma \otimes \mathrm{pr}_Y\sigma$ *and* $B(\tau \otimes \rho) = \tau \times \rho$ *for any singular 0-simplices* σ*,* τ*, and* ρ *in* $X \times Y$*,* X*, and* Y *respectively. Furthermore, for any pair* (X, Y) *the composites*

$$S_\bullet(X \times Y) \xrightarrow{A} S_\bullet X \otimes S_\bullet Y \xrightarrow{B} S_\bullet(X \times Y)$$

and

$$S_\bullet X \otimes S_\bullet Y \xrightarrow{B} S_\bullet(X \times Y) \xrightarrow{A} S_\bullet X \otimes S_\bullet Y$$

are chain homotopic to the identity. Moreover, any two choices of A *(resp.* B*) are naturally chain homotopic.*

In particular, there exist natural isomorphisms

$$H_n(X \times Y) \to H_n(S_\bullet X \otimes S_\bullet Y)$$

for each n*.* □

The proof of this theorem is an easy application of the acyclic models theorem. See the project on the acyclic models theorem at the end of Chapter 3.

The natural transformations A and B are chain homotopy equivalences

$$A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$$

and

$$B : S_\bullet X \otimes S_\bullet Y \to S_\bullet(X \times Y)$$

for any pair of spaces X and Y. We will call these maps the *Eilenberg-Zilber maps*.

The confusing, abstract, but important point is that A and B are not *canonical*, but only *natural*. That is, they are obtained by the method of acyclic models, and so constructed step by step by making certain arbitrary choices. However, these choices are made consistently for all spaces.

In what follows, we will show how a choice of A and B determines natural additional structure, namely products, on the singular complex and homology of a space. But you should keep in mind that all the constructions depend at core on the noncanonical choice of the transformations A and B.

An alternative approach to this material is to just give specific formulas for A and B. It is easy to imagine a chain map $B : S_\bullet X \otimes S_\bullet Y \to S_\bullet(X \times Y)$. Given singular simplices $\sigma : \Delta^p \to X$ and $\tau : \Delta^q \to Y$, there is the product map $\sigma \times \tau : \Delta^p \times \Delta^q \to X \times Y$. Unfortunately the product of simplices is not a simplex, but it can be chopped up into a union of $p+q$-simplices (consider a square chopped into triangles or a prism chopped into tetrahedra). Then one could choose $B(\sigma \otimes \tau)$ to be a sum of singular $p + q$-simplices – the "shuffle product".

The reverse map $A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ can be defined using the front p-face and back q-face idea. For a singular n-simplex $\sigma : \Delta^n \to X \times Y$, one can define

$$A(\sigma) = \sum_{p+q=n} {}_p(p_X \circ \sigma) \otimes (p_Y \circ \sigma)_q$$

where p_X and p_Y are the projection maps to the factors. However, even if one constructs the maps A and B explicitly, they will only be chain homotopy equivalences, not isomorphisms; $S_\bullet(X \times Y)$ is simply bigger than $S_\bullet X \otimes S_\bullet Y$. And one would have to write down the formulas for the chain homotopy equivalences.

In any case, invoking a technical formula can obscure the point of a construction (just look at any page of a differential geometry book for evidence of this principle). Thus for what follows, fix natural transformations A and B whose existence is asserted in Theorem 4.4. Each product on chain complexes constructed below depends on the choice of A or B, but this dependence disappears when passing to homology and cohomology.

4.2.2. The Künneth formula and the homology cross product. Exercise 58 implies that the natural map

$$\times_{\text{alg}} : H_p X \otimes H_q Y \to H_{p+q}(S_\bullet X \otimes S_\bullet Y)$$

given on the chain level by $[\alpha] \otimes [\beta] \mapsto [\alpha \otimes \beta]$ is well-defined. Denote by B_* the isomorphism induced by the Eilenberg-Zilber map on homology, so

$$B_* : H_*(S_\bullet X \otimes S_\bullet Y) \to H_*(S_\bullet(X \times Y)) = H_*(X \times Y).$$

Composing $\times_{\mathrm{alg}}$ with B_*, we obtain

$$\times : H_pX \otimes H_qY \to H_{p+q}(X \times Y).$$

Definition 4.5. If $a \in H_pX,\ b \in H_qY$, the image of $a \otimes b$ under this map is called the *homology cross product* of a and b and is denoted by $a \times b$.

The Eilenberg-Zilber theorem has the following important consequence.

Theorem 4.6 (Künneth formula)**.** *If R is a PID, there exists a split exact sequence*

$$0 \to \bigoplus_{p=0}^{n} H_pX \otimes H_{n-p}Y \to H_n(X \times Y) \to \bigoplus_{p=0}^{n-1} \mathrm{Tor}(H_pX, H_{n-1-p}Y) \to 0.$$

The first map is given by the cross product.

Proof. This follows easily by combining the Künneth exact sequence (Theorem 4.3) to the free chain complexes $S_\bullet X$ and $S_\bullet Y$ with the Eilenberg-Zilber theorem. □

Corollary 4.7. *If R is a PID and H_qY is free for all q (e.g. if R is a field), then $H_*(X \times Y) \cong H_*X \otimes H_*Y$ as graded R-modules.*

Exercise 59. Compute $H_*(\mathbf{R}P^2 \times \mathbf{R}P^2)$, both with $\mathbf{Z}$ and $\mathbf{Z}/2$-coefficients. Give a geometric interpretation of the class coming from the Tor term in the Künneth formula.

The Künneth formula implies that if R is a PID, $a \times b \neq 0$ if $a \neq 0$ and $b \neq 0$.

4.2.3. The cohomology cross product. Let $C_\bullet$ and $D_\bullet$ be chain complexes over a ring R and let $C^\bullet$ and $D^\bullet$ be the dual chain complexes $\mathrm{Hom}_R(C_\bullet, R)$ and $\mathrm{Hom}_R(D_\bullet, R)$ respectively.

Exercise 60. If $C_\bullet, D_\bullet$ are chain complexes, there is a natural map

$$\times^{\mathrm{alg}} : H^pC^\bullet \otimes H^qD^\bullet \to H^{p+q}((C_\bullet \otimes D_\bullet)^*)$$

defined by $[\alpha] \otimes [\beta] \mapsto [\sum z_i \otimes w_i \mapsto \sum \alpha(z_i) \cdot \beta(w_i)]$. In this formula if α and z_i are of different degrees, then $\alpha(z_i)$ is zero, and likewise for $\beta(w_i)$. The notation $\alpha(z_i) \cdot \beta(w_i)$ refers to multiplication in the ring R.

This map is called the *algebraic cohomology cross product*.

Applying this product to the singular complexes, we see that for any spaces X and Y we have a map

$$\times^{\mathrm{alg}} : H^pX \otimes H^qY \to H^{p+q}((S_\bullet X \otimes S_\bullet Y)^*).$$

Using the Eilenberg-Zilber theorem we can further map to $H^{p+q}(X \times Y)$. Explicitly, the dual of the Eilenberg-Zilber map $A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ is a chain homotopy equivalence $A^* : (S_\bullet X \otimes S_\bullet Y)^* \to S^\bullet(X \times Y)$. Passing to cohomology one obtains an isomorphism

$$A^* : H^*((S_\bullet X \otimes S_\bullet Y)^*) \to H^*(X \times Y).$$

This map is independent of the choice of Eilenberg-Zilber map A since any two choices for A are naturally chain homotopic. (We will be somewhat casual with notation and denote by A^* the dual of A as well as the induced map on cohomology. This should not cause any confusion and will keep the notation under control.)

Definition 4.8. If $a \in H^pX,\ b \in H^qY$, the image of $a \otimes b$ under the composite map

$$H^pX \otimes H^qY \xrightarrow{\times^{\mathrm{alg}}} H^{p+q}((S_\bullet X \otimes S_\bullet Y)^*) \xrightarrow{A^*} H^{p+q}(X \times Y)$$

is called the *cohomology cross product* of a and b and is denoted by $a \times b$.

Theorem 4.9 (Cohomology Künneth formula)**.** *If R is a PID and if H_qY is finitely generated for all q, there exists a split exact sequence*

$$0 \to \bigoplus_{p=0}^{n} H^pX \otimes H^{n-p}Y \to H^n(X \times Y) \to \bigoplus_{p=0}^{n+1} \mathrm{Tor}(H^pX, H^{n+1-p}Y) \to 0.$$

The first map is given by the cross product.

See [**45**, pg. 249] for a proof.

4.3. The cup product

The cohomology of a topological space forms a commutative, graded R-algebra H^*X (recall that coefficients in a commutative ring R are understood: $H^nX := H^n(X; R)$.) Multiplication in the cohomology ring is the cup product. There are three different points of view on the definition of the cup product; they are all useful.

- cochains: $(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma)\beta(\sigma_q)$.
- cross product and the diagonal map: $a \cup b = \Delta^*(a \times b)$
- diagonal approximation: $a \cup b = \tau^*(a \times^{\mathrm{alg}} b)$

4.3.1. Cup product via cochains. Recall that

$$S^nX = \text{functions}(\{\text{singular } n\text{-simplices in } X\}, R)$$

and that evaluation on the front p-fact and back q-face gives a bilinear pairing

$$S^pX \times S^qX \to S^{p+q}X$$
$$(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma)\beta(\sigma_q)$$

It is not difficult to see that this cochain cup product is associative. There is also a unit: for any space X, define $1 \in S^0X$ by setting $1(\sigma) = 1 \in R$. In other words, the evaluation of 1 on a 0-chain is the augmentation map.

Thus S^*X is a graded R-algebra and a continuous map $f : X \to Y$ induces a ring map $S^*Y \to S^*X$. Our joy is tempered by two issues. First S^*X is just too big to be useful, and second, the multiplication is not graded commutative (in fact, there are invariants called the Steenrod squares whose definition relies on the noncommutativity of the cochain cup product.

Exercise 61. Show $\delta(\alpha \cup \beta) = \delta\alpha \cup \beta + (-1)^p\alpha \cup \delta\beta$ and deduce that H^*X is a ring (in fact an R-algebra) by showing that $[\alpha \cup \beta] = [\alpha] \cup [\beta]$ is well-defined.

One can show that H^*X is graded commutative by explicit formulas, but we prefer to show this by using diagonal approximations below.

4.3.2. Cup product via the cross product and the diagonal map. The geometric reason why there is a ring structure on cohomology and not on homology is the existence of the diagonal map $\Delta : X \to X \times X$, $x \mapsto (x, x)$. There is no reasonable map $X \times X \to X$ unless, for example, X is a topological group.

The next lemma shows that the cross product determines the cup product and conversely that the cup product determines the cross product. The first two items below imply that $f^* : H^*X \to H^*X'$ is a ring map.

Lemma 4.10. *Let $f : X' \to X$ and $g : Y' \to Y$ be continuous maps. Let $a, b \in H^*X$ and $c \in H^*Y$.*

1. *$f^*(a \cup b) = f^*a \cup f^*b$.*
2. *$f^*1 = 1$.*
3. *$(f \times g)^*(a \times c) = f^*a \times g^*c$.*
4. *$a \cup b = \Delta^*(a \times b)$.*
5. *$a \times c = p_X^*a \cup p_Y^*c$, where p_X and p_Y are the projections in $X \times Y$.*

Proof. 1. Note that the front p-face of $f \circ \sigma$ is $f \circ {}_p\sigma$ and likewise for the back q-face.

2. Clear from the definition.

3. This follows from the naturality of the Eilenberg-Zilber map and the algebraic cohomology cross product with respect to pairs of maps (f, g).

4. The map $(a, b) \mapsto \Delta^*(a \times b)$ is given as the composite

$$H^p X \otimes H^q X \xrightarrow{\times^{\text{alg}}} H^{p+q}((S_\bullet X \otimes S_\bullet X)^*) \xrightarrow{A^*} H^{p+q}(X \times X) \xrightarrow{\Delta^*} H^{p+q} X.$$

If α and β are cochain representatives for $a \in H^p X$ and $b \in H^q X$, and if $\sigma : \Delta^{p+q} \to X$ is a singular $n = p+q$-simplex, then a cocycle representative for the composite is given by

$$\begin{aligned}(\alpha \otimes \beta)(A(\Delta \circ \sigma)) &= (\alpha \otimes \beta)(\sum_{i+j=n} {}_i\sigma \otimes \sigma_j) \\ &= \alpha({}_p\sigma)\beta(\sigma_q) \\ &= (\alpha \cup \beta)(\sigma)\end{aligned}$$

5.

$$\begin{aligned}p_X^* a \cup p_Y^* c &= \Delta_{X\times Y}^*(p_X^* a \times p_Y^* c) \\ &= \Delta_{X\times Y}^*((p_X \times p_Y)^*(a \times c)) \\ &= ((p_X \times p_Y) \circ \Delta_{X\times Y})^*(a \times c) \\ &= \mathrm{Id}_{X\times Y}^*(a \times c).\end{aligned}$$

□

4.3.3. Cup product via diagonal approximation.

Definition 4.11. A *diagonal approximation* τ is a chain map

$$\tau : S_\bullet X \to S_\bullet X \otimes S_\bullet X$$

for every space X, so that

1. $\tau(\sigma) = \sigma \otimes \sigma$ for every 0-simplex σ.
2. τ is natural with respect to continuous maps of spaces.

Condition 1. could be replaced by: τ induces a augmentation preserving map $H_0 X \to H_0 X \times H_0 X$, natural with respect to maps of spaces. Now the functor $S_\bullet X$ is free on the models $\{\Delta^n\}$ and $S_\bullet X \otimes S_\bullet X$ is acyclic on these models, so the acyclic models theorem says that there exists a diagonal approximation and any two such are naturally chain homotopic.

Example 4.12. The *Alexander-Whitney diagonal approximation* is defined by setting

$$\tau(\sigma) = \sum_{p+q=n} {}_p\sigma \otimes \sigma_q$$

for a singular n-simplex $\sigma : \Delta^n \to X$ and extending to $S_n X$ by linearity. It is an exercise (closely related to Exercise 61) to show that τ is a diagonal approximation.

Corollary 4.13. *$a \cup b = \tau^*(a \times^{alg} b)$ for any diagonal approximation τ.*

Proof. Since any two diagonal approximations are chain homotopy equivalent, we can assume τ is the Alexander-Whitney diagonal approximation. Then the formula holds by definition of the cup product. □

Thus there is nothing special about the Alexander-Whitney diagonal approximation and we could define the cup product without it, but it is psychologically reassuring to have a specific formula. In fact, we could have defined the cup product in a number of ways, using the front p and back q-faces, using the Eilenberg-Zilber Theorem and the cross product, or using a diagonal approximation generated by the Acyclic Models Theorem.

A diagonal approximation $\tau : S_\bullet X \to S_\bullet X \otimes S_\bullet X$ determines an Eilenberg-Zilber map $A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ via $A = (p_X \otimes p_Y) \circ \tau$. Conversely an Eilenberg-Zilber map A determines the diagonal approximation $\tau = A \circ \Delta_*$.

Theorem 4.14. *For a topological space X, H^*X is a graded-commutative R-algebra.*

Proof. We have already shown that the Alexander-Whitney diagonal approximation induces a graded R-algebra structure on S^*X and hence on H^*X. The only thing that remains is to show (graded) commutativity. The key observation here is that for chain complexes $C_\bullet$ and $D_\bullet$, the interchange map

$$\begin{gathered} T : C_\bullet \otimes D_\bullet \to D_\bullet \otimes C_\bullet \\ z \otimes w \mapsto (-1)^{|z||w|} w \otimes z \end{gathered}$$

gives an *isomorphism* of chain complexes. Hence if τ is a diagonal approximation, then so is $T \circ \tau$, and hence τ and $T \circ \tau$ are naturally chain homotopic. Thus if $a \in H^p X$ and $b \in H^q X$,

$$\begin{aligned} a \cup b &= \tau^*(a \times^{\mathrm{alg}} b) = \tau^* T^*(a \times^{\mathrm{alg}} b) \\ &= (-1)^{pq} \tau^*(b \times^{\mathrm{alg}} a) \\ &= (-1)^{pq} b \cup a \end{aligned}$$

□

See Vick's book [**51**] for a nice example of computing the cohomology ring of the torus directly using the Alexander-Whitney diagonal approximation.

If X is a CW-complex, then the diagonal map $\Delta : X \to X \times X$ is not cellular (consider X $= [0,1]$). However the cellular approximation theorem says that Δ is homotopic to a cellular map Δ'. If $\Delta'_\bullet(z) = \sum x_i \otimes y_i$, then the cup product on cellular cohomology can be defined by $(\alpha \cup \beta)(z) = \sum \alpha(x_i) \cdot \beta(y_i)$. The geometric root of the Alexander-Whitney diagonal approximation is finding a simplicial map (i.e. takes simplices to simplices and is affine on the simplices) homotopic to the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$.

Notice that the de Rham cochain complex of differential forms on a smooth manifold is graded commutative, since differential forms satisfy $a \wedge b = \pm b \wedge a$. It is possible to give a natural construction of a commutative chain complex over the rationals which gives the rational homology of a space; this was done using rational differential forms on a simplicial complex by Sullivan. This fact is exploited in the subject of rational homotopy theory [**18**]. On the other hand it is impossible to construct a functor from spaces to commutative, associative chain complexes over $\mathbf{Z}$ which gives the integral homology of a space.

Sometimes one wishes to use products on homology and cohomology with coefficients in various R-modules. The following exercise shows how to accomplish this. The basic idea is that multiplication in the ring R was used in the definition of cup products (in fact in the definition of $\times^{\text{alg}}$), and so when passing to more general modules an auxiliary multiplication is needed.

Exercise 62. If M and N are R-modules, construct a cross product

$$\times : H^p(X; M) \times H^q(Y; N) \to H^{p+q}(X \times Y; M \otimes N)$$

and a cup product

$$\cup : H^p(X; M) \times H^q(X; N) \to H^{p+q}(X; M \otimes N).$$

4.3.4. Computation of the cohomology ring and applications of the cup product. Before we give examples of commutative rings, we need the vocabulary to describe the computation. Let's start with an exercise.

Exercise 63. If A and B are commutative rings, then $A \otimes_{\mathbf{Z}} B$ is a commutative ring with $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$. (The main point of the exercise is to show that the multiplication is well-defined.)

Exercise 63 generalizes first to commutative R-algebras and then to graded-commutative R-algebras. If A_* and B_* are graded-commutative R-algebras, so is $A_* \otimes B_*$, where one defines $(a \otimes b)(a' \otimes b') = (-1)^{|b||a'|} aa' \otimes bb'$. This multiplication is consistent with the rule of thumb that in the graded setting, when one interchanges two symbols (here b and a') one pays the price of inserting a factor of -1 raised to the product of the degrees.

One can show that $\otimes$ is the coproduct in the respective categories.

Theorem 4.15. *Let X and Y be topological spaces.*

1. *For $a, a' \in H^*X$ and $b, b' \in H^*Y$,*
$$(a \times b) \cup (a' \times b') = (-1)^{|b||a'|}(a \cup a') \times (b \cup b')$$
2. *The cohomology cross product*
$$H^*X \otimes H^*Y \xrightarrow{\times} H^*(X \times Y)$$
is a homomorphism of graded-commutative rings.

Proof. 1. This follows easily from the relationship between cross and cup products (Lemma 4.10) and the graded-commutativity of cup products (Thereom 4.14).

$$\begin{aligned}(a \times b) \cup (a' \times b') &= p_X^* a \cup p_Y^* b \cup p_X^* a' \cup p_Y^* b' \\ &= (-1)^{|b||a'|} p_X^* a \cup p_X^* a' \cup p_Y^* b \cup p_Y^* b' \\ &= (-1)^{|b||a'|}(a \cup a') \times (b \cup b')\end{aligned}$$

2. All we need to show is that $\times$ preserves multiplication. This follows from part 1. $\square$

The following corollary follows from the Cohomology Künneth Formula 4.9.

Corollary 4.16. *If R is a PID and if H_qY is finitely generated free for all q, then $H^*X \otimes H^*Y$ and $H^*(X \times Y)$ are isomorphic graded R-algebras.*

In fact, the corollary is true for a general commutative ring. An important special case is when R is a field, in which case all modules are free.

Corollary 4.17. *Let $m, n > 0$. Let $p_1 : S^m \times S^n \to S^m$ and $p_2 : S^m \times S^n \to S^n$ be the projection maps. Let $a \in H^m(S^m \times S^n)$ and $b \in H^n(S^m \times S^n)$ be generators of $p_1^*(H^m S^m)$ and $p_2^*(H^n S^n)$ respectively. Then $a \cup b$ is a generator of $H^{m+n}(S^m \times S^n)$.*

Thus $1, a, b, a \cup b$ are additive generators of the cohomology of $S^m \times S^n$. Since $a \cup b = (-1)^{mn} b \cup a$, we now know the cohomology ring of $S^m \times S^n$.

We now turn to the dual result. If A_* and B_* are graded-commutative R-algebras, then so is $A_* \times B_*$ where $(A_* \times B_*)_n = A_n \times B_n$ and $(a, b)(a', b') = (aa', bb')$. One can show that $\times$ is the product in the category of graded-commutative R-algebras.

The proof of the following theorem is clear.

Theorem 4.18. 1. *Let X and Y be topological spaces. Then*

$$H^*(X \amalg Y) \to H^*X \times H^*Y$$

is an isomorphism of graded commutative R-algebras.

2. *Let (X, x_0) and (Y, y_0) be based topological spaces. Then*

$$H^*(X \vee Y) \to H^*X \times H^*Y$$

is a monomorphism of graded commutative R-algebras which is onto in positive degrees and whose image in degree zero is $\{(a,b) \mid \langle a, [x_0]\rangle = \langle b, [y_0]\rangle\}$.

Exercise 64. If $f \simeq g : S^{n-1} \to X$ are homotopic, then the adjunction spaces $X \cup_f D^n$ and $X \cup_g D^n$ are homotopy equivalent.

The following corollary shows how cohomology is more powerful than homology.

Corollary 4.19. *Suppose $m, n > 0$. The attaching map $f : S^{m+n-1} \to S^m \vee S^n$ of the $m+n$-cell of $S^m \times S^n$ is essential (i.e. not homotopy to a constant map).*

Proof. Assume, by contradiction, that f is nullhomotopic. Then

$$S^m \times S^n \simeq S^m \vee S^n \vee S^{m+n}.$$

But the cup product $H^m \otimes H^n \to H^{m+n}$ on the left hand sided is nontrivial but the cup product on the right hand side is trivial. $\square$

A surface (or 2-manifold) is a Hausdorff, second countable, topological space so that every point has a neighborhood homeomorphic to $\mathbf{R}^2$. The connected sum of two connected surfaces M_1 and M_2 is the surface

$$M_1 \# M_2 = (M_1 - \text{int } D_1) \cup_{S^1} (M_2 - \text{int } D_2)$$

where $D_i \subset M_i$ is a subspace homeomorphic to D^2. A compact surface is often called a closed surface. Any connected compact surface is homeomorphic to $T^2 \# T^2 \# \cdots \# T^2$ or $\mathbf{R}P^2 \# T^2 \# \cdots \# T^2$. The number of T^2 summands is called the *genus* g of the surface. Genus zero surfaces are allowed. Surfaces of the first type are called the closed orientable surfaces and surfaces of the second type are called the closed nonorientable surfaces. It is not difficult to show that $\mathbf{R}P^2 \# \mathbf{R}P^2$ is homeomorphic to the Klein bottle and that $\mathbf{R}P^2 \# \mathbf{R}P^2 \# \mathbf{R}P^2$ is homeomorphic to $\mathbf{R}P^2 \# T^2$.

Exercise 65. Let $\Sigma_g = T^2 \# \cdots \# T^2$ be a closed orientable surface of genus g. Show that $H^1(\Sigma_g) \cong \mathbf{Z}^{2g}$ and $H^2(\Sigma_g) \cong \mathbf{Z}$. Choosing a generator μ of $H^2(\Sigma_g)$, show that $H^1(\Sigma_g)$ has a basis $e_1, f_1, e_2, f_2, \ldots, e_g, f_g$ so that $e_i \cup e_j = 0$, $f_1 \cup f_j = 0$, $e_i \cup f_j = \delta_{ij}\mu$ where δ_{ij} is the Kronecker delta symbol. (Hint: Use that $(\Sigma_g, S^1 \amalg S^1 \amalg \cdots \amalg S^1)$ is a good pair and show that $H^*(\Sigma_g) \to H^*(T^2 \vee T^2 \vee \cdots \vee T^2)$ is a ring monomorphism.)

We would like to present a cohomology ring as the quotient of a free graded-commutative ring (or R-algebra). We first discuss quotients. A graded ideal $I_* = \{I_k\}$ of a graded-commutative R-algebra $A_* = \{A_k\}$ is a collection of R-submodules $I_k \subset A_k$ so that $A_k I_l \subset I_{k+l}$. Note that $\oplus I_k$ is a two-sided ideal of the ring $\oplus A_k$. If I_* is a graded ideal of A_*, then $A_*/I_* = \{A_k/I_k\}$ is a graded-commutative R-algebra. If $A_* \to B_*$ is a surjective homomorphism of graded-commutative R-algebras, then the kernel I_* is a graded ideal of A_* and $B_* \cong A_*/I_*$.

A graded set S_* is a collection of sets $\{S_k\}$. A map of graded sets $S_* \to T_*$ is a collection of functions $\{f_k : S_k \to T_k\}$. A graded-commutative R-algebra A_* is free on a graded subset $S_* \subset A_*$ if every map of graded sets $S_* \to B_*$ where B_* is the underlying set of a graded-commutative R-algebra, extends to a unique map $A_* \to B_*$ of graded-commutative R-algebras. For every graded set S_*, there exist a graded-commutative R-algebra $F(S_*)$ free on S_*, unique up to isomorphism. This is constructed by first constructing the tensor algebra. Given an R-module V, the tensor algebra

$$T(V) = R \oplus V \oplus (V \otimes_R V) \oplus (V \otimes_R V \otimes_R V) \oplus \cdots$$

where $(v_1 \otimes \cdots \otimes v_k)(w_1 \otimes \cdots \otimes w_l) = (v_1 \otimes \cdots \otimes v_k \otimes w_1 \otimes \cdots \otimes w_l)$. This is an R-algebra. If V is a graded R-module, then $T(V)$ is a graded R-algebra. If S_* is a (graded) set, then $R[S_*]$ is the (graded) free R-module with basis S (elements are finite linear combinations $r_1 s_1 + \cdots + r_k s_k$). Then

$$F(S_*) = T(R[S_*])/J$$

is a graded-commutative R-algebra with basis S_*, where J is the two-sided ideal generated by $s_k s_l - (-1)^{kl} s_k s_l$ for $s_k \in S_k$ and $s_l \in S_l$. Speaking categorically, F defines a functor

$$F : \text{GrSet} \to \text{GrRalg}$$

which is a left-adjoint for the forgetful functor $U : \text{GrRalg} \to \text{GrSet}$. Then every graded-commutative ring A_* is a quotient $F(S_*)/I_*$ which can be seen by taking a graded set $S_* \subset A_*$ of generators and letting $I_* = \ker(F(S_*) \to A_*)$.

Every graded set can be expressed as the union $S_* = S_{\text{even}} \cup S_{\text{odd}}$ of its even graded elements and its odd graded elements. Note that $F(S_{\text{even}})$ is simply the polynomial algebra $R[S_{\text{even}}]$. If R has characteristic 2 (i.e. $1+1 = 0$), then $F(S_*)$ is also a polynomial ring $R[S_*]$. If $1+1$ is not a divisor of zero (e.g. $R = \mathbf{Z}$), then $F[S_{\text{odd}}]$ is the exterior algebra $\Lambda_R(S_{\text{odd}})$ and $F[S_*] = \Lambda_R(S_{\text{odd}}) \otimes R[S_{\text{even}}]$.

Revising the examples above we see $H^*(S^1 \times S^1) = \Lambda(a,b)$, $H^*(\Sigma_g) = \Lambda(e_1, f_1, e_2, f_2, \ldots, e_g, f_g)/\langle e_1 f_1 - e_2 f_2, \ldots, e_1 f_1 - e_g f_g\rangle$ and $H^*(S^2 \times S^2) = \mathbf{Z}[c,d]/\langle c^2, d^2\rangle$ where $a, b, e_1, f_1, \ldots, e_g f_g$ all have degree 1 and c, d have degree 2.

We have seen that H^*T^n is an exterior algebra on n generators. What about spaces with polynomial cohomology? Examples are provided by $\mathbf{C}P^\infty$ and $\mathbf{R}P^\infty$ (and products of them). It is easy to see that the cellular chain complex of $\mathbf{C}P^\infty = e^0 \cup e^2 \cup e^4 \cup \cdots$ and the mod 2 cellular chain complex of $\mathbf{R}P^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots$ have zero differentials. Thus $H^*\mathbf{C}P^\infty$ is infinite cyclic in even degrees and that $H^*(\mathbf{R}P^\infty; \mathbf{F}_2)$ is $\mathbf{F}_2$ in all nonnegative degrees. (For a prime p, we write $\mathbf{F}_p$ instead of $\mathbf{Z}/p$ when we wish to emphasize the ring/field structure.) The cohomology rings of projective spaces are given by the theorem and corollary below.

Theorem 4.20. 1. *$H^*\mathbf{C}P^\infty = \mathbf{Z}[a]$ where degree $a = 2$.*

2. *$H^*(\mathbf{R}P^\infty; \mathbf{F}_2) = \mathbf{F}_2[b]$ where degree $b = 1$.*

Corollary 4.21. 1. *$H^*\mathbf{C}P^n = \mathbf{Z}[a]/\langle a^{n+1}\rangle$ where degree $a = 2$.*

2. *$H^*(\mathbf{R}P^\infty; \mathbf{F}_2) = \mathbf{F}_2[b]/\langle b^{n+1}\rangle$ where degree $b = 1$.*

The graded rings in the corollary are called *truncated polynomial rings*. One could also deduce the theorem from the corollary.

There are several proofs of the above. An elementary proof is given in Theorem 3.19 of [**19**], we give a proof of the theorem using spectral sequences (see Exercises 205 and 207), and there is a proof of the corollary using Poincaré duality (see Exercise 78).

examples

Find undefined references; multiply-defined labels examples a la Hatcher, applications: Hopf map and borsuk-ulam

4.4. The cap product

Make this into a section(?). Add formula mixing cup and cap product and talk about modules.

Recall that the Kronecker pairing is a natural bilinear evaluation map (sometimes called "integration" by analogy with the de Rham map)

$$\langle\ ,\ \rangle : S^\bullet X \times S_\bullet X \to R$$

defined for $\alpha \in S^q X, z \in S_p X$ by

$$\langle \alpha, z\rangle = \begin{cases} \alpha(z) & \text{if } p = q, \\ 0 & \text{otherwise.} \end{cases}$$

This pairing can be extended to a partial evaluation (or "partial integration") map

$$E : S^{\bullet}X \otimes S_{\bullet}X \otimes S_{\bullet}X \to S_{\bullet}X$$

by evaluating the first factor on the last factor, i.e.

$$E(\alpha \otimes z \otimes w) = \alpha(w) \cdot z.$$

We will define the cap product on the chain level first.

Definition 4.22. The *cap product*

$$S^q(X) \times S_{p+q}(X) \to S_p(X)$$

is defined for $\alpha \in S^q(X),\ z \in S_{p+q}(X)$ by

$$\alpha \cap z = E(\alpha \otimes A \circ \Delta_*(z)).$$

The definition can be given in terms of a diagonal approximation τ instead of the Eilenberg-Zilber map A:

$$\alpha \cap z = E(\alpha \otimes \tau(z)).$$

Lemma 4.23. *For $\alpha \in S^qX$, $z \in S_{p+q}X$,*

$$\partial(\alpha \cap z) = (-1)^p \delta\alpha \cap z + \alpha \cap \partial z.$$

Proof. Suppose $\tau(z) = \sum x_i \otimes y_i$ so that $|x_i| + |y_i| = p + q$. Then since α only evaluates nontrivially on chains in degree q, we have

$$\begin{aligned}
\partial(\alpha \cap z) &= \partial E(\alpha \otimes \tau(z)) \\
&= \partial \sum_{|y_i|=q} \alpha(y_i) \cdot x_i \\
&= \sum_{|y_i|=q} \alpha(y_i) \cdot \partial x_i
\end{aligned}$$

and

$$\begin{aligned}
\delta\alpha \cap z &= E(\delta\alpha \otimes \tau(z)) \\
&= \sum \delta\alpha(y_i) \cdot x_i \\
&= \sum_{|y_i|=q+1} \alpha(\partial y_i) \cdot x_i.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\alpha \cap \partial z &= E(\alpha \otimes \tau(\partial z)) \\
&= E(\alpha \otimes \partial \tau(z)) \\
&= E(\alpha \otimes (\sum \partial x_i \otimes y_i + \sum (-1)^{|x_i|} x_i \otimes \partial y_i)) \\
&= \sum_{|y_i|=q} \alpha(y_i) \cdot \partial x_i + \sum_{|y_i|=q+1} (-1)^{p-1} \alpha(\partial y_i) \cdot x_i \\
&= \partial(\alpha \cap z) + (-1)^{p-1} (\delta\alpha) \cap z.
\end{aligned}$$

□

Lemma 4.23 immediately implies:

Corollary 4.24. *The cap product descends to a well-defined product*

$$\cap : H^q X \times H_{p+q} X \to H_p X$$

$$([\alpha], [z]) \mapsto [\alpha \cap z]$$

after passing to (co)homology. □

Exercise 66. Let $a, b \in H^* X$ and $w, z \in H_* X$. Show that

1. $a \cap (b \cap z) = (a \cup b) \cap z$.
2. $1 \cap z = z$
3. $(a + b) \cap z = a \cap z + b \cap z$
4. $a \cap (z + z') = a \cap z + a \cap z'$

Thus the cap product makes the homology $H_* X$ a module over the ring $H^* X$.

4.4.1. The slant product. We next introduce the slant product which bears the same relation to the cross product as the cap product does to the cup product (this could be on an SAT test). Again we give the definition on the chain level first.

Definition 4.25. The *slant product*

$$\backslash : S^q Y \times S_{p+q}(X \times Y) \to S_p X$$

is defined for $\alpha \in S^q Y$, and $z \in S_{p+q}(X \times Y)$ by

$$\alpha \backslash z = E(\alpha \otimes A(z)) \in S_p X$$

where $A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$ is the Eilenberg-Zilber map.

Similar arguments to those given for the other products given above show that

$$\langle \alpha, \beta \backslash z \rangle = \langle \alpha \times \beta, z \rangle$$

for all $\alpha \in S^q Y$ and that passing to (co)homology one obtains a well-defined bilinear map

$$\backslash : H^q Y \times H_{p+q}(X \times Y) \to H_p X.$$

If M, N, P are R modules and $M \times N \to P$ a bilinear map, one can define cap products

$$\cap : H^q(X; M) \times H_{p+q}(X; N) \to H_p(X; P)$$

and slant products

$$\backslash : H^q(Y; M) \times H_{p+q}(X \times Y; N) \to H_p(X; P).$$

(See Exercise 62.)

There are even more products (the book by Dold [**12**] is a good reference). For example, there is another slant product

$$/ : H^{p+q}(X \times Y) \times H_q Y \to H^p X.$$

Often one distinguishes between *internal products* which are defined in terms of one space X (such as the cup and cap products) and *external products* which involve the product of two spaces $X \times Y$. Of course, one can go back and forth between the two by thinking of $X \times Y$ as a single space and using the two projections p_X and p_Y and the diagonal map $\Delta : X \to X \times X$.

4.5. The Alexander-Whitney diagonal approximation

eliminate

When considered on the chain level, the various products we have defined above *do depend* on the choice of Eilenberg-Zilber map $A : S_\bullet(X \times Y) \to S_\bullet X \otimes S_\bullet Y$. Only by passing to (co)homology does the choice of A disappear. It is nevertheless often useful to work on the chain level, since there is subtle homotopy-theoretic information contained in the singular complex which leads to extra structure, such as Steenrod operations and Massey products.

We will give an explicit formula due to Alexander and Whitney for a specific choice of A. This enables one to write down formulas for the products on the chain level, and in particular gives the singular cochain complex of a space an explicit natural *associative* ring structure.

Recall that for a singular simplex $\sigma : \Delta^n \to X$ and for $n = p + q$ one can define the front p-face ${}_p\sigma : \Delta^p \to X$ and the back q-face $\sigma_q : \Delta^q \to X$ of σ.

Let $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ denote the two projections.

Definition 4.26. The *Alexander-Whitney map*

$$A : S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$$

is the natural transformation given by the formula

$$A(\sigma) = \sum_{p+q=n} {}_p(p_X \circ \sigma) \otimes (p_Y \circ \sigma)_q.$$

Thus

$$A : S_n(X \times Y) \to (S_\bullet X \otimes S_\bullet Y)_n = \bigoplus_{p+q=n} S_p X \otimes S_q Y.$$

The Alexander-Whitney map A is a natural chain map since it is given by a specific formula involving geometric simplices which is independent of the choice of X and Y.

Exercise 67. Show directly that A induces an isomorphism on H_0.

From the uniqueness part of the Eilenberg-Zilber theorem, it follows that A is a chain equivalence and can be used to define cross products and cup products. (This illustrates the power of the acyclic models theorem; the naturality of the Alexander-Whitney map and the map on H_0 suffice to conclude that A is a chain homotopy equivalence.)

To an Eilenberg-Zilber map A one can associate the corresponding diagonal approximation $\tau = A \circ \Delta_\bullet$. Taking A to be the Alexander-Whitney map, one gets the following.

Definition 4.27. The *Alexander-Whitney diagonal approximation* is the map

$$\tau(\sigma) = \sum_{p+q=n} {}_p\sigma \otimes \sigma_q.$$

Exercise 68. Show that the Alexander-Whitney diagonal approximation is a diagonal approximation.

This allows one to define a specific product structure on S^*X: for a cochain $\alpha \in S^pX$ and $\beta \in S^qX$, define $\alpha \cup \beta \in S^{p+q}X$ by

$$(\alpha \cup \beta)(\sigma) = \alpha({}_p\sigma) \cdot \beta(\sigma_q).$$

Exercise 69. By tracing through this definition of the cup product, show that $[\alpha] \cup [\beta] = [\alpha \cup \beta]$.

Exercise 70. Using the Alexander-Whitney diagonal approximation,

1. Prove that $S^*(X)$ is an *associative ring* with unit 1 represented by the cochain $c \in S^0(X) = \mathrm{Hom}(S_0X, R)$ which takes the value 1 on *every* singular 0-simplex in X.

2. Compute cap products: show that if $\alpha \in S^q X$ and σ is a singular $(p+q)$-simplex, then
$$\alpha \cap \sigma = \alpha(\sigma_q) \cdot {}_p\sigma.$$
3. Show that $(\alpha \cup \beta) \cap z = \alpha \cap (\beta \cap z)$, and so the cap product makes $S_*(X)$ into a $S^*(X)$-module.

We have already seen that cohomology is an associative and graded commutative ring with unit in Theorem **??**. However, the methods used there cannot be used to show that $S^*(X)$ is an associative ring; in fact it is not for a random choice of Eilenberg-Zilber map A.

The Alexander-Whitney map is a particularly nice choice of Eilenberg-Zilber map because it does give an associative ring structure on S^*X. This ring structure, alas, is not (graded) commutative (Steenrod squares give obstructions to its being commutative), while the ring structure on H^*X is commutative by Theorem **??**.

Notice that the de Rham cochain complex of differential forms on a smooth manifold is graded commutative, since differential forms satisfy $a \wedge b = \pm b \wedge a$. It is possible to give a natural construction of a commutative chain complex over the rationals which gives the rational homology of a space; this was done using rational differential forms on a simplicial complex by Sullivan. This fact is exploited in the subject of rational homotopy theory [**18**]. On the other hand it is impossible to construct a functor from spaces to commutative, associative chain complexes over $\mathbf{Z}$ which gives the integral homology of a space.

Exercise 71. Give an example of two singular 1-cochains α_1 and α_2 such that $\alpha_1 \cup \alpha_2 \neq -\alpha_2 \cup \alpha_1$ using the Alexander-Whitney diagonal approximation to define the cup product.

See Vick's book [**51**] for a nice example of computing the cohomology ring of the torus directly using the Alexander-Whitney diagonal approximation.

If X is a CW-complex, then the diagonal $\Delta : X \to X \times X$ is not cellular (consider X $= [0,1]$). However the cellular approximation theorem says that Δ is homotopic to a cellular map Δ'. If $\Delta'_\bullet(z) = \sum x_i \otimes y_i$, then the cup product on cellular cohomology can be defined by $(\alpha \cup \beta)(z) = \sum \alpha(x_i) \cdot \beta(y_i)$. The geometric root of the Alexander-Whitney diagonal approximation is finding a simplicial map (i.e. takes simplices to simplices and is affine on the simplices) homotopic to the diagonal map $\Delta^n \to \Delta^n \times \Delta^n$.

Let $\varepsilon : S_0 X \to R$ be the *augmentation map* $\varepsilon(\sum r_i \sigma_i) = \sum r_i$. This passes to homology $\varepsilon : H_0 X \to R$ and is an isomorphism if X is path-connected.

The cap product of a q-dimensional cocycle with a q-dimensional cycle generalizes the Kronecker pairing in the following sense.

Proposition 4.28. *For $a \in H^q X$ and $z \in H_q X$,*

$$\langle a, z\rangle = \varepsilon(\alpha \cap z).$$

Proof. We show that for any cochain $\alpha \in S^q X$ and for any chain $z \in S_q X$, using the Alexander-Whitney definition,

$$\alpha(z) = \varepsilon(\alpha \cap z).$$

By linearity it suffices to check this for $z = \sigma$.

$$\varepsilon(\alpha \cap \sigma) = \varepsilon(\alpha(\sigma_q) \cdot {}_0\sigma)) = \alpha(\sigma).$$

□

Notice that the argument shows that the equation $\langle \alpha, z\rangle = \varepsilon(\alpha \cap z)$ holds even on the chain level.

Can you prove Proposition 4.28 for an arbitrary choice of diagonal approximation using the acyclic models theorem?

Exercise 72. Show $\langle a, b \cap z\rangle = \langle a \cup b, z\rangle$.

4.6. Relative cup and cap products

The constructions of cup and cap products carry over without any difficulty to the singular chains and singular (co)homology of a pair (X, A). Naturality then implies that there is a cup product

$$H^*(X, A) \times H^*(X, B) \to H^*(X, A \cap B). \tag{4.5}$$

However, it turns out that by applying a construction that comes about in proving the excision theorem via acyclic models, one can obtain a very useful form of cup and cap products. For example there is a well-defined natural cup product

$$H^*(X, A) \times H^*(X, B) \to H^*(X, A \cup B), \tag{4.6}$$

provided A and B are open. (Explain to yourself why (4.6) is better than (4.5).)

That the pairing (4.6) exists is not so surprising if you think in terms of the Alexander-Whitney definition of cup product. Recall

$$(a \cup b)\sigma = \sum a({}_p\sigma) \cdot b(\sigma_q).$$

If the image of σ is contained in either A or B, then the sum will be zero, since a is zero on simplices in A and b is zero on simplices in B. However, if A and B are open, then one can subdivide σ so that each piece is contained in A or B. The existence of this cup product follows since subdivision disappears when passing to cohomology. We now give a formal argument.

We begin with some algebraic observations. Suppose (X, A) and (Y, B) are two pairs of spaces. Then

$$\frac{S_\bullet X}{S_\bullet A} \otimes \frac{S_\bullet Y}{S_\bullet B} \cong \frac{S_\bullet X \otimes S_\bullet Y}{S_\bullet X \otimes S_\bullet B + S_\bullet A \otimes S_\bullet Y}. \tag{4.7}$$

This is a natural isomorphism, induced by the surjection

$$S_\bullet X \otimes S_\bullet Y \to \frac{S_\bullet X}{S_\bullet A} \otimes \frac{S_\bullet Y}{S_\bullet B}.$$

Exercise 73. Prove that (4.7) is a natural isomorphism.

Now assume $X = Y$; i.e. let A and B be subsets of X. The diagonal approximation τ satisfies $\tau(S_*A) \subset S_*A \otimes S_*A$ and $\tau(S_*B) \subset S_*B \otimes S_*B$. Thus τ induces a map

$$\tau : \frac{S_*X}{S_*A + S_*B} \to \frac{S_*X \otimes S_*X}{S_*X \otimes S_*A + S_*B \otimes S_*X}.$$

The composite

$$\operatorname{Hom}\left(\frac{S_*X}{S_*A}, R\right) \otimes \operatorname{Hom}\left(\frac{S_*X}{S_*B}, R\right) \xrightarrow{\tau^* \circ \times^{\text{alg}}} \operatorname{Hom}\left(\frac{S_*X}{S_*A + S_*B}, R\right)$$

induces a cup product

$$H^p(X, A) \times H^q(X, B) \to H^{p+q}\left(\operatorname{Hom}\left(\frac{S_*X}{S_*A + S_*B}, R\right)\right). \tag{4.8}$$

Recall that $\{A, B\}$ is an excisive couple if $A, B \subset X$ and if the inclusion map $S_\bullet A + S_\bullet B \subset S_\bullet(A \cup B)$ is a chain equivalence. Recall also that if A and B are open subsets of $A \cup B$, then $\{A, B\}$ is an excisive couple.

If $\{A, B\}$ is an excisive couple , the induced map on cochain complexes

$$S^\bullet(A \cup B) \to \operatorname{Hom}(S_\bullet A + S_\bullet B, R)$$

is also a chain equivalence and hence induces an isomorphism on cohomology.

Suppose $\{A, B\}$ is an excisive couple . Then consider the two short exact sequences of chain complexes

$$\begin{tikzcd}
0 \arrow[r] & S_\bullet A + S_\bullet B \arrow[r] \arrow[d] & S_\bullet X \arrow[r] \arrow[d] & \frac{S_\bullet X}{S_\bullet A + S_\bullet B} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & S_\bullet(A \cup B) \arrow[r] & S_\bullet X \arrow[r] & \frac{S_\bullet X}{S_\bullet(A \cup B)} \arrow[r] & 0
\end{tikzcd}$$

The zig-zag lemma gives a ladder of long exact sequences on cohomology where two-thirds of the vertical arrows are isomorphisms. The five lemma shows that the rest are isomorphisms; in particular, we conclude that if $\{A, B\}$ is an excisive couple , the natural map

$$H^n(X, A \cup B) \to H^n\left(\mathrm{Hom}\left(\frac{S_\bullet X}{S_\bullet A + S_\bullet B}, R\right)\right)$$

is an isomorphism for all n. Combining this fact with the cup product of Equation (4.8) gives a proof of the following theorem.

Theorem 4.29. *If $\{A, B\}$ is an excisive couple, there is a well-defined cup product*

$$\cup : H^p(X, A) \times H^q(X, B) \to H^{p+q}(X, A \cup B).$$

□

Here is a particularly interesting application of Theorem 4.29.

Exercise 74. Show that if X is covered by open, contractible sets U_i, $i = 1, \cdots, n$, then

$$a_1 \cup \cdots \cup a_n = 0$$

for any collection of $a_i \in H^{q_i}(X)$ with $q_i > 0$.

As an example, the torus cannot be covered by two charts, since the cup product of the two 1-dimensional generators of cohomology is nontrivial (by Exercise 78).

Notice that $\{A, A\}$ is always excisive. Thus $H^*(X, A)$ is a ring. Also, $\{A, \phi\}$ is always excisive. This implies the following.

Corollary 4.30. *There is a well-defined natural cup product*

$$\cup : H^p(X, A) \times H^q X \to H^{p+q}(X, A).$$

□

Similar arguments apply to cap products. The final result is:

Theorem 4.31. *If $\{A, B\}$ is an excisive couple , then there is a well-defined cap product*

$$\cap : H^q(X, A) \times H_{p+q}(X, A \cup B) \to H_p(X, B).$$

Proof. (*Special case when $A = \emptyset$, using the Alexander-Whitney map.)*

If $A = \emptyset$, let $a \in S^q X$ be a cocycle, and let $c \in S_{p+q} X$ so that its image in $S_{p+q}(X, B)$ is a cycle, i.e. $\partial c \in S_{p+q-1} B$. Then $a \cap c \in S_q X$. Since $\partial(a \cap c) = \delta a \cap c + (-1)^q a \cap \partial c$, and $\delta a = 0$, it follows that $\partial(a \cap c) = a \cap \partial c$.

Because $\partial c \in S_*B$, $a \cap \partial c \in S_*B$ also. Indeed, if $\partial c = \sum r_i\sigma_i$, $\sigma_i : \Delta^{p+q-1} \to B$, $a \cap \partial c = \sum r_i a({}_q\sigma_i) \cdot \sigma_{i_{p-1}}$, but $\sigma_{i_{p-1}} : \Delta^{p-1} \to B \in S_*B$.

Thus $\partial(a \cap c) \in S_*B$; i.e. $a \cap c$ is a cycle in $S_*(X, B)$.

It is easy to check that replacing a by $a + \delta x$ and c by $c + \partial y$, $y \in S_{p+q+1}(X, B)$ does not change $a \cap c$ in $H_p(X, B)$. □

Exercise 75. Prove Theorem 4.31 when $B = \emptyset$ and $A \neq \emptyset$.

Use z instead of c for cycles. We might give the examples of the cohomology of complex and real projective space and prove the Hopf map is essential and the Borsuk-Ulam theorem. Use the term excisive couple instead of pair.

4.7. Projects: Poincaré duality; Intersection forms

Proof read remaining projects. Maybe delete "Algebraic limits" Maybe add a project on "simplicial sets"

4.7.1. Algebraic limits and the Poincaré duality theorem. Define both the colimit and limit of modules over a directed system (these are also called direct and inverse limit, respectively). Define an n-dimensional manifold. Define the local orientation and the fundamental class of a manifold. Define the compactly supported cohomology of a manifold; then state and prove the Poincaré duality theorem. State the Poincaré-Lefschetz duality for a manifold with boundary. If time permits, state the Alexander duality theorem. A good reference is Milnor and Stasheff's *Characteristic Classes*, [**36**, pg. 276]. Also see [**17**, pg. 217]. For the definition of limits see [**41**]; see also Section 6.5.2.

Let M be a connected manifold of dimension n.

1. If M is noncompact, then $H_nM = 0$. (Just prove the orientable case if the nonorientable case seems too involved.)
2. If M is closed (i.e. compact, connected, and without boundary), then $H_n(M; \mathbf{Z})$ is $\mathbf{Z}$ or 0. It is $\mathbf{Z}$ if and only if M is orientable.
3. Any closed n-dimensional manifold M, orientable or not, satisfies $H_n(M; \mathbf{Z}/2) = \mathbf{Z}/2$.

From these facts you can define the Poincaré duality maps. The following theorem forms the cornerstone of the subject of geometric topology.

Theorem 4.32.

1. (Poincaré duality) *Let M be a closed oriented n-dimensional manifold. Then the orientation determines a preferred generator $[M] \in H_n(M;\mathbf{Z}) \cong \mathbf{Z}$. Taking cap products with this generator induces isomorphisms*
$$\cap[M] : H^p(M;\mathbf{Z}) \to H_{n-p}(M;\mathbf{Z}).$$
2. (Poincaré-Lefschetz duality) *Let M be a compact oriented n-manifold with nonempty boundary ∂M. Then the orientation determines a preferred generator $[M, \partial M] \in H_n(M, \partial M;\mathbf{Z})$. The manifold without boundary ∂M is orientable. Let $[\partial M] = \partial([M, \partial M])$ where $\partial : H_n(M, \partial M) \to H_{n-1}(\partial M)$. Then the diagram*
$$\begin{tikzcd}
\cdots \arrow[r] & H^{p-1}(M) \arrow[r] \arrow[d, "\cap[M,\partial M]"] & H^{p-1}(\partial M) \arrow[r] \arrow[d, "\cap[\partial M]"] & H^p(M, \partial M) \arrow[r] \arrow[d, "\cap[M,\partial M]"] & H^p(M) \arrow[r] \arrow[d, "\cap[M,\partial M]"] & \cdots \\
\cdots \arrow[r] & H_{n-p+1}(M,\partial M) \arrow[r] & H_{n-p}(\partial M) \arrow[r] & H_{n-p}(M) \arrow[r] & H_{n-p}(M,\partial M) \arrow[r] & \cdots
\end{tikzcd}$$
commutes up to sign, where the horizontal rows are the long exact sequences in cohomology and homology for the pairs, and every vertical map is an isomorphism. If ∂M is a disjoint union of path components, $\partial M = \sqcup_{i=1}^k \partial_i M$, then $[\partial M] = \sum_{i=1}^k [\partial_i M] \in H_{n-1}(\partial M) = \oplus_{i=1}^k H_{n-1}(\partial_i M)$.
3. (Alexander duality) *Let M be a closed orientable n-manifold, and let $A \subset B \subset M$ be finite subcomplexes. Then $H^p(A, B)$ is isomorphic to $H_{n-p}(M - B, M - A)$.* □

The integers $\mathbf{Z}$ can be replaced by $\mathbf{Z}/2$ in Theorem 4.32, and all assertions continue to hold. Moreover, with $\mathbf{Z}/2$ coefficients the assertions hold for nonorientable manifolds as well.

The homology groups of a compact manifold are finitely generated. One way to see this is to prove that any compact manifold embeds in $\mathbf{R}^N$ for some N in such a way that it is a retract of a finite subcomplex of $\mathbf{R}^N$. Morse theory gives an easy proof that a *smooth* compact manifold is homotopy equivalent to a CW-complex with finitely many cells.

4.7.2. Exercises on intersection forms. Let M be a compact, closed, oriented n-dimensional manifold. For each p, define a bilinear form
$$H^p(M;\mathbf{Z}) \times H^{n-p}(M;\mathbf{Z}) \to \mathbf{Z}$$
by $a \cdot b = \langle a \cup b, [M] \rangle$.

Exercise 76. $a \cdot b = (-1)^{p(n-p)} b \cdot a$.

Given a finitely generated abelian group A, let $T = T(A) \subset A$ denote the torsion subgroup. Thus A/T is a free abelian group.

Exercise 77. Show that the pairing $(a, b) \mapsto a \cdot b$ passes to a well-defined pairing

$$H^p(M;\mathbf{Z})/T \times H^{n-p}(M;\mathbf{Z})/T \to \mathbf{Z}. \tag{4.9}$$

Show that this pairing is *nonsingular*; i.e. the adjoint

$$H^p(M;\mathbf{Z})/T \to \mathrm{Hom}(H^{n-p}(M;\mathbf{Z})/T, \mathbf{Z})$$

is an isomorphism of free abelian groups. (Hint: Use the universal coefficient theorem and Poincaré duality and the fact that the homology is finitely generated.)

The pairing (4.9) is called the *intersection pairing on M*. In Section 11.7 we will see that the pairing can be described by the intersection of submanifolds of M.

Exercise 78. Compute the cohomology rings $H^*(\mathbf{R}P^n; \mathbf{Z}/2)$, $H^*(\mathbf{C}P^n; \mathbf{Z})$, and $H^*(T^n; \mathbf{Z})$ using Poincaré duality and induction on n. (The first two are truncated polynomial rings; the last one is an exterior algebra.)

If dim $M = 2k$, then

$$H^k(M;\mathbf{Z})/T \times H^k(M;\mathbf{Z})/T \to \mathbf{Z}$$

is called the *intersection form of M*. It is well-defined and unimodular over $\mathbf{Z}$, i.e. has determinant equal to ± 1.

Let $V = H^k(M,\mathbf{Z})/T$. So $(V,\cdot\,)$ is an inner product space over $\mathbf{Z}$. This inner product space can have two kinds of symmetry.

Case 1. k is odd. Thus dim $M = 4\ell + 2$. Then $v \cdot w = -w \cdot v$ for $v, w \in V$, so $(V,\cdot\,)$ is a *skew-symmetric* and unimodular inner product space over $\mathbf{Z}$.

Exercise 79. Prove that there exists a basis $v_1, w_1, v_2, w_2, \cdots, v_r, w_r$ so that $v_i \cdot v_j = 0$ for all i, j; $w_i \cdot w_j = 0$ for all i, j; and $v_i \cdot w_j = \delta_{ij}$. So $(V, \cdot\,)$ has matrix

$$\begin{bmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & 0 & 1 & \\ & & -1 & 0 & \\ & & & & \ddots \end{bmatrix}$$

(all other entries zero) in this basis. Such a basis is called a *symplectic basis*. The closed surface of genus r is an example; describe a symplectic basis geometrically.

Hence unimodular skew-symmetric pairings over $\mathbf{Z}$ are classified by their rank. In other words, the integer intersection form of a $(4k-2)$-dimensional manifold M contains no more information than the dimension of $H^{2k+1}(M)$.

Case 2. k is even. Thus dim $M = 4\ell$. Then $v \cdot w = w \cdot v$, so $(V, \cdot\,)$ is a *symmetric* and unimodular inner product space over $\mathbf{Z}$.

There are 3 invariants of such unimodular symmetric forms:

1. The *rank* of $(V, \cdot\,)$ is the rank of V as a free abelian group.
2. The *signature* of $(V, \cdot\,)$ is the difference of the number of positive eigenvalues and the number of negative eigenvalues in any matrix representation of $(V, \cdot)$. (The eigenvalues of a symmetric real matrix are all real.)

Notice that in any basis $\{v_i\}$ for V, the form $\cdot$ defines a matrix Q with $Q_{ij} = v_i \cdot v_j$. Since Q is symmetric, there exists a basis *over the real numbers* so that in this basis Q is diagonal (with real eigenvalues).

Exercise 80. Show that although the eigenvalues of Q are not well-defined, their signs are well-defined, so that the signature is well-defined. (This is often called Sylvester's Theorem of Inertia.)

3. The *type* (odd or even) of $(V, \cdot)$ is defined to be *even* if and only if $v \cdot v$ is even for all $v \in V$. Otherwise the type is said to be *odd*.

The form $(V, \cdot\,)$ is called *definite* if the absolute value of its signature equals its rank; i.e. the eigenvalues of Q are either all positive or all negative.

The main result about unimodular integral forms is the following theorem, which says that unimodular, symmetric, *indefinite* forms over $\mathbf{Z}$ are determined up to isometry by their rank, signature, and type. For a proof see e.g. [**35**].

Theorem 4.33. *Suppose $(V, \cdot)$ is an indefinite, symmetric, unimodular form. If $(V, \cdot)$ is odd with rank $\ell + m$ and signature $\ell - m$ then it is equivalent to the form with diagonal matrix*

$$\bigoplus_{\ell}(1)\bigoplus_{m}(-1).$$

If $(V, \cdot)$ is even and has signature σ and rank r, let $m = \frac{1}{8}|\sigma|$, $\varepsilon \in \{-1, 0, 1\}$ be the sign of σ, i.e. $\varepsilon = \frac{\sigma}{|\sigma|}$ if $\sigma \neq 0$ and $\varepsilon = 0$ if $\sigma = 0$, and let $\ell = \frac{1}{2}(r - |\sigma|)$, so that $\ell > 0$. Then $(V, \cdot)$ is equivalent to

$$\oplus_\ell \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus_m \varepsilon E8.$$

where

$$E8 = \begin{bmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{bmatrix}$$

(all other entries zero). □

Exercise 81. Prove $E8$ is unimodular and has signature equal to 8.

The classification of definite forms is not known. It is known that:

1. For each rank, there are finitely many isomorphism types.
2. If $(V, \cdot\,)$ is definite and *even*, then $\text{sign}(V, \cdot\,) \equiv 0 \bmod 8$.
3. There are

1	even, positive definite	rank 8	forms
2	”	rank 16	”
24	”	rank 24	”
$\geq 10^7$	”	rank 32	”
$\geq 10^{51}$	”	rank 40	”

   This data is taken from [**35**].

Definition 4.34. The *signature*, sign M, of a compact, oriented $4k$-manifold without boundary M is the signature of its intersection form

$$H^{2k}(M;\mathbf{Z})/T \times H^{2k}(M;\mathbf{Z})/T \to \mathbf{Z}.$$

The following sequence of exercises introduces the important technique of *bordism* in geometric topology. The topic will be revisited from the perspective of algebraic topology in Chapter 9.

Exercise 82.

1. Let M be a closed odd–dimensional manifold. Show that the Euler characteristic $\chi(M) = 0$. Prove it for nonorientable manifolds, too.
2. Let M be a closed, orientable manifold of dimension $4k + 2$. Show that $\chi(M)$ is even.
3. Let M be a closed, oriented manifold of dimension $4k$. Show that the signature sign M is congruent to $\chi(M)$ mod 2.
4. Let M be the boundary of a compact manifold W. Show $\chi(M)$ is even.

5. Let M be the boundary of an compact, oriented manifold W and suppose the dimension of M is $4k$. Show sign $M = 0$.
6. Give examples of manifolds which are and manifolds which are not boundaries.

We have seen that even–dimensional manifolds admit intersection forms on the free part of their middle dimensional cohomology. For odd-dimensional manifolds one can construct the *linking form* on the torsion part of the middle dimensional cohomology as well. The construction is a bit more involved. We will outline one approach. Underlying this construction is the following exercise.

Exercise 83. If M is a compact, closed, oriented manifold of dimension n, show that the torsion subgroups of H^pM and $H^{n-p+1}M$ are isomorphic. (Note: you will use the fact that H_*M is finitely generated if M is a compact manifold.)

Consider the short exact sequence of abelian groups

$$0 \to \mathbf{Z} \to \mathbf{Q} \to \mathbf{Q}/\mathbf{Z} \to 0.$$

For any space X, one can dualize this sequence with the (integer) singular complex to obtain a short exact sequence of cochain complexes

$$0 \to \mathrm{Hom}_{\mathbf{Z}}(S_*X, \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{Z}}(S_*X, \mathbf{Q}) \to \mathrm{Hom}_{\mathbf{Z}}(S_*X, \mathbf{Q}/\mathbf{Z}) \to 0.$$

The zig-zag lemma gives a long exact sequence in cohomology

$$\cdots \to H^{q-1}(X; \mathbf{Q}/\mathbf{Z}) \xrightarrow{\delta} H^q(X; \mathbf{Z}) \xrightarrow{i} H^q(X; \mathbf{Q}) \to \cdots .$$

Exercise 84. Prove that if X is a finite CW-complex, then the map $\delta : H^{q-1}(X; \mathbf{Q}/\mathbf{Z}) \to H^q(X; \mathbf{Z})$ maps onto the torsion subgroup T of $H^q(X; \mathbf{Z})$.

(The map δ is a Bockstein homomorphism; see Section 11.4.)

The bilinear map

$$\mathbf{Q}/\mathbf{Z} \times \mathbf{Z} \to \mathbf{Q}/\mathbf{Z}, \quad (a, b) \mapsto ab$$

is nondegenerate, in fact induces an isomorphism $\mathbf{Q}/\mathbf{Z} \otimes \mathbf{Z} \to \mathbf{Q}/\mathbf{Z}$. This bilinear map can be used to define a cup product

$$H^{q-1}(X; \mathbf{Q}/\mathbf{Z}) \times H^q(X; \mathbf{Z}) \to H^{2q-1}(X; \mathbf{Q}/\mathbf{Z}) \tag{4.10}$$

as in Exercise 62.

Now suppose that M is a closed and oriented manifold of dimension $2k - 1$. Let $T \subset H^k(M; \mathbf{Z})$ denote the torsion subgroup.

Exercise 85. Prove that the *linking pairing of M*

$$T \times T \to \mathbf{Q}/\mathbf{Z}$$

defined by

$$(a, b) \mapsto \langle \delta^{-1}(a) \cup b, [M] \rangle$$

is well-defined. Here $\delta^{-1}(a)$ means any element z in $H^{k-1}(M; \mathbf{Q}/\mathbf{Z})$ with $\delta(z) = a$.

Show that this pairing is skew symmetric if $\dim(M) = 4\ell + 1$ and symmetric if $\dim(M) = 4\ell - 1$.

It is a little bit harder to show that this pairing is nonsingular (the proof uses Exercise 83 in the same way that the corresponding fact for the free part of cohomology is used to show that the intersection pairing is nonsingular).

Chapter 5

Fiber Bundles

Fiber bundles form a nice class of maps in topology, and many naturally occurring maps in geometric topology are fiber bundles. A theorem of Hurewicz says that fiber bundles are fibrations, and fibrations are a natural class of maps to study in algebraic topology, as we will soon see. There are several alternate notions of fiber bundles, and their relationships with one another are somewhat technical. The standard reference is Steenrod's book [**46**].

A fiber bundle is also called a Hurewicz fiber bundle or a locally trivial fiber bundle. The word "fiber" is often spelled "fibre", even by people who live in English speaking countries in the Western hemisphere.

5.1. Fiber bundles with fiber F

Let F be a topological space.

Definition 5.1. A *fiber bundle with fiber* F is a map $p : E \to B$ so that for every $b \in B$, there is a neighborhood U of b and a homeomorphism $\varphi : U \times F \to p^{-1}U$ so that

The diagram

$$\begin{tikzcd} U \times F \arrow[rr, "\varphi", "\cong"'] \arrow[dr, "p_U"'] & & p^{-1}U \arrow[dl, "p"] \\ & U & \end{tikzcd}$$

commutes.

The standard terminology is to call B the *base*, F the *fiber*, E the *total space*, and a map φ a *parameterization* (the inverse of φ is called a *chart*).

Note that p is surjective and the fiber embeds (noncanonically) into the total space, in fact, for every $b \in B$ and for every parameterization φ whose domain contains $\{b\} \times F$, there is an embedding $F \cong \{b\} \times F \xrightarrow{\varphi} p^{-1}\{b\} \subset E$ (check this). Hence $p^{-1}\{b\}$ is also called a fiber. Note that $E = \bigcup_{b \in B} p^{-1}\{b\}$, so that E is really a bundle of fibers!

We will frequently use the notation $F \hookrightarrow E \xrightarrow{p} B$ or

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

to indicate a fiber bundle $p : E \to B$ with fiber F.

5.1.1. Examples of fiber bundles. Here are some examples of fiber bundles. We will revisit these and many more examples in Sections 5.3 and 7.14.

1. The *trivial* bundle is the projection $p_B : B \times F \to B$.
2. If F has the discrete topology, any fiber bundle with fiber F is a covering map; conversely if $p : E \to B$ is a covering map with B path-connected, then p is a fiber bundle with discrete fiber.
3. The Möbius strip mapping onto its core circle is a fiber bundle with fiber $[0, 1]$.
4. The Klein bottle is a fiber bundle over the circle with fiber the circle.
5. If H is a closed subgroup of a Lie group, then $H \hookrightarrow G \to G/H$ is a fiber bundle (see [**52**, Theorem 3.5.8]).
6. The tangent bundle TM of an n-dimensional smooth manifold M is a fiber bundle with fiber $\mathbf{R}^n$. This is easier to define if M is a submanifold of Euclidean space $\mathbf{R}^k$, in which case

$$TM = \{(x, v) \in M \times \mathbf{R}^k \mid \exists\, \gamma : \mathbf{R} \to \mathbf{R}^k, \gamma(0) = x, \gamma'(0) = v\}$$

7. If M is an n-dimensional Riemannian manifold, then a orthogonal frame at $x \in M$ is an ordered orthonormal basis for the tangent space at x. Let $F_O(M) \subset (TM)^n$ be the subspace of orthogonal frames. Then $O(n) \hookrightarrow F_O(M) \to M$ is a fiber bundle.
8. A *smooth submersion* $f : M \to N$ is a smooth map between smooth manifolds so that the differential $df_m : T_m M \to T_{f(m)} N$ is onto for each $m \in M$. The implicit function theorem shows that $f^{-1}\{n\}$ is a smooth submanifold of M for any $n \in N$. Ehresmann's Fibration

Theorem says that a surjective smooth submersion $f : M \to N$ with M compact and N connected is a fiber bundle.

9. If $p : E \to B$ is a fiber bundle with fiber F and $B' \subset B$ is a subspace, then $p^{-1}B' \to B'$ is a fiber bundle with fiber F. It is called the *restriction of* p *to* B'.

Motivated by the example of the trivial bundle, many authors call a fiber bundle a locally trivial bundle or say that the map p is locally a product. Sometimes we will be lazy and say that E is a fiber bundle instead of saying that $p : E \to B$ is a fiber bundle.

A map $s : B \to E$ satisfying $p \circ s = \mathrm{Id}_B$ is called a *section*. The reader should study the above examples and see which ones have sections.

5.2. Fiber bundles with structure group

Suppose $p : E \to B$ is a fiber bundle with fiber F. For each $b \in B$, $p^{-1}\{b\}$ is homeomorphic to the fiber, but the homeomorphism is parameterization dependent, hence two parameterizations give rise to a self-homeomorphism $F \xrightarrow{\cong} p^{-1}\{b\} \xleftarrow{\cong} F$. This is an element of $\mathrm{Homeo}(F)$, the group of homeomorphisms from F to F. More precisely, given two parameterizations $\varphi : U \times F \to p^{-1}U$ and $\varphi' : U' \times F \to p^{-1}U'$ there is a function

$$\theta_{\varphi,\varphi'} : U \cap U' \to \mathrm{Homeo}(F)$$

so that

$$\varphi'(b, f) = \varphi(b, \theta_{\varphi,\varphi'}(b)(f))$$

for all $b \in U \cap U'$. This leads to three points. First, it would be good to endow $\mathrm{Homeo}(F)$ with a topology so that $\theta_{\varphi,\varphi'}$ is continuous. Second, we would like to reconstruct the fiber bundle from the transition functions $\{\theta_{\varphi,\varphi'}\}$, similar to the abstract definition of a smooth manifold. Third, and most important, we want to be able to put restrictions on the homeomorphisms $\theta_{\varphi,\varphi'}(b)$. For example, if $F = \mathbf{R}^n$, and the homeomorphisms happen to be linear maps, then there will be a well-defined vector space structure on each fiber $p^{-1}\{b\}$.

5.2.1. Group actions.

A *topological group* G is a group which is also a topological space so that the multiplication map $\mu : G \times G \to G, \mu(g, h) = gh$ and the inversion map $\iota : G \to G, \iota(g) = g^{-1}$ are continuous.

A topological group is a *Lie group* if it is a smooth manifold and the multiplication and inverse maps are smooth. For example, any closed subgroup of $GL(n, \mathbf{R})$ is a Lie group.

Definition 5.2. A *(left) action of a topological group G on a topological space X* is a continuous map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

so that for the identity element $e \in G$ and for all $x \in X$ and $g, g' \in G$,

$$\begin{aligned} ex &= x \\ g(g'x) &= (gg')x. \end{aligned}$$

This defines a group homomorphism $G \to \mathrm{Homeo}(X)$, $\quad g \mapsto (x \mapsto gx)$. If $\mathrm{Homeo}(X)$ is given the compact-open topology and X is a locally compact Hausdorff space, a group action determines and is determined by a continuous homomorphism

$$G \to \mathrm{Homeo}(X).$$

In some sense (see Theorem 7.5), this is true for general G and X, and henceforth we make no distinction between group actions and such continuous homomorphisms.

The *orbit* of a point $x \in X$ is the space $Gx = \{gx \mid g \in G\}$.

The *orbit space* or *quotient space* X/G is the quotient space $X/{\sim}$, with the equivalence relation $x \sim y$ if and only if there exists a $g \in G$ such that $x = gy$.

The *fixed set* is $X^G = \{x \in X \mid gx = x \text{ for all } g \in G\}$.

An action is called *free* if $gx \neq x$ for all $x \in X$ and for all $g \neq e$.

An action is called *effective* if the homomorphism $G \to \mathrm{Homeo}(X)$ is injective. Thus for any $g \neq e$, there is an $x \in X$ so that $gx \neq x$.

We refer to a space with a G-action as a G-space. A map of G-spaces is a continuous map $f : X \to Y$ so that $f(gx) = gf(x)$ for all $x \in X$ and $g \in G$ (one also says that f is an *equivariant map*).

We have defined a left action of G on X. There is a corresponding notion of right G-action $(x, g) \mapsto xg$. Moreover, any left action determines a right action (and vice versa) by *defining* xg to be $g^{-1}x$.

There is even the notion of a (G, H)-biaction, denoted ${}_GX_H$, which is a space X which is both a left G-space and a right H-space so that the actions commute in the sense that $(gx)h = g(xh)$ for all $g \in G$, $x \in X$, and $h \in H$.

Group actions come up so often in mathematics that it is worth examining their categorical foundations. Assume that G is a group (a topological group with the discrete topology, if you insist). If X is an object in a category $\mathcal{C}$, let $\mathrm{Aut}_{\mathcal{C}} X$ be the group of invertible morphisms $f : X \to X$. An *action of G on X* is a group homomorphism $G \to \mathrm{Aut}_{\mathcal{C}} X$. Familar examples are $\mathcal{C}$ = Set, R-Mod, or complex vector spaces where an action of G on X is also called a G-set, an RG-module, or group representation, respectively. To push the categorical point further, a group is just a category with one object, all of whose morphisms are invertible, a group action is simply a functor $G \to \mathcal{C}$, and an equivariant map is a natural transformation.

We can equip G and $\mathrm{Aut}_{\mathcal{C}}\, X$ with extra structure and require the homomorphism $G \to \mathrm{Aut}_{\mathcal{C}}\, X$ to preserve the structure as in the case of an action of a topological group. For example, one could take the C^∞ topology on $\mathrm{Diff}(X)$ if X is a smooth manifold.

5.2.2. Definition of a fiber bundle with structure group.

Definition 5.3. Let G be a topological group acting on a space F. A *fiber bundle E over B with fiber F and structure group G* is a map $p : E \to B$ together with a collection of homeomorphisms $\mathcal{A} = \{\varphi : U_\varphi \times F \to p^{-1}U_\varphi \mid U_\varphi \subset B\}$ (φ is called a *parameterization*) and a collection of continuous maps $\Theta = \{\theta_{\varphi,\psi} : U_\varphi \cap U_\psi \to G \mid (\varphi, \psi) \in \mathcal{A} \times \mathcal{A}\}$ ($\theta_{\varphi,\psi}$ is called a *transition function*) such that:

1. The diagram

$$\begin{tikzcd} U_\varphi \times F \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}"'] & & p^{-1}U_\varphi \arrow[dl, "p"] \\ & U_\varphi & \end{tikzcd}$$

commutes for each parameterization φ.

2. $\{U_\varphi\}_{\varphi \in \mathcal{A}}$ is an open cover of B.
3. For any two parameterizations φ and ψ, the following equation holds for all $b \in U_\varphi \cap U_\psi$ and all $f \in F$

$$\psi(b, f) = \varphi(b, \theta_{\varphi,\psi}(b) f).$$

4. For any three parameterizations $\varphi, \chi, \psi \in \mathcal{A}$, the following equation holds for all $b \in U_\varphi \cap U_\chi \cap U_\psi$

$$\theta_{\varphi,\psi}(b) = \theta_{\varphi,\chi}(b)\theta_{\chi,\psi}(b) \in G.$$

For shorthand one often abbreviates $(p, E, B, F, G, \mathcal{A}, \Theta)$ by E and calls a fiber bundle E over B with fiber F and structure group G a (G, F)-bundle.

The collection $\mathcal{A}$ is called an *atlas*. Any atlas is contained in a unique maximal atlas $M(\mathcal{A})$ with transition functions $M(\Theta)$. We identify the fiber bundles $(p, E, B, F, G, \mathcal{A}, \Theta)$ and $(p, E, B, F, G, M(\mathcal{A}), M(\Theta))$. Another approach would be to require that a fiber bundle have a maximal atlas, but an advantage of Definition 5.3 is that fiber bundles can be described using small atlases.

Perhaps we should call the concept defined in Definition 5.3 a pre-fiber bundle, then say two pre-fiber bundles are equivalent if they have the same maximal atlas, and define a fiber bundle to be an equivalence class of pre-fiber bundles. But we prefer to be a bit sloppy are refer to both concepts as "a fiber bundle."

It is not hard to see that a fiber bundle with fiber a locally compact Hausdorff space F is the same thing as a fiber bundle with structure group $\mathrm{Homeo}(F)$ given the compact-open topology.

Exercise 86. Show that a fiber bundle with trivial structure group is (isomorphic to) a trivial bundle.

Given a fiber bundle $p : E \to B$ with fiber F, the structure group G is not uniquely determined, indeed it can always be enlarged to $\mathrm{Homeo}(F)$. But one attempts to make the structure group as "small" as possible; this equips the bundle with the maximal structure. Of course making the structure group small means being selective when choosing the set of parameterizations $\mathcal{A}$.

The definition of fiber bundle with structure group is slick, and it helps to play with examples.

Exercise 87. Exhibit each of the examples of fiber bundles in Section 5.1.1 as fiber bundles with structure group, attempting to make the structure group as small as possible.

There are interesting cases of bundles where G does not act effectively on F, for example, when studying local coefficients or Spin structures. But it is usually the case that G acts effectively on F, and in this case the definition of a (G, F)-bundle simplifies since the parameterizations determine the transition functions. We spell this out by giving a definition in this special case.

Definition 5.4. Let G be a topological group acting effectively on a space F. A *fiber bundle E over B with fiber F and structure group G* is a map $p : E \to B$ together with a collection of homeomorphisms $\mathcal{A} = \{\varphi : U_\varphi \times F \to p^{-1}U_\varphi \mid U_\varphi \subset B\}$ such that:

1. The diagram

$$\begin{tikzcd} U_\varphi \times F \arrow[rr, "\varphi"] \arrow[dr, "\mathrm{pr}"'] & & p^{-1}U_\varphi \arrow[dl, "p"] \\ & U_\varphi & \end{tikzcd}$$

commutes for each parameterization $\varphi \in \mathcal{A}$.

2. $\{U_\varphi\}_{\varphi\in\mathcal{A}}$ is an open cover of B.
3. For any two parameterizations φ and ψ, there is a continuous map $\theta_{\varphi,\psi} : U_\varphi \cap U_\psi \to G$ so that for all $b \in U_\varphi \cap U_\psi$ and all $f \in F$
$$\psi(b, f) = \varphi(b, \theta_{\varphi,\psi}(b)f).$$

There is a local point of view of a fiber bundle. It is as a union of spaces of the form $U_i \times F$ where $\{U_i\}$ is an open cover of B, together with a set of rules which describe how to move from one $U_i \times F$ to another. The following definition and exercise outline this point of view.

Definition 5.5. Fix a space B and a topological group G. A *1-cocycle with values in* G is an indexed collection of open sets $\{U_\varphi\}_{\varphi\in\mathcal{A}}$ which gives an open cover and a set of continuous functions
$$\Theta = \{\theta_{\varphi,\psi} : U_\varphi \cap U_\psi \to G \mid (\varphi, \psi) \in \mathcal{A} \times \mathcal{A}\}$$
so that for any $\varphi, \chi, \psi \in \mathcal{A}$, then the restrictions of the θ's to $U_\varphi \cap U_\chi \cap U_\psi$ satisfy
$$\theta_{\varphi,\psi} = \theta_{\varphi,\chi} \cdot \theta_{\chi,\psi}$$
where the $\cdot$ means the pointwise multiplication of functions to G.

Exercise 88.

1. Given a G-action on a space F, an open cover $\{U_\varphi\}_{\varphi\in\mathcal{A}}$ of a space B and a 1-cocycle Θ, then there exists a fiber bundle $p : E \to B$ with structure group G, fiber F, and transition functions Θ.
2. Given a fiber bundle over B with structure group G, then the parameterizations and transition functions determine an open cover $\mathcal{U}$ and 1-cocycle Θ.

The terminology comes from the construction of a variant of cohomology called *Čech cohomology.* A G-valued Čech *1-cochain for an open cover* $\{U_\varphi\}_{\varphi\in\mathcal{A}}$ is a collection of maps $\theta_{\varphi,\psi} : U_\varphi \cap U_\psi \to G$, one for each ordered pair $(\varphi, \psi) \in \mathcal{A} \times \mathcal{A}$. Thus the transition functions for a fiber bundle with structure group G determine a Čech 1-cochain. From this point of view the equation above translates into the requirement that the Čech 1-cochain defined by the $\theta_{\varphi,\psi}$ is in fact a 1-cocycle.

This is a useful method of understanding bundles since it relates them to (Čech) cohomology. Cohomologous cocycles define isomorphic bundles, and so isomorphism classes of bundles over B with structure group G can be identified with $H^1(B; G)$ (this is one starting point for the theory of characteristic classes; we will take a different point of view in a later chapter). One must be extremely cautious when working this out carefully. For example,

G need not be abelian (and so what does $H^1(B;G)$ mean?) Also, one must consider *continuous cocycles* since the $\theta_{\varphi,\psi}$ should be continuous functions. One also needs to allow refinements of open covers. We will not pursue this line of exposition any further in this book.

5.3. More examples of fiber bundles

5.3.1. Vector bundles.

Exercise 89. Let $F = \mathbf{R}^n$, and let $G = GL(n, \mathbf{R}) \subset \text{Homeo}(\mathbf{R}^n)$. A fiber bundle over B with fiber $\mathbf{R}^n$ and structure group $GL(n, \mathbf{R})$ is called a *vector bundle* of *rank* n *over* B. Show that each fiber $p^{-1}\{b\}$ can be given a well-defined vector space structure.

(Similarly, one can take $F = \mathbf{C}^n$, $G = GL(n, \mathbf{C})$ to get a *complex vector bundle*.)

In particular, if M is a differentiable n-manifold and TM is the set of all tangent vectors to M, then $p : TM \to M$ is a vector bundle of dimension n.

5.3.2. S^1-Bundles over S^2.

Note that $SO(2)$ acts on S^1. Note also that the Lie groups $SO(2)$ and S^1 are isomorphic.

For every integer n, construct an S^1-bundle over S^2 with structure group $SO(2)$ as follows. Let $U = \{(x, y, z) \in S^2 \mid z > -\frac{1}{2}\}$ and $V = \{(x, y, z) \in S^2 \mid z < \frac{1}{2}\}$; U and V cover S^2. Form the S^1 bundle over S^2 by taking the transition function $\theta_{U,V} : U \cap V \to SO(2)$ a map of degree n. More precisely, the equator of S^2 is a deformation retract of $U \cap V$. Define $\theta_{U,V}$ to be the composition of this deformation retract with a degree n map from the equator to $SO(2)$. Construct the bundle $E = U \times S^1 \coprod V \times S^1/\sim$ using the solution to Exercise 88 (this is a special case of the clutching construction described below). The resulting bundle $E_n \to S^2$ is said to have *Euler number* n. Every S^1-bundle over S^2 with structure group $SO(2)$ is isomorphic to exactly one $E_n \to S^2$.

For example, when $n = 0$, one can take $\theta_{U,V}$ to be the constant map at $1 \in SO(2)$. This clearly yields the product bundle $p : S^2 \times S^1 \to S^2$. When $n = 1$, one can take $\theta_{U,V}$ to be the composite of the defomation retract to the equator with the identity map $S^1 \to SO(2)$. One can show that the space E_1 is homeomorphic 3-sphere S^3 and $E_1 \to S^2$ is the famous *Hopf fibration* $S^1 \hookrightarrow S^3 \to S^2$.

In general, notice that U and V are open 2-dimensional disks D, and hence E_n is obtained by gluing two copies of $D \times S^1$ together. A simple application of the Seifert-van Kampen theorem shows that $\pi_1(E_n) = \mathbf{Z}/n$.

Careful thought shows that E_n is isomorphic to E_{-n} when considered as bundles with structure group $O(2)$ and fiber S^1.

Exercise 90. Let $S(TS^2)$ be the sphere bundle of the tangent bundle of the 2-sphere, i.e. the tangent vectors of unit length, specifically

$$S(TS^2) = \{(P, v) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid P, v \in S^2 \text{ and } P \cdot v = 0\}.$$

Let $SO(3)$ be the 3-by-3 orthogonal matrices of determinant one (the group of orientation preserving rigid motions of $\mathbf{R}^3$ preserving the origin). This is a topological group. Show that the spaces E_2, $S(TS^2)$, $SO(3)$, and $\mathbf{R}P^3$ are all homeomorphic.

(Hints:

1. Given two perpendicular vectors in $\mathbf{R}^3$, a third one can be obtained by the cross product.
2. On one hand, every element of $SO(3)$ is rotation about an axis. On the other hand $\mathbf{R}P^3$ is $D^3/\!\sim$, where you identify antipodal points on the boundary sphere.)

This gives four incarnations of the S^1-bundle over S^2 with Euler number equal to 2:

1. $p : E_2 \to S^2$, $[u, f] \mapsto u$, $[v, f] \mapsto v$.
2. $p : S(TS^2) \to S^2$, $(P, v) \mapsto P$.
3. $p : SO(3) \to S^2$, $A \mapsto A \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.
4. $p : \mathbf{R}P^3 \to S^2$, $[z_1, z_2] \mapsto z_1/z_2$, viewing $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}, \mathbf{R}P^3 = S^3/(z_1, z_2) \sim (-z_1, -z_2)$, and $S^2 = \mathbf{C} \cup \{\infty\}$.

5.3.3. Clutching. Suppose a topological group G acts on a space F. Let SX be the unreduced suspension of a space X,

$$SX = \frac{X \times I}{(x, 0) \sim (x', 0), (x, 1) \sim (x', 1)}.$$

Then given a map $\beta : X \to G$, define

$$E = \frac{(X \times [0, 1/2] \times F) \amalg (X \times [1/2, 1] \times F)}{\sim}$$

where the equivalence relation is given by identifying $(x, 0, f) \sim (x', 0, f)$, $(x, 1, f) \sim (x', 1, f)$, and $(x, 1/2, f) \sim (x, 1/2, \beta(x) f)$; the last relation glues the summands of the disjoint union. Projecting to the first two factors defines a map $E \to SX$ called the *bundle over SX with clutching function* $\beta : X \to G \subset \mathrm{Homeo}(F)$.

Exercise 91. Show that projection onto the first two coordinates gives a fiber bundle $p : E \to SX$ with fiber F and structure group G. Give some examples with $X = S^0$ and $X = S^1$. In particular, show that the S^1-bundle over $S^2 = S(S^1)$ with Euler number equal to n is obtained by clutching using a degree n map $S^1 \to S^1$.

Clutching provides a good way to describe fiber bundles over spheres. For X a CW-complex, all bundles over SX arise by this clutching construction, in fact isomorphism classes of fiber bundles over SX with fiber F and structure group G are in bijective correspondence with homotopy classes $[X, G]$. This follows from the fact that any fiber bundle over a contractible CW-complex is trivial (this can be proven using obstruction theory). Since SX is the union of two contractible spaces, $X \times [0, \frac{1}{2}]/\sim$ and $X \times [\frac{1}{2}, 1]/\sim$, any bundle over SX is obtained by clutching.

5.3.4. Bundle of discrete abelian groups. An important type of fiber bundle is the following. Let A be a group and G a subgroup of the automorphism group $\mathrm{Aut}(A)$. Then any fiber bundle E over B with fiber A and structure group G has the property that each *fiber* $p^{-1}\{b\}$ has a group structure. This group is isomorphic to A, but the isomorphism is not canonical in general.

We have already run across an important case of this, namely vector bundles, where $A = \mathbf{R}^n$ and $G = GL(n, \mathbf{R})$. Here is another situation, which will be important in specifying a local coefficient system in Chapter 6.

Definition 5.6. Let A be an abelian group, topologized with the discrete topology. A *bundle of discrete abelian groups with fiber* A is a fiber bundle with fiber A and structure group $\mathrm{Aut}(A)$, given the discrete topology. A *bundle of discrete abelian groups* is a bundle of discrete abelian groups with fiber A for some A.

Exercise 92. Define a bundle of R-modules, generalizing the case of $\mathbf{Z}$-modules above.

The basic principle at play here is: *if the structure group preserves a certain structure on F, then every fiber $p^{-1}\{b\}$ has this structure.* For example, in a bundle of discrete abelian groups each fiber $p^{-1}\{b\}$ is an abelian group. A rank n vector bundle corresponds to the case when the structure group is group of linear automorphisms of a rank n vector space, in which case each fiber $p^{-1}\{b\}$ is a rank n vector space.

5.3.5. Other structures. Other examples of fibers with a structure include the following.

1. $F = \mathbf{R}^n$ equipped with an inner product; $G = O(n) \subset GL(n, \mathbf{R})$ consists of those linear isomorphisms which preserve the inner product. Then each fiber $p^{-1}\{b\}$ inherits an inner product.
2. Similarly one can define a *Hermitian vector bundle* by taking $F = \mathbf{C}^n$ and $G = U(n)$.
3. Let F be a Riemannian manifold and suppose that G acts isometrically on F. Then each fiber $p^{-1}\{b\}$ in a bundle with structure group G and fiber F is (noncanonically) isometric to F.
4. Take F to be a smooth manifold and G a subgroup of the diffeomorphism group of F (with the C^∞ strong topology, say). Then each fiber in a fiber bundle with structure group G will be diffeomorphic to F.

Exercise 93. Invent your own examples of fibers with structure and the corresponding fiber bundles.

A *Euclidean vector bundle* is a vector bundle $p : E \to B$ equipped with a continuous map $E \times_B E \to \mathbf{R}$ (see Definition 1.3), which restricts to an inner product on each fiber. Thus for every $b \in B$, the restricted map $p^{-1}\{b\} \times p^{-1}\{b\} \to \mathbf{R}$ is an inner product. The map $\| \; \| : E \to B, \quad \|x\| = \sqrt{\langle x, x\rangle}$ is called a bundle metric. Note that a Riemannian manifold is a smooth manifold with an Euclidean tangent bundle.

Exercise 94. Show for any rank n Euclidean vector bundle the set of local parameterizations which are fiberwise isometries forms an atlas (hint: use the Gram-Schmidt method), and hence that any rank n Euclidean vector bundle admits the structure of an $(O(n), \mathbf{R}^n)$-bundle. Conversely, show that any $(O(n), \mathbf{R}^n)$-bundle can be given the structure of a Euclidean vector bundle so that the local parameterizations are isometries, and the metric is unique up to scaling if the base space is connected.

5.4. Principal bundles

Principal bundles are special cases of fiber bundles, but nevertheless can be used to construct any fiber bundle. Conversely any fiber bundle determines a principal bundle. A principal bundle is technically simpler, since the fiber is just $F = G$ with a canonical action.

Let G be a topological group. It acts on itself by *left translation*.

$$G \to \mathrm{Homeo}(G), \quad g \mapsto (x \mapsto gx).$$

Definition 5.7. A *principal G-bundle over B* is a fiber bundle $p : P \to B$ with fiber $F = G$ and structure group G acting by left translations.

Proposition 5.8. *If $p : P \to B$ is a principal G-bundle, then G acts freely on P on the right with orbit space B (i.e. p is a quotient map and the fibers of p are orbits.)*

Proof. Notice first that G acts on the local trivializations *on the right*:

$$(U \times G) \times G \to U \times G$$
$$(b, g) \cdot g' = (b, gg').$$

This commutes with the action of G on itself by left translation (i.e. $(g''g)g' = g''(gg')$, so one gets a well-defined right action of G on E using the identification provided by a parameterization

$$\varphi : U \times G \to p^{-1}(U).$$

More explicitly, define $\varphi(b, g) \cdot g' = \varphi(b, gg')$. If φ' is another parameterization over U, then

$$\varphi'(b, g) = \varphi(b, \theta_{\varphi,\varphi'}(b)g),$$

and $\varphi'(b, gg') = \varphi(b, \theta_{\varphi,\varphi'}(b)(gg')) = \varphi'(b, (\theta_{\varphi,\varphi'}(b)g)g')$, so the action is independent of the choice of parameterization. The action is free, since the local action $(U \times G) \times G \to U \times G$ is free, and since $(U \times G)/G = U$ it follows that $P/G = B$. □

In fact, one can give an alternate definition of principal G-bundles, avoiding atlases and transition functions:

Exercise 95. If a topological group acts G freely on the right on a space P, and if $P \to P/G$ is a fiber bundle with fiber G, then it is a principal G-bundle.

Remark. A (G, H)-*bispace* is a space F with a left G-action and a right H-action which commute in the sense that $(gx)h = g(xh)$ for all $g \in G$, $x \in F$, and $h \in H$. Sometimes one writes ${}_GF_H$. If $p : E \to B$ is a fiber bundle with fiber a (G, H)-bispace F and structure group G, then E admits a right H-action so that the parameterizations are equivariant with respect to the right H-action. Note that G is a (G, G)-bispace via left and right translation.

The converse to Proposition 5.8 holds in some important cases. We state the following fundamental theorems without proof, referring you to [**7**, Theorem II.5.8] and [**39**, Section 4.1].

Theorem 5.9 (Gleason)**.** *Suppose that P is a compact Hausdorff space, and G is a compact Lie group acting freely on P. Then the orbit map*

$$P \to P/G$$

is a principal G-bundle.

Theorem 5.10. *Suppose P is a topological group and G is a closed Lie subgroup (i.e. G is closed in P, is a subgroup of P, and admits the structure of a smooth manifold so that multiplication and inversion are smooth maps). Then the orbit map*

$$P \to P/G$$

is a principal G-bundle.

Exercise 96. A map $f : X \to Y$ has a *section* if there is a map $s : Y \to X$ so that $f \circ s = \mathrm{Id}$. A map $f : X \to Y$ has *local sections* if every $y \in Y$ has a neighborhood U so that $f^{-1}U \to U$ has a section.

1. Show that any fiber bundle has local sections.
2. If a principal G-bundle has a section, then it is a trivial bundle. (Can you think of nontrivial fiber bundles with sections?)

We illustrate the ideas of Exercise 98 and Theorem 5.9 with free S^1 actions on S^3. Identify $\mathbf{R}^4$ with $\mathbf{C}^2$, so that $S^3 = \{(z_1, z_2) \in \mathbf{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}$. The group of unit complex numbers $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ acts freely on S^3 by

$$(z_1, z_2) \cdot z = (z_1 z, z_2 z).$$

The quotient S^3/S^1 is homeomorphic to S^2. An explicit identification is given by observing that $S^2 = \mathbf{C} \cup \infty$ via stereographic projection; the quotient map $p : S^3 \to S^2$ is then given by $(z_1, z_2) \mapsto z_1/z_2$. Theorem 5.9 shows that $p : S^3 \to S^2$ is a fiber bundle with fiber S^1, this is again the Hopf fibration, which has Euler number equal to 1.

For $n \neq 0$, consider the cyclic subgroup C_n of S^1 of order n generated by the primitive n-th root of unity $\zeta_n = e^{2\pi i/n}$. The Hopf fibration $S^3 \to S^2$ factors through the quotient map $S^3 \to S^3/C_n$. The group S^1/C_n acts freely on S^3/C_n with quotient S^2, and so one obtains a bundle $p_n : S^3/C_n \to S^2$ with fibers S^1/C_n. The map $[z] \mapsto z^n$ gives a group isomorphism $S^1/C_n \cong S^1$ and so p_n is a principal S^1-bundle. It is not hard to show that this is the S^1-bundle over S^2 with Euler number n. Exercise 98 shows that the quotient $S^3 \to S^3/C_n$ is a covering space (i.e. a principal C_n bundle) and in fact S^3/C_n is the the 3-dimensional lens space $L(n, 1)$ (see Section 12.6). The spaces E_n and E_{-n} are homeomorphic, but their structures as bundles over S^2 are different: the structure group acts on S^1 by counterclockwise rotation for E_n and clockwise rotation for E_{-n}. They become isomorphic if we view them as bundles with structure group $O(2)$ rather than $SO(2) = S^1$.

5.4.1. Covering spaces.

As a familiar example of a principal bundle, any regular covering space $p : E \to B$ is a principal G-bundle with $G = \pi_1 B/p_* \pi_1 E$. Here G is given the discrete topology. Recall that a regular covering space $p : E \to B$ is a covering space where E and B are path-connected and where $p_*\pi_1 E$ is a normal subgroup of $\pi_1 B$. A nonregular covering space is not a principal bundle.

Exercise 97 guides you through the details. It relies on the following fundamental result about covering spaces (see e.g. [**28**, Theorem 5.5.1]):

Theorem 5.11. *Let $\widetilde{X}$, X, and Y be path connected and locally path connected spaces, and $p : \widetilde{X} \to X$ a covering map. Suppose that $\phi : Y \to X$ is a continuous map. Let $y_0 \in Y, x_0 \in X, \tilde{x}_0 \in \widetilde{X}$ be points satisfying $p(\tilde{x}_0) = x_0$ and $\phi(y_0) = x_0$.*

Then there exists a map $\tilde{\phi} : (Y, y_0) \to (\widetilde{X}, \tilde{x}_0)$ satisfying $p \circ \tilde{\phi} = \phi$ if and only if $\phi_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\widetilde{X}, \tilde{x}_0))$. If $\tilde{\phi}$ exists, it is unique.

Exercise 97. Let X be a path connected and locally path connected space which admits a universal cover $p : \widetilde{X} \to X$. Fix base points $\tilde{x}_0 \in \widetilde{X}$ and $x_0 \in X$ so that $p(\tilde{x}_0) = x_0$. Let $\pi = \pi_1(X, x_0)$ denote the fundamental group of X based at x_0, with the multiplication defined so that $\gamma_1\gamma_2$ denotes the loop γ_1 followed by γ_2. Let $F = p^{-1}(x_0)$ denote the fiber of p over the base point x_0.

1. Given $\gamma \in \pi$ and $f \in F$, define $f\gamma$ to be the endpoint of the unique lift of γ starting at f. Show that this defines a free right action of π on F.
2. Show that this action is *transitive*, that is, given any $f_1, f_2 \in F$, there exists a $\gamma \in \pi$ so that $f_2 = f_1\gamma$. Conclude that the map $\pi \to F$ given by $\gamma \to \tilde{x}_0\gamma$ is a bijection.

Let $\mathrm{Aut}(p)$ be the group of homeomorphisms $h : \widetilde{X} \to \widetilde{X}$ satisfying $p \circ h = p$. Elements of $\mathrm{Aut}(p)$ are called *covering transformations* or *deck transformations*. Show that:

3. Any continuous map $h : \widetilde{X} \to \widetilde{X}$ satisfying $p \circ h = p$ is a covering transformation.
4. Composition of functions, $(h_1, h_2) \mapsto h_1 \circ h_2$, defines a group structure on $\mathrm{Aut}(p)$.
5. The group $\mathrm{Aut}(p)$ acts freely on $\widetilde{X}$ on the left via $hx := h(x)$, with orbit space X.
6. Given $\gamma \in \pi$ and $z \in \widetilde{X}$, choose a path $\alpha : (0,1) \to \widetilde{X}$ starting at z and ending at $\tilde{x}_0$. Define γz to be the endpoint of the unique lift of $p(\alpha)\gamma p(\alpha^{-1})$ starting at z. Prove that $h_\gamma : \widetilde{X} \to \widetilde{X}$ defined by $h_\gamma(z) = \gamma z$ lies in $\mathrm{Aut}(p)$, and that the resulting function

$$\pi \to \mathrm{Aut}(p), \ \gamma \mapsto h_\gamma$$

is an isomorphism of groups. Conclude that π acts freely on the left on $\widetilde{X}$ with orbit space X.

7. If $N \subset \pi$ is a *normal* subgroup, show that π/N acts freely on $\widetilde{X}/N$ with quotient X.

Since, by convention, principal G-bundles are equipped with *right* free G-actions, *in this book we choose* $\pi_1(X, x_0)$ *to act on the right on its universal cover* $\tilde{X}$, by converting the left action of Part 6 of Exercise 97 to a right action via $z\gamma := \gamma^{-1}z$.

This contrasts with the more common convention of having $\pi_1(X, x_0)$ act on the left, but the payoff is that this allows a unified treatment of principal G bundles and regular covering spaces.

Exercise 98. Any free (right) action of a finite group G on a Hausdorff space P gives a regular cover and hence a principal G-bundle $P \to P/G$.

More generally, if P is a space and G a discrete group acting freely and properly discontinuously on P, then the quotient map $P \to P/G$ is a principal G-bundle and a regular cover. Conversely, every regular cover arises as the quotient of a free, properly discontinuous action. For details see [**38**, Theorem 81.5].

5.5. Associated bundles

Exercise 88 shows that the transition functions $\theta : U \cap V \to G$ and the action of G on F determine a fiber bundle over B with fiber F and structure group G.

As an application note that if a topological group G acts on spaces F and F', and if $p : E \to B$ is a fiber bundle with fiber F and structure group G, then one can use the transition functions from p to define a fiber bundle $p' : E' \to B$ with fiber F' and structure group G with *exactly the same transition functions.*

Thus there is a one-to-one correspondence

$$\{\text{isomorphism classes of } (G, F, B)\text{-bundles}\} \longleftrightarrow \\ \{\text{isomorphism classes of } (G, F', B)\text{-bundles}\}$$

This is called *changing the fiber from F to F'.* This can be useful because the topology of E and E' may change. For example, take $G = GL(2, \mathbf{R})$,

$F = \mathbf{R}^2$, $F' = \mathbf{R}^2 - \{0\}$ and the tangent bundle of the 2-sphere:

$$\begin{tikzcd} \mathbf{R}^2 \arrow[r] & TS^2 \arrow[d, "p"] \\ & S^2 \end{tikzcd}$$

After changing the fiber from $\mathbf{R}^2$ to $\mathbf{R}^2 - \{0\}$, we obtain

$$\begin{tikzcd} \mathbf{R}^2 - \{0\} \arrow[r] & TS^2 - z(S^2) \arrow[d, "p"] \\ & S^2 \end{tikzcd}$$

where $z : S^2 \to TS^2$ denotes the zero section.

With the second incarnation of the bundle the twisting becomes revealed in the homotopy type, because the total space of the first bundle has the homotopy type of S^2, while the total space of the second has the homotopy type of the sphere bundle $S(TS^2)$ and hence of $\mathbf{R}P^3$ according to Exercise 90.

A fundamental case of changing fibers occurs when one lets the fiber F' be the group G itself, with the left translation action. Then the transition functions for the fiber bundle

$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

determine, via the construction of Exercise 88, a principal G-bundle

$$\begin{tikzcd} G \arrow[r] & P(E) \arrow[d, "p"] \\ & B. \end{tikzcd}$$

We call this the *principal G-bundle underlying the fiber bundle $p : E \to B$ with structure group G.*

Conversely, to a principal G-bundle and an action of G on a space F one can associate a fiber bundle, again using Exercise 88. But a better construction is given in the following definition.

Definition 5.12. Let $p : P \to B$ be a principal G-bundle. Suppose G acts on the left on a space F; i.e. an action $G \times F \to F$ is given. Define the *Borel construction*

$$P \times_G F$$

to be the quotient space $P \times F/\sim$ where

$$(xg, f) \sim (x, gf).$$

(Equivalently, $P \times_G F$ is the orbit space of the free left G-action on $P \times F$ given by $(x, f) \mapsto (xg^{-1}, gf)$.)

Let $[x, f] \in P \times_G F$ denote the equivalence class of (x, f). Define a map

$$q : P \times_G F \to B$$

by the formula $[x, f] \mapsto p(x)$.

The following important exercise shows that the two ways of going from a principal G-bundle to a fiber bundle with fiber F and structure group G are the same.

Exercise 99. If $p : P \to B$ is a principal G-bundle and F is a left G-space, then

$$\begin{tikzcd} F \arrow[r] & P \times_G F \arrow[d, "q"'] \\ & B \end{tikzcd}$$

(where $q[x, f] = p(x)$) is a fiber bundle over B with fiber F and structure group G which has the same transition functions as $p : P \to B$.

We say $q : E \times_G F \to B$ is *the fiber bundle associated to the principal bundle $p : E \to B$ via the action of G on F.*

Given that there is an atlas and transition-free definition of a principal bundle (see Exercise 95), the above exercise means that the Borel construction gives an atlas and transition-free definition of a bundle with structure group G and fiber F.

Principal bundles are more basic than fiber bundles, in the sense that the fiber and its G-action are explicit, namely G acting on itself by left translation. Moreover, any fiber bundle with structure group G is associated to a principal G-bundle by specifying an action of G on a space F. Many properties of bundles become more visible when stated in the context of principal bundles.

The following exercise gives a different method of constructing the principal bundle underlying a vector bundle, without using transition functions.

Exercise 100. Let $p : E \to B$ be a vector bundle with fiber $\mathbf{R}^n$ and structure group $GL(n, \mathbf{R})$. Define a space $F(E)$ to be the space of *frames* in E, so that a point in $F(E)$ is a pair $(b, \mathbf{f})$ where $b \in B$ and $\mathbf{f} = (f_1, \ldots, f_n)$ is an ordered basis for the vector space $p^{-1}\{b\}$. In other words

$$F(M) = \{(b, (f_1, \ldots, f_n)) \in B \times E^n \mid \{f_1, \ldots, f_n\} \text{ is a basis for } p^{-1}\{b\}\}.$$

There is an obvious map $q : F(E) \to B$.

Prove that $q : F(E) \to B$ is a principal $GL(n, \mathbf{R})$-bundle, and that

$$E = F(E) \times_{GL(n,\mathbf{R})} \mathbf{R}^n$$

where $GL(n, \mathbf{R})$ acts on $\mathbf{R}^n$ in the usual way.

For example, given a representation of $GL(n, \mathbf{R})$, that is, a homomorphism $\rho : GL(n, \mathbf{R}) \to GL(k, \mathbf{R})$, one can form a new vector bundle

$$F(E) \times_\rho \mathbf{R}^k$$

over B.

An important set of examples comes from this construction by starting with the tangent bundle of a smooth manifold M. The principal bundle $F(TM)$ is called the *frame bundle* of M. Any representation of $GL(n, \mathbf{R})$ on a vector space V gives a vector bundle with fiber isomorphic to V. Important representations include the *alternating representations* $GL(n, \mathbf{R}) \to GL\big(\Lambda^p(\mathrm{Hom}(\mathbf{R}^n, \mathbf{R}))\big)$ from which one obtains the vector bundles of differential p-forms over M.

Here is an application of the Borel construction.

Proposition 5.13. *Every bundle of discrete abelian groups with fiber A over a path-connected (and semi-locally simply connected) space B is of the form*

$$\begin{tikzcd} A \arrow[r] & \widetilde{B} \times_{\pi_1 B} A \arrow[d, "q"] \\ & B \end{tikzcd}$$

i.e. the bundle is associated to the principal $\pi_1 B$-bundle given by the universal cover $\widetilde{B}$ of B and a homomorphism $\pi_1 B \to \mathrm{Aut}(A)$.

In other words the group $\mathrm{Aut}(A)$ can be replaced by the discrete group $\pi_1 B$. Notice that in general one cannot assume that the homomorphism $\pi_1 B \to \mathrm{Aut}(A)$ is injective, and so this is a point where it is necessary to consider noneffective actions. Alternatively, one can take the structure group to be $\pi_1 B / \ker(\phi)$ where $\phi : \pi_1 B \to \mathrm{Aut}(A)$ is the corresponding representation, and so associate the bundle of discrete abelian groups to the cover $\widetilde{B}/\ker(\phi) \to B$ rather than the universal cover.

Sketch of proof. Suppose that $p : E \to B$ is a bundle of discrete abelian groups with a discrete abelian group A as fiber and structure group $\mathrm{Aut}(A)$. Since A is discrete, $p : E \to B$ is a covering space. Let b_0 denote the base point of B. As in Exercise 97, there is a right $\pi = \pi_1(B, b_0)$-action on $p^{-1}\{b_0\}$. Convert this to a left action by setting $[\gamma] \cdot z = z[\gamma]^{-1}$. Explicitly, $[\gamma] \cdot z$ is the ending point of the lift of γ^{-1} starting at z. The fibers of

$p : E \to B$ are abelian groups, and hence the lift of γ^{-1} starting at $z_1 + z_2$ is the pointwise sum of the lifts starting at z_1 and z_2. Hence this action gives a representation $\pi \to \mathrm{Aut}(p^{-1}\{b_0\})$. Fix a parameterization $\varphi : U \times A \to p^{-1}U$ with $b_0 \in U$ and use it to identify $A \cong \{b_0\} \times A$ with $p^{-1}\{b_0\}$. Composing with this identification gives a left representation $\pi \to \mathrm{Aut}(A)$.

Let $u : \widetilde{B} \to B$ denote the universal cover and choose $\tilde{b}_0 \in \widetilde{B}$ so that $u(\tilde{b}_0) = b_0$. Given $a \in A$, let f_a be the unique solution to

$$\begin{tikzcd} (\widetilde{B}, \tilde{b}_0) \arrow[rr, dotted, "f_a"] \arrow[dr, "u"'] & & (E, \varphi(b_0, a)) \arrow[dl, "p"] \\ & (B, b_0) & \end{tikzcd}$$

and define $\tilde{f} : \widetilde{B} \times A \to E$ by $\tilde{f}(\tilde{b}, a) = f_a(b)$. One checks that $\tilde{f}(\tilde{b}g, a) = \tilde{f}(\tilde{b}, ga)$, where $g \in \pi$ acts on $\tilde{b} \in \widetilde{B}$ as a covering transformation (see Exercise 97) and on $a \in A$ as described in the previous paragraph. Thus $\tilde{f}$ descends to give a well-defined map $f : \widetilde{B} \times_\pi A \to E$ so that the diagram commutes:

$$\begin{tikzcd} \widetilde{B} \times_\pi A \arrow[rr, dotted, "f"] \arrow[dr, "q"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

with $q[\tilde{b}, a] = u(\tilde{b})$. The map f induces an isomorphism of abelian groups $q^{-1}\{b\} \to p^{-1}\{b\}$ for each $b \in B$. Looking at charts shows that f provides an identification of bundles of discrete abelian groups $p : E \to B$ and $q : \widetilde{B} \times_\pi A \to B$.

$\square$

5.6. Reducing the structure group

Let H be a subgroup of a topological group G. For simplicity, we assume that G acts effectively on F. A *reduction* of the structure group of a bundle $(p, E, B, F, G, \mathcal{A})$ from G to H is an atlas $\mathcal{B}$ for the bundle whose transition functions all take values in H. Two such reductions $\mathcal{B}$ and $\mathcal{B}'$ will be considered the same if there are contained in a common atlas whose transition functions also land in H. Such a reduction may not exist (e.g. if the bundle is nontrivial and $H = 1$) and may not be unique (e.g. the Hopf bundle $(p, S^3, S^2, S^1, O(2))$ and $H = SO(2)$).

A *partition of unity* subordinate to an open cover $\{U_\alpha\}_{\alpha \in \mathcal{A}}$ of a topological space X is a collection of continuous functions $\{\varphi_\alpha : X \to [0,1]\}_{\alpha \in \mathcal{A}}$ so that for all $\alpha \in \mathcal{A}$, $\varphi_\alpha^{-1}(0,1] \subset U_\alpha$ and for all $x \in X$, $\{\alpha \in \mathcal{A} \mid \varphi_\alpha(x) \neq 0\}$

is finite and $\sum_{\alpha\in\mathcal{A}}\varphi_\alpha(x) = 1$. A *paracompact* space is a Hausdorff space where every open cover admits a partition of unity. Many familiar spaces are paracompact, such as compact Hausdorff spaces, metric spaces, and CW-complexes.

A (G,F)-bundle is *numerable* if there is an atlas $\mathcal{A} = \{\varphi_\alpha : U_\varphi \to p^{-1}U_\varphi\}$ so that the open cover $\{U_\varphi\}_{\varphi\in\mathcal{A}}$ admits a partition of unity. Note that any bundle over a paracompact space is numerable.

Exercise 101. Show that the structure group of any rank n vector bundle (i.e. fiber bundle with structure group $GL(n,\mathbf{R})$ acting on $\mathbf{R}^n$ in the usual way) over a paracompact base can be reduced to the orthogonal group $O(n)$. (Hint: Read Exercise 94. Use a partition of unity to construct the fiberwise inner product.)

In fact, there is a unique reduction (see [**36**, p. 24]) in this case.

Another example giving both lack of existence and lack of uniqueness of reductions concerns orientability and orientation of vector bundles.

Definition 5.14. A real vector bundle is called *orientable* if its structure group $GL(n,\mathbf{R})$ can be reduced to the subgroup $GL^+(n,\mathbf{R})$ of matrices with positive determinant. Such a reduction is called an *orientation* of the vector bundle.

A smooth manifold is orientable if and only if its tangent bundle is orientable. A more detailed discussion of orientability for manifolds and vector bundles can be found in Section 9.7.3.

Exercise 102. Prove that an orientable vector bundle can be oriented in two incompatible ways; that is, the structure group can be reduced from $GL(n,\mathbf{R})$ to $GL^+(n,\mathbf{R})$ (or, using Exercise 101, from $O(n)$ to $SO(n)$) in two ways so that the identity map Id: $E \to E$ is a not a map of fiber bundles with structure group $GL^+(n,\mathbf{R})$ (or $SO(n)$).

One often needs a more general version of reduction of structure group. We discuss this only for principal bundles and leave it to the reader to translate to associated bundles.

Proposition 5.15. *Let $f : H \to G$ be a continuous homomorphism of topological groups. Let $q : Q \to B$ be a principal H-bundle. Note that G is an (H,G)-space via $h\cdot g\cdot g' = f(h)gg'$. Then $p : Q\times_H G \to B, \quad p[x,g] = q(x)$ is a principal G-bundle.*

Exercise 103. Prove Proposition 5.15.

One calls $p : Q \times_H G \to B$ the *induced bundle along f*. A suitable notation would be $p : f_*Q \to B$.

Definition 5.16. Let $f : H \to G$ be a continuous homomorphism of topological groups. A *reduction of the structure group of a principal G-bundle $p : P \to B$ along f* is an isomorphism with a bundle induced along on f. In other words, there is a principal H-bundle $Q \to B$ and a commutative diagram with r a G-map

$$\begin{tikzcd} Q \times_H G \arrow[rr, "r", "\cong"'] \arrow[dr, "q"'] & & P \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Identify two reductions (q, r) and (q', r') of p if there is a an isomorphism of principal H-bundles

$$\begin{tikzcd} Q \arrow[rr, "h", "\cong"'] \arrow[dr, "q"'] & & Q' \arrow[dl, "q'"] \\ & B & \end{tikzcd}$$

inducing a commutative diagram

$$\begin{tikzcd} f_*Q \arrow[rr, "f_*h", "\cong"'] \arrow[dr, "r"'] & & f_*Q' \arrow[dl, "r'"] \\ & P & \end{tikzcd}$$

Here are some examples where it is necessary to reduce a structure group along a map which is not an inclusion. One can show that the fundamental group of $SO(n)$ $(n > 1)$ is cyclic. The Lie group $\mathrm{Spin}(n)$ is the double cover $\mathrm{Spin}(n) \to SO(n)$. A *spin structure* on a smooth manifold is a reduction of the structure group of the tangent bundle along the composite map $f : \mathrm{Spin}(n) \to GL(n, \mathbf{R})$.

A second example is given by Proposition 5.15 which states that a bundle of discrete abelian groups with fiber A admits a (unique) reduction along a map $\pi_1(B, b_0) \to \mathrm{Aut}(A)$.

5.7. Maps of bundles and pullbacks

The concept of a morphism between fiber bundles is subtle. Rather than try to work in the greatest generality, we assume (at first) that we are studying morphisms between (G, F)-bundles where G acts effectively on F.

Definition 5.17. A *morphism of fiber bundles with structure group G and fiber F* from $p : E' \to B'$ to $p : E \to B$ is a pair of continuous maps

$\tilde{f} : E' \to E$ and $f : B' \to B$ so that the diagram

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

commutes and for any parameterizations $\varphi : U_\varphi \times F \to p^{-1}U_\varphi$ and $\psi : U_\psi \times F \to (p')^{-1}U_\psi$, there is a continuous map $\theta_{\varphi,\psi} : f^{-1}U_\varphi \cap U_\psi \to G$ so that for all $b \in U_\varphi \cap f^{-1}U_\psi$ and for all $x \in F$,

$$\tilde{f}(\psi(b, x)) = \varphi(f(b), \theta_{\varphi,\psi}(b)x).$$

A *morphism of fiber bundles with structure group G, fiber F and base space B* is a morphism as above, with $f = \mathrm{Id} : B \to B$. In this case we write the square above as a triangle.

$$\begin{tikzcd} E' \arrow[rr, "\hat{f}"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

We call the first sort of morphism a (G, F)-bundle map and the second sort of morphism a (G, F, B)-bundle map. Notice that the fibers are mapped homeomorphically by both sort of morphisms.

Exercise 104. Show that any (G, F, B)-bundle map is an isomorphism.

The set of isomorphism classes of (G, F, B)-bundles can be an important invariant of a space and a group – see Project 5.8.2.

One important type of an (G, F, B)-bundle map is a *gauge transformation.* This is a bundle map from a bundle to itself which covers the identity map of the base; i.e. the following diagram commutes:

$$\begin{tikzcd} E \arrow[rr, "g"] \arrow[dr, "p"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

By definition g restricts to an homeomorphism given by the action of an element of the structure group on each fiber. The set of all gauge transformations forms a group. For example if $E \to B$ is the universal cover of a space the group of gauge transformations is isomorphic to the fundamental group.

We now give three examples of (G, F)-bundle morphisms. First, a morphism where $(G, F) = (GL(n, \mathbf{R}), \mathbf{R}^n)$ is a commutative diagram as in the definition above which induces linear isomorphisms $p'^{-1}\{b'\} \to p^{-1}\{f(b')\}$

for all $b \in B$. The second example is a (G, G)-bundle map, where G acts on itself by translation, i.e. a morphism of principal bundles, where one requires that the map $\tilde{f}$ is a map of right G-spaces. The third example of a morphism of fiber bundles arises from a pullback construction.

Definition 5.18. Suppose that a fiber bundle $p : E \to B$ with fiber F and structure group G is given, and that $f : B' \to B$ is some continuous function. Define the *pullback of* $p : E \to B$ *by* f to be the space

$$f^*E = \{(b', e) \in B' \times E \mid \; f(b') = p(e)\}.$$

Let $q : f^*E \to B'$ be the restriction of the projection $B' \times E \to B$ to f^*E. Notice that there is a commutative diagram

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Theorem 5.19. *The map* $q : f^*E \to B'$ *is a fiber bundle with fiber* F *and structure group* G*. The map* $f^*E \to E$ *is a map of fiber bundles.*

Proof. This is not hard. The important observation is that if φ is a parameterization over $U \subset B$, then $f^{-1}U$ is open in B' and φ induces a homeomorphism $f^{-1}U \times F \to f^*(E)_{|f^{-1}U}$. We leave the details as an exercise. $\square$

The following exercise shows that any map of fiber bundles is given by a pullback.

Exercise 105. Suppose we have (G, F)-bundles p' and p and a map f

$$\begin{tikzcd} E' \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Give a bijection from the set of (G, F, B)-isomorphisms

$$\begin{tikzcd} E' \arrow[rr, dashed, "\cong"] \arrow[dr] & & f^*E \arrow[dl] \\ & B & \end{tikzcd}$$

to the set of (G,F)-morphisms

$$\begin{tikzcd} E' \arrow[r, dashed] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

Suppose now that G does not effectively on F. A morphism of (G,F,B)-bundles is defined to be an isomorphism – it should be clear to you what this means. To define a morphism with different base spaces we need to discuss pullbacks. Let $(p,E,B,F,G,\mathcal{A},\Theta)$ be a (G,F)-bundle and let $f:B'\to B$ be a continuous function. By taking the pullback of $B'\to B\leftarrow E$, there is a commutative diagram of spaces

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, "q"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"] & B \end{tikzcd}$$

We just need to convince ourselves that $f^*E\to E$ is a (G,F)-bundle. There are two ways to see this, but we will leave the details to the reader. The first way is to pullback the parameterizations $\mathcal{A}$ and the transition functions Θ along f to get parameterizations and transition functions for $f^*E\to E$. The second way is to consider the principal G-bundle $P(E)\to B$ associated to p. Then one shows $f^*P(E)\to B'$ is a principal G-bundle, with associated (G,F)-bundle $f^*P(E)\times_G F\to B'$ given by the Borel construction. It is not difficult to see that induced map $f^*P(E)\times_G F\to f^*E$ is a homeomorphism. Use this homeomorphism to give q the structure of a (G,F)-bundle.

A morphism of (G,F)-bundles from $p':E'\to B'$ to $p:E\to B$ along a map $f:B'\to B$ is defined to be an isomorphism of (G,F,B')-bundles from p' to the pullback q. Alternatively, one could give a direct definition avoiding pullbacks by using either one of the "ways" above.

Here is something else to ponder. Let $f:M\to N$ be a differentiable function between differentiable manifolds. Then the differential $df_x : T_xM\to T_{f(x)}N$ can be bundled together to give a commutative diagram

$$\begin{tikzcd} TM \arrow[r, "df"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"] & N \end{tikzcd}$$

which is linear on fibers. Does this fit into our framework of morphisms of (G,F)-bundles? Not really, since the fibers are different unless $\dim M =$

$\dim N$, and, even if this is the case, df_x may not be invertible. Instead of viewing df as a morphism of bundles, one can view df as a section of the vector bundle $\operatorname{Hom}(TM, f^*TN) \to M$.

$$\begin{array}{ccc} \operatorname{Hom}_{\mathbf{R}}(T_xM, T_{f(x)}M) & \hookrightarrow & \operatorname{Hom}(TM, f^*TN) \\ & & \downarrow \\ & & M \end{array}$$

The pullback construction can be used to define a sum in the category of vector bundles over a space X.

Definition 5.20. The *Whitney sum*, $E \oplus E' \to X$, of two vector bundles $\gamma : E \to X, \gamma' : E' \to X$ over a space X is defined to be the pullback $\Delta^*(E \times E')$ where $E \times E' \to X \times X$ is the product bundle and $\Delta : X \to X \times X$ is the diagonal map. Informally, this is the vector bundle over X whose fiber $(E \oplus E')_x$ over $x \in X$ is the vector space sum $E_x \oplus E'_x$.

5.8. Projects: Fiber bundles are fibrations: Classifying spaces

5.8.1. Fiber bundles over paracompact bases are fibrations. State and prove the theorem of Hurewicz (Theorem 7.8) which says that a map $p : E \to B$ with B paracompact Hausdorff is a fibration (see Definition 7.7) provided that B has an open cover $\{U_i\}$ so that $p : p^{-1}U_i \to U_i$ is a fibration for each i. In particular, any fiber bundle over a paracompact space is a fibration. A reference for the proof is [**14**, Chapter XX, §3-4] or [**45**, Chapter 2, §7].

There is an easier approach, which suffices for most applications. Define a *Serre fibration* to be a map which has the homotopy lifting property with respect to maps of disks. Prove that any fiber bundle is a Serre fibration. A reference for this proof is [**8**, Theorem 6.11].

With either approach, Theorem 7.11 gives an nice application: if $f \simeq g : B \to B'$ are homotopic maps and $E \to B$ is a (G, F)-bundle over B', then the pullback bundles $f^*E \to B$ and $g^*E \to B$ are isomorphic bundles over B.

5.8.2. Classifying spaces. For any topological group G there is a space BG and a numerable principal G-bundle $EG \to BG$ so that given any paracompact Hausdorff space B, the pullback construction induces a bijection between the set $[B, BG]$ of homotopy classes of maps from B to BG and isomorphism classes of numerable principal G-bundles over B. Explain the construction of the bundle $EG \to BG$. Prove the theorem by proving the following three facts. First, for any bundle $E \to B$ there is a morphism to $EG \to BG$. Second, given any two morphisms $(f, \tilde{f})$ and $(g, \tilde{g})$ from $E \to B$ to $EG \to BG$, the maps $f, g : B \to BG$ are homotopic. Third, given homotopic maps $f, g : B \to B'$, and a bundle $E' \to B'$, the pullback bundles $f^*E' \to B$ and $g^*E \to B$ are isomorphic. Show that given any action of G on F, any numerable fiber bundle $E \to B$ with structure group G and fiber F is isomorphic to the pullback

$$f^*(EG \times_G F)$$

where $f : B \to BG$ classifies the principal G-bundle underlying $E \to B$. Define characteristic classes for principal bundles as elements in the cohomology $H^*(BG)$ and discuss their naturality properties.

Show that the assignment $G \mapsto BG$ is functorial with respect to continuous homomorphisms of topological groups. Show that a principal G-bundle P is of the form $Q \times_H G$ (as in Proposition 5.15) if and only if the classifying map $f : B \to BG$ lifts to BH

$$\begin{tikzcd}
 & BH \arrow[d] \\
B \arrow[ur, dotted] \arrow[r, "f"] & BG
\end{tikzcd}$$

See Theorem 9.15 and Corollary 7.55 for more on this important topic.

A reference for this material is [**23**]. We will use these basic facts about classifying spaces throughout this book, notably when we study bordism.

Chapter 6

Homology with Local Coefficients

Include the point of view that a local coefficient system is a functor from the fundamental groupoid to abelian groups.

When studying the homotopy theory of non-simply connected spaces, one is often led to consider an action of the fundamental group on some abelian group. Local coefficient systems are a tool to organize this information. The theory becomes more complicated by the fact that one must consider noncommutative rings. It is possible to learn a good deal of homotopy theory by restricting only to simply connected spaces, but fundamental group issues are ubiquitous in geometric topology.

There are two approaches to constructing the complexes giving the homology and cohomology of a space with local coefficients. The first is more algebraic and takes the point of view that the fundamental chain complex associated to a space X is the singular (or cellular) complex of the universal cover $\tilde{X}$, viewed as a chain complex over the group ring $\mathbf{Z}[\pi_1 X]$. From this point of view local coefficients are nothing more than modules over the group ring $\mathbf{Z}[\pi_1 X]$.

The second approach is more topological; one takes a local coefficient system over X (i.e. a fiber bundle over X whose fibers are abelian groups and whose transition functions take values in the automorphisms of the group) and defines a chain complex by taking the chains to be formal sums of singular simplices (or cells) such that the coefficient of a simplex is an element in the fiber over that simplex (hence the terminology *local coefficients*). Each of these two points of view has its strengths; Proposition 5.13 is the basic result which identifies the two.

In this chapter we will work with $\mathbf{Z}$-modules, (i.e. abelian groups) and modules over integral group rings $\mathbf{Z}\pi$. Everything generalizes appropriately for R-modules and $R\pi$-modules for any commutative ring R.

6.1. Definition of homology with twisted coefficients

We begin with the definition of a group ring.

Definition 6.1. The *group ring* $\mathbf{Z}\pi$ is a ring associated to a group π. Additively it is the free abelian group on π; i.e. elements are (finite) linear combinations of the group elements

$$m_1 g_1 + \cdots + m_k g_k \qquad m_i \in \mathbf{Z}, \quad g_i \in \pi.$$

Multiplication is given by the distributive law and multiplication in π:

$$\big(\sum_i m_i g_i\big)\big(\sum_j n_j h_j\big) = \sum_{i,j} (m_i n_j)(g_i h_j).$$

In working with group rings the group π is always written multiplicatively, and if e is the identity of the group, e is written as 1, since this element forms the unit in the ring $\mathbf{Z}\pi$. To avoid confusing notation we will sometimes write $\mathbf{Z}[\pi]$ instead of $\mathbf{Z}\pi$.

Two examples of group rings (with their standard notation) are

$$\mathbf{Z}[\mathbf{Z}] = \mathbf{Z}[t, t^{-1}] = \{a_{-j}t^{-j} + \cdots + a_0 + \cdots + a_k t^k \mid a_n \in \mathbf{Z}\}$$

(this ring is called the ring of *Laurent polynomials*) and

$$\mathbf{Z}[\mathbf{Z}/2] = \mathbf{Z}[t]/(t^2 - 1) = \{a + bt \mid a, b \in \mathbf{Z}\}.$$

We will work with modules over $\mathbf{Z}\pi$. If π is a nonabelian group, the ring $\mathbf{Z}\pi$ is not commutative, and so one must distinguish between left and right modules.

Let A be an abelian group and

$$\rho : \pi \to \operatorname{Aut}(A)$$

be a homomorphism. (The standard terminology is to call either ρ or A a *representation* of π.) The representation ρ endows A with the structure of a *left* $\mathbf{Z}\pi$-module by taking the action

$$\big(\sum_{g\in\pi} m_g g\big) \cdot a = \sum_{g\in\pi} m_g \ \rho(g)(a).$$

Conversely if A is a left module over a group ring $\mathbf{Z}\pi$, there is a homomorphism

$$\rho : \pi \to \operatorname{Aut}(A)$$

given by $(g \mapsto (a \mapsto ga))$ where ga is multiplying $a \in A$ by $g \in \mathbf{Z}\pi$. Thus *a representation of a group π on an abelian group is the same thing as a $\mathbf{Z}\pi$-module.*

Exercise 106. Let A be a finitely generated (left) module over $\mathbf{Z}[\mathbf{Z}/2]$ so that, as an abelian group, A is finitely generated and torsion free. Show that A is a direct sum of modules of the form $\mathbf{Z}_+$, $\mathbf{Z}_-$, and $\mathbf{Z}[\mathbf{Z}/2]$. Here $\mathbf{Z}_+$ is the trivial $\mathbf{Z}[\mathbf{Z}/2]$-module corresponding to the trivial homomorphism $\rho : \mathbf{Z}/2 \to \operatorname{Aut}(\mathbf{Z}) = \{\pm 1\} \cong \mathbf{Z}/2$, and $\mathbf{Z}_-$ corresponds to the nontrivial homomorphism.

We briefly outline the definition of the tensor product in the noncommutative case.

Definition 6.2. If R is a ring (possibly noncommutative), M is a right R-module, and N is a left R-module (sometimes one writes M_R and ${}_RN$), then the *tensor product* $M \otimes_R N$ is an abelian group satisfying the adjoint property

$$\operatorname{Hom}_{\mathbf{Z}}(M \otimes_R N, A) \cong \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbf{Z}}(N, A))$$

for any abelian group A. The corresponding universal property is that there is a $\mathbf{Z}$-bilinear map $\phi : M \times N \to M \otimes_R N$, so that $\phi(mr, n) = \phi(m, rn)$, and this map is initial in the category of $\mathbf{Z}$-bilinear maps $\overline{\phi} : M \times N \to A$, satisfying $\overline{\phi}(mr, n) = \overline{\phi}(m, rn)$.

The tensor product is constructed by taking the free *abelian* group on $M \times N$ and modding out by the expected relations. Elements of $M \otimes_R N$ are denoted by

$$\sum m_i \otimes n_i.$$

The relation $mr \otimes n = m \otimes rn$ holds. (This is why we take a right module tensored with a left module.)

Exercise 107. Compute the abelian group $\mathbf{Z}_+ \otimes_{\mathbf{Z}[\mathbf{Z}/2]} \mathbf{Z}_-$ (see Exercise 106).

The starting point in the algebraic construction of homology with local coefficients is the observation that *the singular chain complex of the universal cover of a space X is a right $\mathbf{Z}[\pi_1(X)]$-module.*

To proceed, fix a path-connected and locally path-connected space X with a base point which admits a universal cover. For notational ease set $\pi = \pi_1 X$. Let $\widetilde{X} \to X$ be the universal cover of X, with its *right* free π-action (see Section 5.4.1 and the comments following Exercise 97). Then the singular complex $S_\bullet(\widetilde{X})$ of the universal cover (with integer coefficients) is a right $\mathbf{Z}\pi$-module; the action of $g \in \pi$ on a singular simplex $\sigma : \Delta^k \to \widetilde{X}$ is the singular simplex $\sigma \cdot g$ defined as the composite of σ and the covering transformation $g : \widetilde{X} \to \widetilde{X}$. This is extended from π to $\mathbf{Z}\pi$ by linearity.

We can now give the algebraic definition of homology with local coefficients.

Definition 6.3. Given a $\mathbf{Z}\pi$-module A, form the tensor product

$$S_\bullet(X;A) = S_\bullet(\widetilde{X}) \otimes_{\mathbf{Z}\pi} A.$$

This is a chain complex whose homology is called the *homology of X with local coefficients in A* and is denoted by $H_*(X;A)$.

Notice that since the ring $\mathbf{Z}\pi$ is noncommutative (except if π is abelian), the tensored chain complex only has the structure of a chain complex over $\mathbf{Z}$, not $\mathbf{Z}\pi$. Thus the homology group $H_*(X;A)$ is only a $\mathbf{Z}$-module.

If the $\mathbf{Z}\pi$-module is specified by a representation $\rho : \pi_1 X \to \mathrm{Aut}(A)$ for some abelian group A, and we wish to emphasize the representation, we will sometimes embellish A with the subscript ρ and write $H_*(X;A_\rho)$ for the homology with coefficients in A. It is also common to call $H_*(X;A_\rho)$ the *homology of X twisted by $\rho : \pi_1 X \to \mathrm{Aut}(A)$*.

Before we look at examples, we will give the corresponding definition of cohomology. A new wrinkle which appears is that since the functor $\mathrm{Hom}_{\mathbf{Z}\pi}(-,-)$ is defined on the category of pairs of right R-modules or of pairs of left R-modules, we need to either change $S_\bullet(\widetilde{X})$ to a left $\mathbf{Z}\pi$-module or consider coefficients in right $\mathbf{Z}\pi$-modules. We opt for the former and hence transform $S_\bullet(\widetilde{X})$ into a *left* $\mathbf{Z}\pi$-module by the (standard) procedure: $g \cdot z :=_{\mathrm{def}} z \cdot g^{-1}$.

Definition 6.4. Given a left $\mathbf{Z}\pi$-module A, form the cochain complex

$$S^\bullet(X;A) = \mathrm{Hom}_{\mathbf{Z}\pi}(S_\bullet(\widetilde{X}), A).$$

(This means the set of group homomorphisms $f : S_\bullet(\widetilde{X}) \to A$ which satisfy $f(rz) = rf(z)$ for all $r \in \mathbf{Z}\pi$ and $z \in S_\bullet(\widetilde{X})$.)

The cohomology of this complex is called the *cohomology of X with local coefficients in A* and is denoted by

$$H^*(X;A).$$

If the module A is defined by a representation $\rho : \pi_1 X \to \mathrm{Aut}(A)$ for an abelian group A, the cohomology with local coefficients may be denoted by $H^*(X;A_\rho)$ and is often called the *cohomology of X twisted by ρ.*

6.2. Examples and basic properties

The (ordinary) homology and cohomology groups are just special cases of the homology and cohomology with local coefficients corresponding to twisting by the *trivial* representations ρ as we now show.

If $\rho : \pi_1 X \to \mathrm{Aut}(A)$ is the trivial homomorphism, then the definition of tensor product gives a chain map

$$S_\bullet(\widetilde{X}) \otimes_{\mathbf{Z}\pi} A_\rho \ \to\ S_\bullet X \otimes_{\mathbf{Z}} A$$

which we will see is an isomorphism. (In the chain complex on the right A is considered only as an abelian group.) This follows since both $S_\bullet(\widetilde{X})$ and S_X are chain complexes of *free* modules, so it is easy to compute tensor products. The complex $S_\bullet(\widetilde{X})$ is a free $\mathbf{Z}\pi$-chain complex since π acts freely on $\widetilde{X}$, and hence on the set of all singular simplices in $\widetilde{X}$. We obtain a $\mathbf{Z}\pi$ basis by choosing a representative simplex for each orbit. Better yet, for each singular simplex $\sigma : \Delta^n \to X$, choose a single lift $\tilde{\sigma} : \Delta^n \to \widetilde{X}$. Then the set $\{\tilde{\sigma}\}$ gives a basis for $S_\bullet(\widetilde{X})$ over $\mathbf{Z}\pi$, and it follows that $S_\bullet(\widetilde{X}) \otimes_{\mathbf{Z}\pi} A_\rho \ \to\ S_\bullet X \otimes_{\mathbf{Z}} A$ is an isomorphism of graded abelian groups; from this description it is not hard to check that this isomorphism is a chain map, and so $H_k(X; A_\rho) = H_k(X; A)$, the usual homology with coefficients in (the underlying $\mathbf{Z}$-module) A.

Similarly, when ρ is trivial,

$$\mathrm{Hom}_{\mathbf{Z}\pi}(S_\bullet\widetilde{X}, A_\rho) \ \cong\ \mathrm{Hom}_{\mathbf{Z}}(S_\bullet X, A)$$

so $H^k(X; A_\rho) \ \cong\ H^k(X; A)$, the usual cohomology with coefficients in A.

Things are a bit confused with the following exercises. Should their order be reversed? Don't we claim that we have solved the homology exercise? A key point is that $S(\widetilde{X}) \to S(X)$ is a map of $\mathbf{Z}\pi$ complexes.

Exercise 108. Show that if ρ is trivial, the natural map

$$\mathrm{Hom}_{\mathbf{Z}}(S_\bullet X, A) \ \to\ \mathrm{Hom}_{\mathbf{Z}\pi}(S_\bullet\widetilde{X}, A_\rho)$$

is a chain isomorphism.

At the other extreme we consider what happens if A is a (finitely generated) free $\mathbf{Z}\pi$-module. Since the tensor product and Hom functors respect direct sums, it suffices to consider the case when $A = \mathbf{Z}\pi$.

Then,

$$S_\bullet\widetilde{X} \otimes_{\mathbf{Z}\pi} \mathbf{Z}\pi \ =\ S_\bullet\widetilde{X},$$

and therefore

$$H_k(X; \mathbf{Z}\pi) \ =\ H_k(\widetilde{X}; \mathbf{Z}),$$

the (untwisted) integral homology of the universal cover.

In other words, the homology with local coefficients given by the *regular representation* $\rho : \pi \to \mathrm{Aut}(\mathbf{Z}\pi)$

$$\rho(g) = (\sum m_h h \mapsto \sum m_h g h)$$

equals the homology of $\widetilde{X}$ with (untwisted) $\mathbf{Z}$ coefficients.

Exercise 109. Let M be an abelian group. Let $A = \mathbf{Z}\pi \otimes_{\mathbf{Z}} M$; notice that A has a left $\mathbf{Z}\pi$-module structure defined by $g \cdot (x \otimes m) = (gx) \otimes m$. Show that the homology $H_*(X; A)$ is just the (ordinary) homology of $\widetilde{X}$ with coefficients in M.

Exercise 110. (Shapiro's Lemma) Show that if $H \subset \pi$ is a subgroup and $A = \mathbf{Z}[\pi/H]$, viewed as a left π-module, then the corresponding homology is isomorphic to the homology of the H-cover of X. Generalize this as in the previous exercise to include other coefficients. (Hint: Try the case when H is normal first.)

These examples and the two exercises show that the (untwisted) homology of any cover of X with any coefficients can be obtained as a special case of the homology of X with appropriate local coefficients.

One might ask whether the same facts hold for cohomology. They do not without some modification. If $A = \mathbf{Z}\pi$, then the cochain complex $\mathrm{Hom}_{\mathbf{Z}\pi}(S_\bullet\widetilde{X}, A)$ is not in general isomorphic to $\mathrm{Hom}_{\mathbf{Z}}(S_\bullet\widetilde{X}, \mathbf{Z})$ and so $H^k(X; \mathbf{Z}\pi)$ is not isomorphic to $H^k(\widetilde{X}; \mathbf{Z})$. If X is a CW-complex with finitely many cells in each dimension then $H^k(X; \mathbf{Z}\pi)$ is isomorphic to $H^k_c(\widetilde{X}; \mathbf{Z})$, the *compactly supported* cohomology of $\widetilde{X}$.

6.2.1. Cellular methods. If X is a (connected) CW-complex, then homology and cohomology with local coefficients can be defined using the cellular chain complex; this is much better for computations. If $p : \widetilde{X} \to X$ is the universal cover, then $\widetilde{X}$ inherits a CW-structure from X: the cells of $\widetilde{X}$ are the path components of the inverse images of cells of X. The action of $\pi = \pi_1 X$ on $\widetilde{X}$ gives $C_\bullet\widetilde{X}$ the structure of a $\mathbf{Z}\pi$-chain complex. For each oriented cell e of X, choose an oriented cell $\widetilde{e}$ in $\widetilde{X}$ which lies above e. The collection of oriented cells $\{\tilde{e} \mid e \text{ is a cell of } X\}$ forms a $\mathbf{Z}\pi$-basis for $C_\bullet\widetilde{X}$.

For a $\mathbf{Z}\pi$-module A, (co)homology with local coefficients can be computed using the cellular chain complex:

$$H_k(X; A) = H_k(C_\bullet\widetilde{X} \otimes_{\mathbf{Z}\pi} A)$$
$$H^k(X; A) = H^k(\mathrm{Hom}_{\mathbf{Z}\pi}(C_\bullet\widetilde{X}, A)).$$

For example, let $X = \mathbf{R}P^n$, $n > 1$, with universal cover $\widetilde{X} = S^n$. Then $X = e^0 \cup e^1 \cup \cdots \cup e^n$. The corresponding cell decomposition is

$$S^n = e^0_+ \cup e^0_- \cup e^1_+ \cup e^1_- \cup \cdots \cup e^n_+ \cup e^n_-$$

with $e^i_\pm$ being the upper and lower hemispheres of the i-sphere. A basis for the free (rank 1) $\mathbf{Z}\pi$-module $C_i\widetilde{X}$ is e^i_+. With this choice of basis the

$\mathbf{Z}\pi$-chain complex $C_\bullet\widetilde{X}$ is isomorphic to

$$\mathbf{Z}[\mathbf{Z}/2] \to \cdots \xrightarrow{1-t} \mathbf{Z}[\mathbf{Z}/2] \xrightarrow{1+t} \mathbf{Z}[\mathbf{Z}/2] \xrightarrow{1-t} \mathbf{Z}[\mathbf{Z}/2] \to 0.$$

Writing down this complex is the main step in the standard computation of $H_*(\mathbf{R}P^n)$ as in [**51**]: first use the homology of S^n and induction on n to compute $C_\bullet\widetilde{X}$ as a $\mathbf{Z}[\mathbf{Z}/2]$-chain complex, then compute $C_\bullet(\mathbf{R}P^n) = C_\bullet\widetilde{X} \otimes_{\mathbf{Z}\pi} \mathbf{Z}$.

The following exercises are important in gaining insight into what information homology with local coefficients captures.

Exercise 111. Compute the cellular chain complex $C_\bullet(\widetilde{S^1})$ as a $\mathbf{Z}[t,t^{-1}]$-module. Compute $H_k(S^1;A_\rho)$ and $H^k(S^1;A_\rho)$ for any abelian group A and any homomorphism $\rho : \pi_1 S^1 = \mathbf{Z} \to \mathrm{Aut}(A)$.

Exercise 112. For $n > 1$, let $\rho : \pi_1(\mathbf{R}P^n) \xrightarrow{\cong} \mathrm{Aut}(\mathbf{Z}) = \{\pm 1\}$. Compute $H_k(\mathbf{R}P^n;\mathbf{Z}_\rho)$ and $H^k(\mathbf{R}P^n;\mathbf{Z}_\rho)$ and compare to the untwisted homology and cohomology.

Exercise 113. Let p and q be a relatively prime pair of integers and denote by $L(p,q)$ the 3-dimensional Lens space $L(p,q) = S^3/(\mathbf{Z}/p)$, where $\mathbf{Z}/p = \langle t\rangle$ acts on $S^3 \subset \mathbf{C}^2$ via

$$t(Z,W) = (\zeta Z, \zeta^q W)$$

$(\zeta = e^{2\pi i/p})$. Let $\rho : \mathbf{Z}/p \to \mathrm{Aut}(\mathbf{Z}/n) \cong \mathbf{Z}/(n-1)$ for n *prime*. Compute $H_k(L(p,q);(\mathbf{Z}/n)_\rho)$ and $H^k(L(p,q);(\mathbf{Z}/n)_\rho)$.

Exercise 114. Let K be the Klein bottle. Compute $H_n(K;\mathbf{Z}_\rho)$ for all twistings ρ of $\mathbf{Z}$ (i.e. all $\rho : \pi_1 K \to \mathrm{Aut}(\mathbf{Z}) = \{\pm 1\}$).

6.2.2. The orientation double cover and Poincaré duality. An important application of local coefficients is their use in studying the algebraic topology of nonorientable manifolds.

Add definition of orientation

Theorem 6.5. *Any n-dimensional manifold M has a double cover*

$$p : M_O \to M$$

where M_O is an oriented manifold. Moreover, for any point $x \in M$, if $p^{-1}\{x\} = \{x_1, x_2\}$, then the orientations $\mu_{x_1} \in H_n(M_O, M_O - \{x_1\})$ and $\mu_{x_2} \in H_n(M_O, M_O - \{x_2\})$ map (by the induced homomorphism p_*) to the two generators of $H_n(M, M - \{x\})$.

Proof. As a set $M_O = \{a \in H_n(M, M - \{x\}) \mid a$ is a generator and $x \in M\}$. As for the topology, let V be an open set in M and $z \in Z_n(M, M - V)$ a relative cycle. Then let

$$V_z = \{\mathrm{im}[z] \in H_n(M, M - \{x\}) \mid x \in V \text{and } \mathbf{Z} \cdot \mathrm{im}[z] = H_n(M, M - \{x\})\}.$$

Then $\{V_z\}$ is a basis for the topology on M_O. For more details see [**28**]. $\square$

For example, consider $\mathbf{R}P^n$ for n even. The orientation double cover is S^n; the deck transformation reverses orientation. For $\mathbf{R}P^n$ for n odd, the orientation double cover is a disjoint union of two copies of $\mathbf{R}P^n$, oriented with the opposite orientations.

If M is a connected manifold, define the *orientation character* or *the first Stiefel–Whitney class*

$$w : \pi_1 M \to \{\pm 1\}$$

by setting $w[\gamma] = 1$ if γ lifts to a loop in the orientation double cover and setting $w[\gamma] = -1$ if γ lifts to a path which is not a loop. Intuitively, $w[\gamma] = -1$ if going around the loop γ reverses the orientation. M is orientable if and only if w is trivial. Clearly w is a homomorphism.

Corollary 6.6. *Any manifold with $H^1(M; \mathbf{Z}/2) = 0$ is orientable.*

Proof. This is because

$$H^1(M; \mathbf{Z}/2) \cong \mathrm{Hom}(H_1 M; \mathbf{Z}/2) \cong \mathrm{Hom}(\pi_1 M; \mathbf{Z}/2),$$

where the first isomorphism follows from the universal coefficient theorem and the second from the Hurewicz theorem

$$H_1 M \cong \pi_1 M / [\pi_1 M, \pi_1 M].$$

$\square$

Notice that $\mathrm{Aut}(\mathbf{Z}) = \{\pm 1\}$ and so the orientation character defines a representation $w : \pi_1 X \to \mathrm{Aut}(\mathbf{Z})$. The corresponding homology and cohomology $H_k(X; \mathbf{Z}_w)$, $H^k(X; \mathbf{Z}_w)$ are called the homology and cohomology of X *twisted by the orientation character* w, or with *local coefficients in the orientation sheaf.*

The Poincaré duality theorem (Theorem 4.32) has an extension to the nonorientable situation.

Theorem 6.7 (Poincaré duality theorem)**.** *If X is an n-dimensional manifold, connected, compact and without boundary, then*

$$H_n(X; \mathbf{Z}_w) \cong \mathbf{Z},$$

and if $[X]$ denotes a generator, then

$$\cap [X] : H^k(X; \mathbf{Z}_w) \to H_{n-k}(X; \mathbf{Z})$$

and

$$\cap [X] : H^k(X; \mathbf{Z}) \to H_{n-k}(X; \mathbf{Z}_w)$$

are isomorphisms. (This statement of Poincaré duality applies to nonorientable manifolds as well as orientable manifolds.) $\square$

Add discussion of cup and cap products with local coefficients.

The Poincaré–Lefschetz duality theorem also holds in this more general context.

The cap products in Theorem 6.7 are induced by the bilinear maps on coefficients $\mathbf{Z} \times \mathbf{Z}_w \to \mathbf{Z}_w$ and $\mathbf{Z}_w \times \mathbf{Z}_w \to \mathbf{Z}$ as in Exercise 62.

Exercise 115. Check that this works for $\mathbf{R}P^n$, n even.

More generally, for a connected closed manifold X and any right $\mathbf{Z}\pi$-module A given by a representation $\rho : \pi \to \text{Aut } A$, let A_w be the module given by the representation $\rho_w : \pi \to \text{Aut } (A), \quad g \mapsto w(g)\rho(g^{-1})$. Then a stronger form of Poincaré duality says

$$\cap[X] : H^k(X; A) \to H_{n-k}(X; A_w)$$

is an isomorphism.

6.2.3. H_0 and H^0 with local coefficients.

If X is a path-connected space and A is an abelian group, then $H_0(X; A) = A = H^0(X; A)$. One might wonder what happens with local coefficients.

Proposition 6.8. *Let X be a path-connected space, $\pi = \pi_1(X, x_0)$, and A a left $\mathbf{Z}\pi$-module. Then*

1. *Let A_π be the* coinvariants *of A, the quotient of A by the subgroup generated by the elements $\{a - ga \mid a \in A, \ g \in \pi \}$. Then*
$$H_0(X; A) = A_\pi.$$
2. *Let A^π be the* invariants *of A, the subgroup of A consisting of elements fixed by π, i.e. $A^\pi = \{a \in A \mid ga = a \text{ for all } g \in \pi\}$ (the group V^π is called the group of* invariants*). Then*
$$H^0(X; A) = A^\pi.$$

Proof. Since $\widetilde{X}$ is path-connected, there is an exact sequence

$$S_1\widetilde{X} \to S_0\widetilde{X} \xrightarrow{\varepsilon} \mathbf{Z} \to 0$$

By right exactness of the tensor product

$$S_1\widetilde{X} \otimes_{\mathbf{Z}\pi} A \to S_0\widetilde{X} \otimes_{\mathbf{Z}\pi} A \xrightarrow{\varepsilon} \mathbf{Z} \otimes_{\mathbf{Z}\pi} A \to 0$$

is exact. The maps $\mathbf{Z} \otimes_{\mathbf{Z}\pi} A \to A_\pi, \quad n \otimes a \mapsto [na]$ and $A_\pi \to \mathbf{Z} \otimes_{\mathbf{Z}\pi} A, \quad [a] \mapsto 1 \otimes a$ are mutual inverses. The result follows. $\square$

Exercise 116. Prove the second assertion in Proposition 6.8.

Exercise 117. Compute $H_0(\mathbf{R}P^{2n};\mathbf{Z}_-)$ and $H^0(\mathbf{R}P^{2n};\mathbf{Z}_-)$ in two ways, using the above proposition and using Poincaré duality.

Coinvariants and invariants leads to the subject of homology and cohomology of groups. For a $\mathbf{Z}\pi$-module A, one defines $H_0(\pi;A) = A_\pi$ and $H^0(\pi;A) = A^\pi$. This then leads to "derived functors" $H_n(\pi;A) = \operatorname{Tor}_n^{\mathbf{Z}\pi}(\mathbf{Z},A)$ and $H^n(\pi;A) = \operatorname{Ext}^n_{\mathbf{Z}\pi}(\mathbf{Z},A)$.

6.3. Definition of homology with a local coefficient system

The previous (algebraic) definition of homology and cohomology with local coefficients may appear to depend on base points, via the representation

$$\rho : \pi_1(X,*) \to \operatorname{Aut}(A),$$

and the identification of $\pi_1 X$ with the covering translations of $\widetilde{X}$. In fact, it does not. We now give an alternative definition, which takes as input only the local coefficient system itself, i.e. the fiber bundle with discrete abelian group fibers. This definition is more elegant in that it does not depend on the arbitrary choice of a base point, but it is harder to compute with.

Let $p : E \to X$ be a system of local coefficients with fiber a discrete abelian group A and structure group $G \subset \operatorname{Aut}(A)$. Denote the fibers $p^{-1}\{x\}$ by E_x; for each x this is an abelian group noncanonically isomorphic to A.

We construct a chain complex as follows. Let $S_k(X;E)$ denote the set of formal sums

$$\sum_{i=1}^{m} a_i\sigma_i$$

where:

1. $\sigma_i : \Delta^k \to X$ is a singular k-simplex, and
2. a_i is an element of the group $E_{\sigma_i(e_0)}$ where $e_0 \in \Delta^k$ is the *base point* $(1,0,0,\cdots,0)$ of Δ^k. More precisely, $\sigma_i(e_0) \in X$, and we require $a_i \in E_{\sigma_i(e_0)} = p^{-1}(\sigma_i(e_0))$.

The obvious way to add elements of $S_k(X;E)$ makes sense and is well-defined. Thus $S_k(X;E)$ is an abelian group. This is somewhat confusing since the coefficients lie in different groups depending on the singular simplex. One way to lessen the confusion is to recall that $p : E \to X$ is a covering space, and that a k-simplex is simply connected. Hence given any singular k-simplex $\sigma : \Delta^k \to X$ and $a \in E_{\sigma(e_0)}$, there exists a unique lifting of σ to E taking e_0 to a. This point of view is the key to the proof of Theorem 6.9 below.

Think of $S_\bullet(X;E)$ as a graded abelian group. We next describe the differential. The formula would be the usual one were it not for the fact that given any k-simplex, one of its faces does not contain the base point. We will use the local coefficient system to identify fibers over different points of the simplex to resolve this problem.

Recall there are *face maps* $f^k_m : \Delta^{k-1} \to \Delta^k$ defined by

$$f^k_m(t_0, t_1, \cdots, t_{k-1}) \;=\; (t_0, \cdots, t_{m-1}, 0, t_m, \cdots, t_{k-1}).$$

Note that $f^k_m(e_0) = e_0$ if $m > 0$, but

$$f^k_0(e_0) = f^k_0(1, 0, \cdots, 0) \;=\; (0, 1, 0, \cdots, 0).$$

This will make the formulas for the differential a little bit more complicated than usual, since this one face map does not preserve base points.

Given a singular simplex $\sigma : \Delta^k \to X$, let $\gamma_\sigma : [0, 1] \to X$ be the path $\sigma(t, 1-t, 0, 0, \cdots, 0)$. Then because $p : E \to X$ is a covering space (the fiber is discrete), the path γ_σ defines an isomorphism of groups $\gamma_\sigma : E_{\sigma(0,1,\cdots,0)} \to E_{\sigma(1,0,0,\cdots,0)}$ via path lifting.

Thus, define the differential $\partial : S_k(X; E) \to S_{k-1}(X; E)$ by the formula

$$a\sigma \mapsto \gamma_\sigma^{-1}(a)(\sigma \circ f^k_0) \;+\; \sum_{m=1}^{k} (-1)^m \; a \; (\sigma \circ f^k_m).$$

Theorem 6.9. *This is a differential, i.e. $\partial^2 = 0$. Moreover the homology $H_k(S_\bullet(X; E), \partial)$ equals $H_k(X; A_\rho)$, where $\rho : \pi_1 X \to \mathrm{Aut}(A)$ is the homomorphism determined by the local coefficient system $p : E \to X$ as in Proposition 5.13.* □

Exercise 118. Prove Theorem 6.9.

The homology of the chain complex $(S_k(X; E), \partial)$ is called the *homology with local coefficients in E*. Theorem 6.9 says that this is isomorphic to the homology with coefficients twisted by ρ. Notice that the definition of homology with local coefficients does not involve a choice of base point for X. It follows from Theorem 6.9 that the homology twisted by a representation ρ also does not depend on the choice of base point.

Similar constructions apply to cohomology, as we now indicate. Let $S^k(X; E)$ be the set of all functions, c, which assign to a singular simplex $\sigma : \Delta^k \to X$ an element $c(\sigma) \in E_{\sigma(e_0)}$. Then $S^k(X; E)$ is an abelian group and has coboundary operator $\delta : S^k(X; E) \to S^{k+1}(X; E)$ defined by

$$(\delta c)\;(\sigma) = (-1)^k \left(\gamma_\sigma^{-1}(c\;(\partial_0 \sigma)) \;+\; \sum_{i=1}^{k+1} (-1)^i \; c\;(\partial_i \sigma) \right).$$

Then $\delta^2 = 0$ and,

Theorem 6.10. *The cohomology of the chain complex $(S^\bullet(X;E),\delta)$ equals the cohomology $H^*(X;A_\rho)$, where $\rho:\pi_1X\to \mathrm{Aut}(A)$ is the homomorphism determined by the local coefficient system $p:E\to X$.* □

For the proof see [**54**].

Here is the example involving orientability of manifolds, presented in terms of local coefficients instead of the orientation representation. Let M be an n-dimensional manifold. Define a local coefficient system $E\to M$ by setting

$$E=\bigcup_{x\in M} H_n(M,M-\{x\}).$$

A basis for the topology of E is given by

$$V^z=\{\mathrm{im}[z]\in H_n(M,M-\{x\})\mid x\in V\}$$

where V is open in X and $z\in Z_n(M,M-V)$ is a relative cycle. Then $E\to X$ is a local coefficient system with fibers $H_n(M,M-\{x\})\cong \mathbf{Z}$, called the *orientation sheaf of M*. (Note the orientation double cover M_O is the subset of E corresponding to the subset $\pm1\in\mathbf{Z}$.) Then $H_*(M;E)$ can be identified with $H_*(M;\mathbf{Z}_w)$.

6.4. Functoriality

The functorial properties of homology and cohomology with local coefficients depend on more than just the spaces involved; they also depend on the coefficient systems.

Definition 6.11. A *morphism* $(E\to X)\to(E'\to X)$ *of local coefficients over* X is a commutative diagram

$$\begin{tikzcd} E \arrow[rr, "f"] \arrow[dr] & & E' \arrow[dl] \\ & X & \end{tikzcd}$$

so that for each point $x\in X$, the restriction of $f:E\to E'$ to the fibers $f_{|E_x}:E_x\to E'_x$ is a group homomorphism.

Notice that we do not require the maps on fibers to be isomorphisms, and so this is more general than the concept of bundle map we introduced in Section 5.7.

It follows immediately from the definition of pullbacks that a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & E' \arrow[d, "p'"] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

with $\tilde{f}$ inducing homomorphisms on fibers induces a morphism of local coefficients $(E \to X) \to (f^*(E') \to X)$ over X.

Theorem 6.12. *Homology with local coefficients is a functor from a category $\mathcal{L}$ of pairs of spaces (X, A) with the following extra structure.*

1. *The objects of $\mathcal{L}$ are pairs (X, A) (allowing A empty) with a system of local coefficients $p : E \to X$.*
2. *The morphisms of $\mathcal{L}$ are the continuous maps $f : (X, A) \to (X', A')$ together with a morphism of local coefficients*

$$\begin{tikzcd} E \arrow[rr] \arrow[dr] & & f^*E' \arrow[dl] \\ & X & \end{tikzcd}$$

where f^*E' denotes the pullback of E' via f.

Sketch of proof. The basic idea comes from looking at the definition of the chain complex. Given a formal sum $\sum_i a_i\sigma_i$ with $a_i \in E_{\sigma_i(e_0)}$, the simplices σ_i push forward to simplices $f \circ \sigma$ in X'. Thus one needs a way to assign to a_i an element b_i' in $E'_{f(\sigma_i(e_0))}$. This is exactly what the morphism of local coefficients does. □

Cohomology with local coefficients is a functor on a slightly different category, owing to the variance of cohomology with respect to coefficients.

Theorem 6.13. *Cohomology with local coefficients is a contravariant functor on the category $\mathcal{L}^*$, where:*

1. *The objects of $\mathcal{L}^*$ are the same as the objects of $\mathcal{L}$, i.e. pairs (X, A) with a local coefficient system $p : E \to X$.*
2. *A morphism in $\mathcal{L}^*$ from $(p : E \to X)$ to $(p' : E' \to X')$ is a continuous map $f : (X, A) \to (X', A')$ together with a morphism of local coefficients*

$$\begin{tikzcd} f^*E' \arrow[rr, "\tilde{f}"] \arrow[dr] & & E \arrow[dl] \\ & X & \end{tikzcd}$$

In other words, $\tilde{f}$ induces a group homomorphism from $E'_{f(x)}$ to E_x for all $x \in X$.

Sketch of proof. This is similar to the previous argument. A cochain c in $S^k(X'; E')$ is a function that assigns to each singular simplex $\sigma : \Delta^k \to X'$ an element $c(\sigma)$ in $E'_{\sigma(e_0)}$.

We need to construct $f^*(c) \in S^k(X; E)$. Given a simplex $\tau : \Delta^k \to X$, compose with f to get $f \circ \tau : \Delta^k \to X'$. Next apply c to get an element $c(f \circ \tau(e_0)) \in E'_{f(\tau(e_0))}$. Finally apply $\tilde{f}$ to get

$$f^*(c)(\tau) = \tilde{f}(c(f \circ \tau(e_0))) \in E_{\tau(e_0)}.$$

□

Exercise 119. Give an alternative description of these two functoriality properties in terms of representations using the algebraic definition of homology and cohomology with local coefficients. More precisely, if $\rho : \pi_1 X \to \mathrm{Aut}(A)$ is a representation defining the homology of X with coefficients in A_ρ, and similarly $\rho' : \pi_1 X' \to \mathrm{Aut}(A')$ defines the homology of X' with coefficients in $A'_{\rho'}$, construct a commutative diagram which must exist for the homology of X with coefficient in ρ to map to the homology of X' with coefficients in $A'_{\rho'}$. Do the same for cohomology.

A straightforward checking that all the usual constructions continue to hold with local coefficients proves the following theorem.

Theorem 6.14. *Homology with local coefficients forms a homology theory on $\mathcal{L}$. More precisely, for any object in $\mathcal{L}$ there exists a connecting homomorphism and a natural long exact sequence. The excision and homotopy axioms hold.*

Similarly cohomology with local coefficients forms a cohomology theory on the category $\mathcal{L}^*$. □

In particular, there is a Mayer-Vietoris sequence for homology with local coefficients which gives a method for computing. Some care must be taken in using this theorem because local coefficients do not always extend. For example, given a homomorphism $\rho : \pi_1(X - U) \to \mathrm{Aut}(A)$ and an inclusion of pairs $(X - U, B - U) \to (X, B)$, excision holds (i.e. the inclusion of pairs induces isomorphisms in homology with local coefficients) only if ρ extends over $\pi_1 X$. In particular the morphism of local coefficients must (exist and) be isomorphisms on fibers.

6.5. Projects: Hopf degree theorem; Limits and colimits

6.5.1. The Hopf degree theorem. This theorem states that the degree of a map $f : S^n \to S^n$ determines its homotopy class. See Theorems 7.72 and 9.5. Prove the theorem using the simplicial approximation theorem. One place to find a proof is [**54**, pp. 13–17]. A proof use differential topology is given in [**33**].

6.5.2. Colimits and Limits. The categorical point of view involves defining an object in terms of its (arrow theoretic) properties and showing that the properties uniquely define the object up to isomorphism. Colimits and limits are important categorical constructions in algebra and topology. Special cases include the notions of a cartesian product, a disjoint union, a pullback, a pushout, a quotient space X/A, and the topology of a CW-complex.

Define a product and coproduct of two objects in a category, and show that cartesian product and disjoint union give the product and coproduct in the category of topological spaces. Define the colimit of a sequence of topological spaces

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_3 \to \cdots,$$

show that it is unique up to homeomorphism, and show existence by taking

$$\operatorname*{colim}_{i\to\infty} X_i = \frac{\coprod X_i}{(x_i \sim f_i(x_i))}.$$

If all the X_i are subsets of a set A and if all the f_i's are inclusions of subspaces, show that the colimit can be taken to be $X = \cup X_i$. The topology is given by saying $U \subset X$ is open if and only if $U \cap X_i$ is open in X_i for all i. Thus such a colimit can be thought of as some sort of generalization of a union. Define the limit of a sequence of topological spaces

$$\cdots \to X_3 \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0,$$

and show existence by taking

$$\lim_{\leftarrow} X_i = \{(x_i) \in \prod X_i \mid f_i(x_i) = x_{i-1} \quad \text{for all } i > 0\}.$$

Interpret the limit as a generalized form of intersection.

Now let $\mathcal{I}$ be a category and let $\mathcal{T}$ be the category of topological spaces. Let $X : \mathcal{I} \to \mathcal{T}$, $i \mapsto X_i$ be a functor, so you are given a topological space for every object i, and the morphisms of $\mathcal{I}$ give oodles of maps between the X_i satisfying the same composition laws as the morphisms in $\mathcal{I}$ do. Define

$$\operatorname*{colim}_{\mathcal{I}} X_i \qquad \text{and} \qquad \lim_{\mathcal{I}} X_i.$$

Consider the categories $\{\cdot \to \cdot \to \cdot \to \cdot \to \cdots\}$, $\{\cdots \leftarrow \cdot \leftarrow \cdot \leftarrow \cdot \leftarrow\}$, $\{\cdot \to \cdot \leftarrow \cdot\}$, $\{\cdot \leftarrow \cdot \to \cdot\}$, $\{\cdot \quad \cdot\}$, and discuss how colimits and limits over these categories give the above colimit, the above limit, the pullback, the pushout, the cartesian product and the disjoint union.

A definition of a CW-complex can be given in terms of colimits. A CW-complex is a space X together with an increasing sequence of subspaces $X^0 \subset X^1 \subset X^2 \subset \cdots$, where X^{-1} is the empty set, each X^i is the pushout of

$$\begin{tikzcd} \coprod S^{i-1} \arrow[r, "\text{inc}"] \arrow[d] & \coprod D^i \arrow[d] \\ X^{i-1} \arrow[r] & X^i \end{tikzcd}$$

and $X = \operatorname{colim}_{i\to\infty} X^i$. This definition incorporates all the properties of the topology of a CW-complex that you use in practice. Show that this definition is equivalent to your favorite definition of a CW-complex.

Finally suppose that Y is a CW-complex and $Y_0 \subset Y_1 \subset Y_2 \subset Y_3 \subset \cdots$ is an increasing union of subcomplexes whose union is Y. Show that

$$H_n Y = \operatorname*{colim}_{i\to\infty} H_n(Y_i).$$

Define Milnor's $\lim^1$ (see [**31**] or [**53**]) and show that there is an exact sequence

$$0 \to \underset{\leftarrow}{\lim}{}^1 H^{n-1}(Y_i) \to H^n Y \to \underset{\leftarrow}{\lim} H^n(Y_i) \to 0.$$

We are using the more modern terminology of colimit. Some authors use the terms "direct limit" or "inductive limit", and restrict the categories they consider. Other authors use the words "inverse limit" or "projective limit", while we just use the term "limit".

For an old-fashioned approach to limits in the special case of a directed system see [**41**], and for the more modern approach see [**53**].

Add a discussion of $H^{n+1}(K(\mathbf{Q}, n))$ as an example.

Chapter 7

Fibrations, Cofibrations and Homotopy Groups

The material in this chapter forms the topological foundation for algebraic topology.

7.1. Compactly generated spaces

Given a map $f : X \times Y \to Z$, we would like to topologize the set of continuous functions $C(Y, Z)$ in such a way that f is continuous if and only if the adjoint

$$\tilde{f} : X \to C(Y, Z), \qquad \tilde{f}(x)(y) = f(x, y)$$

is continuous. Here are three examples:

1. We would like an action of a topological group $G \times Z \to Z$ to correspond to a continuous function $G \to \mathrm{Homeo}(Z)$, where $\mathrm{Homeo}(Z)$ is given the subspace topology inherited from $C(Z, Z)$.
2. We would like a homotopy $f : I \times Y \to Z$ to correspond to a path $\tilde{f} : I \to C(Y, Z)$ of functions.
3. The evaluation map

   $$C(Y, Z) \times Y \to Z, \qquad (f, y) \mapsto f(y)$$

   should be continuous. (Is the evaluation map an adjoint?)

The above adjoint correspondence holds when Y is locally compact Hausdorff and when $C(Y, Z)$ is given the compact open topology. But not even all

CW-complexes are locally compact Hausdorff, for example an infinite wedge of circles (also called a bouquet of circles). Unfortunately, such a topology on $C(Y, Z)$ is not possible, even for general Hausdorff topological spaces, unless you bend your point of view. Although many of the constructions we will give are set-theoretically simple, the issue of how to appropriately topologize these sets can become a nuisance. The category of compactly generated spaces is a framework which permits one to make such constructions without worrying about these technical issues. The original reference for the material in this section is Steenrod's paper "A convenient category of topological spaces" [**47**].

Definition 7.1. A topological space X is said to be *compactly generated* if a subset $A \subset X$ is closed if and only if $A \cap K$ is closed in K for every compact Hausdorff $K \subset X$.

Examples of compactly generated Hausdorff spaces include:

1. locally compact Hausdorff spaces (e.g. manifolds),
2. metric spaces,
3. CW-complexes, and
4. The product of two CW-complexes, one of which is locally finite, in other words, every point has a neighborhood which intersects a finite number of cells.

We will use the notation CGH for the category of compactly generated Hausdorff spaces. (This is taken as a *full* subcategory of the category of all topological spaces; i.e. every continuous function between compactly generated spaces is a morphism in CGH.)

Any Hausdorff space can be turned into a compactly generated Hausdorff space by the following trick.

Definition 7.2. L $k(X)$ be the set X with the new topology defined by declaring a subset $A \subset X$ to be closed in $k(X)$ if and only if $A \cap K$ is closed in X for all $K \subset X$ compact Hausdorff.

Thus $k(X)$ is the underlying set of X topologized with more closed sets (and hence more open) sets than X. Hence the set-theoretic identity map $k(X) \to X$ is continuous.

Exercise 120. $k : \mathrm{Haus} \to \mathrm{CGH}$ is a functor.

The above exercise is not easy. Start by showing that if X is Hausdorff, then so is $k(X)$. Then show that for any compact subset K of X, and for any subset A of K, then A is closed in X if and only if A is closed in $k(X)$. Conclude that for a subset K of X, then K is compact in X if and only if K is compact in $k(X)$.

Exercise 121. Show that k is a left inverse and a right adjoint for the inclusion functor $i : \mathsf{CGH} \to \mathsf{Haus}$.

7.1.1. Basic facts about compactly generated spaces.

1. The compact sets in X and $k(X)$ are the same.
2. If $X \in \mathsf{CGH}$, then $k(X) = X$.
3. If X is compactly generated Hausdorff and Y is Hausdorff, then a function $f : X \to Y$ is continuous if and only if $f|_K : K \to Y$ is continuous for each compact $K \subset X$.
4. Let $C(X, Y)$ denote the set of continuous functions from X to Y. Then $k_* : C(X, k(Y)) \to C(X, Y)$ is a bijection if X is in CGH.
5. The singular chain complexes of a Hausdorff space Y and the space $k(Y)$ are the same.
6. The homotopy groups (see Definition 7.48) of Y and $k(Y)$ are the same.
7. Suppose that $X_0 \subset X_1 \subset \cdots \subset X_n \subset \cdots$ is an expanding sequence of compactly generated spaces so that X_n is closed in X_{n+1}. Topologize the union $X = \cup_n X_n$ by defining a subset $K \subset X$ to be closed if $K \cap X_n$ is closed for each n. Then if X is Hausdorff, it is compactly generated. In this case every compact subset of X is contained in some X_n.

7.1.2. Products in CGH. Unfortunately, the product of compactly generated spaces need not be compactly generated. An example is given by $\vee_\omega S^1 \times \vee_{\omega_1} S^1$, where ω is a countable ordinal and ω_1 is an uncountable ordinal. This example also shows that the cartesian product of two CW-complexes need not be a CW-complex.

However, this causes little concern, as we now see.

Definition 7.3. Let X, Y be compactly generated Hausdorff spaces. The *categorical product* of X and Y is the space $k(X \times Y)$.

The following useful facts hold about the categorical product.

1. $k(X \times Y)$ is in fact a product in the category CGH.
2. If X is locally compact and Y is compactly generated, then $X \times Y = k(X \times Y)$. In particular, $I \times Y = k(I \times Y)$. Thus the notion of homotopy is unchanged.
3. If X and Y are CW-complexes, so is $k(X \times Y)$.

From now on, if X and Y are compactly generated Hausdorff, we will denote $k(X \times Y)$ by $X \times Y$.

7.1.3. Function spaces. The standard way to topologize the set of functions $C(X,Y)$ is to use the compact-open topology.

Definition 7.4. If X and Y are compactly generated spaces, let $C(X,Y)$ denote the set of continuous functions from X to Y, topologized with the *compact-open topology*. The following collection of sets is a subbasis

$$U(K,W) = \{f \in C(X,Y) \mid f(K) \subset W\}$$

where K is a compact set in X and W an open set in Y.

If Y is a metric space, this is the notion, familiar from complex analysis, of uniform convergence on compact sets. Unfortunately, even for compactly generated spaces X and Y, $C(X,Y)$ need not be compactly generated. We know how to handle this problem: define

$$\mathrm{Map}(X,Y) \;=\; k(C(X,Y)).$$

As a *set*, $\mathrm{Map}(X,Y)$ is the set of continuous maps from X to Y, but its topology is slightly different from the compact–open topology.

Theorem 7.5 (adjoint theorem)**.** *For X, Y, and Z compactly generated Hausdorff, $f(x,y) \mapsto \tilde{f}(x)(y)$ gives a natural homeomorphism*

$$\mathrm{Map}((X \times Y), Z) \to \mathrm{Map}(X, \mathrm{Map}(Y,Z)).$$

Thus $- \times Y$ and $\mathrm{Map}(Y,-)$ are adjoint functors from CGH *to* CGH*.* □

The following useful properties of $\mathrm{Map}(X,Y)$ hold.

1. Let $e : \mathrm{Map}(X,Y) \times X \to Y$ be the *evaluation* $e(f,x) = f(x)$. Then if $X,Y \in$ CGH, e is continuous.
2. If $X,Y,Z \in$ CGH, then:
   (a) $\mathrm{Map}(X, Y \times Z)$ is homeomorphic to $\mathrm{Map}(X,Y) \times \mathrm{Map}(X,Z)$.
   (b) Composition defines a continuous map

$$\mathrm{Map}(X,Y) \times \mathrm{Map}(Y,Z) \to \mathrm{Map}(X,Z).$$

We will also use the notation $\mathrm{Map}(X,A;Y,B)$ (or $\mathrm{Map}((X,A),(Y,B))$) to denote the subspace of $\mathrm{Map}(X,Y)$ consisting of those functions $f : X \to Y$ which satisfy $f(A) \subset B$. A variant of this notation is $\mathrm{Map}(X,x_0;Y,y_0)$ denoting the subspace of basepoint preserving functions.

7.1.4. Quotient maps. We discuss yet another convenient property of compactly generated Hausdorff spaces. For topological spaces, one can give an example of quotient maps $p : W \to Y$ and $q : X \to Z$ so that $p \times q : W \times X \to Y \times Z$ is not a quotient map. However, one can show the following.

Theorem 7.6.

1. *If $p : W \to Y$ and $q : X \to Z$ are quotient maps, and X and Z are locally compact Hausdorff, then $p \times q$ is a quotient map.*
2. *If $p : W \to Y$ and $q : X \to Z$ are quotient maps and all spaces are compactly generated Hausdorff, then $p \times q$ is a quotient map, provided we use the categorical product.* □

From now on, *we assume all spaces are compactly generated Hausdorff.* If we ever meet a space which is not compactly generated, we immediately apply k. Thus, for example, if X and Y are Hausdorff spaces, then by our convention $X \times Y$ really means $k(k(X) \times k(Y))$. By this convention, we lose no information concerning homology and homotopy, but we gain the adjoint theorem.

7.1.5. Compactly generated weak Hausdorff spaces. In the years since Steenrod's paper [**47**] appeared, another category has been developed with slightly better technical properties than CGH. This is the category CGWH of compactly generated weak Hausdorff spaces. We will just touch on this briefly; more details can be found in May's book [**29**].

A topological space X is *weak Hausdorff* if $f(K)$ is closed in X for every continuous map $f : K \to Y$ with K compact Hausdorff. A Hausdorff space is weak Hausdorff. The category CGWH is the full subcategory of Top whose objects are the compactly generated weak Hausdorff spaces. Then, as above, one can define a functor k from the category of weak Hausdorff spaces to the category of compactly generated weak Hausdorff spaces which is both a left inverse and a right adjoint for the inclusion functor. The technical advantage of this category is if $\sim$ is a closed equivalence relation on a compactly generated weak Hausdorff X, then $X/{\sim}$ is also compactly generated weak Hausdorff. As a consequence if X and Y are compactly generated weak Hausdorff, A is a closed subset of X, and $f : A \to Y$ is continuous, then the pushout $X \cup_f Y$ is also compactly generated weak Hausdorff.

7.2. Fibrations

There are two kinds of maps of fundamental importance in algebraic topology: fibrations and cofibrations. Geometrically, fibrations are more complicated than cofibrations. However, your garden variety fibration tends to be a fiber bundle, and fiber bundles over paracompact spaces are always fibrations, so that we have seen many examples so far.

In the notation below we leave off the braces for a singleton set, writing, for example, 0 instead of $\{0\}$.

Definition 7.7. A continuous map $p : E \to B$ is a *fibration* if it has the *homotopy lifting property (HLP)*; i.e. the problem

$$\begin{tikzcd} Y \times 0 \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d, "p"] \\ Y \times I \arrow[r, "G"'] \arrow[ur, dotted, "\tilde{G}"] & B \end{tikzcd}$$

has a solution for every space Y.

In other words, given the continuous maps p, G, $\tilde{g}$, and the inclusion $Y \times 0 \to Y \times I$, the problem is to find a continuous map $\tilde{G}$ making the diagram commute. (Recall that whenever a commutative diagram is given with one dotted arrow, we consider it as a problem whose solution is a map which can be substituted for the dotted arrow to give a commutative diagram.)

A covering map is a fibration. In studying covering space theory this fact is called the covering homotopy theorem. For covering maps the lifting is unique, but this is not true for an arbitrary fibration.

Exercise 122. Show that the projection to the first factor $p : B \times F \to B$ is a fibration. Show by example that the liftings need not be unique.

The following theorem of Hurewicz says that if a map is locally a fibration, then it is so globally.

Theorem 7.8. *Let $p : E \to B$ be a continuous map. Suppose that B is paracompact and suppose that there exists an open cover $\{U_\alpha\}$ of B so that $p^{-1}(U_\alpha) \to U_\alpha$ is a fibration for each U_α.*

Then $p : E \to B$ is a fibration. □

Proving this theorem is one of the projects for Chapter 5. The corollary of most consequence for us is the following.

Corollary 7.9. *If $p : E \to B$ is a fiber bundle over a paracompact Hausdorff space B, then p is a fibration.*

Proof. Exercise 122 says that the projection $U \times F \to U$ is a fibration. Since fiber bundles have this local product structure, Theorem 7.8 implies that a fiber bundle is a fibration. □

A nice application of this corollary is to prove the homotopy invariance of fiber bundles. We need a preliminary lemma.

Lemma 7.10. *Let $p' : E' \to B$ and $p : E \to B$ be principal G-bundles over B. Then there is a fiber bundle $p'' : E'' \to B$ whose sections s correspond bijectively to principal G-bundle isomorphisms $f : E' \to E$ over B.*

Proof. Note that $E' \times_B E \to B, (e', e) \mapsto p'(e') = p(e)$ is a principal $G \times G$-bundle. Note that $G \times G$ acts on G via $((h, k), g) \mapsto hgk^{-1}$. Then let $p'' : E'' \to B$ be the associated $(G \times G, G)$-bundle $(E' \times_B E) \times_{G \times G} G \to B, [(e', e), g] \mapsto p'(e') = p(e)$. Then, letting $1 \in G$ denote the identity, the equation

$$[(e', f(e')), 1] = s(p'(e'))$$

indicates how, given a section s of p'', to define a unique G-map $f : E' \to E$, and, given a G-map $f : E' \to E$, how to define a unique section s of p''. A map $E' \to E$ is G-map if and only if it is a principal G-bundle isomorphism. $\square$

Theorem 7.11. *Let B' be a paracompact Hausdorff space.*

1. *Any (G, F)-bundle $p : E' \to B' \times I$ is isomorphic to the (G, F)-bundle $E'_0 \times I \to B' \times I$ where $E'_0 \to B'$ is the original bundle restricted to $B' = B' \times 0$, Furthermore, the isomorphism restricts to the identity on the inverse image of $B' \times 0$.*
2. *Let $f \simeq g : B' \to B$ be homotopic maps and let $E \to B$ be a (G, F)-bundle. Then the (G, F, B')-bundles $f^*E \to B$ and $g^*E \to B$ are isomorphic.*

Proof. 1. By passing to the associated principal bundle we may assume, without loss of generality, that all the bundles are principal G-bundles. Let $p'' : E'' \to B' \times I$ be the fiber bundle provided by Lemma 7.10 whose sections correspond bijectively with principal G-bundle isomorphisms $E' \to E'_0 \times I$ over $B' \times I$. Consider the homotopy lifting problem

$$\begin{tikzcd} B' \times 0 \arrow[r, "s_0"] \arrow[d] & E'' \arrow[d, "P"] \\ B' \times I \arrow[r, "\mathrm{Id}"'] \arrow[ur, dotted] & B' \times I \end{tikzcd}$$

where s_0 is the section corresponding to the identification of principal G-bundles $E'_0 \to E'_0 \times 0$ over $B' \times 0$. Since B' is paracompact and I is compact, $B' \times I$ is paracompact, so P is a fibration by Corollary 7.9. Hence the dotted line in the above commutative diagram exists, and this provides a section of P. Hence the bundles are isomorphic by Lemma 7.10.

To prove Part 2, let $H : B' \times I \to B$ be the homotopy between f and g. Then $f^*E = (H^*E)_0$ is isomorphic to $g^*E = (H^*E)_1$ by Part 1. $\square$

Exercise 123. Give an example of a fibration which is not a fiber bundle.

Maps between fibrations are analogous to (and simpler than) maps of fiber bundles.

Definition 7.12. If $p' : E' \to B'$ and $p : E \to B$ are fibrations, then a *map of fibrations* is a pair of maps $f : B' \to B$, $\tilde{f} : E' \to E$ so that the diagram

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d] & E \arrow[d] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

Pullbacks make sense and exist in the world of fibrations.

Definition 7.13. If $p : E \to B$ is a fibration, and $f : X \to B$ a continuous map, define the *pullback of* $p : E \to B$ *by* f to be the map $f^*E \to X$ where

$$f^*E = \{(x,e) \in X \times E \mid f(x) = p(e)\} \subset X \times E$$

and the map $f^*E \to X$ is the restriction of the projection $X \times E \to X$.

The following exercise is a direct consequence of the universal property of pullbacks.

Exercise 124. Show that $p' : f^*E \to X$, $p'(x,e) = x$ is a fibration, that the diagram

$$\begin{tikzcd} f^*E \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ X \arrow[r, "f"'] & B \end{tikzcd}$$

commutes, where $\tilde{f}(x,e) = e$, and that for each $x \in X$, $\tilde{f}$ induces a homeomorphism $\tilde{f} : p'^{-1}(x) \cong p^{-1}(f(x))$.

7.3. The fiber of a fibration

A fibration need not be a fiber bundle. Indeed, the definition of a fibration is less rigid than that of a fiber bundle, and it is not hard to alter a fiber bundle slightly to get a fibration which is not locally trivial. Nevertheless, a fibration has a well-defined fiber *up to homotopy type.* The following theorem asserts this and also states that a fibration has a substitute for the structure group of a fiber bundle, namely the group of homotopy classes of self-homotopy equivalences of the fiber.

It is perhaps at first surprising that the homotopy lifting property in itself is sufficient to endow a map with the structure of a "fiber bundle up to homotopy". But as we will see, the notion of a fibration is central in studying spaces up to homotopy.

For a map $p : E \to B$, let $F_b = p^{-1}\{b\}$ denote the fiber above a point $b \in B$. Recall $[X, Y]$ denotes the homotopy class of maps from a space X to a space Y.

Theorem 7.14 (Fiber Transport Theorem). *Let $p : E \to B$ be a fibration.*

1. *A path $\alpha : I \to B$ determines a well-defined homotopy class $\alpha_* \in [F_{\alpha(0)}, F_{\alpha(1)}]$.*
2. *If $\alpha \simeq \beta$ (rel $\{0,1\}$), then $\alpha_* = \beta_*$.*
3. *If α and β are paths with $\alpha(1) = \beta(0)$, then $(\alpha\beta)_* = \beta_* \circ \alpha_*$*
4. *The representatives of α_* are homotopy equivalences.*

Corollary 7.15. *Let $p : E \to B$ be a fibration with B path-connected. Then all fibers F_b are homotopy equivalent.*

Corollary 7.16. *For any fibration $p : E \to B$ and $b_0 \in B$, there exists a well-defined group homomorphism*

$$[\alpha] \mapsto (\alpha^{-1})_*$$

$$\pi_1(B, b_0) \to \textit{Homotopy classes of self-homotopy equivalences of } F_{b_0}.$$

Remark. The reason why we use $\alpha \mapsto (\alpha^{-1})_*$ instead of $\alpha \mapsto \alpha_*$ is because by convention, multiplication of paths in B is defined so that $\alpha\beta$ means first follow α, then β. This implies that $(\alpha\beta)_* = \beta_* \circ \alpha_*$, and so we use the inverse to turn this anti-homomorphism into a homomorphism.

Proof. Let $b_0, b_1 \in B$ and let α be a path in B from b_0 to b_1. The inclusion $F_{b_0} \hookrightarrow E$ completes to a diagram

$$\begin{tikzcd}
F_{b_0} \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
F_{b_0} \times I \arrow[r, "H"'] & B
\end{tikzcd}$$

where $H(e, t) = \alpha(t)$. Since $E \to B$ is a fibration, H lifts to E; i.e. there exists a map $\widetilde{H}$ such that

$$\begin{tikzcd}
F_{b_0} \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
F_{b_0} \times I \arrow[r, "H"'] \arrow[ur, "\widetilde{H}"] & B
\end{tikzcd}$$

commutes.

Notice that the homotopy at time $t = 0$, $\widetilde{H}_0 : F_{b_0} \to E$, is just the inclusion of the fiber F_{b_0} in E. Furthermore, $p \circ \widetilde{H}_t$ is the constant map at $\alpha(t)$, so the homotopy $\widetilde{H}$ at time $t = 1$ is a map $\widetilde{H}_1 : F_{b_0} \to F_{b_1}$. We will let $\alpha_* = [\widetilde{H}_1]$ denote the homotopy class of this map. Since $\widetilde{H}$ is not unique, we need to show that another choice of lift gives a homotopic map. We will in fact show something more general. Suppose $\alpha' : I \to B$ is another path homotopic to α rel end points. Then as before, we obtain a solution $\widetilde{H}'$ to the problem

$$\begin{tikzcd}
F_{b_0} \times 0 \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
F_{b_0} \times I \arrow[r, "H'"'] \arrow[ur, "\widetilde{H}'"'] & B
\end{tikzcd}$$

(where $H' = \alpha' \circ \mathrm{proj}_I$) and hence a map $\widetilde{H}'_1 : F_{b_0} \to F_{b_1}$.

Claim. *$\widetilde{H}_1$ is homotopic to $\widetilde{H}'_1$.*

Proof of Claim. Since α is homotopic rel end points to α', there exists a map $\Lambda : F_{b_0} \times I \times I \to B$ such that

$$\Lambda(e,s,t) = F(s,t)$$

where $F(s,t)$ is a homotopy rel end points of α to α'. (So $F_0 = \alpha$ and $F_1 = \alpha'$.) The solutions $\widetilde{H}$ and $\widetilde{H}'$ constructed above give a diagram

$$\begin{tikzcd}
(F_{b_0} \times I) \times \{0,1\} \cup (F_{b_0} \times 0) \times I \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
(F_{b_0} \times I) \times I \arrow[r, "\Lambda"'] & B
\end{tikzcd}$$

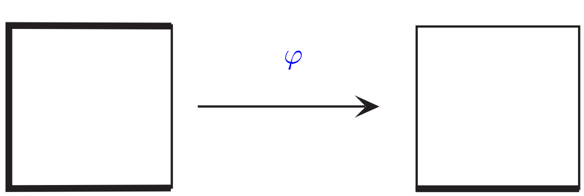
where

$$\Gamma(e,s,0) = \widetilde{H}(e,s)$$

$$\Gamma(e,s,1) = \widetilde{H}'(e,s), \text{ and}$$

$$\Gamma(e,0,t) = e.$$

Let $U = I \times \{0,1\} \cup 0 \times I \subset I \times I$ There exists a homeomorphism $\varphi : I^2 \to I^2$ taking U to $I \times 0$ as indicated in the following picture.



Thus the diagram

$$\begin{tikzcd}
F_{b_0} \times I \times 0 \arrow[d] & F_{b_0} \times U \arrow[l, "\mathrm{Id}\times\varphi"'] \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
F_{b_0} \times I \times I & F_{b_0} \times I^2 \arrow[l, "\varphi"] \arrow[r, "\Lambda"'] & B
\end{tikzcd}$$

has the left two horizontal maps homeomorphisms. Since the homotopy lifting property applies to the outside square, there exists a lift $\widetilde{\Lambda} : F_{b_0} \times I^2 \to E$ so that

$$\begin{tikzcd}
F_{b_0} \times U \arrow[r, "\Gamma"] \arrow[d] & E \arrow[d, "p"] \\
F_{b_0} \times I^2 \arrow[r, "\Lambda"'] \arrow[ur, "\widetilde{\Lambda}"] & B
\end{tikzcd}$$

commutes.

But then $\widetilde{\Lambda}$ is a homotopy from $\widetilde{H} : F_{b_0} \times I \to E$ to $\widetilde{H}' : F_{b_0} \times I \to E$. Restricting to $F_{b_0} \times 1$ we obtain a homotopy from $\widetilde{H}_1$ to $\widetilde{H}'_1$. Thus the homotopy class $\alpha_* = [\widetilde{H}_1]$ depends only on the homotopy class of α rel end points, establishing both 1. and 2. Part 3. is clear. Choosing $\beta = \alpha^{-1}$, 4. follows. $\square$

The following exercise makes sense of the group in Corollary 7.16.

Exercise 125. Show that the set of homotopy classes of homotopy equivalences of a space X forms a group under composition. That is, show that multiplication and taking inverses are well-defined.

Corollary 7.15 asserts that the fibers $p^{-1}\{b\} = F_b$ for $b \in B$ are homotopy equivalent. Thus we will abuse terminology slightly and refer to any space in the homotopy equivalence class of the space F_b for any $b \in B$ as *the fiber F of the fibration $p : E \to B$.*

Since homotopy equivalences induce isomorphisms in homology or cohomology, a fibration with fiber F gives rise to local coefficients systems whose fiber is the homology or cohomology of F, as the next corollary asserts.

Corollary 7.17. *Let $p : E \to B$ be a fibration and let $F = p^{-1}\{b_0\}$. Then p gives rise to local coefficient systems over B with fiber $H_n(F)$ or $H^n(F)$ for any n. These local coefficients are obtained from the representations via the composite homomorphism*

$$\pi_1(B,b_0) \to \left\{ \begin{array}{l} \textit{Homotopy classes of self-homotopy} \\ \textit{equivalences } F \to F \end{array} \right\} \to \mathrm{Aut}(A)$$

where $A = H_n(F)$ *or* $A = H^n(F)$.

Proof. The maps $f_* : H_n(F) \to H_n(F)$ and $f^* : H^n(F) \to H^n(F)$ induced by a homotopy equivalence $f : F \to F$ are isomorphisms which depend only on the homotopy class of f. Thus there is a function from the group of homotopy classes of homotopy equivalences of F to the group of automorphisms of A. This is easily seen to be a homomorphism. The corollary follows. □

We see that a fibration gives rise to many local coefficient systems, by taking homology or cohomology of the fiber. More generally one obtains a local coefficient system given any homotopy functor from spaces to abelian groups (or R-modules), such as the generalized homology theories which we introduce in Chapter 9.

With some extra hypotheses one can also apply this to homotopy functors on the category of based spaces. For example, we will see below that if F is simply connected, or more generally "simple', then taking homotopy groups $\pi_n F$ also gives rise to a local coefficient system. For now, however, observe that the homotopy equivalences constructed by Corollary 7.16 need not preserve base points.

7.4. Path space fibrations

An important type of fibration is the path space fibration. Path space fibrations will be useful in replacing arbitrary maps by fibrations and then in extending a fibration to a fiber sequence.

Definition 7.18. Let (Y, y_0) be a based space. The *path space* $P_{y_0}Y$ is the space of paths in Y starting at y_0, i.e.

$$P_{y_0}Y = \mathrm{Map}(I{,}0; Y{,}y_0) \subset \mathrm{Map}(I{,}Y),$$

topologized as in the previous subsection, i.e. as a compactly generated space. The *loop space* $\Omega_{y_0}Y$ is the space of all loops in Y based at y_0, i.e.

$$\Omega_{y_0}Y = \mathrm{Map}(I{,}\{0{,}1\}; Y{,}\{y_0\}).$$

Often the subscript y_0 is omitted in the above notation. Let $Y^I = \mathrm{Map}(I, Y)$. This is called the *free path space.* Let $p : Y^I \to Y$ be the evaluation at the end point of a path: $p(\alpha) = \alpha(1)$.

By our conventions on topologies, $p : Y^I \to Y$ is continuous. The restriction of p to $P_{y_0}Y$ is also continuous.

Exercise 126. Let y_0, y_1 be two points in a path-connected space Y. Prove that $\Omega_{y_0}Y$ and $\Omega_{y_1}Y$ are homotopy equivalent.

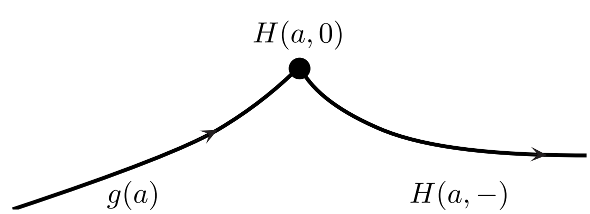
Theorem 7.19.

1. *The map $p : Y^I \to Y$, where $p(\alpha) = \alpha(1)$, is a fibration. Its fiber over y_0 is the space of paths which end at y_0, a space homeomorphic to $P_{y_0}Y$.*
2. *The map $p : P_{y_0}Y \to Y$ is a fibration. Its fiber over y_0 is the loop space $\Omega_{y_0}Y$.*
3. *The free path space Y^I is homotopy equivalent to Y. The projection $p : Y^I \to Y$ is a homotopy equivalence.*
4. *The space of paths in Y starting at y_0, $P_{y_0}Y$, is contractible.*

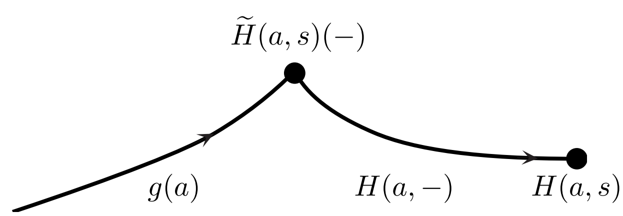
Proof. 1. Let A be a space, and suppose a homotopy lifting problem

$$\begin{tikzcd} A \times 0 \arrow[r, "g"] \arrow[d] & Y^I \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y \end{tikzcd}$$

is given. We write $g(a)$ instead of $g(a, 0)$. For each $a \in A$, $g(a)$ is a path in Y which ends at $p(g(a)) = H(a, 0)$. This point is the start of the path $H(a, -)$.



We will define $\widetilde{H}(a, s)(t)$ to be a path running along the path $g(a)$ and then partway along $H(a, -)$, ending at $H(a, s)$.



Define

$$\widetilde{H}(a,s)(t) = \begin{cases} g(a)((1+s)t) & \text{if } 0 \le t \le 1/(1+s), \\ H(a,(1+s)t-1) & \text{if } 1/(1+s) \le t \le 1. \end{cases}$$

Then $\widetilde{H}(a,s)(t)$ is continuous as a function of (a,s,t), so $\widetilde{H}(a,s) \in Y^I$ and by our choice of topologies $\widetilde{H} : A \times I \to Y^I$ is continuous. Also $\widetilde{H}(a,0) = g(a)$ and $p(\widetilde{H}(a,s)) = \widetilde{H}(a,s)(1) = H(a,s)$. Thus the lifting problem is solved and so $p : P_{y_0}Y \to Y$ is a fibration. The fiber $p^{-1}\{y_0\}$ consists of all paths ending at y_0, and the path space $P_{y_0}Y$ consists of all paths starting at y_0. A homeomorphism is given by

$$\alpha(t) \mapsto \overline{\alpha}(t) = \alpha(1-t).$$

This proves 1.

2. has the same proof; the fact that $g(a)$ *starts at* y_0 means that $\widetilde{H}(a,s)$ also starts at y_0.

To prove 3., let $i : Y \to Y^I$ be the map taking y to the constant path at y. Then $p \circ i = \mathrm{Id}_Y$. Let $F : Y^I \times I \to Y^I$ be given by

$$F(\alpha,s) = (t \mapsto \alpha(s+t-st)).$$

Then $F(\alpha,0) = \alpha$ and $F(\alpha,1)$ is the constant path at $\alpha(1)$ which in turn equals $i \circ p(\alpha)$. Thus F shows that the identity is homotopic to $i \circ p$. Hence p and i are homotopy inverses.

4. has the same proof as 3. □

7.5. Fiber homotopy

A map of fibrations $(p' : E' \to B)$ to $(p : E \to B)$ over B is a commutative diagram

$$\begin{tikzcd} E' \arrow[rr, "f"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Definition 7.20. A *fiber homotopy* between two morphisms (f_i, f_i) $i = 0,1$ of fibrations over B is a commutative diagram

$$\begin{tikzcd} E' \times I \arrow[rr, "H"] \arrow[dr, "p'"'] & & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

with $H_0 = f_0$ and $H_1 = f_1$.

Given two fibrations over B, $p' : E' \to B$ and $p : E \to B$, we say they have the same *fiber homotopy type* if there exists a map f from E' to E covering the identity map of B, and a map g from E to E' covering the identity map of B, such that the composites

$$\begin{tikzcd} E' \arrow[rr, "g\circ f"] \arrow[dr] & & E' \arrow[dl] \\ & B & \end{tikzcd} \qquad \begin{tikzcd} E \arrow[rr, "f\circ g"] \arrow[dr] & & E \arrow[dl] \\ & B & \end{tikzcd}$$

are each fiber homotopic to the identity via a homotopy which is the identity on B. Similarly for $g \circ f$. One calls f a *fiber homotopy equivalence.* The maps f and g are called *fiber homotopy inverses.*

A fiber homotopy equivalence $f : E' \to E$ is, in particular, a homotopy equivalence. But in addition, a fiber homotopy equivalence induces a homotopy equivalence $F'_{b_0} \to F_{b_0}$ on fibers. The following theorem, due to Dold, shows that any map of fibrations which is also a homotopy equivalence is in fact a fiber homotopy equivalence.

Theorem 7.21. *Let $p : E \to B$ and $p' : E' \to B$ be fibrations and suppose there exists a homotopy equivalence $h : E \to E'$ so that $p' \circ h = p$. Then p' and p are fiber homotopy equivalent. In particular, the restriction of h to fibers*

$$h|_{p^{-1}(y_0)} : p^{-1}(y_0) \to p'^{-1}(y_0)$$

is a homotopy equivalence.

Proof. Let $h' : E' \to E$ be a homotopy inverse for h. Let $F : E' \times [0,1] \to E'$ be a homotopy between hh' and $\mathrm{Id}_{E'}$. Let $\bar{G} = p'F$, so $\bar{G} : E' \times [0,1] \to B$. Since $\bar{G}(e',0) = p'(F(e',0)) = p'(h(h'(e'))) = p(h'(e'))$, the HLP for p implies that there is a lift $G : E' \times [0,1] \to E$ of $\bar{G}$ with $G(e',0) = h'(e')$. Then $p(G(e',1)) = \bar{G}(e',1) = p'(F(e',1)) = p'(e')$. In other words, if we define $h'' : E' \to E$ to be $G(-,1)$, i.e. $h''(e') = G(e',1)$, then h'' is a homotopy inverse for h which preserves fibers, i.e. $ph'' = p'$. We will show that h'' is a *fiber homotopy* inverse of h.

Given homotopies $R, S : X \times [0,1] \to Y$ let R^{-1} denote the *reverse* homotopy, i.e. $R^{-1}(x,t) = R(x, 1-t)$ and let $R * S$ denote the *composite* homotopy (assuming $R(x,1) = S(x,0)$)

$$R * S(x,t) = \begin{cases} R(x,2t) & \text{if } t \leq 1/2, \\ S(x,2t-1) & \text{if } t \geq 1/2. \end{cases}$$

Let $H : E' \times [0,1] \to E'$ be the composite $H = (hG)^{-1} * F$, which is defined since $hG(e',0) = hh'(e') = F(0)$. Thus H is a homotopy from hh'' to $\mathrm{Id}_{E'}$. Since $p'F = \bar{G} = pG = p'hG$, $p'H(e',t) = p'H(e',1-t)$. In other words, viewing $p'H$ as a loop $[0,1] \to \mathrm{Map}(E',B)$, this loop is obtained by traveling along a path and then returning along the same path. There is an obvious nullhomotopy obtained by traveling less and less along the path and returning. Precisely, define $\bar{K} : E' \times [0,1] \times [0,1]$ by

$$\bar{K}(e',t,s) = \begin{cases} p'H(e',(1-s)t) & \text{if } t \leq 1/2, \\ p'H(e',(1-s)(1-t)) & \text{if } t \geq 1/2. \end{cases}$$

Then $\bar{K}(e',t,0) = p'H(e',t)$, $\bar{K}(e',t,1) = p'(e')$, $\bar{K}(e',0,s) = p'(e')$, and $\bar{K}(e',1,s) = p'(e')$.

We will use the HLP to lift $\bar{K}$ to a fiber preserving homotopy. Let $U \subset I \times I$ be the union of the three sides

$$U = \{(t,s) \mid s = 0\} \cup \{(t,s) \mid t = 0\} \cup \{(t,s) \mid t = 1\}.$$

Let $K : E' \times U \to E'$ be the map

$$K(e',t,s) = \begin{cases} H(e',t) & \text{if } s = 0, \\ h(h''(e')) & \text{if } t = 0, \\ e' & \text{if } t = 1. \end{cases}$$

Since there is a homeomorphism $I \times I \cong I \times I$ taking U to $I \times 0 = \{(t,s) \mid s = 0\}$, the HLP implies that K extends to a map $K : E' \times I \times I \to E'$ satisfying $p'K = \bar{K}$. Let $D : E' \times I \to E'$ be the endpoint of this map, i.e. $D(e',t) = K(e',t,1)$. Then $D(e',0) = h(h''(e'))$, $D(e',1) = e'$, and $p'(D(e',t)) = \bar{K}(e',t,1) = p'(e')$. In other words, D is a fiber preserving homotopy between hh'' and $\mathrm{Id}_{E'}$.

Now repeat the entire argument to h'' to find a map $h''' : E \to E'$ and a fiber preserving homotopy between $h''h'''$ and Id_E. Use the notation "$\simeq_F$" for fiber preserving homotopic. Then

$$h''h \simeq_F h''hh''h''' \simeq_F h''h''' \simeq_F \mathrm{Id}_E.$$

In other words, $h : E \to E'$ and $h'' : E' \to E$ are fiber homotopy inverses. □

7.6. Replacing a map by a fibration

Let $f : X \to Y$ be a continuous map. We will replace X by a homotopy equivalent space P_f and obtain a map $P_f \to Y$ which is a fibration. In short, every map is homotopy equivalent to a fibration. If f is a fibration to begin with, then the construction gives a fiber homotopy equivalent fibration. We assume that Y is path-connected and X is nonempty.

Let $q : Y^I \to Y$ be the path space fibration, with $q(\alpha) = \alpha(0)$; evaluation at the starting point.

Definition 7.22. The pullback $P_f = f^*(Y^I)$ of the path space fibration along f is called the *mapping path space.*

$$\begin{tikzcd} P_f = f^*(Y^I) \arrow[r] \arrow[d] & Y^I \arrow[d, "q"] \\ X \arrow[r, "f"'] & Y \end{tikzcd} \tag{7.1}$$

An element of P_f is a pair (x, α) where α is a path in Y and x is a point in X which maps via f to the starting point of α.

The *mapping path fibration*

$$p : P_f \to Y$$

is obtained by evaluating at the end point

$$p(x, \alpha) = \alpha(1).$$

Theorem 7.23. *Suppose that $f : X \to Y$ is a continuous map.*

1. *There exists a homotopy equivalence $h : X \to P_f$ so that the diagram*

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & P_f \arrow[dl, "p"] \\ & Y & \end{tikzcd}$$

   commutes.

2. *The map $p : P_f \to Y$ is a fibration.*
3. *If $f : X \to Y$ is a fibration, then h is a fiber homotopy equivalence. In particular, for every point $y \in Y$, h induces a homotopy equivalence $f^{-1}(y) \to p^{-1}(y)$ on the fibers.*

Proof. 1. Let $h : X \to P_f$ be the map

$$h(x) = (x, \mathrm{const}_{f(x)})$$

where $\mathrm{const}_{f(x)}$ means the constant path at $f(x)$. Then $f = p \circ h$, so the triangle commutes. The homotopy inverse of h is $p_1 : P_f \to X$, projection on the X-component. Then $p_1 \circ h = \mathrm{Id}_X$. The homotopy from $h \circ p_1$ to Id_{P_f} is given by

$$F((x, \alpha), s) = (x, \alpha_s),$$

where α_s is the path $t \mapsto \alpha(st)$. (We have embedded X in P_f via h and have given a deformation retraction of P_f to X by contracting a path to its starting point.)

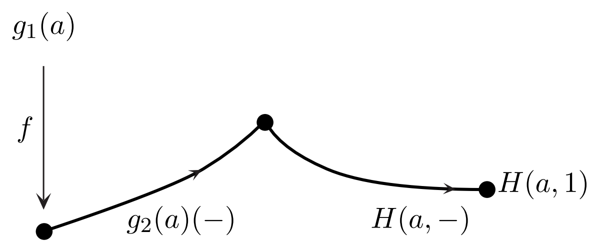
2. Let the homotopy lifting problem

$$\begin{tikzcd} A \times 0 \arrow[r, "g"] \arrow[d] & P_f \arrow[d, "p"] \\ A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & Y \end{tikzcd}$$

be given. For $a \in A$, we write $g(a)$ instead of $g(a, 0)$. Furthermore $g(a)$ has an X-component and a Y^I-component, and we write

$$g(a) = (g_1(a), g_2(a)) \in P_f \subset X \times Y^I.$$

Note that since $g(a)$ is in the pullback, $g_1(a)$ maps via f to the starting point of the path $g_2(a)$ and the square above commutes, so the end point of the path $g_2(a)$ is the starting point of the path $H(a, -)$. Here is a picture of $g(a)$ and $H(a, -)$.

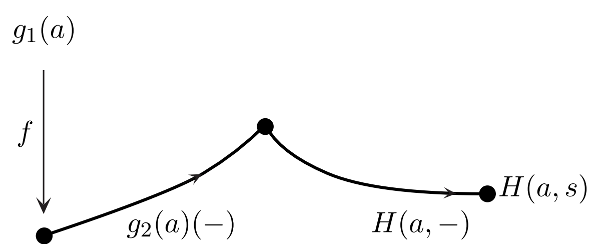


The lift $\widetilde{H}$ will have two components. The X-component will be constant in s,

$$\widetilde{H}_1(a, s) = g_1(a).$$

The Y^I-component of the lift will be a path running along the path $g_2(a)$ and then partway along $H(a, -)$, ending at $H(a, s)$.

Here is a picture of $\widetilde{H}(a, s)$.



A formula is given by

$$\widetilde{H}(a, s) = (g_1(a), \widetilde{H}_2(a, s)(-)) \in P_f \subset X \times Y^I,$$

where

$$\widetilde{H_2}(a,s)(t) = \begin{cases} g_2(a)((1+s)t) & \text{if } 0 \le t \le 1/(1+s), \\ H(a,(1+s)t-1) & \text{if } 1/(1+s) \le t \le 1. \end{cases}$$

We leave it to the reader to check that $\widetilde{H}$ is continuous and that it is a lift of H extending the map g. Thus we have shown that the mapping path fibration is a fibration.

3. Finally, if $f : X \to Y$ is a fibration, then Theorem 7.21 shows that $p : P_f \to Y$ is fiber homotopy equivalent to $f : X \to Y$.

□

Thus any map $f : X \to Y$ can be "replaced" by a fibration $f' : X' \to Y$ in the sense that there exists a homotopy equivalence $h : X \to X'$ so that the diagram

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

commutes. Theorem 7.21 shows that the resulting fibration is unique up to fiber homotopy equivalence, independently of how $f' : X' \to Y$ is constructed. In particular its fiber $f'^{-1}\{y\}$ is well-defined up to homotopy equivalence.

It is common to be sloppy and say "F is the homotopy fiber of f", or "$F \hookrightarrow X \to Y$ is a fibration" to mean that after replacing $f : X \to Y$ by a fibration $f' : X' \to Y$, the fiber $f'^{-1}\{y\}$ is a space of the homotopy type of F.

Notice that the space Y is untouched in this discussion. It is possible to sharpen Theorem 7.21 as follows.

Theorem 7.24. *Suppose that*

$$\begin{tikzcd} X \arrow[r, "m"] \arrow[d, "f"'] & X' \arrow[d, "f'"] \\ Y \arrow[r, "\ell"'] & Y' \end{tikzcd}$$

is a diagram which commutes up to homotopy, and that the horizontal maps ℓ, m are homotopy equivalences. Suppose further that $p : E \to Y$ is a fibration with fiber $F = p^{-1}(y_0)$, $p' : E' \to Y'$ is a fibration with fiber

$F' = p'^{-1}(f(y_0))$, and that $h : X \to E$, $h' : X' \to E'$ are homotopy equivalences so that the diagrams

$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & E \arrow[dl, "p"] \\ & Y & \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[rr, "h'"] \arrow[dr, "f'"'] & & E' \arrow[dl, "p'"] \\ & Y' & \end{tikzcd}$$

commute. Let $k : E \to X$ be a homotopy inverse for h and $k' : E' \to X$ a homotopy inverse for h'.

Then there exists a homotopy equivalence $n : F \to F'$ so that the diagram

$$\begin{tikzcd} F \arrow[r, "n"] \arrow[d, "k|_F"'] & F' \arrow[d, "k'|_{F'}"] \\ X \arrow[r, "m"'] & X' \end{tikzcd}$$

commutes up to homotopy. NOT QUITE RIGHT

7.7. Cofibrations

show that the inclusion of the boundary of a manifold is a cofibration

Definition 7.25. A map $i : A \to X$ is called a *cofibration*, or *satisfies the homotopy extension property (HEP)*, if the following diagram has a solution for any space Y

$$\begin{tikzcd} A \arrow[r, "\mathrm{inc}_0"] \arrow[d, "i"] & A \times I \arrow[d, "i\times \mathrm{Id}"] \arrow[ddr, bend left] & \\ X \arrow[r, "\mathrm{inc}_0"] \arrow[drr, bend right] & X \times I \arrow[dr, dotted] & \\ & & Y \end{tikzcd}$$

where the maps labeled inc_0 send a to $(a, 0)$ and x to $(x, 0)$.

The diagram says that given maps $X \to Y$ and $A \times I$ which agree on A, there is a map $X \times I \to Y$ making the diagram commute. Note the similarity of this diagram with that of a pushout diagram, with the difference being that the dotted map need not be unique. Note also that cofibration is a dual notion to fibration, using the adjointness of the functors $- \times I$ and $-^I$ and reversing the arrows. To see the duality between fibration and cofibration, write the diagram defining a fibration as

$$\begin{tikzcd}
B & B^I \arrow[l, "e_0"'] & \\
E \arrow[u, "f"] & E^I \arrow[l, "e_0"'] \arrow[u] & \\
& & Y \arrow[ull, bend left] \arrow[uul, bend right] \arrow[ul, dotted]
\end{tikzcd}$$

where $e_0(\gamma) = \gamma(0)$ is evaluation at 0.

A few delicate point-set issues arise from the definition of a cofibration. The first is that a cofibration $i : A \to X$ is always an embedding, a homeomorphism onto its image. The idea in verifying this is to define the mapping cylinder $M_i = \frac{(A\times I)\amalg X}{(a,1)\sim i(a)}$. It is not difficult to see that the map $A \to M_i, a \mapsto [a, 0]$ is an embedding. Then one uses the cofibration property to extend the maps $X \to M_i, x \mapsto [x]$ and $A \times I \to M,(a, t) \mapsto (a, 1 - t)$ to a map $\pi : X \times I \to M_i$. Then the maps

$$i(A) \xrightarrow{\text{inc}_1} X \times I \xrightarrow{\pi} M_i \hookleftarrow A$$

give the inverse map to $i : A \to i(A)$. If $i : A \to X$ is a cofibration, the quotient space $X/i(A)$ is called the *cofiber* of i.

We say a pair (X, A) is a *cofibration pair* if the inclusion map is a cofibration. Note that (X, A) is a cofibration pair if and only if the problem

$$\begin{tikzcd}
X \times 0 \,\cup\, A \times I \arrow[r, "f\cup h"] \arrow[d, "i"'] & Y \\
X \times I \arrow[ur, dotted] &
\end{tikzcd}$$

has a solution for all spaces Y, maps $f : X \to Y$ and homotopies $h : A \times I \to Y$ extending $f_{|A}$. Hence the name homotopy extension property: given a homotopy on A and an extension of one end of the homotopy to X, the whole homotopy extends to X.

We showed that any cofibration $i : A \to X$ is an embedding. If X is Hausdorff, then one can show that $i(A)$ is closed in X.

Exercise 127. Suppose X is Hausdorff. A pair (X, A) is a cofibration pair if and only if $X \times 0 \,\cup\, A \times I$ is a retract of $X \times I$.

The exercise is actually true without the Hausdorff hypothesis, but is fairly subtle, see [**19**].

Here is the most basic example of a cofibration.

Lemma 7.26. *(D^n, S^{n-1}) is a cofibration pair.*

Proof. Suppose there is a light source at $(0, \ldots, 0, 2) \in \mathbf{R}^{n+1}$. Then a retraction $r : D^n \times I \to D^n \times 0 \cup S^{n-1} \times I$ is given by sending a point to its shadow. Thus $r(\mathbf{x}, t)$ is the intersection of the line through $(\mathbf{0}, 2)$ and $(\mathbf{x}, t)$ with $D^n \times 0 \,\cup\, S^{n-1} \times I$. $\square$

We next establish that a pushout of a cofibration is a cofibration; this is dual to the fact that pullback of a fibration is a fibration. The word dual here is used in the sense of reversing arrows.

Recall from Definition 1.4 that the pushout of $B \xleftarrow{f} A \xrightarrow{g} C$ can be concretely realized as

$$P = \frac{B \amalg C}{f(a) \sim g(a)}.$$

Lemma 7.27. *If $g : A \to C$ is a cofibration and*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d] \\ C \arrow[r] & P \end{tikzcd}$$

is a pushout diagram, then $B \to P$ is a cofibration. $\square$

The proof is obtained by reversing the arrows in the dual argument for fibrations. We leave it as an exercise.

Exercise 128. Prove Lemma 7.27.

The key example of a cofibration pair is a CW-pair, or more generally, a relative CW-complex.

Corollary 7.28. *A relative CW-complex (X, A) is a cofibration pair.*

Proof. Recall that the definition of a relative CW-complex (X, A) is a filtration

$$A = (X, A)^{-1} \subset (X, A)^0 \subset (X, A)^1 \subset \cdots$$

with $X = \operatorname{colim}(X, A)^n$ and with pushout diagrams

$$\begin{tikzcd} \amalg S^{n-1} \arrow[r] \arrow[d] & (X, A)^{n-1} \arrow[d] \\ \amalg D^n \arrow[r] & (X, A)^n \end{tikzcd}$$

for all $n \geq 0$. The last two lemmas imply that any map $X \times 0 \cup A \times I \to Y$ extends to $X \times 0 \cup (X, A)^n \times I \to Y$ for all n with compatible restrictions.

Part 1 of Theorem 7.6 implies that $\operatorname{colim}(X, A)^n \times I = X \times I$. This gives the desired homotopy extension. $\square$

Exercise 129. If (X, A) and (Y, B) are cofibrations, so is their product

$$(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y).$$

The analogue for cofibrations of Theorem 7.21 is the following theorem.

Theorem 7.29. *Suppose that $i : A \hookrightarrow X$ and $i' : A \hookrightarrow X'$ are cofibrations and $g : X \to X'$ is a homotopy equivalence so that the diagram*

$$\begin{tikzcd} & A \arrow[dl, "i"'] \arrow[dr, "i'"] & \\ X \arrow[rr, "g"'] & & X' \end{tikzcd}$$

commutes up to homotopy. Then g is homotopic to a map $h : X \to X'$ so that $h \circ i = i'$ and h is a homotopy equivalence relative to A. In particular, h induces a homotopy equivalence of based spaces $X/i(A) \to X'/i'(A)$.

Proof. The proof mirrors the proof of Theorem 7.21. Using the homotopy extension property one can find a homotopy of g to a map h satisfying $hi = i'$. Thus h is a also a homotopy equivalence and induces a map on quotients $X/i(A) \to X'/i'(A)$ which we will show is a homotopy equivalence. To simplify notation, identify A with $i(A) \subset X$, so that $i(a) = a$.

Let $h' : X' \to X$ be a homotopy inverse for h. Let $F : X \times I \to X$ be a homotopy from $h'h$ to Id_X. Let $\bar{G} : A \times I \to X$ be defined by $\bar{G}(a, t) = F(a, t)$. Then $\bar{G}(a, 0) = h'h(a) = h'(i'(a))$. Since $i' : A \to X'$ is a cofibration, there exists an extension $G : X' \times I \to X$ of G so that $G(x, 0) = h'(x)$ and $G(i'(a), t) = \bar{G}(a, t) = F(a, t)$. Let $h'' = G(-, 1) : X' \to X$. Since h'' is homotopic to h', h'' is also a homotopy inverse for h. By construction, $h''i'(a) = a$ for $a \in A$.

We will show that $h''h$ is homotopic to the identity by a homotopy which is stationary on A. First construct a homotopy $H : X \times I \to X$ from $h''h$ to Id_X as the composite of the reverse of $G \circ h$ followed by F. More precisely

$$H(x, t) = \begin{cases} G(h(x), 1 - 2t) & \text{if } t \leq \frac{1}{2} \\ F(x, 2t - 1) & \text{if } t \geq \frac{1}{2}. \end{cases}$$

These match up since $G(h(x), 0) = h'h(x)$ and $F(x, 0) = h'h(x)$.

Observe that when $t \leq \frac{1}{2}$,

$$H(a, t) = G(h(a), 1 - 2t) = G(i'(a), 1 - 2t) = \bar{G}(a, 1 - 2t) = F(a, 1 - 2t)$$

and when $t \geq \frac{1}{2}$

$$H(a, t) = F(a, 2t - 1).$$

In other words, viewing the restriction to A of the homotopy H as a loop in $\mathrm{Map}(A, X)$ based at Id_A, we see that this loop is the composite of a loop and its reverse. A nullhomotopy of this loop provides a map $K : (A \times I) \times I \to X$ so that $K(a, t, 0) = H(a, t)$ and $K(a, t, 1) = a$. Since (X, A) is a cofibration, so is $(X \times I, A \times I)$. Therefore the map $K \cup H : (A \times I) \times I \cup (X \times I) \times 0 \to X$ extends to $X \times I \times I$. The restriction $X \times I \times 1 \to X$ of this extension is a homotopy $H' : X \times I \to X$ from $h''h$ to Id_X which is stationary on A, i.e. $H'(a, t) = a$ for all $a \in A$.

Write $\simeq_A$ for homotopies that are stationary on A. Thus $h''h \simeq_A \mathrm{Id}_X$.

Repeat the argument to find a map $h''' : X \to X'$ so that $h'''h'' \simeq_A \mathrm{Id}_{X'}$. Then

$$hh'' \simeq_A h'''h''hh'' \simeq_A h'''h'' \simeq_A \mathrm{Id}_{X'}.$$

Since $h''h \simeq_A \mathrm{Id}_X$ and $hh'' \simeq_A \mathrm{Id}_{X'}$, the induced maps on quotients $h : X/A \to X'/i'(A)$ and $h'' : X'/i'(A) \to X/A$ are homotopy inverses, finishing the proof. □

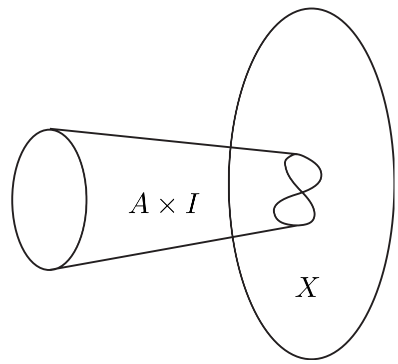
In other words, the cofibrations i and i' have the same cofiber homotopy type.

7.8. Replacing a map by a cofibration

Let $f : A \to X$ be a continuous map. We will replace X by a homotopy equivalent space M_f and obtain a map $A \to M_f$ which is a cofibration. In short, every map is equivalent to a cofibration. If f is a cofibration to begin with, then the construction gives a homotopy equivalent cofibration relative to A. It follows from Theorem 7.29 that defining the cofiber of a map $f : A \to X$ to be the cofiber of any cofibration equivalent to f gives a well-defined space up to homotopy equivalence.

Definition 7.30. The *mapping cylinder* of a map $f : A \to X$ is the space

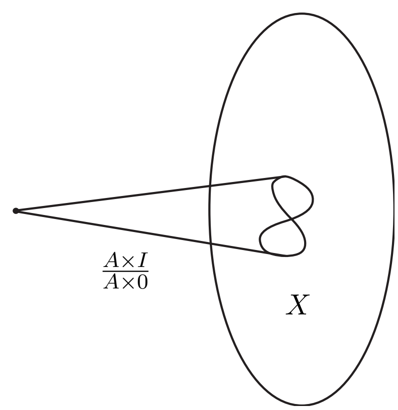
$$M_f = \frac{(A \times I) \amalg X}{(a, 1) \sim f(a)}.$$



The mapping cylinder M_f

The *mapping cone* of $f : A \to X$ is

$$C_f = \frac{M_f}{A \times 0}.$$



The mapping cone C_f

Note that the mapping cylinder M_f can also be defined as the pushout of $A \times I \leftarrow A \times 1 \xrightarrow{f} X$. This shows the analogue with the mapping path fibration P_f more clearly. Sometimes P_f is called the *mapping cocylinder* by those susceptible to categorical terminology. Similarly the mapping cone can be described as the pushout of $\mathrm{cone}(A) \leftarrow A \times 1 \xrightarrow{f} X$, where

$$\mathrm{cone}(A) = \frac{A \times I}{A \times 0}.$$

The dual result to Theorem 7.23 is the following.

Theorem 7.31. *Let $f : A \to X$ be a map. Let $i : A \to M_f$ be the inclusion $i(a) = [a, 0]$.*

1. *There exists a homotopy equivalence* $h : M_f \to X$ *so that the diagram*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "i"] & \\ X & & M_f \arrow[ll, "h"'] \end{tikzcd}$$

   commutes.
2. *The inclusion* $i : A \to M_f$ *is a cofibration with cofiber* C_f*.*
3. *If* $f : A \to X$ *is a cofibration, then* h *is a homotopy equivalence rel* A*. In particular* h *induces a based homotopy equivalence of the cofibers* $C_f \to X/f(A)$*.*

Proof. 1. Let $h : M_f \to X$ be the map

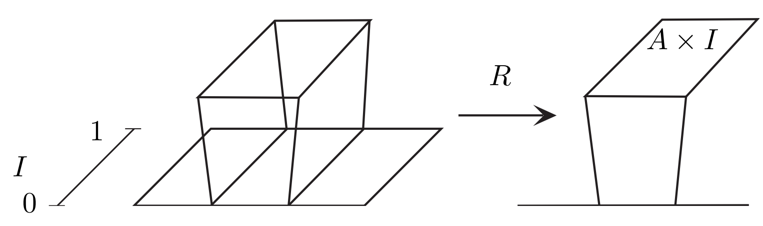
$$h[a, s] = f(a), \qquad h[x] = x.$$

Then $f = h \circ i$ so the diagram commutes. The homotopy inverse of h is the inclusion $j : X \to M_f$. In fact, $h \circ j = \mathrm{Id}_X$, and the homotopy from Id_{M_f} to $j \circ h$ squashes the mapping cylinder onto X and is given by

$$F([a, s], t) = [a, s + t - st]$$

$$F([x], t) = [x].$$

2. By Exercise 127 we need to construct a retraction $R : M_f \times I \to M_f \times 0 \cup A \times I$.



$M_f \times I$ $\qquad\qquad$ $M_f \times 0 \cup A \times I$

Let

$$r : I \times I \to I \times 0 \cup 0 \times I$$

be a retraction so that $r(1 \times I) = \{(1, 0)\}$. (First retract the square onto 3 sides and then contract a side to a point.) Define $R([a, s], t) = [a, r(s, t)]$ and $R([x], t) = ([x], 0)$. Thus $i : A \to M_f$ is a cofibration with cofiber $M_f/A = C_f$.

3. This follows from the first two parts and Theorem 7.29. $\square$

Thus any map $f : X \to Y$ can be replaced by a cofibration $f' : X \hookrightarrow Y'$ in the sense that there exists a homotopy equivalence $h : Y' \to Y$ so that the diagram

$$\begin{tikzcd} & X \arrow[dl, "f'"'] \arrow[dr, "f"] & \\ Y' \arrow[rr, "h"'] & & Y \end{tikzcd}$$

commutes. Theorem 7.29 shows that the resulting cofibration is unique up to homotopy and its cofiber is well-defined up to homotopy equivalence. We say "C is the homotopy cofiber of $f : X \to Y$" to mean that after replacing f by a cofibration $f' : X \hookrightarrow Y'$, the cofiber $Y'/f(X)$ is a space of the homotopy type of C.

Recall that a pair (X, A) is good if $H_*(X, A) \to \widetilde{H}_*(X/A)$ is an isomorphism.

Corollary 7.32. *A cofibration pair (X, A) is good. In particular there is a long exact sequence in reduced homology*

$$\cdots \to \widetilde{H}_n A \to \widetilde{H}_n X \to \widetilde{H}_n(X/A) \to \widetilde{H}_{n-1} A \to \dots$$

Proof. Let $i : A \to X$ be the inclusion. Then A is a strong deformation retract of a neighborhood in M_i. It follows (see Lemma 1.16) that (M_i, A) is a good pair. But (X, A) and (M_i, A) are homotopy equivalent relative to A by Theorem 7.31, so the result follows from the homotopy invariance of homology. □

By Corollary 7.28 this applies to a relative CW-complex.

7.9. Sets of homotopy classes of maps

Recall that if X, Y are spaces, then $[X,Y]$ denotes the set of homotopy classes of maps from X to Y, i.e.

$$[X,Y] = \mathrm{Map}(X,Y)/\simeq$$

where $f \simeq g$ if f is homotopic to g.

Notice that if Y is path-connected, then the set $[X,Y]$ contains a distinguished class of maps, namely the unique class containing all the constant maps. We will use this as a base point for $[X,Y]$ if one is needed.

If X has a base point x_0 and Y has a base point y_0, let $[X,Y]_0$ denote the *based homotopy classes of based maps*, where a *based map* is a map $f : (X,x_0) \to (Y,y_0)$. Then $[X,Y]_0$ has a distinguished class, namely the class of the constant map at y_0. (In the based context, it is not necessary to assume Y is path-connected to have this distinguished class.) Given a map $f : X \to Y$ let $[f]$ denote its homotopy class in $[X, Y]$ or $[X, Y]_0$. Notice that if X and Y are based spaces, there is a forgetful map $[X,Y]_0 \to [X,Y]$. This map need not be injective or surjective.

The notion of an exact sequence of sets is a useful generalization of the corresponding concept for groups.

Definition 7.33. A sequence of functions

$$A \xrightarrow{f} B \xrightarrow{g} C$$

of *sets* (not spaces or groups) *with base points* is called *exact at* B if

$$f(A) = g^{-1}\{c_0\}$$

where c_0 is the base point of C.

All that was necessary here was that C be based. Notice that if A, B, C are groups, with base points the identity element, and f, g homomorphisms, then $A \to B \to C$ is exact *as a sequence of sets* if and only if it is exact as a sequence of groups.

The following two theorems form the cornerstone of constructions of exact sequences in algebraic topology.

Theorem 7.34 (basic property of fibrations)**.** *Let $p : E \to B$ be a fibration with fiber $F = p^{-1}\{b_0\}$. Let Y be any space. Then the sequence of sets*

$$[Y,F] \xrightarrow{i_*} [Y,E] \xrightarrow{p_*} [Y,B]$$

is exact.

Proof. Clearly $p_*(i_*[g]) = [\mathrm{const}_{b_0}]$.

Suppose $f : Y \to E$ so that $p_*[f] = [\mathrm{const}_{b_0}]$; i.e. $p \circ f : Y \to B$ is null homotopic. Let $G : Y \times I \to B$ be a null homotopy, and then let $H : Y \times I \to E$ be a solution to the lifting problem

$$\begin{tikzcd}
Y \times 0 \arrow[r, "f"] \arrow[d] & E \arrow[d, "p"] \\
Y \times I \arrow[r, "G"] \arrow[ur, dotted, "H"] & B
\end{tikzcd}$$

Since $p \circ H(y,1) = G(y,1) = b_0$, $H(y,1) \in F = p^{-1}\{b_0\}$. Thus f is homotopic to a map with image in the fiber, so $[f] = i_*[H(-,1)]$. $\square$

Theorem 7.35 (basic property of cofibrations)**.** *Let $i : A \hookrightarrow X$ be a cofibration, with cofiber X/A. Let $q : X \to X/A$ denote the quotient map. Let Y be any path-connected space. Then the sequence of sets*

$$[X/A,Y] \xrightarrow{q^*} [X,Y] \xrightarrow{i^*} [A,Y]$$

is exact.

Proof. Clearly $i^*(q^*([g])) = [g \circ q \circ i] = [\text{const}]$.

Suppose $f : X \to Y$ is a map and suppose that $f_{|A} : A \to Y$ is nullhomotopic. Let $h : A \times I \to Y$ be a null homotopy. The solution F to the problem

$$\begin{tikzcd} X \times 0 \,\cup\, A \times I \arrow[r, "f \cup h"] \arrow[d, "i"'] & Y \\ X \times I \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

defines a map $f' = F(-,1)$ homotopic to f whose restriction to A is constant, i.e. $f'(A) = \{y_0\}$. Therefore the diagram

$$\begin{tikzcd} X \arrow[r, "f'"] \arrow[d, "q"'] & Y \\ X/A \arrow[ur, dotted, "g"'] & \end{tikzcd}$$

can be completed, by the definition of quotient topology. Thus $[f] = [f'] = q^*[g]$. $\square$

7.10. Adjoint of loops and suspension; smash products

Definition 7.36. Define CGH_* to be the category of compactly generated Hausdorff spaces with a *nondegenerate base point*, i.e. (X,x_0) is an object of CGH_* if the inclusion $\{x_0\} \subset X$ is a cofibration. Note that if x_0 is the strong deformation retract of a neighborhood, then $\{x_0\} \subset X$ is a cofibration. The morphisms in CGH_* are the base point preserving continuous maps.

We will often omit the base point and write that X is a based space instead of (X, x_0). This is a tad disconcerting, but no more so that when we write that X is a CW-complex instead of $(X, \{X^n\})$.

Exercise 130. Show that any point in a CW-complex is nondegenerate.

Exercise 131. Prove the base point versions of the previous two theorems:

1. If $F \hookrightarrow E \to B$ is a base point preserving fibration, then for any $Y \in \mathsf{CGH}_*$
$$[Y,F]_0 \to [Y,E]_0 \to [Y,B]_0$$
is exact.

2. If $A \hookrightarrow X \to X/A$ is a base point preserving cofibration, then for any based space Y,

$$[X/A,Y]_0 \to [X,Y]_0 \to [A,Y]_0$$

is exact.

Most exact sequences in algebraic topology can be derived from Theorems 7.34, 7.35, and Exercise 131. We will soon use this exercise to establish exact sequences of homotopy groups. To do so, we need to be careful about base points and adjoints. Recall that if (X, x_0) and (Y, y_0) are based spaces, then $\mathrm{Map}(X, Y)_0$ is the set of maps of pairs $(X, x_0) \to (Y, y_0)$ with the compactly generated topology.

Definition 7.37. The *smash product* of based spaces is

$$X \wedge Y = \frac{X \times Y}{X \vee Y} = \frac{X \times Y}{X \times \{y_0\} \cup \{x_0\} \times Y}.$$

Note that the smash product $X \wedge Y$ is a based space. Contrary to popular belief, the smash product is *not* the product in the category $\mathbf{CGH}_*$, although the *wedge sum*

$$X \vee Y = (X \times \{y_0\}) \cup (\{x_0\} \times Y) \subset X \times Y$$

is the coproduct in $\mathbf{CGH}_*$. The smash product is the adjoint of the based mapping space. The following theorem follows from the unbased version of the adjoint theorem (Theorem 7.5), upon restricting to based maps.

Theorem 7.38 (adjoint theorem)**.** *There is a (natural) homeomorphism*

$$\mathrm{Map}(X \wedge Y, Z)_0 \cong \mathrm{Map}(X, \mathrm{Map}(Y, Z)_0)_0.$$

□

Definition 7.39. The *(reduced) suspension* of a based space (X, x_0) is $S(X, x_0) = S^1 \wedge X$. The *(reduced) cone* is $C(X, x_0) = I \wedge X$. Here the circle is based by $1 \in S^1 \subset \mathbf{C}$ and the interval by $0 \in I$.

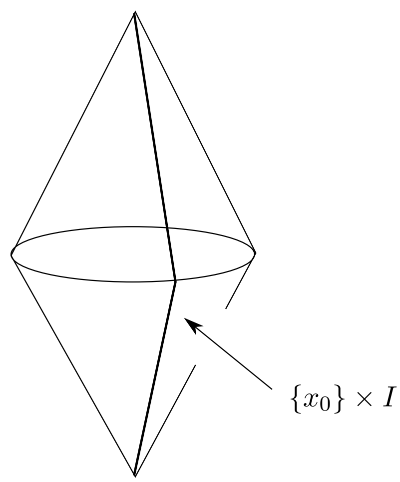
Using the usual identification $I/\{0, 1\} = S^1$ via $t \mapsto e^{2\pi i t}$, one sees

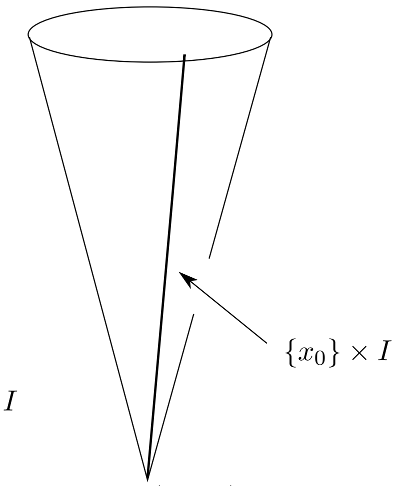
$$S(X, x_0) = \frac{X \times I}{X \times \{0, 1\} \cup \{x_0\} \times I}.$$

In other words, if SX is the unreduced suspension and CX is the unreduced cone $(= (X \times I)/(X \times 0)\ )$, then there are quotient maps

$$SX \to S(X, x_0) \qquad CX \to C(X, x_0)$$

given by collapsing $\{x_0\} \times I$, as indicated in the following figures.





Notice that taking reduced suspensions and reduced cones is functorial. Reduced suspensions and cones are more useful than the unreduced variety since they have canonical base points and satisfy adjoint properties. Nonetheless, it is reassuring to connect them with the more familiar unreduced versions.

Exercise 132. If $X \in \mathrm{CGH}_*$, then the quotient maps $SX \to S(X, x_0)$ and $CX \to C(X, x_0)$ are homotopy equivalences.

We now revert to our earlier convention of omitting mention of base points. Thus the symbol SX can mean two different things, the unreduced suspension when X is a topological space or the reduced suspension when X is a based space. The meaning of SX and CX should be clear from context and not lead to confusion.

Proposition 7.40. *The reduced suspension SS^n is homeomorphic to S^{n+1}, and the reduced cone CS^n is homeomorphic to D^{n+1}.* □

Exercise 133. Prove Proposition 7.40. This shows in a special case that the smash product is associative. Prove associativity of the smash product in general.

Corollary 7.41. *$S^i \wedge S^j$ is homeomorphic to S^{i+j}.* □

We defined loop spaces by $\Omega X = \Omega_{x_0} X = \mathrm{Map}(I, \{0,1\}; X, \{x_0\})$, but by using the identification of the circle as a quotient space of the interval, one sees

$$\Omega X = \mathrm{Map}(S^1, X)_0.$$

Then a special case of Theorem 7.38 shows the following.

Theorem 7.42 (loops and suspension are adjoints)**.** *The spaces*

$$\mathrm{Map}(SX, Y)_0$$

and

$$\mathrm{Map}(X, \Omega Y)_0$$

are naturally homeomorphic. □

7.11. Fibration and cofibration sequences

We will see eventually that the homotopy type of a fiber of a fibration measures how far the fibration is from being a homotopy equivalence. (For example, if the fiber is contractible, then the fibration is a homotopy equivalence.) More generally, given a map $f : X \to Y$, one can turn it into a fibration $f' : X' \to Y$ as above; the fiber of this fibration measures how far f is from a homotopy equivalence.

After turning $f : X \to Y$ into a fibration $f' : X' \to Y$, one then has an inclusion of the fiber $F \subset X'$. Why not turn this into a fibration and see what happens? Now take the fiber of the resulting fibration and continue the process ....

Similar comments apply to cofibrations. Theorem 7.44 below identifies the resulting iterated fibers and cofibers. We introduce some terminology, which is justified by Theorems 7.21, 7.23, 7.29, and 7.31.

Definition 7.43. If $f : X \to Y$ is a map, the *homotopy fiber of* f is any fiber of any fibration obtained by turning f into a fibration. The homotopy fiber is a space, well-defined up to homotopy equivalence, and is equipped with a homotopy class of maps to X. It is denoted by $\mathrm{hofiber}(f)$. Usually one is lazy and just calls this the *fiber of* f. A specific model for the homotopy fiber is $p^{-1}\{x\}$ where $p : P_f \to Y$ is the mapping path fibration.

Similarly, the *homotopy cofiber of* $f : X \to Y$ is any cofiber of any cofibration obtained by turning f into a cofibration. The homotopy cofiber is a space, well-defined up to homotopy equivalence, and is equipped with a homotopy class of maps from Y. It is denoted by $\mathrm{hocofiber}(f)$. A specific model for the homotopy cofiber is the mapping cone $C_f = M_f/X$.

Theorem 7.44.

1. *Let $F \hookrightarrow E \to B$ be a fibration. Let Z be the homotopy fiber of $F \hookrightarrow E$, so $Z \to F \to E$ is a fibration (up to homotopy). Then Z is homotopy equivalent to the loop space ΩB.*
2. *Let $A \hookrightarrow X \to X/A$ be a cofibration sequence. Let W be the homotopy cofiber of $X \to X/A$, so that $X \to X/A \to W$ is a cofibration*

(up to homotopy). Then W is homotopy equivalent to the (unreduced) suspension SA .

Proof. 1. Let $f : E \to B$ be a fibration with fiber $F = f^{-1}\{b_0\}$. Choose a base point $e_0 \in F$. In Section 7.6 we constructed a fibration $p : P_f \to B$ with

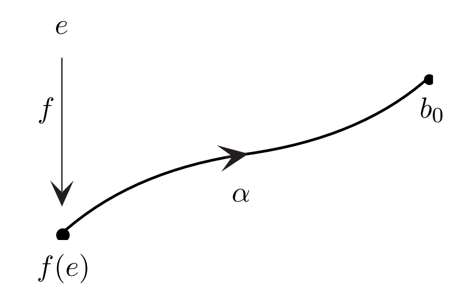
$$P_f = \{(e,\alpha) \in E \times B^I \mid f(e) = \alpha(0)\}$$

and $p(e,\alpha) = \alpha(1)$, and such that the map $h : E \to P_f$ given by $h(e) = (e, \mathrm{const}_{f(e)})$ is a fiber homotopy equivalence.

Let $(P_f)_0 = p^{-1}\{b_0\}$, so $(P_f)_0 \hookrightarrow P_f \xrightarrow{p} B$ is a fibration equivalent to $F \hookrightarrow E \xrightarrow{f} B$.

Define $\pi : (P_f)_0 \to E$ by $\pi(e,\alpha) = e$. Notice that

$$(P_f)_0 = \{(e,\alpha) \mid f(e) = \alpha(0),\ \alpha(1) = b_0\}.$$

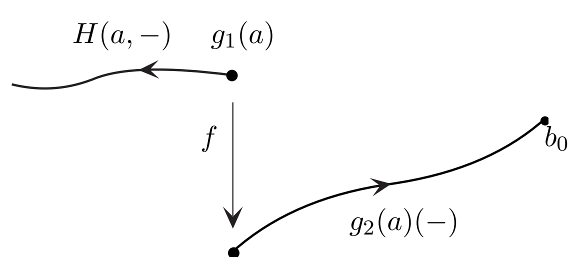


Claim. $\pi : (P_f)_0 \to E$ is a fibration with fiber $\Omega_{b_0} B$.

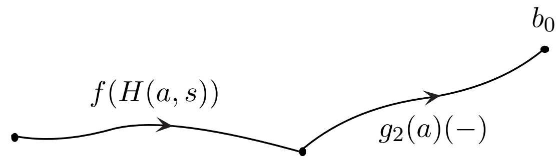
Proof of claim. Clearly $\pi^{-1}\{e_0\} = \{(e_0,\alpha) \mid \alpha(0) = \alpha(1) = b_0\}$ is homeomorphic to the loop space, so we just need to show π is a fibration. Given the problem

$$\begin{tikzcd}
A \times 0 \arrow[r, "g"] \arrow[d] & (P_f)_0 \arrow[d, "\pi"] \\
A \times I \arrow[r, "H"'] \arrow[ur, dotted, "\widetilde{H}"] & E
\end{tikzcd}$$

the picture is



Hence we can set $\widetilde{H}(a,s) = (H(a,s), \widetilde{H}_2(a,s))$ where $\widetilde{H}_2(a,s))(-)$ has the picture



and is defined by

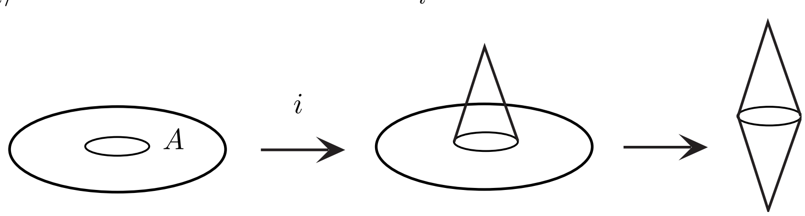
$$\widetilde{H}_2(a,s))(t) = \begin{cases} f(H(y, -(1+s)t+s)) & \text{if } 0 \le t \le s/(s+1), \\ g_2(a)((s+1)t-s) & \text{if } \ s/(s+1) \le t \le 1. \end{cases}$$

The map $F \hookrightarrow (P_f)_0$ is a homotopy equivalence, since $E \to P_f$ is a fiber homotopy equivalence. Thus the diagram

$$\begin{tikzcd} F \arrow[d, "\simeq"'] \arrow[dr] & \\ & E \\ (P_f)_0 \arrow[ur, "\pi"'] & \end{tikzcd}$$

shows that the fibration $\pi : (P_f)_0 \to E$ is obtained by turning $F \hookrightarrow E$ into a fibration, and that the homotopy fiber is $\Omega_{b_0} B$.

2. The map $X \to X/A$ is equivalent to the cofibration $X \hookrightarrow C_i = X \cup \text{cone}(A)$ where $i : A \hookrightarrow X$. The following picture makes clear that $C_i/X = SA$. The fact that $X \to C_i$ is a cofibration is left as an exercise.



X $\quad$ $C_i = X \cup \text{cone}(A) \simeq X/A$ $\quad$ $SA = C_i/X$

□

Exercise 134. Show that $X \hookrightarrow C_i = X \cup \text{cone}(A)$ is a cofibration.

Let X be a based space and ΩX be its loop space, the space of paths in X which start and end at the base point. The loop space is itself a based space with base point the constant loop at the base point of X. Let $\Omega^n X$ denote the n-fold loop space of X. Similarly the reduced suspension SX of X is a based space. Let $S^n X$ denote the n-fold suspension of X.

The previous theorem can be restated in the following convenient form.

Theorem 7.45.

1. *Let $A \hookrightarrow X$ be a cofibration. Then any two consecutive maps in the sequence*

$$A \to X \to X/A \to SA \to SX \to \cdots \to S^n A \to S^n X \to S^n(X/A) \to \cdots$$

have the homotopy type of a cofibration followed by projection onto the cofiber. Here S is the unreduced suspension.

1′. *Let $A \hookrightarrow X$ be a base point preserving cofibration. Then any two consecutive maps in the sequence*

$$A \to X \to X/A \to SA \to SX \to \cdots \to S^n A \to S^n X \to S^n(X/A) \to \cdots$$

have the homotopy type of a cofibration followed by projection onto the cofiber. Here S is the reduced suspension.

2. *Let $E \to B$ be a fibration with fiber F. Then any two consecutive maps in the sequence*

$$\cdots \to \Omega^n F \to \Omega^n E \to \Omega^n B \to \cdots \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

have the homotopy type of a fibration preceded by the inclusion of its fiber.

□

To prove 1′ one must use reduced mapping cylinders and reduced cones.

7.12. Puppe sequences

Lemma 7.46. *Let X and Y be spaces in CGH_*.*

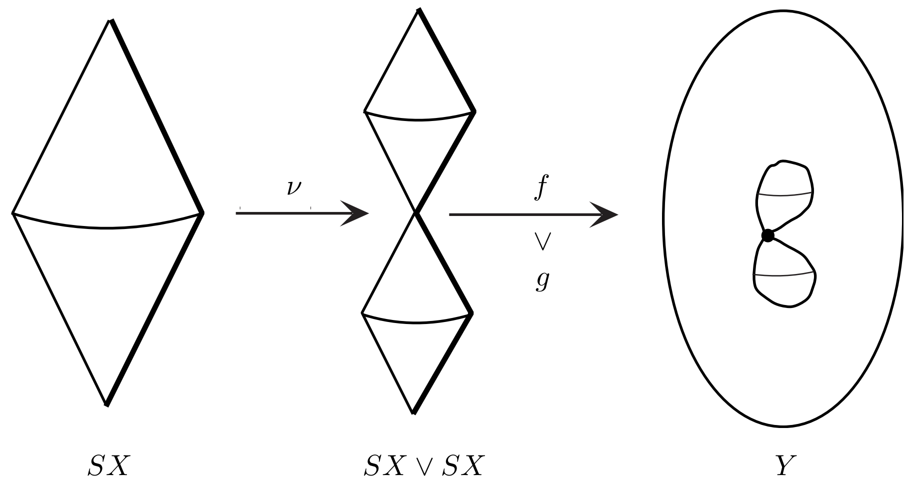
1. *$[X,\Omega Y]_0 = [SX,Y]_0$ is a group.*
2. *$[X,\Omega(\Omega Y)]_0 = [SX, \Omega Y]_0 = [S^2X, Y]_0$ is an abelian group.*

Sketch of proof. The equalities follow from Theorem 7.42, the adjointness of loops and suspension. The multiplication can be looked at in two ways: first on $[SX, Y]_0$ as coming from the map

$$\nu : SX \to SX \vee SX$$

given by collapsing out the "equator" $X \times 1/2$. Then define

$$fg \underset{\text{def}}{:=} (f \vee g) \circ \nu.$$

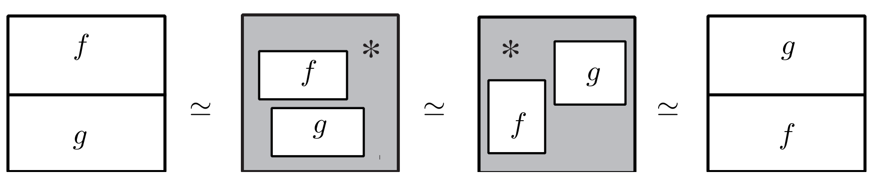


The second interpretation of multiplication is on $[X, \Omega Y]_0$ and comes from composition of loops

$$* : \Omega Y \times \Omega Y \to \Omega Y$$

with $(fg)x = f(x) * g(x)$.

The proof of 2. is obtained by meditating on the following sequence of pictures.



□

Exercise 135. Convince yourself that the two definitions of multiplication on $[X,\Omega Y]_0 = [SX,Y]_0$ are the same and that $\pi_1(Y, y_0) = [SS^0, Y]_0$.

The last lemma sits in a more general context. A loop space is an example of an H-group and a suspension is an example of a co-H-group. See [**45**] or [**54**] for precise definitions, but here is the basic idea. An H-group Z is a based space with a *multiplication* map $\mu : Z \times Z \to Z$ and an *inversion* map $\varphi : X \to X$ which satisfy the axioms of a group up to homotopy (e.g. is associative up to homotopy). For a topological group G and any space X, $\mathrm{Map}(X, G)$ is a group; similarly for an H-group Z, $[X, Z]_0$ is a group. To define a co-H-group, one reverses all the arrows in the definition of H-group, so there is a co-multiplication $\nu : W \to W \vee W$ and a co-inversion $\psi : W \to W$. Then $[W, X]_0$ is a group. Finally, there is a formal, but occasionally very useful result. If W is a co-H-group and Z is an H-group, then the two multiplications on $[W,Z]_0$ agree and are abelian. Nifty, huh? One consequence of this is that $\pi_1(X,x_0)$ of an H-group (e.g. a topological group) is abelian.

Combining Lemma 7.46 with Theorem 7.45 and Exercise 131 yields the proof of the following fundamental theorem.

Theorem 7.47 (Puppe sequences)**.** *Let $Y \in$ CGH$_*$.*

1. *If $F \to E \to B$ is a fibration, the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$).*
$$\cdots \to [Y,\Omega^i F]_0 \to [Y,\Omega^i E]_0 \to [Y,\Omega^i B]_0 \to$$
$$\cdots \to [Y,\Omega B]_0 \to [Y,F]_0 \to [Y,E]_0 \to [Y,B]_0$$
where $\Omega^i Z$ denotes the iterated loop space
$$\Omega(\Omega(\cdots(\Omega Z)\cdots)).$$
2. *If (X,A) is a cofibration, the following sequence is a long exact sequence of sets ($i \geq 0$), groups ($i \geq 1$), and abelian groups ($i \geq 2$).*
$$\cdots \to [S^i(X/A),Y]_0 \to [S^i X,Y]_0 \to [S^i A,Y]_0 \to$$
$$\cdots \to [SA,Y]_0 \to [X/A,Y]_0 \to [X,Y]_0 \to [A,Y]_0.$$

□

This theorem is used as the basic tool for constructing exact sequences in algebraic topology.

7.13. Homotopy groups

We now define the homotopy groups of a based space. We will give the sphere the base point $p = (1,0,\cdots,0) \in S^n$.

Definition 7.48. Suppose that X is a space with base point x_0. Then the n^{th} *homotopy group of X based at x_0* is the group (set if $n = 0$, abelian group if $n \geq 2$)
$$\pi_n(X,x_0) = [S^n,X]_0.$$

(We will usually only consider $X \in$ CGH$_*$.)

Notice that

$$\pi_n(X,x_0) = [S^n,X]_0 = [S^k \wedge S^{n-k},X]_0 = \pi_{n-k}(\Omega^k(X)). \tag{7.2}$$

In particular,
$$\pi_n X = \pi_1(\Omega^{n-1}X).$$

There are other ways of looking at homotopy groups which are useful. For example, to get a handle on the group structure for writing down a proof, use $\pi_n(X, x_0) = [(I^n, \partial I^n), (X, x_0)]$. For the proof of the exact sequence of a pair (coming later) use $\pi_n(X, x_0) = [(D^n, S^{n-1}), (X, x_0)]$. For finding a geometric interpretation of the boundary map in the homotopy long exact sequence of a fibration given below, use

$$\pi_n(X, x_0) = [(S^{n-1} \times I, (S^{n-1} \times \partial I) \cup (* \times I)), (X, x_0)].$$

A useful observation is that the set $\pi_0(X, x_0)$ is in bijective correspondence with the path components of X. A based map $f : S^0 = \{\pm 1\} \to X$ corresponds to the path component of $f(-1)$. In general π_0 is just a based set, unless X is an H-space, e.g. a loop space or a topological group.

Also useful is the fact that $[X, Y]_0 = \pi_0(\mathrm{Map}(X, Y)_0)$, the set of path components of the function space $\mathrm{Map}(X, Y)_0$. In particular, Equation (7.2) shows that $\pi_n(X, x_0)$ is the set of path components of the n-fold loop space of X.

Homotopy groups are the most fundamental invariant of algebraic topology. For example, we will see below that a CW-complex is contractible if and only if all its homotopy groups vanish. More generally we will see that a map $f : X \to Y$ of CW-complexes is a homotopy equivalence if and only if it induces an isomorphism on all homotopy groups. Finally, the homotopy type of a CW-complex X is determined by the homotopy groups of X together with a cohomological recipe (the k-invariants) for assembling these groups. (The homotopy groups by themselves do not usually determine the homotopy type of a space.)

Exercise 136. Show that $\pi_n(X \times Y) = \pi_n X \times \pi_n Y$.

As an application of the Puppe sequences (Theorem 7.47) we immediately get the extremely useful long exact sequence of homotopy groups associated to any fibration.

Corollary 7.49 (long exact sequence of a fibration)**.** *Let $F \hookrightarrow E \to B$ be a fibration. Then the sequence*

$$\begin{aligned} \cdots \to \pi_n F \to \pi_n E \to \pi_n B \to \pi_{n-1} F \to \pi_{n-1} E \to \cdots \\ \to \pi_1 F \to \pi_1 E \to \pi_1 B \to \pi_0 F \to \pi_0 E \to \pi_0 B \end{aligned}$$

is exact. □

In Corollary 7.49, one must be careful with exactness at the right end of this sequence since $\pi_1 F$, $\pi_1 E$, and $\pi_1 B$ are nonabelian groups and $\pi_0 F$, $\pi_0 E$, and $\pi_0 B$ are merely sets.

Taking F discrete in Corollary 7.49 and using the fact that covering spaces are fibrations, one concludes the following important theorem.

Theorem 7.50. *Let $\tilde{X} \to X$ be a connected covering space of a connected space X. Then the induced map*

$$\pi_n \tilde{X} \to \pi_n X$$

is injective if $n = 1$, and an isomorphism if $n > 1$. □

Exercise 137. Give a covering space proof of Theorem 7.50.

7.14. Examples of fibrations

Many examples of fibrations and fiber bundles arise naturally in mathematics. Getting a feel for this material requires getting one's hands dirty. For that reason many facts are left as exercises. We will use the following theorem from equivariant topology to conclude that certain maps are fibrations. This is a special case of Theorem 5.9.

Theorem 7.51 (Gleason)**.** *Let G be a compact Lie group acting freely on a compact manifold X. Then*

$$X \to X/G$$

is a principal fiber bundle with fiber G.

7.14.1. Hopf fibrations.

The first class of examples we give are the famous *Hopf fibrations.* These were invented by Hopf to prove that there are non-nullhomotopic maps $S^n \to S^m$ when $n > m$.

There are four Hopf fibrations (these are fiber bundles):

$$S^0 \hookrightarrow S^1 \to S^1$$

$$S^1 \hookrightarrow S^3 \to S^2$$

$$S^3 \hookrightarrow S^7 \to S^4$$

and

$$S^7 \hookrightarrow S^{15} \to S^8.$$

These are constructed by looking at the various division algebras over $\mathbf{R}$.

Let $K = \mathbf{R}, \mathbf{C}, \mathbf{H}$, or $\mathbf{O}$ (the real numbers, complex numbers, quaternions, and octonions). Each of these has a *norm* $N : K \to \mathbf{R}_+$ so that

$$N(xy) = N(x)N(y)$$

and $N(x) > 0$ for $x \neq 0$.

More precisely,

1. If $K = \mathbf{R}$, then $N(x) = |x| = \sqrt{x\overline{x}}$ where $\overline{x} = x$.
2. If $K = \mathbf{C}$, then $N(x) = \sqrt{x\overline{x}}$ where $\overline{a + ib} = a - ib$.
3. If $K = \mathbf{H}$, then $N(x) = \sqrt{x\overline{x}}$, where $\overline{a + ib + jc + kd} = a - ib - jc - kd$.

4. The octonions (also called Cayley numbers) are defined to be $\mathbf{O} = \mathbf{H} \oplus \mathbf{H}$. Conjugation is defined by the rule: if $x = (a, b)$, then $\overline{x} = (\overline{a}, -b)$. A nonassociative multiplication is defined by
$$(a, b)(c, d) = (ac - \overline{d}b, b\overline{c} + da).$$
The norm on the octonions is defined by
$$N(x) = \sqrt{x\overline{x}}.$$
More explicitly, $(a, b)\overline{(a, b)} = (|a|^2+|b|^2, 0)$, and $N(a, b) = \sqrt{|a|^2 + |b|^2}$. In particular, if $x \in \mathbf{O}$ is nonzero, $N(x)^{-2}\bar{x}$ is a 2-sided inverse for x. Moreover, $\overline{xy} = \bar{y}\bar{x}$ and $N(xy) = N(x)N(y)$.

Let $G_K = \{x \in K \mid N(x) = 1\}$ and let
$$E_K = \{(x, y) \in K \oplus K \mid N(x)^2 + N(y)^2 = 1\}.$$

Note that K is isomorphic as a normed real vector space to $\mathbf{R}^{r+1}$ for $r = 0, 1, 3$ and 7. Hence E_K consists of the unit vectors in $\mathbf{R}^{2r+2}$ and so $E_K = S^{2r+1}$ for $r = 0, 1, 3, 7$. Similarly G_K consists of the unit vectors in $\mathbf{R}^{r+1}$ and so $E_K = S^r$ for $r = 0, 1, 3, 7$.

Define a map $f : E_K \to K \oplus \mathbf{R} \cong \mathbf{R}^{r+2}$ by
$$f(x, y) = (2\overline{x}y, N(x)^2 - N(y)^2).$$

Exercise 138. Prove that the image of $f : E_K \to K \oplus \mathbf{R}$ is the $(r + 1)$-sphere S^{r+1}, and that the map $f : E_K \to S^{r+1}$ is a fiber bundle with fiber S^r.

Exercise 139. For $K = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, G_K is a compact Lie group which acts freely on E_K by $g \cdot (x, y) = (gx, gy)$. For $K = \mathbf{O}$, G_K is not a group; associativity fails.

It is easy to see that for $K = \mathbf{R}, \mathbf{C}$, or $\mathbf{H}$, the principal G_K-bundle $E_K \to E_K/G_K$ given by Gleason's theorem is isomorphic to the Hopf fibration $S^r \hookrightarrow S^{2r+1} \xrightarrow{f} S^{r+1}$.

Using the Hopf fibrations and the long exact sequence of a fibration (Corollary 7.49), one obtains exact sequences
$$\cdots \to \pi_n S^1 \to \pi_n S^3 \to \pi_n S^2 \to \pi_{n-1} S^1 \to \cdots$$
$$\cdots \to \pi_n S^3 \to \pi_n S^7 \to \pi_n S^4 \to \pi_{n-1} S^3 \to \cdots$$
$$\cdots \to \pi_n S^7 \to \pi_n S^{15} \to \pi_n S^8 \to \pi_{n-1} S^7 \to \cdots .$$

Since $\pi_n S^1 = 0$ for $n > 1$ (the universal cover of S^1 is contractible and so this follows from Theorem 7.50), it follows from the first sequence that

$\pi_n S^3 = \pi_n S^2$ for $n > 2$. The Hopf degree Theorem (Corollary 7.72 and a project for Chapter 6) implies that $\pi_n S^n = \mathbf{Z}$. In particular,

$$\pi_3 S^2 = \mathbf{Z}. \tag{7.3}$$

This is our second nontrivial calculation of $\pi_m S^n$ (the first being $\pi_n S^n = \mathbf{Z}$).

The quickest way to obtain information from the other sequences is to use the cellular approximation theorem. This is an analogue of the simplicial approximation theorem. Its proof is one of the projects for Chapter 2.

Theorem 7.52 (cellular approximation theorem). *Let* (X, A) *and* (Y, B) *be relative CW-complexes, and let* $f : (X, A) \to (Y, B)$ *be a continuous map. Then* f *is homotopic rel* A *to a cellular map.* □

Applying this theorem with $(X, A) = (S^n, x_0)$ and $(Y, B) = (S^m, y_0)$, one concludes that

$$\pi_n S^m = 0 \text{ if } \; n < m.$$

Returning to the other exact sequences, it follows from the cellular approximation theorem that $\pi_n S^4 = \pi_{n-1} S^3$ for $n \leq 6$ (since $\pi_n S^7 = 0$ for $n \leq 6$) and that $\pi_n S^8 = \pi_{n-1} S^7$ for $n \leq 14$. We will eventually be able to say more.

7.14.2. Projective spaces. The Hopf fibrations can be generalized by taking G_K acting on K^n for $n > 2$ at least for $K = \mathbf{R}, \mathbf{C}$, and $\mathbf{H}$.

For $K = \mathbf{R}$, $G_K = \mathbf{Z}/2$ acts on S^n with quotient real projective space $\mathbf{R}P^n$. The quotient map $S^n \to \mathbf{R}P^n$ is a covering space and in particular a fibration.

Let S^1 act on

$$S^{2n-1} = \{(z_1, \dots, z_n) \in \mathbf{C}^n \mid \Sigma |z_i|^2 = 1\}$$

by

$$t(z_1, \cdots, z_n) = (tz_1, \cdots, tz_n)$$

if $t \in S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$.

Exercise 140. Prove that S^1 acts freely.

The orbit space is denoted by $\mathbf{C}P^{n-1}$ and is called *complex projective space*. The projection $S^{2n-1} \to \mathbf{C}P^{n-1}$ is a fibration with fiber S^1. (Can you prove directly that this is a fiber bundle?) In fact, if one uses the map $p : S^{2n-1} \to \mathbf{C}P^{n-1}$ to adjoin a $2n$-cell, one obtains $\mathbf{C}P^n$. Thus complex projective space is a CW-complex.

Notice that $\mathbf{C}P^n$ is a subcomplex of $\mathbf{C}P^{n+1}$, and in fact $\mathbf{C}P^{n+1}$ is obtained from $\mathbf{C}P^n$ by adding a single $2n+2$-cell. One defines infinite complex projective space $\mathbf{C}P^\infty$ to be the union of the $\mathbf{C}P^n$, with the CW-topology.

Exercise 141. Using the long exact sequence for a fibration, show that $\mathbf{C}P^\infty$ is an Eilenberg–MacLane space of type $K(\mathbf{Z}, 2)$, i.e. a CW-complex with π_2 the only nonzero homotopy group and $\pi_2 \cong \mathbf{Z}$.

Similarly, there is a fibration

$$S^3 \hookrightarrow S^{4n-1} \to \mathbf{H}P^{n-1}$$

using quaternions in the previous construction. The space $\mathbf{H}P^{n-1}$ is called *quaternionic projective space*.

Exercise 142.

1. Calculate the cellular chain complexes for $\mathbf{C}P^k$ and $\mathbf{H}P^k$.
2. Compute the *ring* structure of $H^*(\mathbf{C}P^k; \mathbf{Z})$ and $H^*(\mathbf{H}P^k; \mathbf{Z})$ using Poincaré duality.
3. Examine whether $\mathbf{O}P^k$ can be defined this way, for $k > 1$.
4. Show these reduce to Hopf fibrations for $k = 1$.

7.14.3. More general homogeneous spaces and fibrations.

Definition 7.53.

1. The *Stiefel manifold* $V_k(\mathbf{R}^n)$ is the space of orthonormal k-frames in $\mathbf{R}^n$:
$$V_k(\mathbf{R}^n) = \{(v_1, v_2, \ldots, v_k) \in (\mathbf{R}^n)^k \mid v_i \cdot v_j = \delta_{ij}\}$$
given the topology as a subspace of $(\mathbf{R}^n)^k = \mathbf{R}^{nk}$.
2. The *Grassmann manifold* or *grassmannian* $G_k(\mathbf{R}^n)$ is the space of k-dimensional subspaces (a.k.a. k-planes) in $\mathbf{R}^n$. It is given the quotient topology using the surjection $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ taking a k-frame to the k-plane it spans.

Let G be a compact Lie group. Let $H \subset G$ be a closed subgroup (and hence a Lie group itself). The quotient G/H is called a *homogeneous space*. The (group) quotient map $G \to G/H$ is a principal H-bundle since H acts freely on G by right translation. If H has a closed subgroup K, then H acts on the homogeneous space H/K. Changing the fiber of the above bundle results in a fiber bundle $G/K \to G/H$ with fiber H/K.

For example, if $G = O(n)$ and $H = O(k) \times O(n-k)$ with $H \hookrightarrow G$ via

$$(A, B) \mapsto \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

let $K \subset O(n)$ be $O(n-k)$, with

$$A \mapsto \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}.$$

Exercise 143. Identify G/H with the grassmannian and G/K with the Stiefel manifold. Conclude that the map taking a frame to the plane it spans defines a principal $O(k)$ bundle $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$.

Let

$$\gamma_k(\mathbf{R}^n) = \{(p, V) \in \mathbf{R}^n \times G_k(\mathbf{R}^n) \mid p \text{ is a point in the } k\text{-plane } V\}.$$

There is a natural map $\gamma_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ given by projection on the second coordinate. The fiber bundle so defined is a vector bundle with fiber $\mathbf{R}^k$ (a *k-plane bundle*)

$$\mathbf{R}^k \hookrightarrow \gamma_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n).$$

It is called the canonical (or tautological) vector bundle over the grassmannian.

Exercise 144. Identify the canonical bundle with the bundle obtained from the principal $O(k)$ bundle $V_k(\mathbf{R}^n) \to G_k(\mathbf{R}^n)$ by changing the fiber to $\mathbf{R}^k$.

Exercise 145. Show there are fibrations

$$O(n-k) \hookrightarrow O(n) \to V_k(\mathbf{R}^n)$$

$$O(n-1) \hookrightarrow O(n) \to S^{n-1}$$

taking a matrix to its last k columns. Deduce that

$$\pi_i(O(n-1)) \cong \pi_i(O(n)) \quad \text{for} \quad i < n-2, \tag{7.4}$$

and

$$\pi_i(V_k(\mathbf{R}^n)) = 0 \quad \text{for} \quad i < n-k-1.$$

The isomorphism of Equation (7.4) is an example of "stability" in algebraic topology. In this case it leads to the following construction. Consider the infinite orthogonal group

$$O = \operatorname*{colim}_{n\to\infty} O(n) = \bigcup_{n=1}^{\infty} O(n),$$

where $O(n) \subset O(n+1)$ is given by the continuous monomorphism

$$A \to \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Topologize O as the expanding union of the $O(n)$. Then any compact subset of O is contained in $O(n)$ for some n; hence $\pi_i O = \operatorname*{colim}_{n\to\infty} \pi_i(O(n)) = \pi_i(O(n))$ for any $n > i+2$.

A famous theorem of Bott says:

Theorem 7.54 (Bott periodicity)**.**

$$\pi_k O \cong \pi_{k+8} O \quad \textit{for } k \in \mathbf{Z}_+.$$

Moreover the homotopy groups of O are computed to be

$k \pmod 8$	0	1	2	3	4	5	6	7
$\pi_k O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$

□

An element of $\pi_k O$ is given by an element of $\pi_k(O(n))$, for some n, which by clutching (see Section 5.3.3) corresponds to a bundle over S^{k+1} with structure group $O(n)$. (Alternatively, one can show that $\pi_{k+1}(BO(n)) \cong \pi_k(O(n))$ by using the long exact sequence of homotopy groups of the fibration $O(n) \hookrightarrow EO(n) \to BO(n)$). The generators of the first eight homotopy groups of O are given by Hopf bundles.

Similarly one can consider stable Stiefel manifolds and stable grassmanians. Let $V_k(\mathbf{R}^\infty) = \operatorname*{colim}_{n\to\infty} V_k(\mathbf{R}^n)$ and $G_k(\mathbf{R}^\infty) = \operatorname*{colim}_{n\to\infty} G_k(\mathbf{R}^n)$. Then $\pi_i(V_k(\mathbf{R}^\infty)) = \operatorname*{colim}_{n\to\infty} \pi_i(V_k(\mathbf{R}^n))$ and $\pi_i(G_k(\mathbf{R}^\infty)) = \operatorname*{colim}_{n\to\infty} \pi_i(G_k(\mathbf{R}^n))$. In particular $\pi_i(V_k(\mathbf{R}^\infty)) = 0$.

A project for Chapter 5 was to show that for every topological group G, there is a principal G-bundle $EG \to BG$ where EG is contractible.

This bundle classifies principal G-bundles in the sense that given a principal G-bundle $p : G \hookrightarrow E \to B$ over a CW-complex B (or more generally a paracompact space), there is a map of principal G-bundles

$$\begin{tikzcd} E \arrow[r, "\tilde f"] \arrow[d, "p"'] & EG \arrow[d] \\ B \arrow[r, "f"'] & BG \end{tikzcd}$$

and the homotopy class $[f] \in [B, BG]$ is uniquely determined. It follows that the (weak) homotopy type of BG is uniquely determined.

Corollary 7.55. *The infinite grassmannian $G_k(\mathbf{R}^\infty)$ is a model for $BO(k)$. The principal $O(k)$ bundle*

$$O(k) \hookrightarrow V_k(\mathbf{R}^\infty) \to G_k(\mathbf{R}^\infty)$$

is universal and classifies principal $O(k)$-bundles. The canonical bundle

$$\mathbf{R}^k \hookrightarrow \gamma_k(\mathbf{R}^\infty) \to G_k(\mathbf{R}^\infty)$$

classifies $\mathbf{R}^k$-vector bundles with structure group $O(k)$ (i.e. $\mathbf{R}^k$-vector bundles equipped with a metric on each fiber which varies continuously from fiber to fiber). □

The fact that the grassmannian classifies orthogonal vector bundles makes sense from a geometric point of view. If $M \subset \mathbf{R}^n$ is a k-dimensional smooth submanifold, then for any point $p \in M$, the tangent space T_pM defines a k-plane in $\mathbf{R}^n$, and hence a point in $G_k(\mathbf{R}^n)$. Likewise a tangent vector determines a point in the canonical bundle $\gamma_k(\mathbf{R}^n)$. Thus there is a bundle map

$$\begin{tikzcd} TM \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & \gamma_k(\mathbf{R}^n) \arrow[d] \\ M \arrow[r, "f"'] & G_k(\mathbf{R}^n) \end{tikzcd}$$

Moreover, $G_k(\mathbf{R}^\infty)$ is also a model for $BGL_k(\mathbf{R})$ and hence is a classifying space for k-plane bundles over CW-complexes. This follows either by redoing the above discussion, replacing k-frames by sets of k-linearly independent vectors, or by using the fact that $O(k) \hookrightarrow GL_k(\mathbf{R})$ is a homotopy equivalence, with the homotopy inverse map being given by the Gram-Schmidt process.

Similar statements apply in the complex setting to unitary groups $U(n)$. Let

$$G_k(\mathbf{C}^n) = \text{complex } k\text{-planes in } \mathbf{C}^n$$

$$G_k(\mathbf{C}^n) = U(n)/(U(k) \times U(n-k)), \quad \text{the complex grassmannian}$$

$$V_k(\mathbf{C}^n) = U(n)/U(n-k), \quad \text{the unitary Stiefel manifold.}$$

There are principal fiber bundles

$$U(n-k) \hookrightarrow U(n) \to V_k(\mathbf{C}^n)$$

and

$$U(k) \hookrightarrow V_k(\mathbf{C}^n) \to G_k(\mathbf{C}^n).$$

Moreover, $V_1(\mathbf{C}^n) \cong S^{2n-1}$. Therefore

$$\pi_k(U(n)) \cong \pi_k(U(n-1)) \quad \text{if} \quad k < 2n-2.$$

So letting

$$U = \operatorname*{colim}_{n \to \infty} U(n),$$

we conclude that

$$\pi_k U = \pi_k(U(n)) \quad \text{for } n > 1 + \frac{k}{2}.$$

Bott periodicity holds for the unitary group; the precise statement is the following.

Theorem 7.56 (Bott periodicity)**.**

$$\pi_k U \cong \pi_{k+2} U \text{ for } k \in \mathbf{Z}_+.$$

Moreover,

$$\pi_k U = \begin{cases} \mathbf{Z} & \text{if } k \text{ is odd, and} \\ 0 & \text{if } k \text{ is even.} \end{cases}$$

Exercise 146. Prove that $\pi_1 U = \mathbf{Z}$ and $\pi_2 U = 0$.

Taking determinants gives fibrations $SO(n) \hookrightarrow O(n) \xrightarrow{\det} \{\pm 1\}$ and $SU(n) \hookrightarrow U(n) \xrightarrow{\det} S^1$. In particular, $SO(n)$ is the identity path component of $O(n)$, so $\pi_k(SO(n)) = \pi_k(O(n))$ for $k \geq 1$. Similarly, since $\pi_k(S^1) = 0$ for $k > 1$, $\pi_1(SU(n)) = 0$ and $\pi_k SU(n) = \pi_k(U(n))$ for $k > 1$.

Exercise 147. Prove that $SO(2) = U(1) = S^1$, $SO(3) \cong \mathbf{R}P^3$, $SU(2) \cong S^3$, and that the map $p : S^3 \times S^3 \to SO(4)$ given by $(a, b) \mapsto (v \mapsto av\bar{b})$ where $a, b \in S^3 \subset \mathbf{H}$ and $v \in \mathbf{H} \cong \mathbf{R}^4$ is a 2-fold covering map.

Exercise 148. Using Exercise 147 and the facts:

1. $\pi_n S^n = \mathbf{Z}$ (Hopf degree Theorem),
2. $\pi_k S^n = 0$ for $k < n$ (Hurewicz theorem),
3. $\pi_k S^n \cong \pi_{k+1} S^{n+1}$ for $k < 2n - 1$ (Freudenthal suspension theorem),
4. There is a covering $\mathbf{Z} \hookrightarrow \mathbf{R} \to S^1$,
5. $\pi_n S^{n-1} = \mathbf{Z}/2$ for $n > 3$ (this theorem is due to V. Rohlin and G. Whitehead; see Corollary 10.30),

compute as many homotopy groups of S^n's, $O(n)$, Grassmann manifolds, Stiefel manifolds, etc., as you can.

7.15. Relative homotopy groups

Let (X, A) be a pair, with base point $x_0 \in A \subset X$. Let $p = (1, 0, \cdots, 0) \in S^{n-1} \subset D^n$.

Definition 7.57. The *relative homotopy group (set if $n = 1$) of the pair* (X, A) is

$$\pi_n(X, A, x_0) = [D^n, S^{n-1}, p; X, A, x_0],$$

the set of based homotopy classes of base point preserving maps from the pair (D^n, S^{n-1}) to (X, A). This is a functor from pairs of spaces to sets ($n = 1$), groups ($n = 2$), and abelian groups ($n > 2$).

Thus, representatives for $\pi_n(X, A, x_0)$ are maps $f : D^n \to X$ such that $f(S^{n-1}) \subset A,\ f(p) = x_0$ and f is equivalent to g if there exists a homotopy $F : D^n \times I \to X$ so that for each $t \in I$, $F(-, t)$ is base point preserving and takes S^{n-1} into A, and $F(-, 0) = f,\ F(-, 1) = g$.

(Technical note: associativity is easier to see if instead one takes

$$\pi_n(X, A, x_0) = [D^n, S^{n-1}, P; X, A, x_0]$$

where P is one-half of a great circle, running from p to $-p$, e.g.

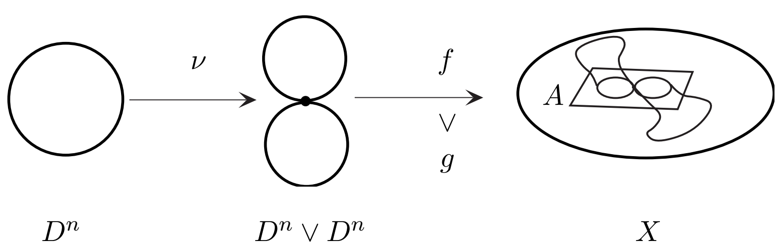
$$P = \{(\cos\theta, \sin\theta, 0, \cdots, 0) \mid \theta \in [0, \pi]\}.$$

This corresponds to the previous definition since the reduced cone on the sphere is the disk.)

Theorem 7.58 (homotopy long exact sequence of a pair)**.** *The homotopy set $\pi_n(X, A)$ is a group for $n \geq 2$ and is abelian for $n \geq 3$. Moreover, there is a long exact sequence*

$$\cdots \to \pi_n A \to \pi_n X \to \pi_n(X, A) \to \pi_{n-1} A \to \cdots \to \pi_1(X, A) \to \pi_0 A \to \pi_0 X.$$

Proof. The proof that $\pi_n(X, A)$ is a group is a standard exercise, with multiplication based on the idea of the following picture.



The connecting homomorphism $\pi_n(X, A) \to \pi_{n-1} A$ assigns to a homotopy class of maps $(D^n, S^{n-1}) \to (X, A)$ its restriction to the boundary S^{n-1}.

Exercise 149. Concoct an argument from the picture and use it to figure out why $\pi_1(X, A)$ is not a group. Also use it to prove that the long exact sequence is exact.

$\square$

In contrast to homology groups, it is not true that $\pi_n(X, A) \cong \pi_n(X/A)$. For example, taking $X = D^2$ and $A = \partial D^2 = S^1$, X/A is homeomorphic to S^2. Hence $\pi_3(X/A) \cong \pi_3(S^2) \cong \mathbf{Z}$ (see Equation (7.3). Hoever, since X is contractible, $\pi_3(X) = 0 = \pi_2(A)$, and so by Theorem 7.58, $\pi_3(X, A) = 0$.

Lemma 7.59. *Let $f : E \to B$ be a fibration with fiber F. Let $A \subset B$ be a subspace, and let $G = f^{-1}A$, so that $F \hookrightarrow G \xrightarrow{f} A$ is a fibration. Then f induces isomorphims $f_* : \pi_k(E,G) \to \pi_k(B,A)$ for all k. In particular, taking $A = \{b_0\}$ one obtains the commuting ladder*

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_k F \arrow[r] \arrow[d, "\mathrm{Id}"] & \pi_k E \arrow[r] \arrow[d, "\mathrm{Id}"] & \pi_k(E,F) \arrow[r] \arrow[d, "f_*"] & \pi_{k-1}F \arrow[r] \arrow[d, "\mathrm{Id}"] & \cdots \\
\cdots \arrow[r] & \pi_k F \arrow[r] & \pi_k E \arrow[r] & \pi_k(B) \arrow[r] & \pi_{k-1}F \arrow[r] & \cdots
\end{tikzcd}$$

with all vertical maps isomorphisms, taking the long exact sequence of the pair (E,F) to the long exact sequence in homotopy for the fibration $F \hookrightarrow E \to B$.

Proof. This is a straightforward application of the homotopy lifting property. Suppose that $h_0 : (D^k, S^{k-1}) \to (B,A)$ is a map. Viewed as a map $D^k \to B$ it is nullhomotopic, i.e. homotopic to the constant map $c_{b_0} = h_1 : D^k \to B$. Let H be a homotopy, and let $\tilde{h}_1 : D^k \to G \subset E$ be the constant map at the base point of G. Since $f \circ \tilde{h}_1 = h_1 = H(-,1)$, the homotopy lifting property implies that there is a lift $\widetilde{H} : D^k \times I \to E$ with $f \circ \widetilde{H}(-,0) = h_0$. This proves that $f_* : \pi_k(E,G) \to \pi_k(B,A)$ is surjective. A similar argument shows that $f_* : \pi_k(E,G) \to \pi_k(B,A)$ is injective.

The only square in the diagram for which commutativity is not obvious is

$$\begin{tikzcd}
\pi_k(E,F) \arrow[r] \arrow[d, "f_*"] & \pi_{k-1}F \arrow[d, "\mathrm{Id}"] \\
\pi_k B \arrow[r] & \pi_{k-1}F
\end{tikzcd} \tag{7.5}$$

We leave this as an exercise. □

Exercise 150. Prove that the diagram (7.5) commutes. You will find the constructions in the proof of Theorem 7.44 useful. Notice that the commutativity of this diagram and the fact that f_* is an isomorphism give an alternative definition of the connecting homomorphism $\pi_k B \to \pi_{k-1}F$ in the long exact sequence of the fibration $F \hookrightarrow E \to B$.

An alternative and useful perspective on Theorem 7.58 is obtained by replacing a pair by a fibration as follows.

Turn $A \hookrightarrow X$ into a fibration, with A' replacing A and $L(X,A)$ the fiber. Using the construction of Section 7.6 we see that

$$\begin{aligned} L(X,A) &= \{(a,\alpha) \mid \alpha : I \to X, \alpha(0) = a \in A, \alpha(1) = x_0\} \\ &= \mathrm{Map}((I,0,1),(X,A,x_0)). \end{aligned}$$

This shows that if $\Omega X \hookrightarrow PX \xrightarrow{e} X$ denotes the path space fibration, then $L(X,A) = PX|_A = e^{-1}(A)$. Thus Lemma 7.59 shows that e induces an isomorphism $e_* : \pi_k(PX, L(X,A)) \to \pi_k(X,A)$ for all k. Since PX is contractible, using the long exact sequence for the pair $(PX, L(X,A))$ gives an isomorphism $\partial : \pi_k(PX, L(X,A)) \xrightarrow{\cong} \pi_{k-1}(L(X,A))$. Therefore the composite

$$\pi_{k-1}(L(X,A)) \xrightarrow{e_* \circ \partial^{-1}} \pi_k(X,A)$$

is an isomorphism which makes the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \pi_{k+1}X \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k(L(X,A)) \arrow[r] \arrow[d] & \pi_k A \arrow[r] \arrow[d, "\mathrm{Id}"'] & \pi_k X \arrow[r] \arrow[d, "\mathrm{Id}"'] & \cdots \\
\cdots \arrow[r] & \pi_{k+1}X \arrow[r] & \pi_{k+1}(X,A) \arrow[r] & \pi_k A \arrow[r] & \pi_k X \arrow[r] & \cdots
\end{tikzcd}$$

commute, where the top sequence is the long exact sequence for the fibration $L(X,A) \hookrightarrow A \to X$ and the bottom sequence is the long exact sequence of the pair (X,A).

Homotopy groups are harder to compute and deal with than homology groups, essentially because excision fails for relative homotopy groups. In Chapter 9 we will discuss stable homotopy and generalized homology theories, in which (properly interpreted) excision does hold. Stabilization is a procedure which looks at a space X only in terms of what homotopy information remains in $S^n X$ as n gets large. The fiber $L(X,A)$ and cofiber X/A are stably homotopy equivalent.

7.16. The action of the fundamental group on homotopy sets

The question which arises naturally when studying based spaces is, what is the difference between the based homotopy classes $[X,Y]_0$ and the unbased classes $[X,Y]$? Worrying about base points can be a nuisance. It turns out that for simply connected spaces one need not worry; the based and unbased homotopy sets are the same. In general, the fundamental group acts on the based set as we will now explain.

Let X be in CGH_*, so it is a based space with a nondegenerate base point x_0. Suppose Y is a based space.

Definition 7.60. Let $f_0, f_1 : X \to Y$. Let $u : I \to Y$ be a path and suppose there is a homotopy $F : X \times I \to Y$ from f_0 to f_1 so that $F(x_0, t) = u(t)$. Then we say f_0 *is freely homotopic to* f_1 *along* u, and write

$$f_0 \underset{u}{\simeq} f_1.$$

Notice that if $f_0, f_1 : (X, x_0) \to (Y, y_0)$, then u is a loop. Thus a *free* homotopy of *based maps* gives rise to an element of $\pi_1(Y, y_0)$.

Lemma 7.61.

1. *(Existence) Given a map* $f_0 : X \to Y$ *and a path* u *in* Y *starting at* $f_0(x_0)$, $f_0 \underset{u}{\simeq} f_1$ *for some* f_1.
2. *(Uniqueness) Suppose* $f_0 \underset{u}{\simeq} f_1$, $f_0 \underset{v}{\simeq} f_2$ *and* $u \simeq v$ *(rel* ∂I*). Then* $f_1 \underset{\text{const}}{\simeq} f_2$.
3. *(Multiplicativity)* $f_0 \underset{u}{\simeq} f_1,\ f_1 \underset{v}{\simeq} f_2 \Longrightarrow f_0 \underset{uv}{\simeq} f_2$.

Proof. 1. There exists a free homotopy $F : X \times I \to Y$ with $F(x_0, t) = u(t),\ F(-, 0) = f_0$, since (X, x_0) is a cofibration pair :

$$\begin{tikzcd}
X \times 0 \cup \{x_0\} \times I \arrow[r, "f_0 \cup u"] \arrow[d] & Y \\
X \times I \arrow[ur, dotted, "F"'] &
\end{tikzcd}$$

Then let $f_1 = F(-, 1)$.

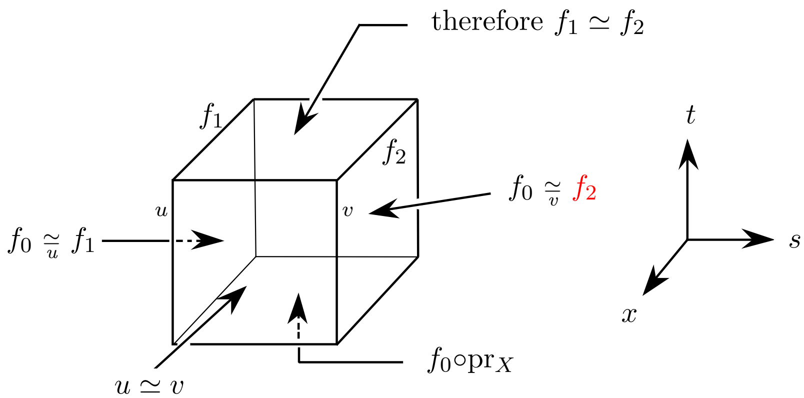
2. Since $(I, \partial I), (X, x_0)$ are cofibrations, so is their product $(X \times I, X \times \partial I \cup x_0 \times I)$ (see Exercise 129), and so the following problem has a solution:

$$\begin{tikzcd}
X \times I \times 0 \cup X \times \{0,1\} \times I \cup \{x_0\} \times I \times I \arrow[r] \arrow[d] & Y \\
X \times I \times I \arrow[ur, dotted, "H"'] &
\end{tikzcd}$$

In this diagram,

1. $X \times I \times 0 \to Y$ is the map $(x, s, 0) \mapsto f_0(x)$.
2. $X \times 0 \times I \to Y$ is the homotopy of f_0 to f_1 along u.
3. $X \times 1 \times I \to Y$ is the homotopy of f_0 to f_2 along v.
4. $\{x_0\} \times I \times I \to Y$ is the path homotopy of u to v.

The situation is represented in the following picture of a cube $X \times I \times I$.



Then $H(-,-,1)$ is a homotopy of f_1 to f_2 along a constant path.

3. This is clear. □

In light of Lemma 7.61, we can define a right action of $\pi_1(Y, y_0)$ on $[X, Y]_0$ by the following recipe.

For $[u] \in \pi_1(Y, y_0)$ and $[f] \in [X, Y]_0$, define $[f][u]$ to be $[f_1]$, where f_1 is any map so that $f \underset{u}{\simeq} f_1$.

Theorem 7.62. *This defines an action of $\pi_1(Y, y_0)$ on the based set $[X, Y]_0$, and $[X, Y]$ is the quotient set of $[X, Y]_0$ by this action if Y is path-connected.*

Proof. We need to verify that this action is well-defined. It is independent of the choice of representative of $[u]$ by Lemma 7.61, Part 2. Suppose now $[f] = [g] \in [X, Y]_0$, $f \underset{u}{\simeq} f_1$, and $g \underset{u}{\simeq} g_1$. Then

$$f_1 \underset{u^{-1}}{\simeq} f \underset{\text{const}}{\simeq} g \underset{u}{\simeq} g_1$$

so that f_1 and g_1 are based homotopic by Lemma 7.61, Parts 2 and 3.

This is an action of the group $\pi_1(Y, y_0)$ on the set $[X, Y]_0$ by Lemma 7.61, Part 3. Let

$$\Phi : [X, Y]_0 \to [X, Y]$$

be the forgetful functor. Clearly $\Phi([f][u]) = [f]$, and if $\Phi[f_0] = \Phi[f_1]$, then there is a u so that $[f_0][u] = [f_1]$. Finally Φ is onto by Lemma 7.61, Part 3, and the fact that Y is path-connected. □

Corollary 7.63. *A based map of path-connected spaces is nullhomotopic if and only if it is based nullhomotopic.*

Proof. If $c : X \to Y$ is the constant map, then clearly $c \underset{u}{\simeq} c$ for any $u \in \pi_1(Y, y_0)$. Thus $\pi_1(Y, y_0)$ fixes the class in $[X, Y]_0$ containing the constant map. $\square$

Corollary 7.64. *Let $X, Y \in$ CGH$_*$. If Y is a simply connected space then the forgetful functor $[X, Y]_0 \to [X, Y]$ is bijective.* $\square$

Since $\pi_n(Y, y_0) = [S^n, Y]_0$, we have the following corollary.

Corollary 7.65. *For any space Y, for all n, $\pi_1(Y, y_0)$ acts on $\pi_n(Y, y_0)$ with quotient $[S^n, Y]$, the set of free homotopy classes.* $\square$

7.16.1. Alternative description in terms of covering spaces. Suppose Y is path-connected, and X is *simply connected*. Then covering space theory says that any map $f : (X, x_0) \to (Y, y_0)$ lifts to a *unique* map $\tilde{f} : (X, x_0) \to (\tilde{Y}, \tilde{y}_0)$, where $\tilde{Y}$ denotes the universal cover of Y. Moreover based homotopic maps lift to based homotopic maps. Thus the function

$$p_* : [X, \tilde{Y}]_0 \to [X, Y]_0$$

induced by the cover $p : (\tilde{Y}, \tilde{y}_0) \to (Y, y_0)$ is a *bijection*. On the other hand, since $\tilde{Y}$ is simply connected, Corollary 7.64 shows that the function $[X, \tilde{Y}]_0 \to [X, \tilde{Y}]$ induced by the inclusion is a bijection.

Now $\pi_1(Y, y_0)$ can be identified with the group of covering transformations of $\tilde{Y}$; our convention is that this is a right action. Thus, $\pi_1(Y, y_0)$ acts on the right $[X, \tilde{Y}]$ by post composition.

A standard exercise in covering space theory shows that if $\alpha \in \pi_1(Y, y_0)$, the diagram

$$\begin{tikzcd}
{[X,Y]_0} \arrow[d, "\alpha"'] & {[X,\tilde{Y}]_0} \arrow[l, "\cong"'] \arrow[r, "\cong"] & {[X,\tilde{Y}]} \arrow[d, "\alpha"] \\
{[X,Y]_0} & {[X,\tilde{Y}]_0} \arrow[l, "\cong"'] \arrow[r, "\cong"] & {[X,\tilde{Y}]}
\end{tikzcd}$$

commutes, where the action on the left is via the action from Theorem 7.62, and the action on the right is the action induced by the covering translation corresponding to α, and the two left horizontal bijections are induced by the covering projection. Thus the two notions of action agree.

One could restrict to simply connected spaces Y and never worry about the distinction between based and unbased homotopy classes of maps into Y. This is not practical in general, and so instead one can make a dimension-by-dimension definition.

In the formulas below, we often omit the base point from a homotopy group in situations where the choice of base point is not relevant.

Definition 7.66. We say a space Y is *n-simple* if it is path-connected and $\pi_1 Y$ acts trivially on $\pi_n Y$. We say Y is *simple* if Y is n-simple for all n.

Thus, simply connected spaces are simple.

Exercise 151. Prove that the action of $\pi_1 Y$ on itself is just given by conjugation, so that Y is 1-simple if and only if $\pi_1 Y$ is abelian.

Exercise 152. Show that a topological group is simple. (In fact H-spaces are simple.)

Exercise 153. Let $n > 0$. Show that $\mathbf{R}P^n$ is simple if and only if n is odd. What is the map $[S^{2k}, \mathbf{R}P^{2k}]_0 \to [S^{2k}, \mathbf{R}P^{2k}]$? Can you identify two different elements of $\pi_{2k}(\mathbf{R}P^{2k})$ which become freely homotopic and see the homotopy geometically?

Proposition 7.67. *If F is n-simple, then the fibration $F \hookrightarrow E \to B$ defines a local coefficient system over B with fiber $\pi_n F$.*

(A good example to think about is the Klein bottle mapping onto the circle.)

Proof. Corollary 7.15 shows that given any fibration $F \hookrightarrow E \to B$, there is a well-defined homomorphism

$$\pi_1 B \to \left\{ \begin{array}{l} \text{Homotopy classes of self-homotopy} \\ \text{equivalences } F \to F \end{array} \right\}.$$

A homotopy equivalence induces a bijection

$$[S^n, F] \xrightarrow{\cong} [S^n, F].$$

But, since we are assuming that F is n-simple, this is the same as an automorphism

$$\pi_n F \to \pi_n F.$$

Thus, we obtain a homomorphism

$$\rho : \pi_1 B \to \operatorname{Aut}(\pi_n(F)),$$

i.e. a local coefficient system over B. □

Theorem 7.68. *The group $\pi_1 A$ acts on $\pi_n(X, A)$, $\pi_n X$, and $\pi_n A$ for all n. Moreover, the long exact sequence of the pair*

$$\cdots \to \pi_n A \to \pi_n X \to \pi_n(X, A) \to \pi_{n-1} A \to \cdots$$

is $\pi_1 A$-equivariant.

Proof. Let $h : (I, \partial I) \to (A, x_0)$ represent $[h] \in \pi_1 A$. Let $f : (D^n, S^{n-1}, p) \to (X, A, x_0)$. Then since (S^{n-1}, p) is a cofibration pair , the problem

$$\begin{tikzcd} S^{n-1} \times 0 \cup \{p\} \times I \arrow[r, "f_{|S^{n-1}} \cup h"] \arrow[d] & A \\ S^{n-1} \times I \arrow[ur, dotted, "H"'] & \end{tikzcd}$$

has a solution H. Since (D^n, S^{n-1}) is a cofibration, the problem

$$\begin{tikzcd} D^n \times 0 \cup S^{n-1} \times I \arrow[r, "f \cup H"] \arrow[d] & X \\ D^n \times I \arrow[ur, dotted, "F"'] & \end{tikzcd}$$

has a solution F. By construction, $F(x, 0) = f(x)$, and also $F(-, 1)$ takes the triple (D^n, S^{n-1}, p) to (X, A, x_0). Taking $[f][h] = [F(-, 1)]$ defines the action of $\pi_1 A$ on $\pi_n(X, A)$. It follows immediately from the definitions that the maps in the long exact sequence are $\pi_1 A$-equivariant. $\square$

Definition 7.69. A *pair* (X, A) is n*-simple* if $\pi_1 A$ acts trivially on $\pi_n(X, A)$.

7.17. The Hurewicz and Whitehead theorems

Perhaps the most important result of homotopy theory is the Hurewicz Theorem. We will state the general relative version of the Hurewicz theorem and its consequence, the Whitehead theorem, in this section.

Recall that D^n is oriented as a submanifold of $\mathbf{R}^n$; i.e., the chart $D^n \hookrightarrow \mathbf{R}^n$ determines the local orientation at any $x \in D^n$ via the excision isomorphism $H_n(D^n, D^n - \{x\}) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - \{x\})$. This determines the fundamental class $[D^n, S^{n-1}] \in H_n(D^n, S^{n-1})$. The sphere S^{n-1} is oriented as the boundary of D^n; i.e. the fundamental class $[S^{n-1}] \in H_{n-1}(S^{n-1})$ is defined by $[S^{n-1}] = \partial[D^n, S^{n-1}]$ where $\partial : H_n(D^n, S^{n-1}) \xrightarrow{\cong} H_{n-1}(S^{n-1})$ is the connecting homomorphism in the long exact sequence for the pair (D^n, S^{n-1}).

Definition 7.70. The *Hurewicz map* $\rho : \pi_n X \to H_n X$ is defined by

$$\rho[f] = f_*[S^n],$$

where $f : (S^n, p) \to (X, x_0)$ represents an element of $\pi_n X$, where $f_* : H_n S^n \to H_n X$ is the induced map.

There is also a *relative Hurewicz map* $\rho : \pi_n(X, A) \to H_n(X, A)$ defined by

$$\rho[f] = f_*[D^n, S^{n-1}],$$

where $f_* : H_n(D^n, S^{n-1}) \to H_n(X, A)$ is the homomorphism induced by $[f : (D^n, S^{n-1}, p) \to (X, A, x_0)] \in \pi_n(X, A)$.

Since the connecting homomorphism $\partial : H_n(D^n, S^{n-1}) \to H_{n-1}(S^{n-1})$ takes $[D^n, S^{n-1}]$ to $[S^{n-1}]$, the map of exact sequences

$$\begin{tikzcd} \cdots \arrow[r] & \pi_n A \arrow[r] \arrow[d, "\rho"] & \pi_n X \arrow[r] \arrow[d, "\rho"] & \pi_n(X, A) \arrow[r] \arrow[d, "\rho"] & \pi_{n-1} A \arrow[r] \arrow[d, "\rho"] & \cdots \\ \cdots \arrow[r] & H_n A \arrow[r] & H_n X \arrow[r] & H_n(X, A) \arrow[r] & H_{n-1} A \arrow[r] & \cdots \end{tikzcd} \tag{7.6}$$

commutes.

For $n > 2$, let $\pi_n^+(X, A)$ be the quotient of $\pi_n(X, A)$ by the subgroup generated by $[f] - [f][\alpha]$ for $[f] \in \pi_n(X, A)$ and $[\alpha] \in \pi_1 A$. (This quotient is called the *coinvariants* of the $\mathbf{Z}[\pi_1 A]$-module $\pi_n(X, A)$.) Let $\pi_2^+(X, A)$ be the quotient of $\pi_2(X, A)$ by the subgroup normally generated by elements of the form $[f]([f][\alpha])^{-1}$. Thus $\pi_n^+(X, A) = \pi_n(X, A)$ if $\pi_1 A = 0$, or if (X, A) is n-simple. Clearly ρ factors through $\pi_n^+(X, A)$, since $f_*[D^n, S^n]$ depends only on the free homotopy class of f.

The following theorem is the subject of one of the projects for this chapter. It says that for simply connected spaces, the first non-vanishing homotopy and homology groups coincide. We will give a proof of the Hurewicz theorem for simply connected spaces in Chapter 11.

Theorem 7.71 (Hurewicz theorem)**.**

1. *Let $n > 0$. Suppose that X is path-connected. If $\pi_k X = 0$ for all $k < n$, then $H_k X = 0$ for all $0 < k < n$, and the Hurewicz map*

   $$\rho : \pi_n X \to H_n X$$

   is an isomorphism if $n > 1$, and a surjection with kernel the commutator subgroup of $\pi_1 X$ if $n = 1$.
2. *Let $n > 1$. Suppose X and A are path-connected. If $\pi_k(X, A) = 0$ for all $k < n$ then $H_k(X, A) = 0$ for all $k < n$, and*

   $$\rho : \pi_n^+(X, A) \to H_n(X, A)$$

   is an isomorphism. In particular $\rho : \pi_n(X, A) \to H_n(X, A)$ is an epimorphism. □

Corollary 7.72 (Hopf degree theorem)**.** *The Hurewicz map $\rho : \pi_n S^n \to H_n S^n$ is an isomorphism. Hence two maps $f, g : S^n \to S^n$ are homotopic if and only if they have the same degree.* □

Although we have stated this as a corollary of the Hurewicz theorem, it can be proven directly using only the (easy) simplicial approximation theorem. (The Hopf degree theorem was covered as a project in Chapter 6.)

Definition 7.73.

1. A space X is called *n-connected* if $\pi_k X = 0$ for $k \leq n$. (Thus path-connected is synonymous with 0-connected and simply connected is synonymous with 1-connected.)
2. A pair (X, A) is called *n-connected* if $\pi_k(X, A) = 0$ for $k \leq n$.
3. A map $f : X \to Y$ is called *n-connected* if the pair (M_f, X) is n-connected, where M_f = mapping cylinder of f.

Using the long exact sequence for (M_f, X) and the homotopy equivalence $M_f \simeq Y$, we see that f is *n-connected* if and only if

$$f_* : \pi_k X \to \pi_k Y$$

is an isomorphism for $k < n$ and an epimorphism for $k = n$. Replacing the map $f : X \to Y$ by a fibration and using the long exact sequence for the homotopy groups of a fibration one concludes that f is n-connected if and only if the homotopy fiber of f is $(n-1)$-connected.

Corollary 7.74.

1. *If $f : X \to Y$ is n-connected, then $f_* : H_q X \to H_q Y$ is an isomorphism for all $q < n$ and an epimorphism for $q = n$.*
2. *If X, Y are 1-connected, and $f : X \to Y$ is a map such that*
$$f_* : H_q X \to H_q Y$$
is an isomorphism for all $q < n$ and an epimorphism for $q = n$, then f is n-connected.
3. *(Whitehead theorem) If X, Y are 1-connected spaces and $f : X \to Y$ is a map inducing an isomorphism on $\mathbf{Z}$-homology, then f induces isomorphisms $f_* : \pi_q X \xrightarrow{\cong} \pi_q Y$ for all q.*

Exercise 154. Prove Corollary 7.74.

A map $f : X \to Y$ inducing an isomorphism of $\pi_k X \to \pi_k Y$ for all k is called a *weak homotopy equivalence.* Thus a map inducing a homology isomorphism between simply connected spaces is a weak homotopy equivalence. Conversely a weak homotopy equivalence between two spaces gives a homology isomorphism.

We will see later (Theorem 8.33) that if X, Y are CW-complexes, then $f : X \to Y$ is a weak homotopy equivalence if and only if f is a homotopy equivalence. As a consequence,

Corollary 7.75. *(Whitehead theorem) A continuous map $f : X \to Y$ between simply connected CW-complexes inducing an isomorphism on all $\mathbf{Z}$-homology groups is a homotopy equivalence.* □

There are three closely related results, Corollary 7.74, Part 3, Corollary 7.75, and Theorem 8.33. Following historical tradition, we call each of these results the "Whitehead theorem."

This corollary does not imply that if X, Y are two simply connected spaces with the same homology, then they are homotopy equivalent; one needs a *map* inducing the homology equivalence.

For example, $X = S^4 \vee S^2$ and $Y = \mathbf{C}P^2$ are simply connected spaces with the same homology. They are not homotopy equivalent because their cohomology rings are different. In particular, there does not exist a continuous map from X to Y inducing isomorphisms on homology.

The Whitehead theorem for non-simply connected spaces involves homology with local coefficients: If $f : X \to Y$ is a map, let $\tilde{f} : \tilde{X} \to \tilde{Y}$ be the corresponding lift to universal covers. Recall from Shapiro's lemma (Exercise 110) that

$$H_k(\tilde{X}; \mathbf{Z}) \cong H_k(X; \mathbf{Z}[\pi_1 X]) \quad \text{for all } k$$

and

$$\pi_k \tilde{X} \cong \pi_k X \quad \text{for } k > 1$$

(and similarly for Y).

We obtain (with $\pi = \pi_1 X \cong \pi_1 Y$):

Theorem 7.76. *If $f : X \to Y$ induces an isomorphism $f_* : \pi_1 X \to \pi_1 Y$, then f is n-connected if and only if it induces isomorphisms*

$$H_k(X; \mathbf{Z}[\pi]) \to H_k(Y; \mathbf{Z}[\pi])$$

for $k < n$ and an epimorphism

$$H_n(X; \mathbf{Z}[\pi]) \to H_n(Y; \mathbf{Z}[\pi]).$$

In particular, f is a weak homotopy equivalence (homotopy equivalence if X, Y are CW-complexes) if and only if $f_* : H_k(X; A_\rho) \to H_k(Y; A_\rho)$ is an isomorphism for all local coefficient systems $\rho : \pi \to \mathrm{Aut}(A)$. □

Thus, in the presence of a map $f : X \to Y$, *homotopy equivalences can be detected by homology.*

7.18. Projects: Hurewicz theorem; Freudenthal suspension theorem

7.18.1. The Hurewicz theorem. Give or outline a proof of Theorem 7.71. A reference is [**54**, §IV.4-IV.7]. Another possibility is to give a spectral sequence proof. Chapter 11 contains a spectral sequence proof of the Hurewicz theorem.

7.18.2. The Freudenthal suspension theorem. Give or outline a proof of Theorem 9.7. A good reference for the proof is [**54**, §VII.6-VII.7]. You can find a spectral sequence proof in Section 11.3.

Chapter 8

Obstruction Theory and Eilenberg-MacLane Spaces

8.1. Basic problems of obstruction theory

Obstruction theory addresses the following types of problems. Let (X, A) be a relative CW-complex , Y an arbitrary space, and $p : E \to B$ a fibration.

1. *Extension problem.* Suppose $f : A \to Y$ is a continuous map. When does f extend to all of X? The problem is stated in the following diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & Y \\ X \arrow[ur, dotted] & \end{tikzcd}$$

(Given the two solid arrows, can one find a dotted arrow so that the diagram commutes?)

2. *Homotopy problem.*

$$\begin{tikzcd} X \times \{0,1\} \cup A \times I \arrow[r, "f"] \arrow[d] & Y \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

In words, given two maps $f_0, f_1 : X \to Y$ and a homotopy of the restrictions $f_{0|_A} : A \to Y$ to $f_{1|_A} : A \to Y$, can one find a homotopy from f_0 to f_1 restricting to the given homotopy on A ?

If A is empty, this is just the question of whether two maps f_0 and f_1 are homotopic. This problem is different from the homotopy extension problem (which is always solvable in our context) since in this case f_1 is *specified.*

Notice that the homotopy problem is a special case of the extension problem.

3. *Lifting problem.*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dotted] & B \end{tikzcd}$$

If $f : X \to B$ is given, can we find a lift of f to E? This is a special case of the *relative lifting problem*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d, "p"] \\ X \arrow[r, "f"] \arrow[ur, dotted] & B \end{tikzcd}$$

The extension problem is also a special case of the relative lifting problem (take B to be a point).

4. *Section problem.* Does a fibration $p : E \to B$ have a section $s : B \to E$? This is just a special case of the lifting problem in the case when $X = B$ and $f : X \to B$ is the identity map. Conversely, the lifting problem reduces to finding a section of the pullback bundle $f^*E \to X$.

Exercise 155. For each of the four problems above, if there is a dotted arrow that makes the diagram commute up to homotopy, then there is a dotted arrow that makes the diagram commute. (In the case of the relative lifting problem, this is a bit tough, and you would need to use a skeleton-by-skeleton argument to prove the relative homotopy lifting property.)

Fibrations and cofibrations are easier to work with than arbitrary maps since they have fibers and cofibers. Although we have required that (X, A) be a relative CW-complex and $p : E \to B$ be a fibration, the methods of Chapter 7 show how to work in greater generality, provided that you are willing to settle for solutions up to homotopy. Suppose that X and A are arbitrary CW-complexes and $g : A \to X$ is a continuous map. The cellular approximation theorem (Theorem 7.52) implies that g is homotopic to a cellular map; call it $h : A \to X$. The mapping cylinder M_h is then a CW-complex containing A as a subspace, and (M_h, A) is a CW-pair. Similarly if $p : E \to B$ is not a fibration, replace E by the mapping path space P_p of p to obtain a fibration $P_p \to B$.

Then the following exercise is an easy consequence of the homotopy lifting property, the homotopy extension property, and the method of turning maps into fibrations or cofibrations.

Exercise 156. Each of the four problems stated above is solvable up to homotopy for a continuous map $g : A \to X$ between CW-complexes and for an arbitrary continuous map $p : E \to B$ if and only if it is solvable for the CW-pair (M_h, A) and the fibration $P_p \to B$.

To solve the following exercise, work cell-by-cell one dimension at a time. Obstruction theory is a formalization of this geometric argument.

Exercise 157. (Motivating exercise of obstruction theory)

1. If Y is n-connected and (X, A) is a CW-pair of dimension $n+1$ then any map $A \to Y$ extends to a map $X \to Y$.
2. Any map $X \to Y$ from an n-dimensional CW-complex to an n-connected space is nullhomotopic.

It turns out that if Y is only assumed to be $(n-1)$-connected, there is a single *obstruction* $\gamma(f) \in H^n(X; \pi_n Y)$ which vanishes if and only if the map f is nullhomotopic.

The strategy of obstruction theory is to solve the four problems cell-by-cell and skeleton-by-skeleton. For example, for the extension problem, if the problem is solved over the n-skeleton X^n of X and e^{n+1} is an $(n+1)$-cell of X, the map is defined on ∂e^{n+1} and so the problem is to extend it over e^{n+1}. The obstruction to extending this map is that it be nullhomotopic or, more formally, that the element of $\pi_n Y$ represented by the composite $S^n \to \partial e^{n+1} \to X^n \to Y$ equals zero.

In this way we obtain a *cellular cochain* which assigns to $e^{n+1} \subset X$ the element in $\pi_n Y$. If this cochain is the zero cochain, then the map can be extended over the $(n+1)$-skeleton of X. It turns out this cochain is in fact a cocycle and so represents a cohomology class in $H^{n+1}(X; \pi_n Y)$.

The remarkable result is that if this cocycle represents the zero cohomology class, then by *redefining* the map on the n-skeleton one can then extend it over the $(n+1)$-skeleton of X (if you take one step backward, then you will be able to take two steps forward).

We will deal with the extension and homotopy problems first. The homotopy problem can be viewed as a relative form of the extension problem;

just take

$$(X', A') = (X \times I, X \times \partial I \cup A \times I).$$

Hence the problem of finding a homotopy between $f : X \to Y$ and $g : X \to Y$ is obstructed by classes in $H^{n+1}(X \times I, X \times \{0,1\}; \pi_n Y)$, which is isomorphic to $H^n(X; \pi_n Y)$.

We end this introduction to obstruction theory with some comments to indicate how the results of obstruction theory lead to a major conceptual shift in perspective on what cohomology is.

Let A be an abelian group and $K(A,n)$ be a space such that

$$\pi_k K(A,n) = \begin{cases} A & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

Such a space is called an *Eilenberg-MacLane space of type* (A,n). Then the solution to the homotopy problem for maps into $Y = K(A,n)$ shows that there is a single obstruction $\gamma \in H^n(X; \pi_n K(A,n))$ to homotoping a map $f : X \to K(A,n)$ to another map $g : X \to K(A,n)$. With a little more work one shows that this sets up an isomorphism

$$[X, K(A,n)] \xrightarrow{\cong} H^n(X; A).$$

This suggests one could *define* $H^n(X;A)$ to be $[X, K(A,n)]$. This observation forms the basic link between the homological algebra approach to cohomology and homotopy theory. From this perspective the Puppe sequence (Theorem 7.47) immediately gives the long exact sequence in cohomology, and the other Eilenberg–Steenrod axioms are trivial to verify. But more importantly, it suggests that one could find generalizations of cohomology by replacing the sequence of spaces $K(A,n)$ by some other sequence E_n and defining functors from spaces to sets (or groups, or rings, depending on how much structure one has on the sequence E_n) by

$$X \mapsto [X, E_n].$$

This indeed works and leads to the notion of a *spectrum* $\{E_n\}$ and its corresponding generalized homology and cohomology theories, one of the subjects of Chapter 9.

8.2. The obstruction cocycle

Suppose that (X,A) is a relative CW-complex. Thus X is filtered

$$A = X^{-1} \subset X^0 \subset X^1 \subset \cdots \subset X^n \subset \cdots \subset X.$$

Each X^n is obtained from X^{n-1} by attaching n-cells and X is the union of its skeleta X^n. We refer the reader to Definition 1.18 for the precise definition. Notice that X/A is a CW-complex. The dimension of (X,A)

is defined to be the highest dimension of the cells attached (we allow the dimension to be infinite).

Suppose that $g : A \to Y$ is a continuous map with Y path-connected. We wish to study the question of whether g can be extended to a map $X \to Y$.

Since Y is path-connected, the map $g : A \to Y$ extends over the 1-skeleton X^1 . Thus the zeroth and first step in extending $g : A \to Y$ to X is always possible (when Y is path-connected).

Suppose that $g : A \to Y$ has been extended to $g : X^n \to Y$ for some $n \geq 1$.

We now make a simplifying assumption.

Assumption. Y is n-simple, so that $[S^n, Y] = \pi_n Y$.

(We will indicate later how to avoid this assumption by using local coefficients.)

Theorem 8.1 (Main theorem of obstruction theory)**.** *Let (X, A) be a relative CW-complex, $n \geq 1$, and Y a path-connected n-simple space. Let $g : X^n \to Y$ be a continuous map.*

1. *There is a cellular cochain $\theta(g) \in C^{n+1}(X, A; \pi_n Y)$ which vanishes if and only if g extends to a map $X^{n+1} \to Y$.*
2. *$\theta(g)$ is a cocycle.*
3. *The cohomology class $[\theta(g)] \in H^{n+1}(X, A; \pi_n Y)$ vanishes if and only if the restriction $g_{|_{X^{n-1}}} : X^{n-1} \to Y$ extends to a map $X^{n+1} \to Y$.*

The proof of this theorem will occupy several sections.

8.3. Construction of the obstruction cocycle

Recall that if I_n indexes the n-cells of (X, A),

$$\begin{aligned} C^{n+1}(X, A; \pi_n Y) &= \mathrm{Hom}_{\mathbf{Z}}(C_{n+1}(X, A), \pi_n Y) \\ &= \mathrm{Funct}(\{e_i^{n+1} \mid i \in I_{n+1}\}, \pi_n Y). \end{aligned}$$

Each $(n+1)$-cell e_i^{n+1} admits a characteristic map

$$\chi_i : (D^{n+1}, S^n) \to (X^{n+1}, X^n)$$

with $e_i^{n+1} = \chi_i(\mathrm{int} D^{n+1})$. The restriction of χ_i to the boundary S^n

$$\phi_i = \chi_{i|_{S^n}} : S^n \to X^n$$

is called the attaching map for e_i^{n+1}.

Composing ϕ_i with $g : X^n \to Y$ defines a map

$$S^n \xrightarrow{\phi_i} X^n \xrightarrow{g} Y.$$

This defines an element $[g \circ \phi_i] \in [S^n, Y]$, which equals $[S^n, Y]_0 = \pi_n Y$, since Y is assumed to be n-simple.

Definition 8.2. Define the *obstruction cochain* $\theta^{n+1}(g) \in C^{n+1}(X, A; \pi_n Y)$ on the basis of oriented $(n+1)$-cells by the formula

$$\theta^{n+1}(g)(e_i^{n+1}) = [g \circ \phi_i]$$

and extend by linearity.

We will often drop the superscript and write $\theta(g)$ instead of $\theta^{n+1}(g)$.

A map $h : S^n \to Y$ is homotopically trivial if and only if h extends to a map $D^{n+1} \to Y$. The following lemma follows from this fact and the definition of attaching cells.

Lemma 8.3. *$\theta(g) = 0$ if and only if g extends to a map $X^{n+1} \to Y$.* □

We gave a geometric definition of the obstruction cochain and came to a geometric conclusion. Next we give an algebraic definition, which will allow us to see that the obstruction cochain is actually a cocycle.

Recall from Section 1.6.1 that the cellular chain complex is defined by taking the chain groups to be $C_n(X, A) = H_n(X^n, X^{n-1})$. The differential is defined to be the composite

$$H_n(X^n, X^{n-1}) \xrightarrow{\partial} H_{n-1}X^{n-1} \xrightarrow{i} H_{n-1}(X^{n-1}, X^{n-2}).$$

The following lemma gives the algebraic definition of the obstruction cochain.

Lemma 8.4. *Consider the diagram*

$$\begin{tikzcd}
\pi_{n+1}(X^{n+1}, X^n) \arrow[r, "\rho"] \arrow[d, "\partial"] & H_{n+1}(X^{n+1}, X^n) = C_{n+1}(X, A) \arrow[d, dotted, "\theta(g)"] \\
\pi_n X^n \arrow[r, "g_*"'] & \pi_n Y.
\end{tikzcd}$$

The Hurewicz map ρ is onto and there is a unique map $\theta(g)$ making the diagram commute.

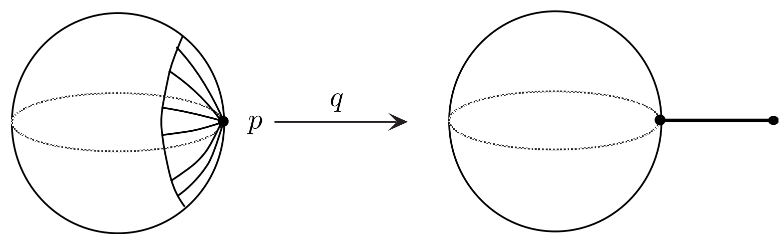
Proof. By the cellular approximation theorem $\pi_n X^n \to \pi_n X^{n+1}$ is onto and when $k < n$, then $\pi_k X^n \to \pi_k X^{n+1}$ is an isomorphism. Hence the pair (X^{n+1}, X^n) is n-connected. By the relative Hurewicz theorem, ρ is onto with kernel K the subgroup of $\pi_{n+1}(X^{n+1}, X^n)$ generated by the set

$$\{x - x\alpha \mid x \in \pi_{n+1}(X^{n+1}, X^n),\ \alpha \in \pi_1 X^n\}.$$

Since ρ is onto, $\theta(g)$ is unique if it exists. To show $\theta(g)$ exists, it suffices to show that $g_*(\partial(K)) = 0$. This holds since $g_*(\partial(x - x\alpha)) = g_*(\partial x) - (g_*(\partial x))(g_*\alpha)$ which equals zero, since Y is n-simple. □

Proposition 8.5. *The geometric and algebraic definitions of $\theta^{n+1}(g)$ agree.*

Proof. We first work on the algebraic definition. Construct a map $q : D^{n+1} \to D^{n+1} \vee I$ as illustrated in the next figure.



D^{n+1} $\qquad$ $D^{n+1} \vee I$

Let $p \in S^n$ and $x_0 \in X^0$ be the base points. Given an oriented $(n+1)$-cell e_i^{n+1}, let $\chi_i : (D^{n+1}, S^n) \to (X^{n+1}, X^n)$ be the characteristic map for the cell. Choose a path u in X^n from $\chi_i(p)$ to x_0. Then $(\chi_i \vee u) \circ q : (D^{n+1}, S^n, p) \to (X^{n+1}, X^n, x_0)$ is a based map which is freely homotopic to the characteristic map χ_i. Thus $\rho((\chi_i \vee u) \circ q)$ is the generator of $H_{n+1}(X^{n+1}, X^n)$ represented by the oriented cell e_i^{n+1}.

By definition, $\partial((\chi_i \vee u) \circ q) \in \pi_n X^n$ is represented by

$$\partial((\chi_i \vee u) \circ q) = (\phi_i \vee u) \circ \overline{q} : S^n \to X^n,$$

where $\overline{q} : S^n \to S^n \vee I$ is obtained by restricting the map q of the previous figure to the boundary and $\phi_i = \chi_{i|_{S^n}}$ is the attaching map for the cell e_i^{n+1}. Hence by the algebraic definition, $\theta(g)(e_i^{n+1}) = g \circ (\phi_i \vee u) \circ \overline{q}$. But this is $(g \circ \phi_i \vee g \circ u) \circ \overline{q}$, which equals $[g \circ \phi_i] \in [S^n, Y] = \pi_n Y$, which in turn is the geometric definition of $\theta(g)(e_i^{n+1})$. □

Theorem 8.6. *The obstruction cochain $\theta^{n+1}(g)$ is a cocycle.*

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
\pi_{n+2}(X^{n+2}, X^{n+1}) \arrow[r] \arrow[d] & H_{n+2}(X^{n+2}, X^{n+1}) \arrow[d] \\
\pi_{n+1}X^{n+1} \arrow[r] \arrow[d] & H_{n+1}X^{n+1} \arrow[d] \\
\pi_{n+1}(X^{n+1}, X^n) \arrow[r] \arrow[d] & H_{n+1}(X^{n+1}, X^n) \arrow[d, "\theta(g)"] \\
\pi_n X^n \arrow[r, "g_*"] & \pi_n Y
\end{tikzcd}$$

The unlabeled horizontal arrows are the Hurewicz maps. The unlabeled vertical arrows come from homotopy or homology exact sequences of the pairs (X^{n+2}, X^{n+1}) and (X^{n+1}, X^n).

The theorem follows by noting that the $\delta\theta(g)$ is the composite of all the right vertical maps, that the top horizontal arrow is onto by the Hurewicz Theorem, and that the composite of the bottom two vertical maps on the left are zero, because they occur in the homotopy exact sequence of the pair (X^{n+1}, X^n). □

8.4. Proof of the extension theorem

Lemma 8.3 says that $\theta(g)$ is the zero cochain if and only if $g : X^n \to Y$ extends over X^{n+1}. Theorem 8.6 says that $\theta(g)$ is a cocycle. To complete the proof of Theorem 8.1, we need to show that $\theta(g)$ is a coboundary if and only if g restricted to X^{n-1} extends to a map $X^{n+1} \to Y$.

To prove this we need two lemmas, the difference cochain lemma and the realization lemma. The difference cochain lemma immediately implies that if g restricted to X^{n-1} extends to X^{n+1}, then $\theta(g)$ is a coboundary. We state a general version of this lemma (involving a homotopy G) for later use in Section 8.6.

Lemma 8.7 (Difference cochain lemma)**.** *Let $f_0, f_1 : X^n \to Y$ be two maps and let $G : X^{n-1} \times I \to Y$ be a homotopy between $f_0|_{X^{n-1}}$ and $f_1|_{X^{n-1}}$. Then there is a cochain $d(f_0, G, f_1) \in C^n(X, A; \pi_n Y)$ so that*

$$\delta d(f_0, G, f_1) = \theta^{n+1}(f_0) - \theta^{n+1}(f_1).$$

We call $d(f_0, G, f_1)$ the *difference cochain*. In the case where G is constant in time (in which case $f_{0|_{X^{n-1}}} = f_{1|_{X^{n-1}}}$), we simply write $d(f_0, f_1)$, omitting the G.

Proof. Let $\hat{X} = X \times I$, $\hat{A} = A \times I$. Then $(\hat{X}, \hat{A})$ is a relative CW-complex, with $\hat{X}^k = X^k \times \partial I \cup X^{k-1} \times I$. Hence a map $\hat{X}^n \to Y$ is a pair of maps $f_0, f_1 : X^n \to Y$ and a homotopy $G : X^{n-1} \times I \to Y$ of the restrictions of f_0, f_1 to X^{n-1}.

Thus one obtains the cocycle

$$\theta(f_0, G, f_1) \in C^{n+1}(\hat{X}, \hat{A}; \pi_n Y)$$

which obstructs finding an extension of $f_0 \cup G \cup f_1$ to $\hat{X}^{n+1}$. From this cocycle one obtains the *difference cochain*

$$d(f_0, G, f_1) \in C^n(X, A; \pi_n Y)$$

which sends the n-cell e^n of (X, A) to the value of θ on the $(n+1)$-cell $e^n \times e^1$ of $(\hat{X}, \hat{A})$; that is,

$$d(f_0, G, f_1)(e_i^n) = (-1)^{n+1}\theta(f_0, G, f_1)(e_i^n \times e^1) \tag{8.1}$$

for each oriented n-cell e_i^n of X. (The reason for the sign will be apparent shortly.)

Using the fact that $\theta(f_0, G, f_1)$ is a cocycle, for all oriented $(n+1)$-cells e_i^{n+1},

$$\begin{aligned}
0 &= (\delta\theta(f_0, G, f_1))(e_i^{n+1} \times e^1) \\
&= \theta(f_0, G, f_1)(\partial(e_i^{n+1} \times e^1)) \\
&= \theta(f_0, G, f_1)(\partial(e_i^{n+1}) \times e^1) \\
&\qquad +(-1)^{n+1}(\theta(f_0, G, f_1)(e_i^{n+1} \times 1) - \theta(f_0, G, f_1)(e^{n+1} \times 0)) \\
&= (-1)^{n+1}(\delta(d(f_0, G, f_1))(e_i^{n+1}) + \theta(f_1)(e_i^{n+1}) - \theta(f_0)(e_i^{n+1})).
\end{aligned}$$

Therefore

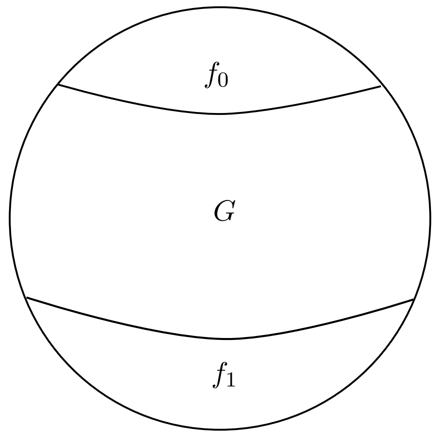
$$\delta d(f_0, G, f_1) = \theta^{n+1}(f_0) - \theta^{n+1}(f_1).$$

□

There is a geometric interpretation of the difference cochain. Identify S^n with $\partial(D^n \times I)$. For an oriented n-cell e_i^n, let $\chi_i : (D^n, S^{n-1}) \to (X^n, X^{n-1})$ be its characteristic map with attaching map $\phi_i = \chi_{i|_{S^{n-1}}}$. Then $\pm d(f_0, G, f_1)(e_i^n) \in \pi_n Y$ is the composite of $f_0 \cup G \cup f_1$ with the attaching map

$$\chi_i \times 0 \cup \phi_i \times \mathrm{Id}_I \cup \chi_i \times 1 : \partial(D^n \times I) = S^n \to (X \times I)^n = X^n \times \{0, 1\} \cup X^{n-1} \times I$$

of $e_i^n \times e^1$, as indicated in the next figure.



What we want now is a converse, which is provided by the realization lemma below.

Lemma 8.8 (Realization lemma). *Given a map $f_0 : X^n \to Y$ and a homotopy $G : X^{n-1} \times I \to Y$ with $G_0 = f_{0|X^{n-1}}$, and an element $d \in C^n(X, A; \pi_n Y)$, there is a map $f_1 : X^n \to Y$ so that $G(-, 1) = f_{1|X^{n-1}}$ and $d = d(f_0, G, f_1)$.*

Given the previous geometric description of the difference cochain, all we really need to prove is:

Lemma 8.9. *For any map $f : D^n \times 0 \cup S^{n-1} \times I \to Y$ and for any element $\alpha \in [\partial(D^n \times I), Y]$, there is a map $F : \partial(D^n \times I) \to Y$ so that F represents α and restricts to f.*

Proof. The proof is easy. Let $K : \partial(D^n \times I) \to Y$ be any map representing α and let $D = D^n \times 0 \cup S^{n-1} \times I$. Since D is contractible, both $K_{|D}$ and f are nullhomotopic, hence homotopic to each other; the map K gives an extension of one end of this homotopy. The homotopy extension property of the pair $(\partial(D^n \times I), D)$ gives a homotopy $H : \partial(D^n \times I) \times I \to Y$ and, $F = H_1$ is the required map. $\square$

Proof of Lemma 8.8. Given an oriented n-cell e_i^n of X^n, let

$$\chi_i : (D^n, S^{n-1}) \to (X^n, X^{n-1})$$

be the characteristic map. Apply Lemma 8.9 to the map

$$f = f_0 \circ \chi_i \cup G \circ (\chi_{i|S^{n-1}} \times \mathrm{Id}_I)$$

and $\alpha = d(e_i^n)$ and let F_i be the map provided by the conclusion of Lemma 8.9. Define $f_1 : X^n \to Y$ on the n-cells by $f_1(\chi_i(x)) = F_i(x, 1)$. The geometric interpretation of the difference cochain shows $d(f_0, G, f_1)(e_i^n) = d(e_i^n)$ as desired. $\square$

Proof of Theorem 8.1. We have already shown that there is a cocycle $\theta(g)$ which vanishes if and only if $g : X^n \to Y$ extends to a map $X^{n+1} \to Y$. What remains to be proven if that $\theta(g)$ is a coboundary if and only if g restricted to X^{n-1} extends to a map $X^{n+1} \to Y$.

Suppose $\theta(g)$ is a coboundary, that is, $\theta(g) = \delta d$. The Realization Lemma 8.8 says that there is a map $f_1 : X^n \to Y$ which agrees with g restricted to X^{n-1} and so that the difference cochain $d(g, f_1)$ equals d. Then the Difference Cochain Lemma 8.7 shows that

$$\theta(g) = \delta d(g, f_1) = \theta(g) - \theta(f_1)$$

and hence $\theta(f_1) = 0$. Thus f_1 extends to X^{n+1} as desired.

Conversely, assume that g restricted to X^{n-1} extends to a map $f_1 : X^{n+1} \to Y$. Then

$$\delta d(g, f_1) = \theta(g) - \theta(f_1) = \theta(g) - 0,$$

so $\theta(g)$ is a coboundary. □

Exercise 158. Find examples of (X, A), Y, and g where:

1. $\theta^{n+1}(g) = 0$.
2. $\theta^{n+1}(g) \neq 0$, but $[\theta^{n+1}(g)] = 0$.
3. $[\theta^{n+1}(g)] \neq 0$.

It is conceivable (and happens frequently) that finding an extension of $g : X^n \to Y$ to X^{n+1} may require g to be redefined not just on the n-cells, but maybe even on the $(n-1)$-cells, or perhaps even on the $(n-k)$-cells for $k = 1, \cdots, r$ for some r.

This suggests that there may be theorems which state “given g, the restriction $g_{|_{X^{n-k}}}$ extends to X^{n+1} if and only if some obstruction vanishes.” Such theorems exist, and working them out leads to the definition of secondary and higher obstructions.

To get a feel for where such obstructions may lie, notice that the obstruction cochain $\theta(g)$ is the obstruction to extending $g : X^n \to Y$ to X^{n+1}, and that its cohomology class $[\theta(g)]$ is the obstruction to extending $g_{|X^{n-1}}$ to X^{n+1}. The cohomology group is a subquotient of the cochain group $C^{n+1}(X, A; \pi_n Y)$. It turns out that the obstructions live in further subquotients, that is, in subquotients of cohomology.

8.5. Obstructions to finding a homotopy

We now turn to the construction of obstructions to finding a homotopy between $f_0 : X \to Y$ and $f_1 : X \to Y$ extending a fixed homotopy on A.

$$\begin{tikzcd} X \times \{0,1\} \cup A \times I \arrow[r, "f"] \arrow[d] & Y \\ X \times I \arrow[ur, dotted] & \end{tikzcd}$$

This is accomplished by viewing the homotopy problem as an extension problem and then applying the Künneth theorem.

Consider the product relative CW-complex:

$$(X^*, A^*) = (X, A) \times (I, \partial I) = (X \times I, X \times \partial I \cup A \times I).$$

Then a map $F : X^{*n} \to Y$ is a pair of maps $f_0, f_1 : X \to Y$ and a homotopy of $f_{0|_{X^{n-1}}}$ to $f_{1|_{X^{n-1}}}$. Therefore the obstruction class $[\theta^{n+1}(F)] \in H^{n+1}(X^*, A^*; \pi_n Y)$ is defined. This group is isomorphic to $H^n(X, A; \pi_n Y)$ by the Künneth theorem (this is really the suspension isomorphism in cohomology). Call the corresponding element $\theta^n(f_0, f_1) \in H^n(X, A; \pi_n Y)$. Then one gets the following theorem.

Theorem 8.10. *Let (X, A) be a relative CW-complex, Y an n-simple space, $f_0, f_1 : X \to Y$ two maps which agree on A, and $F : X^{n-1} \times I \to Y$ a homotopy from $f_{0|_{X^{n-1}}}$ to $f_{1|_{X^{n-1}}}$ (rel A). Then the cohomology class of $\theta^n(f_0, f_1)$ equals 0 if and only if the restriction of F to $X^{n-2} \times I$ extends to a homotopy of $f_{0|_{X^n}}$ to $f_{1|_{X^n}}$.* □

An interesting special case occurs when f_1 is constant (see Exercise 157).

Corollary 8.11. *Any continuous map from an n-dimensional CW-complex to an n-connected space is nullhomotopic.* □

8.6. Primary obstructions

A case where obstruction theory is easy to use occurs if $H^{n+1}(X, A; \pi_n Y) = 0$ for all n. This occurs quite frequently. For example, if (X, A) has dimension a and Y is $(a-1)$-connected, then any map from A to Y extends to X.

The next interesting case occurs when $H^{n+1}(X, A; \pi_n Y)$ is nonzero in only one dimension. Then there is a single obstruction to extending g, and this obstruction sets up a correspondence between extensions and the corresponding cohomology group. As a first step in understanding this correspondence, we have the following theorem.

Theorem 8.12. *Let (X, A) be a relative CW-complex, $n \geq 1$, and Y an $(n-1)$-connected space (if $n = 1$, assume $\pi_1 Y$ is abelian). Let $f : A \to Y$ be a map. Then f extends to a map $g : X^n \to Y$. If g_0, g_1 are extensions of f, then $g_{0|_{X^{n-1}}} \simeq g_{1|_{X^{n-1}}}$ (rel A) and the obstructions $\theta^{n+1}(g_0)$ and $\theta^{n+1}(g_1)$ are cohomologous.*

Proof. Since Y is path-connected, f can be extended over X^1. Since the obstructions to extending f lie in $H^{r+1}(X, A; \pi_r Y)$, f can be extended to X^n. Since the obstructions to finding a homotopy between maps lie in $H^r(X, A; \pi_r Y)$, any two extensions of f are homotopic over X^{n-1}, and as we saw, the difference cochain has coboundary equal to the difference of the obstruction cocycles. □

Definition 8.13. Let (X, A) be a relative CW-complex, $n \geq 1$, and Y an $(n-1)$-connected space (if $n = 1$, assume $\pi_1 Y$ is abelian). Let $f : A \to Y$ be a map. The obstruction to extending f to X^{n+1} is denoted by

$$\gamma^{n+1}(f) \in H^{n+1}(X, A; \pi_n Y).$$

It is called the *primary obstruction to extending* f.

Theorem 8.12 says that the primary obstruction is well-defined and vanishes if and only if f extends over X^{n+1}. We next show that it is homotopy invariant.

Theorem 8.14. *Let (X, A) be a relative CW-complex, $n \geq 1$, and Y an $(n-1)$-connected space (if $n = 1$, assume $\pi_1 Y$ is abelian). Let $f : A \to Y$ be a map. Suppose f' is homotopic to f. Then $\gamma^{n+1}(f') = \gamma^{n+1}(f)$.*

Proof. By Theorem 8.12, f extends to a map $g : X^n \to Y$ and f' extends to a map $g' : X^n \to Y$. Likewise $g_{|X^{n-1}} \simeq g'_{|X^{n-1}}$, since Y is highly connected. Call the homotopy F. Then the difference cochain satisfies

$$\delta d(g, F, g') = \theta(g) - \theta(g')$$

by Lemma 8.7. This shows $\gamma^{n+1}(f)$ and $\gamma^{n+1}(f')$ are cohomologous. □

In the situation of the above theorems, if the primary obstruction vanishes, then the map f extends to $g : X^{n+1} \to Y$. However the next obstruction class $[\theta^{n+2}(g)]$ may depend on the choice of g. So it is usually only the primary obstruction which is computable. Obstruction theory ain't all it's cracked up to be.

To define the primary obstruction for two maps to be homotopic, we apply the above theorems to $(X \times I, X \times \partial I \cup A \times I)$ and obtain the following theorem.

Theorem 8.15. *Let (X, A) be a relative CW-complex, $n \geq 1$, and Y an $(n-1)$-connected space (if $n = 1$ assume $\pi_1 Y$ is abelian). Let $f_0, f_1 : X \to Y$ be two functions which agree on A. Then $f_{0|X^{n-1}} \simeq f_{1|X^{n-1}}$ rel A, and the cohomology class in $H^n(X, A; \pi_n Y)$ of the obstruction to extending this homotopy to X^n is independent of the choice of homotopy on X^{n-1} and depends only on the homotopy classes of f_0 and f_1 relative to A.* □

In light of this theorem, one can make the following definition.

Definition 8.16. Let (X, A) be a relative CW-complex, $n \geq 1$, and Y an $(n-1)$-connected space (if $n = 1$, assume $\pi_1 Y$ is abelian). Let $f_0, f_1 : X \to Y$ two functions which agree on A. The obstruction to constructing a homotopy $f_{0|X^n} \simeq f_{1|X^n}$ rel A is denoted

$$\gamma^n(f_0, f_1) \in H^n(X, A; \pi_n Y)$$

and is called the *primary obstruction to homotoping f_0 to f_1*. It depends only on the homotopy classes of f_0 and f_1 relative to A.

8.7. Eilenberg-MacLane spaces

An important class of spaces is the class of those spaces Y satisfying $\pi_k Y = 0$ for all $k \neq n$.

Definition 8.17. Let n be a positive integer and let π be a group, with π abelian if $n > 1$. A space Y is called a *$K(\pi, n)$-space* if it has the homotopy type of a CW-complex and if

$$\pi_k Y = \begin{cases} 0 & \text{if } k \neq n \\ \pi & \text{if } k = n. \end{cases}$$

We will see later that (π, n) determines the homotopy type of Y; that is, for a fixed pair (π, n), any two $K(\pi, n)$-spaces are homotopy equivalent. A $K(\pi, n)$-space is called an *Eilenberg–MacLane space of type (π, n)* .

Theorem 8.18. *Given any $n > 0$ and any group π with π abelian if $n > 1$, there exists a $K(\pi, n)$-CW-complex.*

Sketch of proof. Let $\langle x_i, i \in I \mid r_j,\ j \in J\rangle$ be a presentation (abelian if $n > 1$) of π. Let K^n be the wedge $\vee_{i \in I} S^n$ of n-spheres, one for each generator of π. Then the Seifert-van Kampen theorem and Hurewicz theorems imply that $\pi_k K_n = 0$ for $k < n$, that $\pi_1 K^1$ is the free group on the generating set I when $n = 1$, and that $\pi_n K^n$ is the free abelian group on the generating set I when $n > 1$.

For each relation, attach an $(n+1)$-cell using the relation to define the homotopy class of the attaching map. This defines a complex K^{n+1} with

$$\pi_k K^{n+1} = \begin{cases} 0 & \text{if } k < n \\ \pi & \text{if } k = n. \end{cases}$$

For $n = 1$, this follows from the Seifert-van Kampen theorem theorem. For $n > 1$ and $k < n$, this follows from the cellular approximation theorem, and for $k = n$ from the Hurewicz theorem.

Attach $(n+2)$-cells to kill $\pi_{n+1} K^{n+1}$. More precisely, choose a set of generators for $\pi_{n+1} K^{n+1}$ and attach one $(n+2)$-cell for each generator, using the generator as the homotopy class of the attaching map. This gives a $(n+2)$-dimensional complex K^{n+2}. By the cellular approximation theorem, the homotopy groups in dimensions less than $n+1$ are unaffected, and there is a surjection $\pi_{n+1} K^{n+1} \to \pi_{n+1} K^{n+2}$. Thus

$$\pi_k K^{n+2} = \begin{cases} 0 & \text{if } k < n \text{ or } k = n+1, \\ \pi & \text{if } k = n. \end{cases}$$

Now attach $(n+3)$-cells to kill π_{n+2}, etc. The union of the K^r with the CW-topology is a CW-complex and a $K(\pi, n)$-space. □

We follow standard abuse of notation and write $K(\pi, n)$ for any $K(\pi, n)$-space. This is not such a crime, since by Corollary 8.22, any two $K(\pi, n)$-spaces have the same homotopy type.

An important property of Eilenberg-MacLane spaces is that they possess fundamental cohomology classes. These classes are extremely useful. They allow us to set up a functorial correspondence between $H^n(X; \pi)$ and $[X, K(\pi, n)]$. They are used to define cohomology operations. They can be used to give the "fibering data" needed to decompose an arbitrary space into Eilenberg-MacLane spaces (Postnikov towers) and also to construct characteristic classes for fiber bundles.

Assume π is abelian, so that $K(\pi, n)$ is simple. Then

$$H^n(K(\pi, n); \pi) \cong \operatorname{Hom}(H_n(K(\pi, n); \mathbf{Z}), \pi) \cong \operatorname{Hom}(\pi, \pi), \tag{8.2}$$

where the first isomorphism is the adjoint of the Kronecker pairing (Exercise 32) and is an isomorphism by the universal coefficient theorem (Theorem 3.29). The second map is the Hurewicz isomorphism (Theorem 7.71).

Definition 8.19. The *fundamental class of the* $K(\pi, n)$,

$$\iota \in H^n(K(\pi, n); \pi),$$

is the class corresponding to the identity map $\mathrm{Id} : \pi \to \pi$ under the isomorphisms of Equation (8.2).

In other words, $\langle \iota, \rho(\alpha)\rangle = \alpha$ for ρ the Hurewicz homomorphism and for $\alpha \in \pi_n(K(\pi,n)) = \pi$.

The fundamental class can be used to define a function

$$\Phi : [X, K(\pi,n)] \to H^n(X;\pi) \tag{8.3}$$

by the formula

$$\Phi[f] = f^*[\iota]$$

for $f : X \to K(\pi,n)$.

The primary obstruction class can be used to define another function

$$\Psi : [X, K(\pi,n)] \to H^n(X;\pi) \tag{8.4}$$

by setting $\Psi[f]$ to be the primary obstruction to homotoping f to the constant map,

$$\Psi[f] = \gamma^n(f, \text{const}).$$

Theorem 8.15 shows that $\Psi[f]$ depends only on the homotopy class of f, and hence is well-defined.

Theorem 8.20. *The functions Φ and Ψ*

$$[X, K(\pi,n)] \to H^n(X;\pi)$$

coincide, are bijections, and are natural with respect to maps $X \to X'$.

Proof. *Step 1. Ψ is injective.* Let $f : X \to K(\pi,n)$ be a continuous map. Obstruction theory says that if $\gamma^n(f,\text{const}) = 0$, then f and the constant map are homotopic over the n-skeleton. But all higher obstructions vanish since they live in zero groups. Hence if $\Psi[f] = 0$, f is nullhomotopic. In other words $\Psi^{-1}[0] = [\text{const}]$.

If we knew that $[X, K(\pi,n)]$ were a group and Ψ a homomorphism, then we could conclude that Ψ is injective. (Of course, this follows from the present theorem.)

Instead, we will outline the argument proving the "addition formula"

$$\gamma^n(f,g) = \gamma^n(f,\text{const}) - \gamma^n(g,\text{const});$$

i.e. $\gamma^n(f,g) = \Psi[f] - \Psi[g]$ for any two functions $f, g : X \to K(\pi,n)$. This implies that Ψ is injective.

We now prove the addition formula. If F is a homotopy from $f_{|_{X^{n-1}}}$ to the constant map and G is a homotopy from $g_{|_{X^{n-1}}}$ to the constant map, then compose F and $\overline{G}$ to get a homotopy from $f_{|_{X^{n-1}}}$ to $g_{|_{X^{n-1}}}$. (Here $\overline{G}$ means the reverse homotopy, i.e. $\overline{G}(x,t) = G(x, 1-t)$.)

Write $F * \overline{G}$ for this homotopy from $f_{|X^{n-1}}$ to $g_{|X^{n-1}}$. Then on an n-cell $e \subset X$ the obstruction $\gamma^n(f,g)(e) \in \pi_n(K(\pi,n))$ is defined to be the homotopy class of the map $S^n \to K(\pi,n)$ defined as follows. Decompose

S^n as a neighborhood of the poles together with a neighborhood of the equator: $S^n = D^n_0 \cup (S^{n-1} \times I) \cup D^n_1$. Then define $\gamma = \gamma^n(f,g)(e) : S^n \to K(\pi,n)$ to be the homotopy class of the map which equals f on D^n_0, $F * \bar{G}$ on $S^{n-1} \times I$, and g on D^n_1. Since this map is constant on the equator $S^{n-1} \times \frac{1}{2}$, the homotopy class of d is clearly the sum of two classes, the first representing $\gamma^n(f, \text{const})(e)$ and the second representing $\gamma^n(\text{const}, g)(e)$. Therefore $\gamma^n(f,g) = \gamma^n(f, \text{const}) - \gamma^n(g, \text{const})$ and so $\Psi[f] = \Psi[g]$ if and only if f is homotopic to g.

Step 2. Ψ is surjective. We do this by proving a variant of the realization lemma (Lemma 8.8) for the difference cochain.

Given $[\alpha] \in H^n(X;\pi)$, choose a cocycle α representing $[\alpha]$. Since the quotient X^n/X^{n-1} is the wedge of n-spheres, one for each n-cell of X, α defines a function (up to homotopy)

$$g : X^n/X^{n-1} \to K(\pi,n)$$

with the restriction of g to the i-th n-sphere representing $\alpha(e^n_i) \in \pi_n(K(\pi,n)) = \pi$.

The function g extends to $X^{n+1}/X^{n-1} \to K(\pi,n)$ because α is a cocycle. In fact, for each oriented $(n+1)$-cell e^{n+1}_i

$$0 = (\delta\alpha)(e^{n+1}_i) = \alpha(\partial e^{n+1}_i),$$

which implies that the composite

$$S^n \to X^n \to X^n/X^{n-1} \xrightarrow{g} K(\pi,n)$$

of the attaching map of e^{n+1}_i and g is nullhomotopic. Thus g extends over the $(n+1)$-skeleton.

Since $H^{n+i+1}(X;\pi_{n+i}(K(\pi,n))) = 0$ for $i \geq 1$, obstruction theory and induction show that there exists an extension of $g : X^n/X^{n-1} \to K(\pi,n)$ to $\tilde{g} : X/X^{n-1} \to K(\pi,n)$. Composing with the quotient map, one obtains a map

$$f : X \to K(\pi,n),$$

constant on the $(n-1)$-skeleton, so that the characteristic map for an n-cell e^n_i induces

$$\alpha(e^n) : D^n/\partial D^n \to X/X^{n-1} \to K(\pi,n).$$

But $\gamma^n(f, \text{const})(e^n)$ was defined to be the map on S^n which equals f on the upper hemisphere and the constant map on the lower hemisphere. Therefore $\gamma^n(f, \text{const})$ equals $\alpha(e^n)$; hence $\Psi[f] = [\alpha]$ and so Ψ is onto.

Step 3. Ψ is natural. This follows from the algebraic definition of the obstruction cocycle, but we leave the details as an exercise.

Exercise 159. Prove that Ψ is natural.

Step 4. $\Psi = \Phi$. We first prove this for the identity map $\mathrm{Id} : K(\pi, n) \to K(\pi, n)$, and then use naturality. In other words, we need to show the primary obstruction to finding a null homotopy of Id is the fundamental class $\iota \in H^n(K(\pi, n); \pi))$. Since $K(\pi, n)$ is $(n-1)$-connected, Id is homotopic to a map, say Id', which is constant on the $(n-1)$-skeleton $K(\pi, n)^{n-1}$. By the universal coefficient and Hurewicz theorems,

$$H^n(K(\pi, n); \pi) \cong \mathrm{Hom}(\rho(\pi_n(K(\pi, n))), \pi).$$

The definition of the fundamental class is equivalent to the formula

$$\langle \iota, \rho[g] \rangle = [g] \in \pi_n(K(\pi, n))$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing. Thus what we need to show is:

$$\langle \gamma^n(\mathrm{Id}', \mathrm{const}), \rho[g] \rangle = [g].$$

This is an equation which can be lifted to the cochain level; i.e. we need to show that if $[g] \in \pi_n(K(\pi, n)^n, K(\pi, n)^{n-1})$, then

$$\gamma^n(\mathrm{Id}', \mathrm{const})(\rho[g]) = \mathrm{Id}'_*[g].$$

In particular, we only need to verify this equation for the characteristic maps $\chi_i : (D^n, S^{n-1}) \to (K(\pi, n)^n, K(\pi, n)^{n-1})$ of the n-cells. But the element $\gamma^n(\mathrm{Id}', \mathrm{const})[\rho(\chi_i)] \in \pi_n(K(\pi, n))$ is represented by the map $S^n \to K(\pi, n)$ given by the characteristic map composed with Id' on the upper hemisphere and the constant map on the lower hemisphere, and this map is homotopic to the characteristic map composed with the identity. Thus

$$\gamma^n(\mathrm{Id}', \mathrm{const})[\rho(\chi_i)] = \mathrm{Id}'_*[\chi_i]$$

as desired. Hence $\Phi[\mathrm{Id}] = \Psi[\mathrm{Id}]$.

Now suppose $[f] \in [X, K(\pi, n)]$. Then naturality of Ψ and Φ means that the diagram

$$\begin{tikzcd}
{[X, K(\pi, n)]} \arrow[d, "{\Psi,\Phi}"'] & {[K(\pi, n), K(\pi, n)]} \arrow[l, "f^*"'] \arrow[d, "{\Psi,\Phi}"] \\
H^n(X; \pi) & H^n(K(\pi, n); \pi) \arrow[l, "f^*"]
\end{tikzcd}$$

commutes when either both vertical arrows are labeled by Ψ or when both are labeled by Φ. Then we have

$$\begin{aligned}\Psi[f] &= \Psi f^*[\mathrm{Id}] \\ &= f^*\Psi[\mathrm{Id}] \\ &= f^*\Phi[\mathrm{Id}] \\ &= \Phi f^*[\mathrm{Id}] \\ &= \Phi[f].\end{aligned}$$

□

Corollary 8.21. *For $n \geq 1$ and for π and π' abelian groups, there is a 1-1 correspondence*

$$[K(\pi, n), K(\pi', n)]_0 = [K(\pi, n), K(\pi', n)] \longleftrightarrow \mathrm{Hom}(\pi, \pi')$$

taking a map $K(\pi, n) \to K(\pi', n)$ to the induced map on homotopy groups.

Proof.

$$\begin{aligned}[K(\pi, n), K(\pi', n)] &\cong H^n(K(\pi, n); \pi') \\ &\cong \mathrm{Hom}(H_n(K(\pi, n)), \pi') \\ &\cong \mathrm{Hom}(\pi, \pi').\end{aligned}$$

This follows from Theorem 8.20, the universal coefficient theorem, and the Hurewicz theorem. The composite is the map induced on the n-th homotopy group. That the based and unbased homotopy sets are the same when $n > 1$ follows from Corollary 7.64 since $K(\pi', n)$ is simply connected. For $n = 1$ we refer to Corollary 8.26. □

Corollary 8.22. *Let $K(\pi, n)$ and $K'(\pi, n)$ be two Eilenberg-MacLane spaces of type (π, n) for $n \geq 1$ and for π abelian. There is a based homotopy equivalence $K(\pi, n) \to K'(\pi, n)$ inducing the identity on the n-th homotopy group. Any two such homotopy equivalences are based homotopic.*

We shall see in the next section that Corollaries 8.21 and 8.22 continue to hold when π and π' are nonabelian and $n = 1$ provided one uses based homotopy classes.

Computing the cohomology of Eilenberg-MacLane spaces is very important, because of connections to cohomology operations.

Definition 8.23. For positive integers n and m and abelian groups π and π', a *cohomology operation of type* (n, π, m, π') is a natural transformation of functors $\theta : H^n(-; \pi) \to H^m(-; \pi')$.

For example $u \mapsto u \cup u$ gives a cohomology operation of type $(n, \mathbf{Z}, 2n, \mathbf{Z})$.

Exercise 160. (Serre) Let $O(n, \pi, m, \pi')$ be the set of all cohomology operations of type (n, π, m, π'). Show that $\theta \leftrightarrow \theta(\iota)$ gives a 1-1 correspondence

$$O(n, \pi, m, \pi') \longleftrightarrow H^m(K(\pi, n); \pi') = [K(\pi, n), K(\pi', m)].$$

We will return to this subject in Section 11.4.

8.8. Aspherical spaces

It follows from our work above that for π abelian, $[X, K(\pi, 1)] = H^1(X; \pi) = \text{Hom}(H_1 X, \pi) = \text{Hom}(\pi_1 X, \pi)$.

For π nonabelian we have the following theorem.

Theorem 8.24. *For a based CW-complex X, taking induced maps on fundamental groups gives a bijection*

$$[X, K(\pi, 1)]_0 \to \text{Hom}(\pi_1 X, \pi).$$

Sketch of proof. By collapsing out a maximal tree, we will assume that the zero-skeleton of X is a single point. Then by the Seifert-van Kampen theorem theorem, $\pi_1 X$ is presented with generators given by the characteristic maps of the 1-cells

$$\overline{\chi_i^1} : D^1/S^0 \to X,$$

and relations given by the attaching maps of the 2-cells

$$\phi_j^2 : S^1 \to X.$$

We will discuss why the above correspondence is onto. Let $\gamma : \pi_1 X \to \pi$ be a group homomorphism. Construct a map $g : X^1 \to K(\pi, 1)$ by defining g on a 1-cell (a circle) e_i^1 to be a representative of $\gamma[\overline{\chi_i^1}]$. The attaching maps ϕ_j^2 are trivial in $\pi_1 X$, and hence $g_*[\phi_j^2] = \gamma[\phi_j^2] = \gamma(e) = e$. Thus g extends over the 2-skeleton. The attaching maps of the 3-cells of X are nullhomotopic in $K(\pi, 1)$, so the map extends over the 3-cells. Continuing inductively, one obtains a map $X \to K(\pi, 1)$ realizing γ on the fundamental group.

The proof that if $g, h : X \to K(\pi, 1)$ are two maps inducing the same homomorphism (i.e. $g_* = h_* : \pi_1 X \to \pi$), then g is based point preserving homotopic to h (rel x_0) is similar in nature and will be omitted. $\square$

Corollary 8.25. *Let $K(\pi, 1)$ and $K'(\pi, 1)$ be two Eilenberg-MacLane spaces of type $(\pi, 1)$. There is a based homotopy equivalence $K(\pi, 1) \to K'(\pi, 1)$ inducing the identity on the fundamental group. Any two such homotopy equivalences are based homotopic.*

Corollary 8.26. *For a based CW-complex X, there is a bijection*

$$\Phi : [X, K(\pi, 1)] \to \mathrm{Hom}(\pi_1 X, \pi)/\sim$$

where Φ is defined by taking a based representative and then taking the induced map on the fundamental group. The equivalence relation is given by conjugation: if $\phi, \psi : \pi_1 X \to \pi$ are two homomorphisms, then $\phi \sim \psi$ if and only if there is an $h \in \pi$ so that for all $g \in \pi_1 X$, $\psi(g) = h^{-1}\phi(g)h$.

Proof. We have already two bijections: one from Theorem 8.24

$$[X, K(\pi, 1)]_0 \to \mathrm{Hom}(\pi_1 X, \pi)$$

given by the fundamental group and one from Theorem 7.62

$$\frac{[X, Y]_0}{\sim} \to [X, Y]$$

given by the forgetful map, where the equivalence relation is given by the action of $\pi_1 Y$. To compare the two bijections, the key fact is that if f and g are homotopic maps from X to Y, then the induced maps $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_* : \pi_1(X, x_0) \to \pi_1(Y, g(x_0))$ are related by $g_*[\gamma] = [\alpha^{-1}(f \circ \gamma)\alpha]$ with $\alpha(t) = H(x_0, t)$ and $\alpha^{-1}(t) = H(x_0, 1-t)$ where H is the homotopy from f to g. In the case where $f(x_0) = g(x_0)$, we see that the maps f_* and g_* are conjugate. In particular, the fundamental group functor

$$[X, Y]_0 \to \mathrm{Hom}(\pi_1 X, \pi_1 Y)$$

is equivariant with respect to the action of $\pi_1 Y$. The corollary follows. □

Proposition 8.27. *Suppose that*

$$1 \to L \xrightarrow{\phi} \pi \xrightarrow{\gamma} H \to 1$$

is an exact sequence of (not necessarily abelian) groups. Then the homotopy fiber of the map $g : K(\pi, 1) \to K(H, 1)$ inducing γ as in Theorem 8.24 is $K(L, 1)$ and the inclusion of the fiber $K(L, 1) \to K(\pi, 1)$ induces the homomorphism ϕ.

If L, π, and H are abelian, the same assertions hold with $K(\pi, 1)$ replaced by $K(\pi, n)$ for any positive integer n. □

Thus short exact sequences of groups correspond exactly to fibrations of Eilenberg–MacLane spaces; the sequence of groups

$$1 \to L \to \pi \to H \to 1$$

is a short exact sequence of groups if and only if the corresponding sequence of spaces and maps

$$K(L, 1) \hookrightarrow K(\pi, 1) \to K(H, 1)$$

is a fibration sequence up to homotopy.

Similarly the sequence of abelian groups

$$0 \to L \to \pi \to H \to 0$$

is exact if and only if for any n the corresponding sequence of spaces and maps

$$K(L,n) \hookrightarrow K(\pi,n) \to K(H,n)$$

is a fibration sequence up to homotopy.

Exercise 161. Prove Proposition 8.27.

Definition 8.28. A space is *aspherical* if its universal cover is contractible.

Corollary 8.33 below implies that a CW-complex is aspherical if and only if it is a $K(\pi,1)$.

Using $K(\pi,1)$ spaces, one can define functors from groups to abelian groups by taking homology and cohomology. The group $H_n(K(\pi,1))$ is called the *nth homology of the group* π and is often denoted by $H_n(\pi)$. Similarly the *nth cohomology of the group* π is defined by $H^n(\pi) = H^n(K(\pi,1))$. We will study these functors in greater detail in Chapter 10. A purely algebraic definition of the (co)homology of groups can also be given:

Exercise 162. Show that $H_n(\pi) = \mathrm{Tor}_n^{\mathbf{Z}\pi}(\mathbf{Z},\mathbf{Z})$ and $H^n(\pi) = \mathrm{Ext}^n_{\mathbf{Z}\pi}(\mathbf{Z},\mathbf{Z})$.

Exercise 163. Using a CW-structure on the circle whose cells are permuted by a free $\mathbf{Z}/n = \langle t\rangle$-action, deduce an exact sequence

$$0 \to \mathbf{Z} \to \mathbf{Z}[\mathbf{Z}/n] \xrightarrow{1-t} \mathbf{Z}[\mathbf{Z}/n] \to \mathbf{Z} \to 0$$

By splicing construct a free $\mathbf{Z}[\mathbf{Z}/n]$-resolution of $\mathbf{Z}$. Use this resolution to compute $H_*(\mathbf{Z}/n)$ and $H^*(\mathbf{Z}/n)$.

Aspherical spaces are ubiquitous. Compact 2-manifolds other than the sphere and projective space are $K(\pi,1)$'s. Also, $K(\mathbf{Z}/2,1) = \mathbf{R}P^\infty$. More generally $K(\mathbf{Z}/n,1) = L_n^\infty$, where L_n^∞ is the infinite lens space given as $S^\infty/(\mathbf{Z}/n)$ where $S^\infty \subset \mathbf{C}^\infty$ is the infinite dimensional sphere and the action is given by multiplication by a primitive n-th root of unity in every coordinate. Since $\pi_n(X \times Y) = \pi_n X \times \pi_n Y$, $K(\mathbf{Z}^n,1) = (S^1)^n$, the *n-torus*. The Cartan–Hadamard Theorem states that if M is a complete Riemannian manifold with sectional curvature everywhere ≤ 0, then for every point $p \in M$, the exponential map

$$\exp : T_pM \to M$$

is a covering map. In particular M is aspherical. Here is an application.

Exercise 164. If M is a complete Riemannian manifold with sectional curvature everywhere ≤ 0, then $\pi_1 M$ is torsion-free.

We also mention the still open

Borel conjecture. *Closed aspherical manifolds with isomorphic fundamental groups are homeomorphic.*

The $K(\pi, 1)$-spaces are important for at least three reasons.

1. If M is a $\mathbf{Z}\pi$-module, then $H_*(K(\pi, 1); M)$ is an important algebraic invariant of the group and the module.
2. $K(\pi, 1) = B\pi$, and hence $[X, B\pi] = \text{Hom}(\pi_1 X, \pi)/(\phi \sim g\phi g^{-1})$ classifies regular covers with deck transformations π.
3. In the study of flat bundles, that is, bundles whose structure group G reduces to a discrete group π, the classifying map $X \to BG$ factors through some $K(\pi, 1)$.

8.9. CW-approximations and Whitehead's theorem

Definition 8.29.

1. A *weak homotopy equivalence* is a map $f : X \to Y$ which induces isomorphisms $\pi_i(X, x) \to \pi_i(Y, f(x))$ for all i and for all base points x in X.
2. A *CW-approximation of a topological space* Y is a weak homotopy equivalence
$$X \to Y$$
where X is a CW-complex.

Theorem 8.30. *Any space Y has a CW-approximation.*

Proof. We may reduce to the case where Y is path-connected by approximating each path component separately. We will inductively construct maps
$$g_n : X^n \to Y$$
which are n-connected, that is, give a surjection on π_n and a bijection on π_i for $i < n$. Also the restriction of g_n to the $(n-1)$-skeleton will be g_{n-1}.

Take X^0 to be a point. Assume inductively the existence of an n-connected map $g_n : X^n \to Y$, where X^n is an n-dimensional CW-complex. Attach an $(n+1)$-cell to X^n for every generator ker $g_{n*} : \pi_n X^n \to \pi_n Y$ to obtain a complex X'. Since the attaching maps are in the kernel, g_n extends to a map $g'_{n+1} : X' \to Y$. By cellular approximation and by construction $g'_{n+1*} : \pi_i X' \to \pi_i Y$ is an isomorphism for $i < n+1$. (One could alternatively use the relative Hurewicz theorem.) Finally, define $X^{n+1} = X' \vee (\bigvee S_i^{n+1})$ with an $(n+1)$-sphere for each generator of the cokernel of $g'_{n+1*} : \pi_{n+1} X' \to \pi_{n+1} Y$. Define the map $g_{n+1} : X^{n+1} \to Y$ by defining the map on S_i^{n+1} to be a representative of the corresponding element of the cokernel.

This shows how to construct the skeleta of $X = \cup X^n$. Topologize X as a CW-complex; see Definition 1.18. □

By the relative Hurewicz theorem, a CW-approximation induces an isomorphism on homology.

Milnor defined a *functorial* CW-approximation using simplicial methods [**32**]. This is done by defining a CW-complex X with an n-cell for each nondegenerate singular n-simplex in Y, where a nondegenerate simplex means that it does not factor through a degeneracy map, i.e. a linear projection onto one of its $n-1$ dimensional faces. Milnor's construction gives a functor from topological spaces to CW-complexes; this functor takes a CW-complex to another complex of the same homotopy type.

A very useful theorem is given by the following.

Theorem 8.31 (cofibrant theorem)**.** *A map $f : Y \to Z$ is a weak homotopy equivalence if and only if for all CW-complexes X,*

$$f_* : [X, Y] \to [X, Z] \qquad [g] \mapsto [f \circ g]$$

is a bijection.

Proof. ($\Longleftarrow$) When Y and Z are simple (e.g. simply connected), then by choosing X to be the n-sphere $n = 0, 1, 2, \ldots$, one sees that f is a weak homotopy equivalence. We omit the proof without the simplicity hypothesis and refer the reader to Whitehead's book [**54**].

($\Longrightarrow$) Philosophically, f_* is a bijection, since it is for spheres and disks, and CW-complexes are built from spheres and disks. An easy proof along these lines can be given using the Puppe sequence for finite–dimensional CW-complexes, but for the general case we need a lemma similar to the motivating exercise (Exercise 157).

Lemma 8.32. *Let $g : (X, A) \to (Z, Y)$ be a map of pairs where (X, A) is a relative CW-complex and $Y \hookrightarrow Z$ is a weak homotopy equivalence. Then $g \simeq h$ (rel A) where $h(X) \subset Y$.*

Proof. We will construct a sequence of maps $h_n : X \to Z$ so that $h_n(X^n) \subset Y$ with $h_{-1} = g$ and $h_{n-1} \simeq h_n$ (rel X^{n-1}). (Slowly drag g into Y.) Then for a point x in an open n-cell, we define $h(x) = h_n(x)$, and the homotopy from g to h is defined by squeezing the homotopy $h_{n-1} \simeq h_n$ into the time interval $[1 - (1/2^n), 1 - (1/2^{n+1})] \subset [0, 1]$.

Assume inductively that $h_{n-1} : X \to Z$ has been constructed. Let

$$\chi_i : (D^n, S^{n-1}) \to (X^n, X^{n-1})$$

be the characteristic map of an n-cell. Since $\pi_n(Z, Y) = 0$, the map

$$h_{n-1} \circ \chi_i : (D^n, S^{n-1}) \to (Z, Y)$$

is homotopic rel S^{n-1} to a map $h_{n,i}$ whose image lies in Y. (See Exercise 165 below for this interpretation of the vanishing of the relative homotopy group.) Then define $h_n : X^n \to Y$ by

$$h_n(\chi_i(e)) = h_{n,i}(e)$$

where $e \in D^n$. Using the above homotopy, one sees $h_{n-1|X^n} \simeq h_n : X^n \to Z$. Apply the homotopy extension theorem to extend this to a homotopy $H : X \times I \to Z$ and define $h_n(x)$ as $H(x, 1)$. □

We now return to the proof of the cofibrant theorem. We have a weak homotopy equivalence $f : Y \to Z$, which we may as well assume is the inclusion of a subspace by replacing Z by a mapping cylinder. We see

$$f_* : [X, Y] \to [X, Z]$$

is onto by applying the lemma to the pair (X, ϕ). We see f_* is injective by applying the lemma to the pair $(X \times I, X \times \{0, 1\})$. □

Exercise 165. Let $Y \subset Z$ be path-connected spaces. If $\pi_n(Z, Y, y_0) = 0$, show that any map $f : (D^n, S^{n-1}) \to (Z, Y)$ is homotopic rel S^{n-1} to a map whose image lies in Y.

Corollary 8.33 (Whitehead theorem)**.** *A weak homotopy equivalence between CW-complexes is a homotopy equivalence.*

Proof. Let $f : Y \to Z$ be a weak homotopy equivalence between CW-complexes. By the surjectivity of $f_* : [Z, Y] \to [Z, Z]$, there is a $g : Z \to Y$ so that $[\mathrm{Id}_Z] = f_*[g] = [f \circ g]$. Then

$$f_*[g \circ f] = [f \circ g \circ f] = [\mathrm{Id}_Z \circ f] = [f \circ \mathrm{Id}_Y] = f_*[\mathrm{Id}_Y].$$

By the injectivity of f_*, $[g \circ f] = [\mathrm{Id}_Y]$, so f and g are homotopy inverses. □

The corollary is a geometric analogue of the fact that a quasi-isomorphism between projective chain complexes is a chain homotopy equivalence (see Lemma 1.7 and Exercise 246.

Corollary 8.34. *Any n-connected CW-complex Y has the homotopy type of a CW-complex X whose n-skeleton is a point.*

Proof. Apply the proof of the CW-approximation to Y to find a weak homotopy equivalence $X \to Y$ where X^n is a point. By Corollary 8.33 it is a homotopy equivalence. □

Theorem 8.35. *Let $f : X \to Y$ be a continuous map. Suppose that CW-approximations $u : X' \to X$ and $v : Y' \to Y$ are given. Then there exists a map f' so that the diagram*

$$\begin{tikzcd} X' \arrow[r, "u"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "v"'] & Y \end{tikzcd}$$

commutes up to homotopy. Furthermore, the map f' is unique up to homotopy. □

The theorem follows from the cofibrant theorem (v_* is a bijection). Applying it to the case where $f = \mathrm{Id}_X$, we see that CW-approximations are unique up to homotopy type. The theorem and the cofibrant theorem imply that for the purposes of homotopy theory, one may as well assume all spaces involved are CW-complexes. A relative version of the cofibrant theorem gives the same result for based homotopy theory.

Exercise 166. Formulate the existence and uniqueness theorem for a CW-approximation of a pair (X, A). (The proof is an easy modification of the absolute case above.)

8.10. Obstruction theory in fibrations

We next turn to the lifting and section problems.

Consider the lifting problem:

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[r, "f"'] \arrow[ur, dotted, "g"] & B \end{tikzcd}$$

where $p : E \to B$ is a fibration. Note that if $f = \mathrm{Id}_B$, then the lifting problem is the same as constructing a section of p.

Suppose g has been defined over the n-skeleton of X. Given an oriented $(n+1)$-cell e^{n+1} of X, the composite of the attaching map and g gives a map $S^n \to X \xrightarrow{g} E$. The composite $S^n \to X \xrightarrow{g} E \xrightarrow{p} B$ is null homotopic since it equals $S^n \to X \xrightarrow{f} B$, which extends over the cell $e^{n+1} \hookrightarrow X$.

The homotopy lifting property of fibrations implies that the composite

$$S^n \to X \xrightarrow{g} E$$

is homotopic to a map $S^n \to F$ by lifting the nullhomotopy in the base (cf. Corollary 7.49).

Thus, to each oriented $(n+1)$-cell of X we have defined, in a highly noncanonical way, a map $S^n \to F$. We would like to say that this defines a cochain on X with values in $\pi_n F$.

If we assume F is n-simple so that $\pi_n F = [S^n, F]$ (unbased maps), then any map $S^n \to F$ defines an element in $\pi_n F$.

However, if $\pi_1 B \neq 0$, then some ambiguity remains; namely it was not necessary that f preserved base points, and hence, *even if F is n-simple*, we do not obtain a cochain in $C^n(X; \pi_n F)$. However, one does get a cochain with local coefficients. Thus obstruction theory for fibrations requires the use of cohomology with local coefficients, as we will now see.

Recall from Proposition 7.67 that if F is n-simple, then the fibration $F \hookrightarrow E \to B$ defines a local coefficient system over B with fiber $\pi_n F$. In fact Proposition 7.67 shows how to associate to each $\alpha \in \pi_1 B$ a homotopy class h_α of self-homotopy equivalences of F. Then h_α induces an automorphism of $[S^n, F]$ by $f \mapsto h_\alpha \circ f$. Since we are assuming F is n-simple, $[S^n, F] = \pi_n F$, and so this shows how the fibration determines a representation $\rho : \pi_1 B \to \mathrm{Aut}(\pi_n F)$.

Now pull back this local coefficient system over X via $f : X \to B$ to obtain a local coefficient system over X. We continue to call it ρ, so

$$\rho : \pi_1 X \xrightarrow{f_*} \pi_1 B \xrightarrow{\rho} \mathrm{Aut}(\pi_n(F)).$$

With these hypotheses, one obtains an obstruction cocycle

$$\theta^{n+1}(g) \in C^{n+1}(X; \pi_n(F)_\rho) = \mathrm{Hom}_{\mathbf{Z}[\pi_1 X]}(C_{n+1}(\widetilde{X}), \pi_n(F)_\rho).$$

One then can prove the following theorem.

Theorem 8.36. *Let X be a CW-complex, $p : E \to B$ be a fibration with fiber F. Let $f : X \to B$ be a map and $g : X^n \to E$ a lift of f on the n-skeleton.*

If F is n-simple, then an obstruction class

$$[\theta^{n+1}(g)] \in H^{n+1}(X; (\pi_n F)_\rho)$$

is defined.

If $[\theta^{n+1}(g)]$ vanishes, then g can be redefined over the n-skeleton (rel the $(n-1)$-skeleton), then extended over the $(n+1)$-skeleton X^{n+1}. □

If the local coefficient system is trivial, for example if $\pi_1 X = 0$ or $\pi_1 B = 0$, then $[\theta^{n+1}(g)] \in H^{n+1}(X; \pi_n F)$ (untwisted coefficients). If $\pi_1 F = 0$, then F is k-simple for all k, so that the hypotheses of Theorem 8.36 hold.

If $\pi_k F = 0$ for $k \leq n-1$, then $[\theta^{n+1}(g)] \in H^{n+1}(X; (\pi_n F)_\rho)$ is called the *primary obstruction to lifting f* and is well-defined; i.e.

1. a lift over the n-skeleton always exists, and
2. $[\theta^{n+1}(g)]$ is independent of the choice of lift to the n-skeleton.

Henceforth we write $\gamma^{n+1}(f)$ for the primary obstruction to lifting f.

The proof of Theorem 8.36 is in many ways similar to the proofs given earlier. In certain important cases one can reduce this theorem to a special case of the extension problem by the following useful device.

Suppose there is a fibration $E \to B$, so that the fiber can be "delooped" in the following sense. Namely there exists a fibration $q : B \to Z$ with fiber E' so that the inclusion $E' \hookrightarrow B$ is equivalent to the fibration $E \to B$. Then we have seen that F is homotopy equivalent to the loop space ΩZ, and the sequence

$$[X, F] \to [X, E] \to [X, B] \xrightarrow{q_*} [X, Z]$$

is exact by Theorem 7.47.

This sequence shows that $f : X \to B$ can be lifted to $g : X \to E$ if and only if $q_*[f]$ is nullhomotopic. Thus the problem of lifting f is equivalent to the problem of nullhomotoping $q \circ f$.

As was explained above, there are obstructions

$$\theta^k(q \circ f, *) \in H^k(X; \pi_k Z)$$

to nullhomotoping $q \circ f$ (provided Z is simple, etc.). But since

$$\pi_k(Z) = \pi_{k-1}(\Omega Z) = \pi_{k-1}F,$$

we can view $\theta^k(q \circ f, *)$ as an element of $H^k(X; \pi_{k-1}F)$. Thus the obstructions to finding sections are in this special case obtainable from the homotopy obstruction theorem.

This point of view works if $E \to B$ is, say, a principal G bundle, since one can take Z to be the classifying space BG.

8.11. Characteristic classes

One application of obstruction theory is to define characteristic classes. For example, suppose $p : E \to B$ is an oriented n-plane vector bundle, i.e. a bundle with fiber $\mathbf{R}^n$ and structure group $GL^+(n, \mathbf{R})$, the group of automorphisms of $\mathbf{R}^n$ with positive determinant. Then the *Euler class* $e(p) \in H^n(B; \mathbf{Z})$ is the primary obstruction to finding a section of the bundle

$$\mathbf{R}^n - 0 \hookrightarrow E_0 \to B$$

where $E_0 = E - i(B)$ is E minus the zero section.

Exercise 167. The primary obstruction to finding a section of $E_0 \to B$ lies in $H^n(B; \pi_{n-1}(\mathbf{R}^n - 0))$. Show that $\mathbf{R}^n - 0$ is n-simple, that $\pi_{n-1}(\mathbf{R}^n - 0) =$

$\mathbf{Z}$, and that $\pi_1 B$ acts trivially on $\pi_{n-1}(\mathbf{R}^n - 0)$; i.e. the local coefficient system is trivial (this uses the fact that the bundle is orientable).

In other words the Euler class is the primary obstruction to finding a nowhere zero section of p. The Euler class is a *characteristic class* in the sense that given a map of oriented n-plane vector bundles which is an isomorphism on fibers

$$\begin{tikzcd} E' \arrow[r] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

then

$$f^* e(p) = e(p').$$

If B is a CW-complex of dimension n, then the primary obstruction is the only obstruction, so there is a nowhere zero section if and only if the Euler class is zero. The Euler class is related to the Euler characteristic of a manifold by the following theorem (see [**36**]).

Theorem 8.37. *If $p : TB \to B$ is the tangent bundle of a closed, oriented n-manifold, then*

$$\langle e(p), [B] \rangle = \chi(B).$$

□

Corollary 8.38 (Poincaré–Hopf theorem)**.** *A closed, oriented n-manifold has a nowhere-zero vector field if and only if its Euler characteristic is zero.* □

For example, you can't comb the hairy ball!

The mod 2 reduction of the Euler class of an $\mathbf{R}^n$-vector bundle $E \to B$ is called the *nth Stiefel–Whitney class* $w_n(E) \in H^n(B; \mathbf{Z}/2)$.

There are many other aspects of obstruction theory: for example the analogue of the homotopy problem in this setting is the problem of finding vertical homotopies between two sections. There are obstructions

$$\theta^n(g, g^1) \in H^n(X; (\pi_n F)_\rho).$$

Here are a few more examples to ponder.

Exercise 168. When we studied the extension problem, we did not come across local coefficient systems. This is because we assumed that Y was n-simple. Use Theorem 8.36 stated above together with the inverse of the delooping method outlined above to find a statement of a theorem about obstructions to extending maps into non-simple spaces.

Exercise 169. Write down a careful statement of a theorem about the obstruction to finding vertical homotopies between sections of a fibration.

8.12. Projects: Postnikov systems

8.12.1. Postnikov systems. The decomposition of a CW-complex into its skeleta has a "dual" construction leading to *Postnikov decompositions* of a space. The word "dual" here is used in the same sense that cofibrations and fibrations are dual. The building blocks for CW-complexes are cells (D^n, S^{n-1}). These have homology $\mathbf{Z}$ in dimension n and zero in other dimensions. The building blocks for Postnikov decompositions are the Eilenberg-MacLane spaces $K(\pi, n)$. For CW-complexes, the attaching maps describe how the cells are put together. For Postnikov decompositions, spaces are described as iterated fibrations with fibers Eilenberg-MacLane spaces and the primary obstruction to finding sections determine how the space is to be assembled from its $K(\pi, n)$s.

For this project, show how to construct a Postnikov tower for a space X.

Theorem 8.39. *If X is a simple path-connected space, there exists a "tower"*

$$\cdots \to X_n \xrightarrow{p_n} X_{n-1} \to \cdots \to X_1 \xrightarrow{p_1} X_0,$$

as well as maps $f_n : X \to X_n$ so that the diagrams

$$\begin{tikzcd} & X_n \arrow[dd, "p_n"] \\ X \arrow[ur, "f_n"] \arrow[dr, "f_{n-1}"'] & \\ & X_{n-1} \end{tikzcd}$$

commute for each n.

For each n, the map $p_n : X_n \to X_{n-1}$ is a fibration with fiber the Eilenberg-MacLane space $K(\pi_n X, n)$. Moreover, $\pi_k X_n = 0$ for $k > n$ and $(f_n)_* : \pi_k X \to \pi_k X_n$ is an isomorphism for all $k \leq n$.

To avoid complications you may assume that X is simply connected.

Each fibration in the tower $p_n : X_n \to X_{n-1}$ has fiber $K(\pi_n X, n)$, and so there is a single obstruction (namely the primary obstruction) to finding a section. This obstruction lies in

$$H^{n+1}(X_{n-1}; \pi_n(K(\pi_n X, n))) = H^{n+1}(X_{n-1}; \pi_n X)$$

and is called the $(n+1)$*st k-invariant of X* and is denoted by k^{n+1}. Using the identification $H^{n+1}(X_{n-1}; \pi_n X) = [X_{n-1}, K(\pi_n, n+1)]$, k^{n+1} can be thought of as a (homotopy class of a) map $X_{n-1} \to K(\pi_n, n+1)$ so that the fibration $K(\pi_n, n) \hookrightarrow X_n \to X_{n-1}$ is the pullback of the path space fibration $K(\pi_n, n) \hookrightarrow P \to K(\pi_n, n+1)$ via k^{n+1}.

Thus to a (simple) path-connected space X this construction associates a collection $\{\pi_n, p_n, k^n\}$ where

1. π_n is an abelian group,
2. $p_n : X_n \to X_{n-1}$ is a fibration with fiber $K(\pi_n, n)$,
3. X_0 is contractible,
4. $k^n \in H^n(X_{n-2}; \pi_{n-1})$ classifies p_{n-1},
5. the inclusion of the fiber induces an isomorphism $\pi_n(K(\pi_n, n)) \to \pi_n X_n$.

This collection has the property that $\pi_n = \pi_n X$.

This data is called the *Postnikov system* or *Postnikov decomposition* for X.

Prove the main result about Postnikov systems.

Theorem 8.40. *The weak homotopy type of X is determined by its Postnikov system. More precisely, given the data $\{\pi_n, p_n, k^n\}$ satisfying conditions 1–5 above, there exists a space X with this data as its Postnikov decomposition. If Y is any space with this Postnikov decomposition, then X and Y are weakly homotopy equivalent.* □

Thus a space is completely determined up to homotopy inductively by the π_n and the k-invariants. More precisely, let $X_1 = K(\pi_1, 1)$ and let $p_1 : X_1 \to X_0 = \text{pt}$ be the constant map. Inductively, suppose $k^3, \cdots, k^n$ determine fibrations $p_\ell : X_\ell \to X_{\ell-1}$ for $\ell \leq n-1$, and suppose $k^{n+1} \in H^{n+1}(X_{n-1}; \pi_n)$ is given. Then define $p_n : X_n \to X_{n-1}$ to be the pullback of the path space fibration $K(\pi_n, n) \to P \to K(\pi_n, n+1)$ via the map $k^{n+1} : X_{n-1} \to K(\pi_n, n+1)$. If $\{\pi_n, k^n\}$ is the Postnikov system for a space X, then X is homotopy equivalent to $\lim X_n$.

A good reference for this material is [**54**, pp. 421–437] and [**18**, pp. 78-82].

If time permits, lecture on the dual exposition of obstruction theory from the point of view of Postnikov decompositions. Spanier's book [**45**] is one place to find this material.

Chapter 9

Bordism, Spectra, and Generalized Homology

Make bordism notation consistent in other chapters

This chapter contains a mixture of algebraic and differential topology and serves as an introduction to generalized homology theories. We will give a precise definition of a generalized homology theory later, but in the meantime you should think of a generalized homology theory as a functor from pairs of spaces to graded abelian groups (or graded R-modules) satisfying all Eilenberg–Steenrod axioms but the dimension axiom.

We introduce generalized homology theories and spectra by focusing on one type of generalized homology theory called bordism, a theory whose origins are due to Pontrjagin and Thom. In the 1950's they noted that in many situations there is a one-to-one correspondence between problems in geometric topology (= manifold theory) and problems in algebraic topology. Usually the algebraic problem is more tractable, and its solution leads to geometric consequences. For example, it is very difficult to classify smooth manifolds up to diffeomorphism, but the bordism classes of compact n-manifolds form an abelian group using disjoint union, and the abelian group can be computed using algebraic topology.

9.1. Manifolds, bundles, and bordism

The material in this chapter will draw on the basic notions and theorems of differential topology, and you should re-familiarize yourself with the notion of smooth maps between smooth manifolds, submanifolds, tangent bundles, orientation of a vector bundle, the normal bundle of a submanifold, Sard's theorem, transversality and the tubular neighborhood theorem. One of the projects for this chapter is to prepare a lecture on these topics. A good reference for this material is Hirsch's book [**22**]; more elementary references include [**33**] and [**20**]. For further information on bordism in general, see [**36**], [**48**], [**49**] and the references therein.

9.1.1. Manifolds, submanifolds, and bundles. In this chapter (in contrast to the rest of this book), the word "manifold" means a *compact, smooth* manifold with or without boundary, and "submanifold" $N \subset M$ means a compact smooth submanifold whose boundary (if nonempty) is contained in the boundary of M in such a way that N meets the boundary of M transversely.

The *normal bundle* of a submanifold $N \subset M$ is, by definition, the quotient bundle $(TM|_N)/TN \to N$. In other words, the vector space assigned to a point $p \in N$ is the quotient of the tangent space of M at p by the subspace of vectors tangent to the submanifold N itself. We use the notation $\nu_{N\subset M}$ or ν_N for the normal bundle. If we wish to emphasize the embedding $i : N \subset M$ we write $\nu(i)$.

If M is a submanifold of $\mathbf{R}^n$, or more generally if M has a Riemannian metric (i.e., the tangent bundle TM is equipped with a metric), then the normal bundle $\nu_{N\subset M}$ can be identified with the subbundle of $TM|_N$ consisting of all tangent vectors in T_pM which are perpendicular to T_pN, where $p \in N$. This can be expressed by saying that the restriction of the tangent bundle of M to N, $TM|_N$, decomposes as a Whitney sum (Definition 5.20)

$$TM|_N = TN \oplus \nu_{N\subset M}.$$

A succinct way to express these notions is that a submanifold $N \subset M$ determines a short exact sequence of vector bundles over N:

$$0 \to TN \to TM|_N \to \nu_{N\subset M} \to 0$$

and a choice of Riemannian metric on M determines a splitting of this sequence.

A *tubular neighborhood* of a submanifold $N \subset M$ is an embedding $f : \nu_{N\subset M} \to M$ which restricts to the identity on (the zero section) N. Informally, we say that the open set $U = f(\nu_N) \subset M$ is a tubular neighborhood of N.

The normal bundle of an embedding is determined up to isomorphism by the isotopy class of the embedding i. More precisely, if $i_0, i_1 : M \subset \mathbf{R}^N$ are isotopic embeddings, then an isotopy i_t determines a homotopy class of vector bundle isomorphisms $\nu_{i_0} \cong \nu_{i_1}$.

9.1.2. Motivating example: representing homology classes by manifolds and unoriented bordism. We introduce bordism informally and explore its relation to homology in this section. A careful definition is given later.

Poincaré duality (Theorem 4.32) asserts that if N is a closed connected n-dimensional manifold, then $H_n(N;\mathbf{Z}/2) \cong \mathbf{Z}/2$; the generator is denoted by $[N]$. Moreover, if W is a compact $(n+1)$-manifold with boundary N, the connecting homomorphism in the long exact sequence for the pair (W,N), $H_{n+1}(W,N;\mathbf{Z}/2) \to H_n(N;\mathbf{Z}/2)$ is an isomorphism, and therefore the inclusion $N \subset W$ induces the zero map $H_n(N;\mathbf{Z}/2) \to H_n(W;\mathbf{Z}/2)$. Hence any continuous map $f : N \to X$ to a space X determines a homology class $f_*[N] \in H_n(X;\mathbf{Z}/2)$, and if f extends to $F : W \to X$, then $f_*[N] = 0$. These assertions hold with $\mathbf{Z}$ coefficients if N and W are oriented.

Exercise 170. Show, more generally, that if W is a compact $(n+1)$-dimensional manifold with two boundary components $\partial W = N_0 \amalg N_1$ and $F : W \to X$ a continuous map, with restrictions $f_i : N_i \to X,\ i = 1, 2$, then $(f_0)_*[N_0] = (f_1)_*[N_1]$ in $H_n(X;\mathbf{Z}/2)$.

Exercise 170 hints at a possible geometric approach to defining the homology of X: one might consider equivalence classes of functions $f : N \to X$ where N is a smooth compact n-dimensional manifold without boundary. Set f_0 to be equivalent to f_1 provided a $F : W \to X$ as as in Exercise 170 exists. This equivalence relation is called *unoriented bordism*. Denote the set of unoriented bordism classes by $\Omega_n^{\mathbf{O}}(X)$.

We will define $\Omega_n^{\mathbf{O}}(X)$ more carefully (and rigorously) below, and show that $\Omega_n^{\mathbf{O}}(X)$ is a 2-torsion abelian group, called the *n-dimensional unoriented bordism group of X.*

Exercise 170 and the discussion which precedes it shows that there is a well-defined function

$$\Omega_n^{\mathbf{O}}(X) \to H_n(X;\mathbf{Z}/2),\ (N,f) \mapsto f_*[N]. \tag{9.1}$$

It is reasonable to ask whether this function is an isomorphism of groups for all n and X, i.e. whether unoriented bordism is equivalent to $\mathbf{Z}/2$ homology. If it were, this would give a manifold-theoretic construction of homology which avoids singular simplices or Eilenberg-MacLane spaces.

However the homomorphism of Equation 9.1 is not an isomorphism. For example, taking $X = \{p\}$ to be a point and $n = 2$, although $H_2(p;\mathbf{Z}/2) = 0$, the set $\Omega_2^{\mathbf{O}}(p)$ contains at least (in fact exactly) two equivalence classes. Indeed the 2-sphere S^2 is the boundary of a 3-ball D^3, but the projective plane $\mathbf{R}P^2$ is not the boundary of any compact 3-manifold, so that the bordism classes of the constant maps $S^2 \to \{p\}$ and $\mathbf{R}P^2 \to \{p\}$ are different.

To see that $\mathbf{R}P^2$ is not a boundary, suppose that $\mathbf{R}P^2 = \partial W$ for some compact 3-manifold W. The long exact sequence in homology with $\mathbf{Z}/2$ coefficients for the pair $(W, \partial W)$ shows that the Euler characteristics satisfy $\chi(\partial W) - \chi(W) + \chi(W, \partial W) = 0$. Poincaré-Lefschetz duality (Theorem 4.32) shows that $\chi(W, \partial W) = -\chi(W)$, so that $\chi(\partial W) = 2\chi(W)$. But this contradicts $\chi(\mathbf{R}P^2) = 1$.

Thus bordism and homology are different functors. So what kind of functor is unoriented bordism? It turns out to be a *generalized homology theory* (defined carefully later in this chapter), which essentially means that it satisfies all the Eilenberg-Steenrod axioms (Definition 2.17) except the Dimension Axiom.

Remark. We use the algebraic topology terminology where *cobordism* is the theory dual (in the Spanier-Whitehead sense) to *bordism.* It is traditional for geometric topologists to call bordant manifolds "cobordant," but we will avoid this terminology in this book.

9.2. Bordism over a vector bundle

Before we return to the topic of bordism as a generalized homology theory, it is useful to first study unstable bordism which, like homotopy groups, is not a generalized homology functor, but contains most of the essential ideas, unencumbered by the notational issues required in the stable case. Rather than working with abstract manifolds, one can work with submanifolds of Euclidean space. The normal bundle, ν_N, of an n-dimensional submanifold $N^n \subset \mathbf{R}^{n+k}$ is an Euclidean $\mathbf{R}^k$-vector bundle over N.

Definition 9.1. Fix a space B and a rank k vector bundle $\gamma : E \to B$. A γ-*structure* on a compact smooth n-dimensional submanifold without boundary, $N^n \subset \mathbf{R}^{n+k}$, is a bundle map (a commutative diagram such that $\tilde{g}$ is a linear isomorphism in each fiber)

$$\begin{tikzcd} \nu_N \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d, "\gamma"] \\ N \arrow[r, "g"'] & B \end{tikzcd}$$

We denote this bundle map by $(\tilde{g}, g)$.

We now introduce the notion of γ-bordism. In parsing the following definition, it is helpful to note that if Z is a $(n+k+1)$-dimensional manifold with boundary and W is an $(n+1)$-dimensional submanifold with boundary (i.e. $\partial W \subset \partial Z$ and W is transverse to ∂Z), then the normal bundle of the restriction $\partial W \subset \partial Z$ is equal to the restriction of ν_W to ∂W:

$$\nu_{\partial W} = \nu_W|_{\partial W}.$$

This observation is applied in the case $Z = \mathbf{R}^{n+k} \times [0,1]$.

Definition 9.2. Fix a $\mathbf{R}^k$-vector bundle $\gamma : E \to B$ over a space B. The γ-*bordism set*, denoted by Ω_n^γ, consists of equivalence classes of pairs $(N, (\tilde{g}, g))$ where N is a compact n-dimensional smooth submanifold of $\mathbf{R}^{n+k}$, and $(\tilde{g}, g)$ a γ-structure on its normal bundle ν_N.

The equivalence relation of bordism is defined as follows. Two pairs $(N_0, (\tilde{g}_0, g_0))$ and $(N_1, (\tilde{g}_1, g_1))$ are called γ-*bordant* provided there exists a pair $(W, (\tilde{G}, G))$, where $W \subset \mathbf{R}^{n+k} \times [0,1]$ is a compact smooth $(n+1)$-dimensional submanifold with boundary

$$\partial W = N_0 \times \{0\} \amalg N_1 \times \{1\}$$

and $(\tilde{G}, G)$ is a γ-structure on ν_W compatible with $(\tilde{g}_i, g_i)$:

$$\begin{tikzcd}
\nu_{N_0} \amalg \nu_{N_1} \arrow[r, "="] \arrow[d] \arrow[rrr, bend left, "\tilde{g}_0 \amalg \tilde{g}_1"] & \nu_{\partial W} \arrow[r, hook] \arrow[d] & \nu_W \arrow[r, "\tilde{G}"] \arrow[d] & E \arrow[d, "\gamma"] \\
N_0 \amalg N_1 \arrow[r, "="] \arrow[rrr, bend right, "g_0 \amalg g_1"'] & \partial W \arrow[r, hook] & W \arrow[r, "G"] & B
\end{tikzcd}$$

Notice that if $\gamma : E \to B$ and $\gamma' : E' \to B'$ are rank k vector bundles and

$$\begin{tikzcd}
E' \arrow[r, "\tilde{f}"] \arrow[d, "\gamma'"'] & E \arrow[d, "\gamma"] \\
B' \arrow[r, "f"] & B
\end{tikzcd} \tag{9.2}$$

a bundle map, then there is an induced function

$$\Omega_n^{\gamma'} \to \Omega_n^\gamma. \tag{9.3}$$

In particular, if $f : B' \to B$ a map, then the pullback diagram

$$\begin{tikzcd}
f^*(E) \arrow[r] \arrow[d, "f^*(\gamma)"'] & E \arrow[d, "\gamma"] \\
B' \arrow[r, "f"] & B
\end{tikzcd}$$

is a bundle map and hence induces a function $\Omega_n^{f^*(\gamma)} \to \Omega_n^\gamma$.

This functoriality is used to define $\Omega_n^\gamma(X)$ for any space X. Let $p : X \times B \to B$ denote the projection onto the second factor, and let $p^*(\gamma) : p^*(E) \to X \times B$ denote the pullback bundle. Explicitly, $p^*(\gamma)$ is the product bundle

$$\mathrm{Id}_X \times \gamma : X \times E \to X \times B.$$

Define

$$\Omega_n^\gamma(X) = \Omega_n^{p^*(\gamma)}.$$

This exhibits Ω_n^γ as a functor Top $\to$ Ab.

9.3. Thom spaces, bordism, and homotopy groups

Definition 9.3. Fix a space B and an Euclidean rank k vector bundle $\gamma : E \to B$. Define its disk bundle $D(\gamma) = \{x \in E \mid \|x\| \leq 1\}$ and its sphere bundle $S(\gamma) = \{x \in E \mid \|x\| = 1\}$.

The *Thom space of* $\gamma : E \to B$ is the based space

$$T(\gamma) := D(\gamma)/S(\gamma),$$

where the equivalence class of $S(\gamma)$ serves as the base point for $T(\gamma)$.

The Thom space does not really depend on the metric. Recall that if the base space of a vector bundle is paracompact (e.g. if the base space is a CW-complex), then every rank k vector bundle admits an $O(k)$-structure, unique up to isomorphism. (See Exercise 101.) Also, any $(O(k), \mathbf{R}^k)$-bundle admits a metric, unique up to scaling if the base space is connected. (See Section 5.3.5.) It follows that the homeomorphism class of the Thom space $T(\gamma)$ is independent of the choice of the metric provided the base space is paracompact.

In fact, one can define the Thom space of an arbitrary vector bundle $\gamma : E \to B$. If B is compact Hausdorff, then one simply defines the Thom space to be the one-point compactification of E. When B is noncompact, one defines the Thom space to be the fiberwise one-point compactification of E modulo the copy of B at ∞. More precisely, identify S^k with $\mathbf{R}^k \cup \{\infty\}$ using stereographic projection. Note that $GL(k, \mathbf{R})$ acts on S^k fixing the points $\{0\}$ and $\{\infty\}$. Change the fiber of γ from $\mathbf{R}^k$ to S^k (see Section 5.5), i.e. consider the S^k-bundle

$$\begin{array}{c} F(E) \times_{GL(k,\mathbf{R})} S^k \\ \big\downarrow \\ B \end{array}$$

where $F(E)$ is the principal $GL(k, \mathbf{R})$-bundle (the frame bundle) associated to the vector bundle γ. Then define $T(\gamma) = (F(E) \times_{GL(k,\mathbf{R})} S^k)/(F(E) \times_{GL(k,\mathbf{R})} \{\infty\})$.

However, it is best to think of the Thom space as $T(\gamma) = D(\gamma)/S(\gamma)$.

Notice that the zero section $z : B \to E$ defines an embedding of B into the Thom space $T(\gamma)$. The construction of Thom spaces is functorial. A bundle map $(\tilde{g}, g)$ as in Equation (9.2), determines a continuous, base point preserving map on Thom spaces

$$T(\gamma') \to T(\gamma)$$

which takes the zero section B' to B.

Exercise 171. Suppose $\gamma : E \to B$ and $\gamma' : E' \to B'$ are vector bundles. Show that $T(\gamma \times \gamma') = T(\gamma) \wedge T(\gamma')$ (Assume the vector bundles are Euclidean for simplicity.) The Thom space of the $\mathbf{R}^0$-bundle $\mathrm{Id} : B \to B$ is $B/\emptyset$, which is defined to be B_+, which denotes the disjoint union of B with a base point $+$. Deduce that the Thom space of the trivial bundle $B \times \mathbf{R}^n \to B$ is the n-fold reduced suspension $S^n(B_+)$, and that if $\underline{\mathbf{R}}^n$ denotes the trivial bundle over B, the Thom space of the Whitney sum $\gamma \oplus \underline{\mathbf{R}}^n$ is $S^n(T(\gamma))$.

For a compact submanifold N of $\mathbf{R}^{n+k}$, there is the *Pontrjagin-Thom collapse map*

$$c : S^{n+k} \to T(\nu_N),$$

which sends everything outside of a closed tubular neighborhood of $N \subset S^{n+k} = \mathbf{R}^{n+k} \cup \{\infty\}$ to the base point. If $f : \nu_N \hookrightarrow S^{n+k}$ is the embedding given by the tubular neighborhood theorem, then for $x \in D(\nu_N)$ one defines $c(f(x)) = [x] \in T(\nu_N) = D(\nu_N)/S(\nu_N)$. When we write $D(\nu_N)$, $S(\nu_N)$, and $T(\nu_N)$, we abuse notation a bit, implicitly using ν_N for both the total space of the normal bundle as well as the projection map.

Now suppose N is a compact submanifold of $\mathbf{R}^{n+k}$, equipped with a γ-structure on its normal bundle ν_N. There is then an induced continuous map on Thom spaces

$$T(\nu_N) \to T(\gamma)$$

which takes the zero section N to B.

Combining the last two paragraphs one concludes that to a pair $(N, (\tilde{g}, g))$ representing a class in Ω_n^γ, one can assign the composite of the Pontrjagin-Thom collapse and the induced map on Thom spaces

$$S^{n+k} \xrightarrow{c} T(\nu_N) \to T(\gamma). \tag{9.4}$$

Theorem 9.4 (Unstable Pontrjagin-Thom construction)**.** *Assigning the map of Equation (9.4) to a pair $(N, (\tilde{g}, g))$ induces a bijection*

$$\Omega_n^\gamma \to \pi_{n+k}(T(\gamma)).$$

Sketch of proof. We outline the proof of Theorem 9.4 in the special case when E and B are smooth manifolds and $\gamma : E \to B$ a smooth vector bundle, that is, when the composites

$$\psi^{-1} \circ \phi : (U_\phi \cap U_\psi) \times \mathbf{R}^k \cong \gamma^{-1}(U_\phi \cap U_\psi) \cong (U_\phi \cap U_\psi) \times \mathbf{R}^k$$

are diffeomorphisms. This special case is sufficient to handle most flavors of bordism.

First, note that a collapse map $S^{n+k} \times [0,1] \to T(\gamma)$ can be similarly defined for a bordism $(W, (\tilde{G}, G))$ with $W \subset S^{n+k} \times [0,1]$, showing that the map of Equation (9.4) is well-defined on bordism classes.

To define the inverse to map (9.4) requires transversality.

Choose $\alpha : S^{n+k} \to T(\gamma)$ representing a homotopy class in $\pi_{n+k}(T(\gamma))$. For each $0 < \delta < 1$, the open disc bundle $D^\circ_\delta(E)$ is an open subset of $T(\gamma)$. Let

$$U^\delta := \alpha^{-1}(D^\circ_\delta(E)) \subset S^{n+k}.$$

Then U^δ is an open set in S^{n+k}, hence a smooth manifold. Fix $0 < \varepsilon < \delta < 1$. The standard transversality and approximation theorems (see e.g. [**22, 33, 20**]) imply that the restriction $\alpha| : U^\delta \to D^\circ_\delta(E)$ is homotopic to a new map α' so that α' agrees with α on $U^\delta - U^\varepsilon$, and the restriction of α' to U^ε is smooth and transverse to the zero section $Z = z(B) \subset D^\circ_\delta(E)$. Extend α' to $\alpha' : S^{n+k} \to T(\gamma)$ by taking $\alpha'(x) = \alpha(x)$ for $x \notin U^\varepsilon$.

Transversality now implies that $N = (\alpha')^{-1}Z \subset U^\varepsilon$ is a smooth compact manifold without boundary and that the differential of α' on U^ε determines a γ-structure $(d\alpha', \alpha')$ on ν_N. This provides an inverse map to the map of Theorem 9.4.

The proof that homotopic maps α_0 and α_1 determine bordant pairs $(N_0, (d\alpha_0, \alpha_0))$, $(N_1, (d\alpha_1, \alpha_1))$ follows by a similar argument applied to a homotopy $\alpha : S^{n+k} \times [0,1] \to T(\gamma)$. □

This method of translating between bordism and homotopy sets is called the *Pontrjagin–Thom construction*.

Remark. The general case of Theorem 9.4 is proven in Chapter II of Stong [**48**]. Here is an outline of the proof. Let $\gamma : E \to B$ be a rank k vector bundle; we will assume that B is a finite CW complex to simplify the argument. The vector bundle is classified by a map to the Grassmannian

$$\begin{tikzcd} E \arrow[r, "\tilde{h}"] \arrow[d] & \gamma_k \arrow[d] \\ B \arrow[r, "h"] & G_k(\mathbf{R}^\infty) \end{tikzcd}$$

where γ_k is the canonical k-plane bundle (see Corollary 7.55). Since $h(B)$ is compact, it intersects a finite number of cells, hence is contained in $G_k(\mathbf{R}^N)$ for some $N < \infty$. Next we assume the map $h : B \to G_k(\mathbf{R}^N)$ is a fibration, by replacing with the mapping path fibration if necessary (see Theorem 7.23.) Fix $0 < \varepsilon < \delta < 1$ and let $U^\varepsilon \subset U^\delta \subset S^{n+k}$ be the inverse images of the open disk bundles (of radii ε and δ) of γ_k under the map $h \circ g$. Then, by transversality, $h \circ g| \simeq l : U^\delta \to D^\circ_\delta(\gamma_k)$ where the homotopy is constant outside U^ε and l is transverse to the zero section $G(k, \mathbf{R}^N) \subset D^\circ_\delta(\gamma_k)$. (Here we used that $G_k(\mathbf{R}^N)$ is a smooth manifold.) Since the restricted map $h^{-1}(D^\circ_\delta(\gamma_k)) \to D^\circ_\delta(\gamma_k)$ is also a fibration, and $g : U^\delta \to h^{-1}(D^\circ_\delta(\gamma_k))$ is a lift of $h \circ g$, the homotopy and hence l lifts to a map $\hat{l} : U^\delta \to h^{-1}(D^\circ_\delta(\gamma_k))$. Then $N = \hat{l}^{-1}B = l^{-1}(G_k(R^N))$ is the desired manifold with a γ-structure.

Note that Theorem 9.4 implies that Ω_n^γ is an abelian group (at least when $n+k \geq 2$). Chasing through the construction of addition in homotopy groups and the Pontrjagin-Thom collapse, one sees that on the bordism side, the zero element is represented by the empty n-dimensional submanifold of $\mathbf{R}^{n+k}$ (the based constant map $\alpha : S^{n+k} \to T(\gamma)$ has $\alpha^{-1}(z(B))$ empty), and the addition $(N_0, (\tilde{g}_0, g_0)) + (N_1, (\tilde{g}_1, g_1))$ is represented by first translating N_1 so that N_0 and N_1 lie in different half-spaces in $\mathbf{R}^{n+k}$ (translation can be achieved by an isotopy, hence a bordism), and then taking the disjoint union of N_0 and (the translated) N_1. The negative of a bordism class $(N, (\tilde{g}, g))$ is represented by $(r(N), (\tilde{g} \circ dr, g \circ r))$, where $r : \mathbf{R}^{n+k} \to \mathbf{R}^{n+k}$ is reflection through a hyperplane which misses N.

Exercise 172. Provide the details that this describes the abelian group structure on Ω_n^γ corresponding, via the Pontrjagin-Thom construction, to addition in the homotopy group $\pi_{n+k}(T(\gamma))$.

Given a space X, the Pontrjagin-Thom construction identifies $\Omega_n^\gamma(X)$ with $\pi_{n+k}(T(p^*(\gamma)))$, where $p : X \times B \to B$ denotes the projection to the second factor. This can be made more explicit using the half-smash.

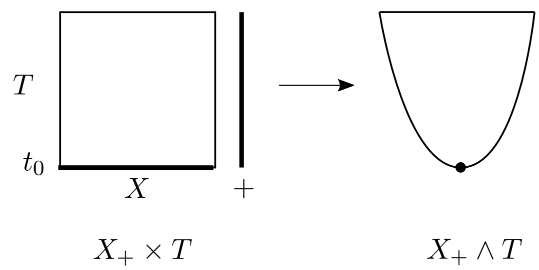
Let X_+ denote the disjoint union of X and a base point $+$. Given any based space (T, t_0), the smash product

$$X_+ \wedge T = (X_+ \times T)/(X_+ \vee T) = (X \times T)/(X \times \{t_0\})$$

is called the *half smash* of X and T and is depicted in the following picture.

Exercise 173. Let $p : X \times B \to B$ denote the projection to the second factor. Show that for any vector bundle γ over B, there is an identification of Thom spaces:

$$T(p^*(\gamma)) = X_+ \wedge T(\gamma).$$



It follows from Exercise 173 and Theorem 9.4 that the Pontrjagin-Thom collapse induces a natural isomorphism:

$$\Omega_n^\gamma(X) \to \pi_{n+k}(X_+ \wedge T(\gamma)). \tag{9.5}$$

There is a homomorphism from the γ-bordism group $\Omega_n^\gamma(X)$ to the $\mathbf{Z}/2$ homology of X defined in the following manner. Given a representative $(N, (\tilde{g}, g))$ of a class in $\Omega_n^\gamma(X)$, the composite $N \xrightarrow{g} X \times B \xrightarrow{\text{proj}_1} X$ determines a class $(\text{proj}_1 \circ g)_*[N] \in H_n(X; \mathbf{Z}/2)$. This defines a natural homomorphism

$$\Omega_n^\gamma(X) \to H_n(X; \mathbf{Z}/2). \tag{9.6}$$

Finally, notice that the entire previous discussion can be carried out, replacing $S^{n+k} = \mathbf{R}^{n+k} \cup \{\infty\}$ by the one-point compactification $\widehat{M}$ of an arbitrary $(n+k)$-dimensional smooth manifold M. The set $\Omega_{n,M}^\gamma(X)$ of γ-bordism classes of compact smooth n-dimensional submanifolds $N \subset M$ is identified, via the Pontrjagin-Thom collapse, with the based homotopy set $[\widehat{M}, X_+ \wedge T(\gamma)]_0$. In general, $\Omega_{n,M}^\gamma(X)$ does not admit a natural group structure.

9.3.1. Example: the trivial bundle. A simple, but quintessential case (a reference is the last section of Milnor's beautiful little book [**33**]) of γ-bordism is to take the (trivial) rank k vector bundle over a point $\text{fr} : \mathbf{R}^k \to \{b\}$. Then the Thom space $T(\text{fr})$ is $D^k/S^{k-1} \cong S^k$, so that Theorem 9.4 implies that

$$\Omega_n^{\text{fr}} = \pi_{n+k}(S^k),$$

and more generally, that

$$\Omega_n^{\text{fr}}(X) = \pi_{n+k}(X_+ \wedge S^k),$$

In this setting, a fr-structure is the same thing as *trivialization of* ν_N, since pulling back the standard basis of $\mathbf{R}^k$ using $\tilde{g}$ yields a *normal trivialization*, that is, k pointwise linearly independent sections of ν_N. If two trivializations of ν_N are joined by a 1-parameter family of trivializations, we call them homotopic, and homotopy classes of trivializations of ν_M are called *normal framings*. Since homotopic trivializations of ν_N determine a trivialization of the normal bundle of $N \times [0,1]$ in $\mathbf{R}^{n+k}$, fr-bordism classes only depend on normal framings, and we call Ω_n^{fr} the group of *normally framed n-dimensional bordism classes in* $\mathbf{R}^{n+k}$.

More generally, if M is a smooth $(n+k)$-dimensional manifold, the set $\Omega_{n,M}^{\text{fr}}(X)$ of normally framed bordism classes of normally framed n-dimensional submanifolds of M is identified, via the Pontrjagin-Thom collapse, with the homotopy set $[\widehat{M}, X_+ \wedge S^k]_0$.

Exercise 174. A submanifold $N^n \subset M^{n+k}$ admits a normal framing if and only if its normal bundle is trivial if and only if the inclusion $N \subset M$ extends to an embedding $\phi : N \times \mathbf{R}^k \hookrightarrow M$.

So for example, if $M = \mathbf{R}P^2$ and $N \subset M$ is a circle representing the nontrivial generator of $\pi_1 M$, then the tubular neighborhood of N is a Möbius band, and hence does not admit a normal framing. In particular it does not represent an element of $\Omega_{1,M}^{\text{fr}}$.

The Pontrjagin-Thom identification $\Omega_n^{\text{fr}} = \pi_{n+k} S^k$ gives a geometric interpretation for elements of the homotopy groups of spheres. Here are some examples (without proof) to help your geometric insight. A normally framed point in $\mathbf{R}^k$ gives, via the Pontrjagin-Thom construction, a map $S^k \to S^k$ which generates $\pi_k S^k \cong \mathbf{Z}$. Any normally framed circle in $\mathbf{R}^2 \subset S^2$ is null-bordant: for example the equator with the obvious framing is the boundary of the 2-disk in the 3-ball. Thus $\pi_2 S^1 = 0$. However, a framed circle $S^1 \subset \mathbf{R}^3$ so that the pushed-off circle $\phi(S^1 \times \{(1,0)\})$ (where $\phi : S^2 \times \mathbf{R}^2 \to \mathbf{R}^3$ is the corresponding embedding of Exercise 174) links the S^1 with linking number 1 represents the generator of $\pi_3 S^2 \cong \mathbf{Z}$. (Can you reinterpret this in terms of the Hopf map? Why can't one see the complexities of knot theory?)

Now $\mathbf{R}^3$ is normally framed in $\mathbf{R}^4$, $\mathbf{R}^4$ in $\mathbf{R}^5$, etc. so we can suspend the linking number 1 framing of S^1 in $\mathbf{R}^3$ to get a normal framing of S^1 in $\mathbf{R}^{k+1}$ for $k > 2$. This represents the generator of $\pi_{k+1} S^k \cong \mathbf{Z}_2$.

More generally, one can produce examples of normally framed manifolds by twisting and suspending. If $(N^n, \phi : N \times \mathbf{R}^k \hookrightarrow M^{n+k})$ is a normally framed submanifold and $\alpha : N \to O(k)$, then the *twist* is the normally framed submanifold $(N, \phi.\alpha)$ where $\phi.\alpha(p,v) = \phi(p, \alpha(p)v)$. The normally framed bordism class depends only on (N, ϕ) and the homotopy class of α. (See Exercise 175 below for more on this construction.)

Next, if $N^n \subset \mathbf{R}^{n+k}$ is normally framed, with corresponding embedding $\phi : N \times \mathbf{R}^k \to \mathbf{R}^{n+k}$, then the *suspension* of (N^n, ϕ) is the framed submanifold $(N^n, S\phi)$ of $\mathbf{R}^{n+k+1}$, defined using the obvious 1-dimensional normal framing of $\mathbf{R}^{n+k}$ in $\mathbf{R}^{n+k+1}$ to extend the normal framing $\phi : N \times \mathbf{R}^k \to$

$\mathbf{R}^{n+k}$ to $S\phi : N \times \mathbf{R}^{k+1} \to \mathbf{R}^{n+k+1}$ The generator of $\pi_3 S^2$ mentioned earlier can be described by first suspending the inclusion of a normally framed circle in $\mathbf{R}^2$ and then twisting by the inclusion of the circle in $O(2)$.

Exercise 175. (The J-homomorphism) Let (N^n, ϕ) be a nonempty normally framed submanifold of M^{n+k}. Use twisting to define a function

$$J_\phi : [N, O(k)] \to [\widehat{M}^{n+k}, S^k]_0.$$

Now let N be the equatorial $S^n \subset S^{n+k}$ with the canonical normal framing coming from the iterated inclusions $S^n \subset S^{n+1} \subset \cdots \subset S^{n+k}$, and show that the function

$$J : \pi_n O(k) \to \pi_{n+k} S^k$$

is a homomorphism provided $n > 0$. It is called the *J-homomorphism* and can be used to construct interesting elements in $\pi_{n+k} S^k$.

Draw an explicit picture of a normally framed circle in $\mathbf{R}^3 = S^3 - \{\infty\}$ representing $J(\iota)$ where $\iota \in \pi_1 O(2) = \mathbf{Z}$ is the generator.

Theorem 9.5 (Hopf degree theorem)**.** *Let M^k be a connected, closed, smooth manifold.*

1. *If M^k is orientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree.*
2. *If M^k is nonorientable, then two maps $M^k \to S^k$ are homotopic if and only if they have the same degree mod 2.*

Exercise 176. Prove the Hopf degree theorem in two ways: obstruction theory and normally framed bordism. (See also Milnor [**33**].)

The result that $\pi_n S^n \cong \mathbf{Z}$ is a nontrivial result in algebraic topology; it is cool that this can be proven using differential topology.

As we have already encountered in Chapter 7, computing $\pi_{n+k} S^k = \Omega_n^{\mathrm{fr}}$ is in general difficult. We have seen that $\pi_k S^k \cong \mathbf{Z}$ and $\pi_3 S^2 \cong \mathbf{Z}$. A few calculations of $\pi_{n+k} S^k$ for various n and k, as well as a proof that these are finitely generated abelian groups, can be found in Chapter 10.

9.4. Suspension and the Freudenthal theorem

Recall that the (reduced) suspension of a space $X \in \mathrm{CGH}_*$ with nondegenerate base point is the space

$$SX = X \times I/\sim$$

where the subspace $(x_0 \times I) \cup (X \times \{0, 1\})$ is collapsed to a point. This construction is functorial with respect to based maps $f : X \to Y$. In particular, the suspension defines a function

$$S : [X, Y]_0 \to [SX, SY]_0.$$

By Proposition 7.40, $SS^k = S^{k+1}$, so that when $X = S^k$, the suspension defines a function, in fact a homomorphism

$$S : \pi_k Y \to \pi_{k+1}(SY)$$

for any space Y. Taking Y to be a sphere one obtains

$$S : \pi_k S^n \to \pi_{k+1} S^{n+1}.$$

The suspension isomorphism in the homology of a space X is the composite of the isomorphisms

$$E : \widetilde{H}_k X \xleftarrow{\partial} H_{k+1}(CX, X) \xrightarrow{c_*} H_{k+1}(SX, *) = \widetilde{H}_{k+1}(SX)$$

where ∂ is the connecting homomorphism for the long exact sequence of the pair (CX, X) and $c : CX \to SX = CX/X$ is the collapse map. Note that $\widetilde{H}_k X = H_k X$ for $k > 0$.

The Hurewicz map ρ from homotopy to homology was discussed in Section 7.17.

Proposition 9.6. *The Hurewicz map commutes with suspension, i.e. for* $k > 0$*, the diagram*

$$\begin{tikzcd} \pi_k X \arrow[r, "S"] \arrow[d, "\rho"] & \pi_{k+1}(SX) \arrow[d, "\rho"] \\ H_k X \arrow[r, "E"] & H_{k+1}(SX) \end{tikzcd}$$

commutes.

Proof. There is a commutative diagram

$$\begin{tikzcd} \pi_k X \arrow[d, "\rho"] & \pi_{k+1}(CX, X) \arrow[l, "\partial", "\cong"'] \arrow[r, "c_*"] \arrow[d, "\rho"] & \pi_{k+1}(SX) \arrow[d, "\rho"] \\ H_k X & H_{k+1}(CX, X) \arrow[l, "\partial", "\cong"'] \arrow[r, "c_*"] & H_{k+1}(SX) \end{tikzcd}$$

Since CX is contractible, the maps labelled ∂ are isomorphisms. We know that the map $c_* \circ \partial^{-1}$ is the suspension in homology; to complete the proof we now claim the same in homotopy. From the definition of $\partial : \pi_{k+1}(CX, X) \to \pi_k X$, one sees that its inverse takes a homotopy class $\alpha : S^k \to X$ to the homotopy class $\partial^{-1}(\alpha) : (D^{k+1}, S^k) \to (CX, X)$ given by $tv \mapsto (f(v), t) \in CX$, for $t \in [0, 1]$, and $v \in S^k$. Composing ∂^{-1} with the collapse map $c : CX \to SX$ yields the suspension homomorphism on homotopy groups, $S : \pi_k X \to \pi_{k+1}(SX)$. $\square$

Note that the suspension map need not give an isomorphism on homotopy groups; this shows that homotopy groups do not satisfy the Eilenberg-MacLane axioms. For example, neither $S : \pi_2 S^1 \to \pi_3 S^2$ ($0 \to \mathbf{Z}$) nor $S : \pi_3 S^2 \to \pi_4 S^3$ ($\mathbf{Z} \to \mathbf{Z}/2$) are isomorphisms.

The following fundamental result is the starting point for the investigation of "stable" phenomena in homotopy theory. We will not give a proof at this time, since a spectral sequence proof is the easiest way to go. The proof is given in Section 11.3.

Theorem 9.7 (Freudenthal suspension theorem)**.** *Suppose that X is an $(n-1)$–connected space $(n \geq 2)$. Then the suspension homomorphism*

$$S : \pi_k X \to \pi_{k+1}(SX)$$

is an isomorphism if $k < 2n-1$ and an epimorphism if $k = 2n-1$. □

In the case where X is a sphere, this has a geometric interpretation. For example, the epimorphism result corresponds to the geometric fact that any closed, normally framed submanifold N^n of $\mathbf{R}^{2n+2}$ is isotopic (and therefore bordant) to a normally framed submanifold of $\mathbf{R}^{2n+1}$.

Exercise 177. Show that for any k-dimensional CW-complex X and for any $(n-1)$-connected space Y $(n \geq 2)$ the suspension map

$$[X, Y]_0 \to [SX, SY]_0$$

is bijective if $k < 2n-1$ and surjective if $k = 2n-1$. (Hint: Instead consider the map $[X, Y]_0 \to [X, \Omega SY]_0$. Convert the map $Y \to \Omega SY$ to a fibration and apply obstruction theory as well as the Freudenthal suspension theorem.)

For a based space X, $\pi^n X = [X, S^n]_0$ is called the *n-th cohomotopy set*. If X is a CW-complex with dim $X < 2n-1$, then Exercise 177 implies that $\pi^n X$ is a group, with group structure given by suspending and using the suspension coordinate in SX. The reader might ponder the geometric meaning (normally framed bordism) of the cohomotopy group structure when X is a manifold.

Definition 9.8. The k-th *stable homotopy group* of a based space X is the colimit

$$\pi_k^S X = \operatorname*{colim}_{\ell \to \infty} \pi_{k+\ell} S^\ell X.$$

The *stable k-stem* is

$$\pi_k^S = \pi_k^S S^0.$$

The computation of the stable k-stem for all k is the holy grail of the field of homotopy theory. Proposition 9.6 implies that there is a well-defined *stable Hurewicz map*

$$\rho : \pi_k^S X \to H_k X.$$

The Hurewicz theorem implies that if X is $(n-1)$-connected, then SX is n-connected, since $\widetilde{H}_\ell(SX) = \widetilde{H}_{\ell-1}X = 0$ if $\ell \leq n$ and $\pi_1(SX) = 0$ if X is path-connected. The following corollary follows from this fact and the Freudenthal theorem.

Corollary 9.9. *If X is path-connected,*

$$\pi_k^S X = \pi_{2k}(S^k X) = \pi_{k+\ell}(S^\ell X) \qquad \text{for } \ell \geq k.$$

For the stable k-stem,

$$\pi_k^S = \pi_{2k+2} S^{k+2} = \pi_{k+\ell} S^\ell \qquad \text{for } \ell \geq k+2.$$

□

Recall from Equation (7.4) that $\pi_k O(n-1) \to \pi_k O(n)$, induced by the inclusion $O(n-1) \hookrightarrow O(n)$, is an isomorphism for $k < n-2$, and therefore letting $O = \operatorname{colim}_{n\to\infty} O(n)$, $\pi_k O = \pi_k O(n)$ for $k < n-2$. It follows from the definitions that the following diagram commutes:

$$\begin{tikzcd}
\pi_k O(n-1) \arrow[r, "J"] \arrow[d, "i_*"'] & \pi_{k+n-1}(S^{n-1}) \arrow[d, "s"] \\
\pi_k O(n) \arrow[r, "J"] & \pi_{k+n}(S^n)
\end{tikzcd}$$

with the horizontal maps the J-homomorphisms, the left vertical map induced by the inclusion, and the right vertical map the suspension homomorphism. If $k < n-2$, then both vertical maps are isomorphisms, and so one obtains the *stable J-homomorphism*

$$J : \pi_k O \to \pi_k^S.$$

9.5. Bordism, stable normal bundles and suspension

Returning to bordism, in Section 9.2 we constructed, for any rank k vector bundle γ, a functor $\Omega_n^\gamma : \mathrm{Top} \to \mathrm{Ab}$. In Section 9.3 the Pontrjagin-Thom collapse was used to identify $\Omega_n^\gamma(X)$ with $\pi_{n+k}(X_+ \wedge T(\gamma))$. Suspension defines a homomorphism $\pi_{n+k}(X_+ \wedge T(\gamma)) \to \pi_{n+k+1}(S(X_+ \wedge T(\gamma)))$. We next describe the stabilization process on the bordism side of the Pontrjagin-Thom construction. Stabilizing leads to an notion of bordism, generalizing the construction of Section 9.1.2, which eliminates need to consider embedded submanifolds, and to get a bordism functor which behaves nicely (i.e. is a generalized homology theory). It also formalizes the notion of suspension discussed in Section 9.3.1 in the context of normally framed bordism.

Notation. In the following, whenever the base space Y is understood, we will use the $\underline{\mathbb{R}}^k$ as shorthand for the trivial rank k (real) vector bundle $\mathbf{R}^k \times Y \to Y$.

A basic consequence of transversality is that every compact n-manifold N smoothly embeds in $\mathbf{R}^{2n+1}$. Moreover, any two embeddings $i_0, i_1 : N \subset \mathbf{R}^{2n+3}$ are *isotopic*, i.e. there exists a path $i_t,\ t \in [0,1]$ of embeddings joining i_0 to i_1.

Given an n-dimensional submanifold $N \subset \mathbf{R}^{n+k}$, the inclusion $\mathbf{R}^{n+k} \subset \mathbf{R}^{n+k+1}$ exhibits N as a submanifold of $\mathbf{R}^{n+k+1}$. The normal bundles are related by the Whitney sum

$$\nu_{N\subset \mathbf{R}^{n+k+1}} \cong \nu_{N\subset \mathbf{R}^{n+k}} \oplus \underline{\mathbb{R}}, \tag{9.7}$$

Here $\underline{\mathbb{R}}$ is the trivial rank 1 subbundle of $\nu_{N\subset \mathbf{R}^{n+k+1}}$ spanned by perpendicular vector pointing positively in the last coordinate. It follows that if $\nu_{N\subset \mathbf{R}^{n+k}}$ is equipped with a γ-structure for some rank k bundle γ, then $\nu_{N\subset \mathbf{R}^{n+k+1}}$ inherits a $\gamma \oplus \underline{\mathbb{R}}$ structure. This defines a natural transformation

$$\Omega_n^{\gamma} \to \Omega_n^{\gamma\oplus\underline{\mathbb{R}}} \tag{9.8}$$

The following exercise shows that this transformation is compatible with suspension homomorphisms on homotopy groups via the Pontrjagin-Thom construction.

Exercise 178. Let γ be a rank k vector bundle.

1. There is an identification $T(\gamma \oplus \underline{\mathbb{R}}) = S(T(\gamma))$,where S denotes reduced suspension.
2. Given any space X, There is an identification $X_+ \wedge S(T(\gamma)) = S(X_+ \wedge T(\gamma))$.
3. The diagram

$$\begin{tikzcd} \Omega_n^{\gamma}(X) \arrow[r] \arrow[d, "c"] & \Omega_n^{\gamma\oplus\underline{\mathbb{R}}}(X) \arrow[d, "c"] \\ \pi_{n+k}(X_+ \wedge T(\gamma)) \arrow[r, "S"'] & \pi_{n+k+1}(S(X_+ \wedge T(\gamma))) \end{tikzcd}$$

commutes, with the vertical maps the Pontrjagin-Thom isomorphisms, and the bottom horizontal map the suspension map.

Exercise 178 immediately implies the following.

Proposition 9.10. *For any rank k vector bundle γ, The Pontrjagin-Thom construction defines a (functorial in X) isomorphism*

$$\operatorname*{colim}_{\ell\to\infty} \Omega_n^{\gamma\oplus\underline{\mathbb{R}}^\ell}(X) \cong \pi_{n+k}^S(X_+ \wedge T(\gamma)).$$

□

For example, taking γ to be the trivial rank 0 vector bundle over a point yields the isomorphism

$$\operatorname*{colim}_{\ell\to\infty} \Omega_n^{\underline{\mathbb{R}}^\ell}(X) \cong \pi_n^S(X_+).$$

When X is a point, this gives a bordism description of the stable n-stem

$$\operatorname*{colim}_{\ell\to\infty} \Omega_n^{\underline{\mathbb{R}}^\ell} \cong \pi_n^S.$$

We need a slight generalization of this construction, which we formalize in a definition.

Definition 9.11. A *stable system of vector bundles* is an integer k_0 and a sequence $\gamma_k : E_k \to B_k$ of rank k vector bundles, one for each integer $k \geq k_0$, linked by bundle maps $(\tilde{g}_k, g_k)$

$$\begin{tikzcd} E_k \oplus \underline{\mathbb{R}} \arrow[r, "\tilde{g}_k"] \arrow[d, "\gamma_k"'] & E_{k+1} \arrow[d, "\gamma_{k+1}"] \\ B_k \arrow[r, "g_k"'] & B_{k+1} \end{tikzcd}$$

We write $\boldsymbol{\gamma} = \{\gamma_k, (\tilde{g}_k, g_k)\}$. We call two systems equivalent provided there exists bundle maps from one to the other making the appropriate diagrams commute for all k large enough.

In the special case when $B_{k+1} = B_k$ and $g_k = \mathrm{Id}$ for all $k \geq k_0$, we call this simply a *stable vector bundle.*

Notice that a stable vector bundle is a special case of a stable system of vector bundles, where all the spaces B_k are the same and all the maps g_k are the identity. Also, any rank k_0 vector bundle γ determines a stable bundle by defining $\gamma_{k_0+\ell} = \gamma \oplus \underline{\mathbb{R}}^\ell$.

Proposition 9.12. *Let $\xi : E \to W$ be a rank k vector bundle over a compact Hausdorff space. Then there exists a rank k' vector bundle $\xi' : E' \to W$ so that the Whitney sum $\xi \oplus \xi'$ is isomorphic to the trivial bundle $\underline{\mathbb{R}}^{k+k'}$.*

Moreover, if $\xi'' : E' \to W'$ is a rank k'' vector bundle satisfying $\xi \oplus \xi'' = \underline{\mathbb{R}}^{k+k''}$, then ξ' and ξ'' are stably isomorphic, i.e. there exist N', N'' so that $\xi' \oplus \underline{\mathbb{R}}^{N'}$ is isomorphic to $\xi'' \oplus \underline{\mathbb{R}}^{N''}$.

Proof. Choose a finite cover $\{U_i\}_{i=1}^{\ell}$ so that the restriction of ξ to each U_i is trivial. Let $\{\phi_i : W \to [0,1]\}$ be a partition of unity associated to this cover.

For each $i = 1, \cdots, \ell$, let $v_{i,j} : U_i \to \xi^{-1}(U_i),\ j = 1 \cdots, k$ be sections which are pointwise linearly independent. Define

$$\Phi : W \times (\mathbf{R}^k)^\ell \to E$$

by

$$\Phi\big(w, (r_{1,1}, \cdots, r_{1,k}), \cdots, (r_{\ell,1}, \cdots, r_{\ell,k})\big) = \sum_{i=1}^{\ell} \phi_i(w) \sum_{j=1}^{k} r_{i,j} v_{i,j}(w)$$

Then Φ is a function from $\underline{\mathbb{R}}^{k\ell}$ to E which is a linear surjection in each vector space fiber. Define ξ' to be the kernel of Φ, this gives a short exact sequence of vector bundles over W

$$0 \to \xi' \to \underline{\mathbb{R}}^{k\ell} \to \xi \to 0.$$

This sequence splits, e.g. by taking the orthogonal complement to ξ'. Hence one obtains an isomorphism of vector bundles over W:

$$\underline{\mathbb{R}}^{k\ell} \cong \xi \oplus \xi',$$

establishing the first assertion.

For the second assertion, since the Whitney sum of vector bundles is commutative and associative,

$$\xi' \oplus \underline{\mathbb{R}}^{k+k''} \cong \xi' \oplus \xi \oplus \xi'' \cong \underline{\mathbb{R}}^{k+k'} \oplus \xi'' = \xi'' \oplus \underline{\mathbb{R}}^{k+k'},$$

completing the proof. □

In light of Proposition 9.12, we call a stable vector bundle ξ' *the stable inverse to* ξ if $\xi \oplus \xi'$ is isomorphic to the trivial bundle.

An important example of a stable vector bundle is the *stable normal bundle* of a manifold. Equation (9.7) shows that given an n-dimensional submanifold $N \subset \mathbf{R}^{n+k}$, viewing N as a submanifold of $\mathbf{R}^{n+k+\ell}$ and taking normal bundles determines the stable vector bundle:

$$\boldsymbol{\nu}_N = \{\nu_{N \subset \mathbf{R}^{n+k+\ell}} \mid \ell = 0, 1, 2, \cdots\}$$

For small values of ℓ, the normal bundles $\nu_{N \subset \mathbf{R}^{n+k+\ell}}$ may depend on the choice of embedding of N in $\mathbf{R}^{n+k}$, but, in light of the Whitney sum decomposition

$$\underline{\mathbb{R}}^{n+k} = T\mathbf{R}^{n+k}|_N \cong TN \oplus \nu_{N \subset \mathbf{R}^{n+k+\ell}},$$

the stable normal bundle is the stable inverse to the tangent bundle TN, which is independent of the choice of embedding of N. Hence the equivalence class of $\boldsymbol{\nu}_N$ is well-defined, i.e. independent of the choice of embedding, and is therefore an invariant of the (abstract) manifold N.

Given a stable sequence of vector bundles $\boldsymbol{\gamma} = \{\gamma_k, (\tilde{g}_k, g_k)\}$, the map $\tilde{g}_k$ determines a bundle isomorphism

$$\gamma_k \oplus \underline{\mathbb{R}} \cong g_k^*(\gamma_{k+1}).$$

This induces a natural transformation (see Equation (9.3)) $\Omega_n^{\gamma_k \oplus \underline{\mathbb{R}}} \to \Omega_n^{\gamma_{k+1}}$, which, composed with the transformation of Equation (9.8), yields the directed system

$$\cdots \to \Omega_n^{\gamma_k} \to \Omega_n^{\gamma_{k+1}} \to \Omega_n^{\gamma_{k+2}} \to \cdots \tag{9.9}$$

Definition 9.13. Given a stable system of vector bundles $\boldsymbol{\gamma} = \{\gamma_k, (\tilde{g}_k, g_k)\}$, define the $\boldsymbol{\gamma}$-*bordism group* as the colimit of the directed system (9.9)

$$\Omega_n^{\boldsymbol{\gamma}}(X) = \operatorname*{colim}_{\ell} \Omega_n^{\gamma_{k+\ell}}(X).$$

Since bundle maps induce maps on Thom spaces, the diagram
(9.10)

$$\begin{tikzcd}
\Omega_n^{\gamma_k}(X) \arrow[r] \arrow[d, "c"] & \Omega_n^{\gamma_k \oplus \underline{\mathbb{R}}}(X) \arrow[r] \arrow[d, "c"] & \Omega_n^{\gamma_{k+1}}(X) \arrow[d, "c"] \\
\pi_{n+k}(X_+ \wedge T(\gamma_k)) \arrow[r, "S"] & \pi_{n+k+1}(S(X_+ \wedge T(\gamma_k))) \arrow[r] & \pi_{n+k+1}(X_+ \wedge T(\gamma_{k+1}))
\end{tikzcd}$$

commutes, with the vertical maps isomorphisms induced by the Pontrjagin-Thom collapse. The bottom edge in Diagram (9.10) defines a directed system

$$\cdots \to \pi_{n+k}(X_+ \wedge T(\gamma_k)) \to \pi_{n+k+1}(X_+ \wedge T(\gamma_{k+1})) \to \cdots$$

Taking colimits immediately proves the following generalization of Proposition 9.10.

Theorem 9.14 (Stable Pontrjagin-Thom construction)**.** *Given a stable system of vector bundles $\boldsymbol{\gamma}$, the Pontrjagin-Thom construction induces a natural isomorphism*

$$\Omega_n^{\boldsymbol{\gamma}}(X) \xrightarrow{\cong} \operatorname*{colim}_{\ell \to \infty} \pi_{n+\ell}(X_+ \wedge T(\gamma_\ell)).$$

$\square$

In Theorem 9.14, "natural isomorphism" means that the Pontrjagin-Thom isomorphism provides a natural transformation between the two functors $\Omega_n^{\boldsymbol{\gamma}}(-)$, $\operatorname{colim}_{\ell \to \infty} \pi_{n+\ell}((-)_+ \wedge T(\gamma_\ell)) : \mathrm{Top} \to \mathrm{Ab}$.

start reading here, but think about stable tangential versus normal structures

9.6. Classifying spaces

Before we give explicit examples of stable systems of vector bundles γ and their associated bordism groups Ω_n^γ, we reinterpret stable systems of vector bundles in the language of classifying spaces, as classifying spaces provide a convenient way to describe important instances of bordism theories, but also figure prominently in the study of characteristic classes, which is taken up in Section 11.8. Classifying spaces are the topic of a project in Chapter 5 and are also discussed in Corollary 7.55.

9.6.1. Classifying spaces and classifying maps for vector bundles.

Recall that a *weakly contractible space* is a space all of whose homotopy groups vanish.

The basic result about classifying spaces is the following (see the projects for Chapter 5).

Theorem 9.15. *Given any topological group G, there exists a principal G-bundle $EG \to BG$ where EG is a weakly contractible space. The construction is functorial, so that any continuous group homomorphism $\alpha : G \to H$ induces a bundle map*

$$\begin{tikzcd} EG \arrow[r, "E\alpha"] \arrow[d] & EH \arrow[d] \\ BG \arrow[r, "B\alpha"] & BH \end{tikzcd}$$

compatible with the actions, so that if $x \in EG, g \in G$,

$$E\alpha(x \cdot g) = (E\alpha(x)) \cdot \alpha(g).$$

The space BG is called a classifying space for G.

The function

$$\Phi : Maps(B, BG) \to \{Principal\ G\text{-bundles over } B\}$$

defined by pulling back (so $\Phi(f) = f^*(EG)$) induces a bijection *from the homotopy set $[B, BG]$ to the set of isomorphism classes of principal G-bundles over B, when B is a CW-complex (or more generally a paracompact space).* □

The long exact sequence for the fibration $G \to EG \to BG$ shows that $\pi_n BG = \pi_{n-1} G$. In fact, ΩBG is (weakly) homotopy equivalent to G, as one can see by taking the extended fiber sequence $\cdots \to \Omega EG \to \Omega BG \to G \to EG \to BG$, computing with homotopy groups, and observing that EG and ΩEG are contractible. Thus the space BG is a *delooping* of G.

The following lemma is extremely useful.

Lemma 9.16. *Let $p : E \to B$ be a principal G-bundle, and let $f : B \to BG$ be the classifying map. Then the homotopy fiber of f is weakly homotopy equivalent to E.*

Proof. Turn $f : B \to BG$ into a fibration $q : B' \to BG$ using Theorem 7.23 and let F' denote the homotopy fiber of $q : B' \to BG$. Thus there is a commutative diagram

$$\begin{tikzcd} B \arrow[rr, "h"] \arrow[dr, "f"'] & & B' \arrow[dl, "q"] \\ & BG & \end{tikzcd}$$

with h a homotopy equivalence. The fact that f is the classifying map for $p : E \to B$ implies that there is a commutative diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "p"'] & EG \arrow[d] \\ B \arrow[r, "f"] & BG \end{tikzcd}$$

and since EG is contractible, $f \circ p = q \circ h \circ p : E \to BG$ is nullhomotopic. By the homotopy lifting property for the fibration $q : B' \to BG$ it follows that $h \circ p : E \to B'$ is homotopic into the fiber F' of $q : B' \to BG$, and so one obtains a homotopy commutative diagram of spaces

$$\begin{tikzcd} E \arrow[r] \arrow[d, "p"'] & F' \arrow[d] \\ B \arrow[r, "h"] \arrow[d, "f"'] & B' \arrow[d, "q"] \\ BG \arrow[r, "="] & BG \end{tikzcd}$$

The left edge is a fibration, h is a homotopy equivalence, and by the five lemma the map $\pi_n(E) \to \pi_n(F')$ is an isomorphism for all n. $\square$

In Lemma 9.16 one can usually conclude that the homotopy fiber of $f : B \to BG$ is in fact homotopy equivalent to E. This would follow if we know that B' is homotopy equivalent to a CW-complex. For most G, this is a consequence of a theorem of Milnor [**30**].

Exercise 179. Show that given a principal G-bundle $E \to B$, there is a fibration

$$\begin{tikzcd} E \arrow[r] & EG \times_G E \arrow[d] \\ & BG \end{tikzcd}$$

where $EG \times_G E$ denotes the Borel construction. How is this fibration related to the fibration of Lemma 9.16?

Recall that to any principal $O(k)$ bundle $P \to B$ one can construct the associated rank k vector bundle $P \times_{O(k)} \mathbf{R}^k$. This vector bundle has structure group $O(k)$ hence is equipped with a metric. Conversely, given a rank k vector bundle with metric $E \to B$, one can form the bundle of orthonormal frames in E, a principal $O(k)$ bundle. As explained in Section 5.5, this sets up a one-to-one correspondence between isomorphism classes of rank k vector bundles with metric over B and isomorphism classes of principal $O(k)$ bundles over B.

Let $u_k : E_{O(k)} \to BO(k)$ denote the rank k vector bundle associated to the principal $O(k)$ bundle $EO(k) \to BO(k)$. Hence

$$E_{O(k)} = EO(k) \times_{O(k)} \mathbf{R}^k.$$

The vector bundle $u_k : E_{O(k)} \to BO(k)$ is called the *universal rank k vector bundle*. The observation of the previous paragraph and Theorem 9.15 implies that for any space B, the assignment

$$\mathrm{Map}(B, BO(k)) \to \{\text{rank } k \text{ vector bundles over } B\},\ f \mapsto f^*(u_k)$$

induces an isomorphism

$$[B, BO(k)] \cong \{\text{rank } k \text{ vector bundles over } B\}/_{\text{isomorphism}} \tag{9.11}$$

Given a vector bundle $\gamma : E \to B$, any map $f : B \to BO(k)$ in the homotopy class corresponding to γ (i.e. so that $f^*(u_k) = \gamma$) is called the *classifying map for γ*.

An application of this correspondence is as follows: given an n-dimensional submanifold $N \subset \mathbf{R}^{n+k}$, its normal bundle $\nu_{N \subset \mathbf{R}^{n+k}}$ is a rank k vector bundle over N. Its classifying map $f_{\nu_N} : N \to BO(k)$ is uniquely determined up to homotopy.

Moreover, if $\gamma : E \to B$ is any vector bundle, with classifying map $f_\gamma : B \to BO(k)$, then a γ-structure on the normal bundle of $N \subset \mathbf{R}^{n+k}$ is the same thing as a factorization up to homotopy

$$f_{\nu_N} = f_\gamma \circ g$$

for some map $g : N \to B$. We state this formally.

Proposition 9.17. *Let $\gamma : E \to B$ be a rank k-vector bundle with metric and let $f_\gamma \in [B, BO(k)]$ denote its classifying map.*

Given an n-dimensional submanifold $N \subset \mathbf{R}^{n+k}$, there is a one-to-one correspondence between equivalence classes of γ-structures on the normal bundle ν_N and the set of $g \in [N, B]$ so that f_{ν_N} is homotopic to $f_\gamma \circ g$. $\square$

Stabilization is simple to express in terms of classifying spaces. Consider the monomorphism

$$O(k) \to O(k+1), \; A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \tag{9.12}$$

Proposition 9.18. *The map $s_k : BO(k) \to BO(k+1)$ induced by the homomorphism of Equation (9.12) satisfies*

$$s_k^*(u_{k+1}) \cong u_k \oplus \underline{\mathbb{R}}.$$

In other words, there is a bundle map

$$\begin{tikzcd} E_{O(k)} \oplus \underline{\mathbb{R}} \arrow[r] \arrow[d, "u_k"'] & E_{O(k+1)} \arrow[d, "u_{k+1}"] \\ BO(k) \arrow[r, "s_k"] & BO(k+1) \end{tikzcd}$$

Sketch of proof. Note that $E_{O(k)} = EO(k) \times_{\rho_k} \mathbf{R}^k$, where $\rho_k : O(k) \to GL(\mathbf{R}^k)$ denotes the standard action, and similarly for $E_{O(k+1)}$. Moreover, $E_{O(k)} \oplus \underline{\mathbb{R}} = EO(k) \times_{\rho_{k+1} \circ i} \mathbf{R}^{k+1}$, where $i : O(k) \to O(k+1)$ is the homomorphism of Equation (9.12). $\square$

Proposition 9.18 shows that the universal bundles $u_k : E_{O(k)} \to BO(k)$ form a stable system of vector bundles (Definition 9.11). More generally, one can construct a stable system of vector bundles given any sequence $B_k, B_{k+1}, \cdots$ $(k > k_0)$ of spaces and a commutative ladder:

$$\begin{tikzcd} \cdots \arrow[r, "g_{k-1}"] & B_k \arrow[r, "g_k"] \arrow[d, "f_k"] & B_{k+1} \arrow[r, "g_{k+1}"] \arrow[d, "f_{k+1}"] & \cdots \\ \cdots \arrow[r, "s_{k-1}"] & BO(k) \arrow[r, "s_k"] & BO(k+1) \arrow[r, "s_{k+1}"] & \cdots \end{tikzcd} \tag{9.13}$$

by pulling back the universal bundles $E_{O(k)} \to BO(k)$ via the maps f_k. Conversely, every stable system of vector bundles arises from a ladder of the form (9.13). We leave the details as an exercise.

Exercise 180. Use the universal property of classifying spaces shows that every stable system of vector bundles can be obtained by pulling back the universal stable system using a commutative ladder of the form (9.13).

9.7. Examples of bordism theories

In light of Exercise 180 and the discussion which precedes it, one can define bordism groups associated to diagrams (9.13), by assigning to such a diagram the corresponding stable system of vector bundles.

One important class of examples of bordism theories constructed from this perspective starts with a sequence of topological groups G_n and a commutative ladder of homomorphisms

$$\begin{tikzcd} \cdots \arrow[r] & G_k \arrow[r] \arrow[d] & G_{k+1} \arrow[r] \arrow[d] & \cdots \\ \cdots \arrow[r] & O(k) \arrow[r] & O(k+1) \arrow[r] & \cdots \end{tikzcd} \tag{9.14}$$

Applying the classifying space functor gives the ladder of the type (9.13)

$$\begin{tikzcd} \cdots \arrow[r] & BG_k \arrow[r] \arrow[d, "f_k"] & BG_{k+1} \arrow[r] \arrow[d, "f_{k+1}"] & \cdots \\ \cdots \arrow[r] & BO(k) \arrow[r] & BO(k+1) \arrow[r] & \cdots \end{tikzcd} \tag{9.15}$$

The bordism functor Ω_n^{γ} determined by the stable system of bundles associated to the diagram is denoted $\Omega_n^{\mathbf{G}}$. Theorem 9.14 says that the Pontrjagin-Thom collapse induces a natural isomorphism

$$\Omega_n^{\mathbf{G}}(X) \xrightarrow{\cong} \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X_+ \wedge T(f_k^*(u_k))) \tag{9.16}$$

where $u_k : E_{O(k)} \to BO(k)$ denotes the universal rank k vector bundle. We give several examples of bordism functors constructed from sequences $\{G_k\}$ as in Equation (9.14).

9.7.1. Unoriented bordism. If one takes $G_k = O(k)$ and the vertical maps in (9.14) identity maps, the resulting bordism functor $\Omega_n^{\mathbf{O}}$ is called *unoriented bordism*.

It follows from Proposition 9.17, taking f_γ to be the identity, and using the fact that the classifying map for the normal bundle of a submanifold is unique up to homotopy, that there is no need to keep track of the γ-structure on normal bundles, and so $\Omega_n^{\mathbf{O}}$ is the functor introduced in Section 9.1.2, that is, $\Omega_n^{\mathbf{O}}(X)$ consists of bordism classes of maps $f : N^n \to X$. (This explains our use of the notation $\Omega_n^{\mathbf{O}}$ for unoriented bordism.)

From Equation (9.16) we know that

$$\Omega_n^{\mathbf{O}}(X) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X_+ \wedge T(E_{O(k)})).$$

In particular taking X to be a point one has

$$\Omega_n^{\mathbf{O}}(p) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(T(E_{O(k)})).$$

This is the starting point for Thom's remarkable theorem (stated in Theorem 11.43 below) computing $\Omega_n^{\mathbf{O}}(p)$ for all n [**50**]. We gave an argument based on the Euler characteristic in Section 9.1.2 that $\Omega_2^{\mathbf{O}}(p)$ is nonzero. We list without proof some low-dimensional calculations:

n	0	1	2	3	4	5	6	7
$\Omega_n^{\mathbf{O}}(p)$	$\mathbf{Z}/2$	0	$\mathbf{Z}/2$	0	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/2$

9.7.2. Framed bordism. Next, we take $G_k = \{1\}$, the trivial group, for all k in (9.14). It is traditional to denote the resulting bordism group Ω_n^{fr} and call it the *framed bordism group.* Then BG_k has the homotopy type of a point, and the maps f_k in (9.15) can all be taken to be the (constant) map from a point. Hence $f_k^*(E_{O(k)})$ is just the trivial $\mathbf{R}^k$ bundle over a point, and so its Thom space is just S^k.

Equation (9.16) tells us

$$\Omega_n^{\mathrm{fr}}(X) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X_+ \wedge S^k) = \pi_n^S(X_+).$$

In other words, framed bordism and stable homotopy are essentially isomorphic functors, although they take as input different categories: framed bordism is a functor on the category of unbased spaces and stable homotopy is a functor on based spaces. This point is described in the discussion of unreduced and reduced homology theories below.

The reader should contrast the definition of the stable bordism group $\Omega_n^{\mathrm{fr}}(X)$ with the unstable bordism group $\Omega_n^{\mathrm{fr}}(X) \cong \pi_{n+k}(X_+ \wedge S^k)$ described in Section 9.3.1.

Here is a list of some computations of $\Omega_n^{\mathrm{fr}}(p) = \pi_n^S(S^0) =: \pi_n^S$ for you to reflect on. (Note: π_n^S has been computed for $n \leq 64$. There is no reasonable conjecture for π_n^S for general n, although there are many results known. For example, in Chapter 11, we show that the groups are finite for $n > 0$; $\pi_0^S = \mathbf{Z}$ by the Hopf degree theorem.)

n	1	2	3	4	5	6
π_n^S	$\mathbf{Z}/2$	$\mathbf{Z}/2$	$\mathbf{Z}/24$	0	0	$\mathbf{Z}/2$
n	7	8	9	10	11	12
π_n^S	$\mathbf{Z}/240$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^3$	$\mathbf{Z}/6$	$\mathbf{Z}/504$	0
n	13	14	15	16	17	18
π_n^S	$\mathbf{Z}/3$	$(\mathbf{Z}/2)^2$	$\mathbf{Z}/480 \oplus \mathbf{Z}/2$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^4$	$\mathbf{Z}/8 \oplus \mathbf{Z}/2$
n	19	20	21	22	23	24
π_n^S	$\mathbf{Z}/264 \oplus \mathbf{Z}/2$	$\mathbf{Z}/24$	$(\mathbf{Z}/2)^2$	$(\mathbf{Z}/2)^2$	†	$(\mathbf{Z}/2)^2$

† π_{23}^S is $\mathbf{Z}/65520 \oplus \mathbf{Z}/24 \oplus \mathbf{Z}/2$.

The reference [**40**] is a good source for the tools to compute π_n^S.

We will give stably framed manifolds representing generators of π_n^S for $n < 9$; you may challenge your local homotopy theorist to supply the proofs. In this range there are (basically) two sources of framed manifolds: normal framings on spheres coming from the image of the stable J-homomorphism $J : \pi_n O \to \pi_n^S$, and tangential framing coming from Lie groups. There is considerable overlap between these sources.

Bott periodicity (Theorem 7.54) computes $\pi_n O$.

n	0	1	2	3	4	5	6	7	8
$\pi_n O$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$

Then $J : \pi_n O \to \pi_n^S$ is an isomorphism for $n = 1$, an epimorphism for $n = 3, 7$, and a monomorphism for $n = 8$.

Another source for framed manifolds are Lie groups. If G is a compact k-dimensional Lie group and $T_eG \cong \mathbf{R}^k$ is an identification of its tangent space at the identity, then one can use the group multiplication to identify $TG \cong G \times \mathbf{R}^k$ and thereby frame the tangent bundle. This is the so-called Lie invariant framing. The generators of the cyclic groups $\pi_0^S, \pi_1^S, \pi_2^S, \pi_3^S, \pi_6^S, \pi_7^S$ are given by $e, S^1, S^1 \times S^1, S^3, S^3 \times S^3, S^7$ with invariant framings. (The unit octonions S^7 fail to be a group because of the lack of associativity, but nonetheless, they do have an invariant framing.)

Finally, the generators of π_8^S are given by S^8 with framing given by the J-homomorphism and the unique exotic sphere in dimension 8. (An exotic sphere is a smooth manifold homeomorphic to a sphere and not diffeomorphic to a sphere.)

9.7.3. Oriented bordism. The next example is to take $G_k = SO(k)$ and $SO(k) \to O(k)$ the inclusion in (9.14). The resulting bordism functor $\Omega_n^{\mathbf{SO}}$ is called *oriented bordism.*

Define $\tilde{u}_k : E_{SO(k)} \to BSO(k)$ to be the associated rank k-vector bundle

$$E_{SO(k)} = ESO(k) \times_{SO(k)} \mathbf{R}^k \xrightarrow{\tilde{u}_k} BSO(k),$$

and call $\tilde{u}_k$ the *universal oriented rank k vector bundle.* It is straightforward to see that there is a pullback diagram of vector bundles:

$$\begin{tikzcd} E_{SO(k)} \arrow[r] \arrow[d, "\tilde{u}_k"] & E_{O(k)} \arrow[d, "u_k"] \\ BSO(k) \arrow[r, "f_k"] & BO(k) \end{tikzcd}$$

Equation (9.16) tells us

$$\Omega_n^{\mathbf{SO}}(X) \cong \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X_+ \wedge (T(\tilde{u}_k))).$$

The name oriented bordism is justified by the fact that manifolds with $\tilde{u}_k$ structures are orientable. To understand this, we digress and discuss the various notions of orientability and orientation for vector bundles and manifolds. Resolving these notions of orientability is necessary, and so our exposition will involve a sequence of exercises to relate the various notions. These exercises are all straightforward, but they can be a bit confusing. A mastery of orientation issues is quite useful for a working mathematician, and you should keep in mind that such a mastery comes only from a thorough understanding of the equivalence between different points of view. (In other words: solve these exercises!)

To begin with, recall that an orientation of a real finite-dimensional vector space V is an equivalence class of bases of V where two bases are considered equivalent if the determinant of the change of basis matrix is positive. Notice that a choice of basis identifies V with $\mathbf{R}^k$ for some k. This in turn induces an isomorphism

$$H_k(V, V-0) \xrightarrow{\cong} H_k(\mathbf{R}^k, \mathbf{R}^k - 0) \cong \mathbf{Z}. \tag{9.17}$$

Exercise 181. Show that changing the orientation of V changes the identification of Equation (9.17) by a sign.

It follows from this exercise that an orientation of V can be defined as a choice of generator of $H_k(V, V-0)$. By choosing the dual generator an orientation of V can also be defined as a choice of orientation of $H^k(V, V-0) = \mathrm{Hom}(H_k(V, V-0), \mathbf{Z})$.

Exercise 182. If V is a k-dimensional vector space, its top exterior power $\Lambda^k V$ is a 1-dimensional vector space. Show that an orientation of V is equivalent to a choice of one of the two path components of $\Lambda^k V - \{0\}$.

Definition 5.14, adapted to vector bundles with structure group $O(k)$ (see Exercise 185), states that a rank k vector bundle is orientable if its structure group can be reduced from $O(k)$ to $SO(k)$, and that an orientation of a vector bundle is a choice of reduction.

Exercise 183. Prove that a rank k vector bundle $E \to B$ is orientable (i.e. its structure group reduces from $O(k)$ to $SO(k)$; see Definition 5.14) if and only if the the local coefficient system

$$\pi_1 B \to \mathrm{Aut}(H^k(\mathbf{R}^k, \mathbf{R}^k - 0; \mathbf{Z})) = \mathrm{Aut}(\mathbf{Z}) \cong \mathbf{Z}/2$$

(determined by Corollary 7.17) is trivial. Notice that by pulling back bundles, it is enough to prove this for $B = S^1$. Prove that a vector bundle over S^1 is orientable if and only if it is trivial. (Hint: use the clutching construction and the fact that $O(k)$ has exactly two path components.)

An orientation of a vector space V uniquely determines an orientation of $V \oplus \mathbf{R}$, by extending the basis by the vector $(0, 1)$. It follows that an orientation of a vector bundle $\xi : E \to W$ uniquely determines an orientation of the Whitney sum $\xi \oplus \underline{\mathbb{R}}$, and so orientability and orientation are well-defined notions for *stable* vector bundles, such as the stable normal bundle of a submanifold $N \subset \mathbf{R}^{n+k}$.

We have come across several notions of orientability for smooth manifolds. One notion is that a smooth manifold N is orientable if its tangent bundle is orientable; i.e. the structure group of TN can be reduced from $O(n)$ to $SO(n)$. An orientation is a choice of such a reduction.

Exercise 184. Show that an orientation in this sense determines an equivalence class of bases at each tangent space T_pN. (Hint: use Exercise 183.) More generally show that a reduction of the structure group of a vector bundle E from $O(k)$ to $SO(k)$ determines an equivalence class of bases in each fiber F_x so that these equivalence classes are compatible with the local trivializations of E; i.e. the homeomorphism $E|_U \cong U \times \mathbf{R}^k$ takes the orientation of F_x to the same orientation of $\mathbf{R}^k$ for all $x \in U$.

Another notion of orientability says that a compact, connected n-manifold N is orientable if $H_n(N, \partial N) \cong \mathbf{Z}$ and that an orientation is a choice $[N, \partial N] \in H_n(N, \partial N)$ of generator, called the *fundamental class* of the oriented manifold N. In the course of the proof of the Poincaré duality theorem one shows that if $[N, \partial N]$ is the fundamental class of N, then for each $p \in N$ the inclusion $H_n(N, \partial N) \to H_n(N, N - p)$ is an isomorphism.

Given $p \in N$ (and a choice of Riemannian metric on N) the exponential map $\exp : T_pN \to N$ restricts to a diffeomorphism in a small ball $W \subset T_pN$, exp: $W \to U \subset N$ and hence gives isomorphisms (the first and third are excision isomorphisms)

$$H_n(T_pN, T_pN - 0) \cong H_n(W, W - 0) \xrightarrow{\exp} H_n(U, U - p) \cong H_n(N, N - p).$$

This shows that the choice of fundamental class $[N, \partial N] \in H_n(N, \partial N)$ orients the tangent space T_pN.

Exercise 185. Prove that this sets up an identification between the two notions of an orientation of a smooth compact, connected manifold (the choice of generator $[N, \partial N]$ of $H_n(N, \partial N)$ and an orientation of the vector bundle TN).

The fundamental class $[N, \partial N] \in H_n(N, \partial N)$ of a compact, connected, and oriented n-manifold determines the dual *cohomology fundamental class* $[N, \partial N]^* \in H^n(N, \partial N)$ (and conversely) by the equation

$$\langle [N, \partial N]^*, [N, \partial N] \rangle = 1$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing.

Lemma 9.19. *A submanifold $N^n \subset \mathbf{R}^{n+k}$ is orientable if and only if its normal bundle $\nu_{N \subset \mathbf{R}^{n+k}}$ is orientable. Moreover a choice of orientation of N determines an orientation of ν_N and conversely.*

Proof. Since $N \subset \mathbf{R}^{n+k}$,

$$T(\mathbf{R}^{n+k})|_N = TN \oplus \nu_{N \subset \mathbf{R}^{n+k}}. \tag{9.18}$$

The tangent bundle of $\mathbf{R}^{n+k}$ is trivial, in fact translation to the origin defines a canonical trivialization $T\mathbf{R}^{n+k} = \mathbf{R}^{n+k} \times \mathbf{R}^{n+k}$.

If V_1 and V_2 are real vector spaces, it is simple to see that orientations of V_1 and $V_1 \oplus V_2$ uniquely determine an orientation on V_2 compatible with the direct sum, i.e. obtained by juxtaposing bases. Since $\mathbf{R}^{n+k}$ is oriented (by the standard basis) the lemma follows by applying this observation to the Whitney sum (9.18), taking $V_1 = T_nN$ and $V_2 = (\nu_N)_n$ so that $V_1 \oplus V_2$ is the oriented vector space $\mathbf{R}^{n+k} = T_n\mathbf{R}^{n+k}$. □

Returning to oriented bordism, we see that $\Omega_n^{\mathbf{SO}}(X)$ consists of oriented bordism classes of maps $f : N \to X$ where N is an oriented compact manifold.

Here are some basic computations of oriented bordism of a point.

1. An oriented closed 0-manifold is just a finite number of signed points (i.e. each point p is equipped with a label $e_p \in \{1, -1\}$). This bounds

a 1-manifold if and only if the sum of the signs is zero. Hence $\Omega_0^{\mathbf{SO}}(p) \cong \mathbf{Z}$. Also, $\pi_k(T(\tilde{u}_k)) = \mathbf{Z}$ for $k \geq 2$.

2. Every oriented closed 1-manifold bounds an oriented 2-manifold, since $S^1 = \partial D^2$. Therefore $\Omega_1^{\mathbf{SO}}(p) = 0$.
3. Every oriented 2-manifold bounds an oriented 3-manifold since any oriented 2-manifold embeds in $\mathbf{R}^3$ with one of the two complementary components compact. Thus $\Omega_2^{\mathbf{SO}}(p) = 0$.
4. A theorem of Rohlin states that every oriented 3-manifold bounds a 4-manifold. Thus $\Omega_3^{\mathbf{SO}}(p) = 0$.
5. An oriented 4-manifold has a signature in $\mathbf{Z}$, i.e. the signature of its intersection form. A good exercise using Poincaré duality (see the projects for Chapter 4) shows that this is an oriented bordism invariant and hence defines a homomorphism $\Omega_4^{\mathbf{SO}}(p) \to \mathbf{Z}$. This turns out to be an isomorphism. More generally the signature defines a map $\Omega_{4k}^{\mathbf{SO}}(p) \to \mathbf{Z}$ for all k. This is a surjection since the signature of $\mathbf{C}P^{2k}$ is 1.
6. It is a fact that away from multiples of 4, the oriented bordism groups are torsion; i.e. $\Omega_n^{\mathbf{SO}}(p) \otimes \mathbf{Q} = 0$ if $n \neq 4k$.
7. For all n, $\Omega_n^{\mathbf{SO}}$ is finitely generated, in fact, isomorphic to a finite direct sum of $\mathbf{Z}$'s and $\mathbf{Z}/2$'s.

Statements 5, 6, and 7 can be proven by computing $\pi_{n+k}(T(\tilde{u}_k))$. How does one do this? A starting point is the *Thom isomorphism theorem*, which says that for all k,

$$H_n(BSO(k)) \cong \tilde{H}_{n+\ell}(T(\tilde{u}_k))$$

(where $\tilde{H}$ denotes reduced cohomology). The cohomology of $BSO(k)$ can be studied in several ways, and so one can obtain information about the cohomology of $T(\tilde{u}_k)$ by this theorem. Combining this with the Hurewicz theorem and other methods leads ultimately to a complete computation of oriented bordism (due to C.T.C. Wall), and this technique was generalized by Adams to a machine called the Adams spectral sequence. We will return to the Thom isomorphism theorem in Chapter 11.

Once the coefficients $\Omega_*^{\mathbf{SO}}(p)$ are understood, techniques such as the Atiyah-Hirzebruch spectral sequence (see Section 10.3) can be applied to get information about $\Omega_*^{\mathbf{SO}}(X)$ for a space X.

Since a compact oriented n-manifold has a fundamental class $[N] \in H_n(N;\mathbf{Z})$, oriented bordism maps to integral homology

(9.19) $$\Omega_n^{\mathbf{SO}}(X) \to H_n(X;\mathbf{Z}),\ (f : N \to X) \mapsto f_*[N].$$

The Hurewicz map $\rho : \pi_n X \to H_n X$ factors through $\Omega_n^{\mathbf{SO}}(X)$. In fact, since S^n is an oriented manifold, a homotopy class of maps $\alpha : S^n \to X$

determines an oriented bordism class. This assignment defines a homomorphism $\rho_1 : \pi_n(S^n) \to \Omega_n^{\mathbf{SO}}(X)$. It follows immediately from the definition of the Hurewicz map (Definition 7.70) that composing ρ_1 with the map of (9.19) yields the Hurewicz map ρ.

9.7.4. Complex bordism. Consider $G_k = U([\frac{k}{2}])$ where $[\frac{k}{2}]$ denotes the greatest integer less than or equal to $\frac{k}{2}$. Since G_{2k} acts complex linearly on $\mathbf{C}^k$, forgetting the complex structure means that G_{2k} acts linearly on $\mathbf{R}^{2k}$. This action is easily computed to be orthogonal, defining a monomorphism $G_{2k} = U(k) \to O(2k)$, and one can map G_{2k+1} to $O(2k+1)$ by taking the composite $G_{2k+1} = U(k) \to O(2k) \subset O(2k+1)$, yielding a system (9.14). The resulting bordism functor $\Omega_n^{\mathbf{U}}$ is called *complex bordism.*

In contrast to the terminology used for oriented bordism, it is not true that manifolds representing complex bordism classes are complex manifolds (i.e. manifolds admitting an atlas so that the transition functions are biholomorphic). For one thing, complex manifolds always have even dimension.

Call a manifold N *almost complex* if its tangent bundle TN is equipped with the structure of a complex vector bundle. More generally, call a manifold *stably almost complex* if $TN \oplus \underline{\mathbb{R}}^k$ is equipped the structure of a complex vector bundle for some $k \geq 0$.

The proof of Proposition 9.12 works identically in the context of complex vector bundles, and so, given a compact manifold N of dimension n and a complex vector bundle structure on $TN \oplus \underline{\mathbb{R}}^k$ (note $n+k$ is even), there exists a complex vector bundle $\xi : E \to N$ so that $(TN \oplus \underline{\mathbb{R}}^k) \oplus \xi$ is isomorphic to the trivial complex vector bundle $\underline{\mathbb{C}}^\ell = N \times \mathbf{C}^\ell$ for some ℓ. The underlying real vector bundles satisfy

$$TN \oplus \underline{\mathbb{R}}^k \oplus \xi \cong \underline{\mathbb{R}}^{2\ell}.$$

Therefore, the uniqueness part of Proposition 9.12 shows that $\underline{\mathbb{R}}^k \oplus \xi$ is stably isomorphic to the stable normal bundle $\boldsymbol{\nu}_N$, or more precisely, that there exists some k', k'', m so that

$$\underline{\mathbb{R}}^{k'} \oplus \underline{\mathbb{R}}^k \oplus \xi \cong \underline{\mathbb{R}}^{k''} \oplus \nu_{N \subset \mathbf{R}^{n+m}}$$

for some embedding $N \subset \mathbf{R}^{n+m}$. By increasing k', k'' each by one if needed, we may assume that $k' + k$ is even. Then

$$\underline{\mathbb{R}}^{k+k'} \cong \underline{\mathbb{C}}^{(k+k')/2} \text{ and } \underline{\mathbb{R}}^{k''} \oplus \nu_{N \subset \mathbf{R}^{n+m}} \cong \nu_{N \subset \mathbf{R}^{n+m+k''}}$$

so that

$$\underline{\mathbb{C}}^{(k+k')/2} \oplus \xi \cong \nu_{N \subset \mathbf{R}^{n+m+k''}}.$$

In other words, a complex structure on the stable tangent bundle of N determines one on the complex stable normal bundle of N. The converse is true, by the same proof.

In other words, complex bordism classes are represented by stably almost complex manifolds. This class includes complex and almost complex manifolds, but also some manifolds of odd dimension: for example the tangent bundle of S^1 is trivial since S^1 is a Lie group, and hence $TS^1 \oplus \underline{\mathbb{R}} \cong S^1 \times \mathbf{R}^2 = S^1 \times \mathbf{C}$.

An almost complex manifold may or may not admit the structure of a complex manifold. (It can be shown that S^6 is an almost complex manifold, but whether or not S^6 is a complex manifold is still an open question.)

The inclusion $U([\frac{k}{2}]) \subset O(k)$ factors through $SO(k)$, and so manifolds representing complex bordism classes are oriented.

9.7.5. Spin bordism. Recall that $SO(n)$ is path connected for $n > 1$, $\pi_1(SO(2)) \cong \mathbf{Z}$, and $\pi_1(SO(n)) \cong \mathbf{Z}/2$ for $n > 2$. Moreover, $\pi_1(SO(n)) \to \pi_1(SO(n+1))$ is onto for $n = 2$ and an isomorphism for $n > 2$.

Let $\mathrm{Spin}(n) \to SO(n)$ be the connected double cover. Take $G_k = \mathrm{Spin}(k)$ and the composite $\mathrm{Spin}(k) \to SO(k) \to O(k)$ in (9.14). The resulting bordism functor $\Omega_n^{\mathbf{Spin}}$ is called *Spin bordism*.

A *Spin manifold* is a manifold whose tangent bundle has a spin structure. Spin structures come up in differential geometry and index theory.

9.7.6. Other examples. There are many examples of **G**-structures. As a perhaps unusual example, one could take G_n to be $O(n)$ or $SO(n)$ with the *discrete* topology. This structure arises in the study of flat bundles and algebraic K-theory.

9.8. Relative bordism

To complete the picture of bordism, we sketch the construction of the *relative bordism groups* $\Omega_n^\gamma(X,A)$ associated to a pair (X,A). The definition is most easily understood if one keeps in focus the requirement that the sequence

$$\Omega_n^\gamma(X) \to \Omega_n^\gamma(X,A) \to \Omega_{n-1}^\gamma(A) \tag{9.20}$$

be exact.

Definition 9.20. Given a rank k vector bundle $\gamma : E \to B$ and a pair (X,A), $\Omega_n^\gamma(X,A)$ consists of *relative bordism* classes of pairs $((N,\partial N),(\tilde{g},g))$ where N is a smooth compact n-dimensional submanifold of $\mathbf{R}^{n+k-1} \times [0,\infty)$ with boundary $\partial N = N \cap (\mathbf{R}^{n+k-1} \times \{0\})$, and $g : (N,\partial N) \to (X \times B, A \times B)$

a continuous map of pairs, covered by a bundle map of pairs $\tilde{g}$

$$\begin{tikzcd}
(\nu_N, \nu_{\partial N}) \arrow[r, "\tilde{g}"] \arrow[d] & (\pi_X^*(E), \pi_A^*(E)) \arrow[d, "\pi_X^*\gamma"'] \\
(N, \partial) \arrow[r, "g"'] & (X \times B, A \times B)
\end{tikzcd}$$

where $\pi_X : X \times B \to B, \pi_A : A \times B \to B$ are the projections to the second factor.

The relative bordism equivalence relation in Definition 9.20 is straightforward, if awkward to define, in terms of *manifolds with corners*, or *triads*. We outline the idea and leave the details as an exercise.

An $(n+1)$-dimensional *manifold triad* (W, N, C) consists of a compact topological $(n+1)$-dimensional manifold with boundary equipped with charts to open sets in $\mathbf{R}^{n-1} \times [0, \infty) \times [0, \infty)$ with smooth transition functions on the overlaps. The $(n-1)$ dimensional compact manifold with empty boundary C corresponds to points whose charts map to $\mathbf{R}^{n-1} \times \{(0,0)\}$. The subset N consists of points mapped to $\mathbf{R}^{n-1} \times \big([0,\infty) \times \{0\} \cup \{0\} \times [0,\infty)\big)$. Hence W is a topological manifold with boundary N and C is a topological submanifold of N, but smoothly, W is said to have *corners* along B.

Two smooth n-dimensional smooth compact manifolds with boundary, $(N_0, \partial N_0)$ and $(N_1, \partial N_1)$ are called *relatively bordant* provided there exists a $(n+1)$-dimensional manifold triad (W, N, C) with

1. $C = \partial N_0 \amalg \partial N_1$,
2. $N = N_0 \cup_{\partial N_0} N_+ \cup_{\partial N_1} N_1$ with N_+ a bordism from ∂N_0 to ∂N_1.

Exercise 186. Complete Definition 9.20 by making the relative γ-bordism relation explicit, and prove that the sequence (9.20) is exact.

9.9. Spectra

We motivate the introduction of spectra by looking at some common features of three examples of sequences of functors from spaces to abelian groups which are studied in this and previous chapters. For now, the reader should not pay too close attention to base point issues, as we discuss these more carefully below when we distinguish between reduced and unreduced homology theories.

Start with singular cohomology. In Chapter 8, obstruction theory was used to show that singular homology with coefficients in an abelian group

π is naturally isomorphic to homotopy classes of maps to the Eilenberg-MacLane space $K(\pi, n)$, explicitly

$$H^n(X;\pi) = [X, K(\pi, n)].$$

Moreover, the path space fibration shows that $\Omega_x(K(\pi, n+1))$, the based loop space of a $K(\pi, n+1)$ space, is a $K(\pi, n)$ space. In other words the sequence of spaces $\{K(\pi, n)\}_{n=0}^{\infty}$ (take $K(\pi, 0)$ to be the group π with the discrete topology) has the property that there exists homotopy equivalences

$$K(\pi, n) \sim \Omega_x(K(\pi, n+1)).$$

The adjoints of these homotopy equivalences are maps

$$s_n : SK(\pi, n) \to K(\pi, n+1).$$

Hence singular cohomology naturally leads to the data

$$\mathbf{K}(\pi) := \{K(\pi, n), s_n : SK(\pi, n) \to K(\pi, n+1)\}.$$

making use of the Puppe sequences (Theorem 7.47) one can recover the long exact sequence of a cofibration in singular cohomology from the data $\mathbf{K}(\pi)$. In fact, the following theorem, which generalizes Theorem 8.20, shows that singular homology and cohomology is entirely determined by $\mathbf{K}(\pi)$. The directed systems used to form the colimits are obtained by composing the suspension maps with the maps s_n.

Theorem 9.21. *For any space X,*

1. $H_n(X;\pi) = \operatorname{colim}_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge K(\pi, \ell))$,
2. $H^n(X;\pi) = \operatorname{colim}_{\ell\to\infty} [S^\ell(X_+), K(\pi, n+\ell)]_0$.

Sketch of proof. Recall that the suspension isomorphism in cohomology is the composite of isomorphisms

$$E^* : H^{n+1}(SX) \xrightarrow{c^*} H^{n+1}(CX/C) \xleftarrow{\delta} H^n X.$$

For any pair of spaces X, Y, denote by $A : [X, \Omega Y]_0 \to [SX, Y]_0$ the adjoint isomorphism, given by passing to path components in Theorem 7.42.

Proposition 9.22. *The diagram*

$$\begin{tikzcd}
 & {[X, \Omega K(\mathbf{Z}, n)]_0} \arrow[dr, "A"] & \\
{[X, K(\mathbf{Z}, n)]_0} \arrow[ur, "h_n"] \arrow[r, "S"] \arrow[d, "\cong"] & {[SX, SK(\mathbf{Z}, n)]_0} \arrow[r, "k_n"] & {[SX, K(\mathbf{Z}, n+1)]_0} \arrow[d, "\cong"] \\
H^n(X;\mathbf{Z}) & & H^{n+1}(SX;\mathbf{Z}) \arrow[ll, "E^*"]
\end{tikzcd}$$

commutes.

Proof. That the top part of the diagram commutes is easily checked from the definitions. Theorem 8.20 states that the isomorphism $[X, K(\mathbf{Z}, n)] \to H^k(X; \mathbf{Z})$ takes $\alpha : X \to K(\mathbf{Z}, n)$ to $\alpha^*(\iota_n)$, where $\iota_n \in H^n(K(\mathbf{Z}, n); \mathbf{Z})$ denotes the fundamental class. Thus by naturality, it suffices to consider the case when $X = K(\mathbf{Z}, n)$, and $\alpha : K(\mathbf{Z}, n) \to K(\mathbf{Z}, n)$ equals the identity. Hence what must be shown is that $E^* \circ k_n^*(\iota_{n+1}) = \iota_n$. This follows from the universal coefficient theorem and the fact that the diagram

$$\begin{tikzcd}
\pi_n(K(\mathbf{Z}, n)) \arrow[r, "S"] \arrow[d, "\rho"] & \pi_{n+1}(SK(\mathbf{Z}, n)) \arrow[r, "k_n"] \arrow[d, "\rho"] & \pi_{n+1}(K(\mathbf{Z}, n+1)) \arrow[d, "\rho"] \\
H_n(K(\mathbf{Z}, n)) \arrow[r, "E_*"] & H_{n+1}(SK(\mathbf{Z}, n)) \arrow[r, "k_n"] & H_{n+1}(K(\mathbf{Z}, n+1))
\end{tikzcd}$$

commutes by Proposition 9.6, with all vertical Hurewicz maps isomorphisms by the Hurewicz theorem. $\square$

Proposition 9.22 (and being careful with base points, as in Section 7.62) implies that the composites

$$k_{n+\ell} \circ S : [S^\ell(X_+), K(\mathbf{Z}, n+\ell)]_0 \to [S^{\ell+1}(X_+), K(\mathbf{Z}, n+\ell+1)]_0$$

are isomorphisms. This verifies the second statement of Theorem 9.21 The first statement can be proven by starting with the statement for cohomology and using Spanier-Whitehead duality. $\square$

Next consider stable homotopy groups. The collection of spheres, $\{S^n\}_{n=0}^\infty$, together with the maps (in fact homeomorphisms)

$$s_n : S(S^n) \xrightarrow{\cong} S^{n+1}$$

forms a system of spaces

$$\mathbf{S} := \{S^n, s_n : SS^n \to S^{n+1}\}.$$

From $\mathbf{S}$ one can construct the stable homotopy groups $\pi_n^S(X)$ of a based space X by

$$\pi_n^S(X) = \operatorname*{colim}_{\ell \to \infty} \pi_{n+\ell}(S^\ell \wedge X)$$

where the maps s_n are used to define the directed system by composing with the suspension map:

$$\pi_{n+\ell}(S^\ell \wedge X) \xrightarrow{S} \pi_{n+\ell+1}(SS^\ell \wedge X) \xrightarrow{s_\ell} \pi_{n+\ell+1}(S^{\ell+1} \wedge X).$$

Motivated by Theorem 9.21, one might define *stable cohomotopy* of a based space by

$$\pi^n_S(X) = \operatorname*{colim}_{\ell\to\infty}[S^\ell(X), S^{n+\ell}]_0.$$

Finally, consider bordism. Given a stable system of vector bundles $\boldsymbol{\gamma} = \{\gamma_n\}$, the sequence of Thom spaces $T(\gamma_n)$ are equipped with maps

$$s_n : ST(\gamma_n) \to T(\gamma_{n+1})$$

defining the data

$$\mathbf{M}(\boldsymbol{\gamma}) := \{T(\gamma_n), s_n : ST(\gamma_n) \to T(\gamma_{n+1})\}.$$

Theorem 9.14 shows that the Pontrjagin-Thom construction induces a natural isomorphism

$$\Omega^\gamma_n(X) \xrightarrow{\cong} \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge T(\gamma_\ell)).$$

The reader will by now anticipate that *cobordism* might be defined by

$$\Omega^n_\gamma(X) = \operatorname*{colim}_{\ell\to\infty}[S^\ell(X_+), T(\gamma_{n+\ell})]_0.$$

The notion of a spectrum abstracts from these three examples and introduces a category which measures "stable" phenomena, that is, phenomena which are preserved by suspending. Recall that $\widetilde{H}^n(X) = \widetilde{H}^{n+1}(SX)$. By definition $\pi^S_n(X) = \pi^S_{n+1}(SX)$. Similarly

$$\Omega^\gamma_n(X) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge T(\gamma_\ell)) = \Omega^\gamma_{n+1}(SX).$$

Thus cohomology, stable homotopy, and stable $\boldsymbol{\gamma}$-bordism groups measure stable information about a space X.

Definition 9.23. A *spectrum* is a sequence of pairs $\{K_n, k_n\}$ where the K_n are based spaces and $k_n : SK_n \to K_{n+1}$ are base point preserving maps, where SK_n denotes the suspension.

Thus $\mathbf{K}(\pi)$ is called the *Eilenberg-MacLane spectrum* for (co)homology with coefficients in π, $\mathbf{S}$ is called the *sphere spectrum* for stable homotopy, and $\mathbf{M}(\boldsymbol{\gamma})$ is called the *Thom spectrum* for $\boldsymbol{\gamma}$-bordism.

The examples above motivate to the following definition of homology and cohomology for based spaces and pairs of spaces.

Recall that X_+ denotes the space X with a disjoint base point. In particular, if $A \subset X$, then $(X_+/A_+) = X/A$ if A is nonempty and equals X_+ if A is empty.

Definition 9.24. Let $\mathbf{K} = \{K_n, k_n\}$ be a spectrum. Define the *(unreduced) homology and cohomology with coefficients in the spectrum* $\mathbf{K}$ to be the functor taking a space X to the abelian group

$$H_n(X;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell}(X_+ \wedge K_\ell)$$

and

$$H^n(X;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} [S^\ell(X_+); K_{n+\ell}]_0,$$

the *reduced homology and cohomology with coefficients in the spectrum* $\mathbf{K}$ to be the functor taking a based space X to the abelian group

$$\widetilde{H}_n(X;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell}(X \wedge K_\ell)$$

and

$$\widetilde{H}^n(X;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} [S^\ell X; K_{n+\ell}]_0,$$

and the *homology and cohomology of a pair with coefficients in the spectrum* $\mathbf{K}$ to be the functor taking a pair of spaces (X, A) to the abelian group

$$H_n(X, A;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} \pi_{n+\ell}((X_+/A_+) \wedge K_\ell)$$

and

$$H^n(X, A;\mathbf{K}) = \operatorname*{colim}_{\ell\to\infty} [S^\ell(X_+/A_+); K_{n+\ell}]_0.$$

It is a theorem that these are *generalized (co)homology theories*; they satisfy all the Eilenberg–Steenrod axioms except the dimension axiom. We will discuss this in more detail later.

For example, stable homotopy theory $\widetilde{H}_n(X;\mathbf{S}) = \pi_n^S X$ is a reduced homology theory; framed bordism $H_n(X;\mathbf{S}) = \pi_n^S X_+ = \Omega_n^{\mathrm{fr}}(X)$ is an unreduced homology theory.

Note that $H_n(\mathrm{pt};\mathbf{K})$ can be nonzero for $n \neq 0$, for example $H_n(\mathrm{pt};\mathbf{S}) = \pi_n^S$. Ordinary homology is characterized by the fact that $H_n(\mathrm{pt}) = 0$ for $n \neq 0$ (see Theorem 2.21). The groups $H_n(\mathrm{pt};\mathbf{K})$ are called the *coefficients* of the spectrum.

There are many relationships between reduced homology, unreduced homology, suspension, and homology of pairs, some of which are obvious and some of which are not. We list some facts for homology.

- For a based space X, $\widetilde{H}_n(X;\mathbf{K}) = \widetilde{H}_{n+1}(SX;\mathbf{K})$.
- For a space X, $H_n(X;\mathbf{K}) = \widetilde{H}_n(X_+;\mathbf{K})$.
- For a pair of spaces, $H_n(X, A;\mathbf{K}) \cong \widetilde{H}_n(X/A;\mathbf{K})$.
- For a CW-pair, $H_n(X, A;\mathbf{K})$ fits into the long exact sequence of a pair.

9.10. Generalized homology theories

We have several functors from (based) spaces to graded abelian groups: singular homology and cohomology, stable homotopy $\pi_n^S X$, $\boldsymbol{\gamma}$-bordism $\Omega_n^{\boldsymbol{\gamma}} X$, and, more generally, homology of a space with coefficients in a spectrum $H_n(X;\mathbf{K})$. These are examples of *generalized homology theories*. Generalized homology theories come in two (equivalent) flavors, *reduced* and *unreduced*. Unreduced theories apply to unbased spaces and pairs. Reduced theories are functors on based spaces. The equivalence between the two points of view is obtained by passing from (X, A) to X/A and from X to X_+.

There are three high points to look out for in our discussion of homology theories.

- The axioms of a (co)homology theory are designed for computations. One first computes the coefficients of the theory (perhaps using the Adams spectral sequence) and then computes the homology of a CW-complex X, using excision, Mayer–Vietoris, or a generalization of cellular homology discussed in the next chapter, the Atiyah–Hirzebruch spectral sequence.
- There is a uniqueness theorem. A natural transformation of (co)homology theories inducing an isomorphism on coefficients induces an isomorphism for all CW-complexes X.
- A (co)homology theory is given by (co)homology with coefficients in a spectrum $\mathbf{K}$.

9.10.1. Reduced homology theories.

Let CGH_* be the category of compactly generated spaces with nondegenerate base points.

Definition 9.25. A *reduced homology theory* is

1. A family of functors
$$h_n : \mathrm{CGH}_* \to \mathrm{Ab} \ \text{ for } n \in \mathbf{Z}.$$
(Remark: We *do not* assume h_n is zero for $n < 0$.)
2. A family of natural transformations
$$e_n : h_n \to h_{n+1} \circ \mathcal{S}$$
where $S : \mathrm{CGH}_* \to \mathrm{CGH}_*$ is the (reduced) suspension functor.

These must satisfy the three following axioms:

A1. *(Homotopy)* If $f_0, f_1 : X \to Y$ are homotopic, then
$$h_n(f_0) = h_n(f_1) : h_n X \to h_n Y.$$

A2. *(Exactness)* For $f : X \to Y$, let C_f be the mapping cone of f, and $j : Y \hookrightarrow C_f$ the inclusion. Then

$$h_n X \xrightarrow{h_n(f)} h_n Y \xrightarrow{h_n(j)} h_n(C_f)$$

is exact for all $n \in \mathbf{Z}$.

A3. *(Suspension)* The homomorphism

$$e_n(X) : h_n X \to h_{n+1}(SX)$$

given by the natural transformation e_n is an isomorphism for all $n \in \mathbf{Z}$.

Exercise 187. Show that ordinary singular homology defines a homology theory in this sense by taking $h_n X$ to be the reduced homology of X.

There are two other "nondegeneracy" axioms which a given generalized homology theory may or may not satisfy.

A4. *(Additivity)* If X is a wedge sum $X = \bigvee_{j \in J} X_j$, then

$$\bigoplus_{j \in J} h_n X_j \to h_n X$$

is an isomorphism for all $n \in \mathbf{Z}$.

A5. *(Isotropy)* If $f : X \to Y$ is a weak homotopy equivalence, then $h_n(f)$ is an isomorphism for all $n \in \mathbf{Z}$.

If we work in the category of based CW-complexes instead of CGH_*, then Axiom A5 follows from Axiom A1 by the Whitehead theorem. Given a reduced homology theory on based CW-complexes, it extends uniquely to an isotropic theory on CGH_*, by taking a CW-approximation.

For any reduced homology theory, $h_n(\mathrm{pt}) = 0$ for all n, since

$$h_n(\mathrm{pt}) \to h_n(\mathrm{pt}) \to h_n(\mathrm{pt}/\mathrm{pt}) = h_n(\mathrm{pt})$$

is exact, but also each arrow is an isomorphism. Thus the reduced homology of a point says nothing about the theory; instead one makes the following definition.

Definition 9.26. The *coefficients* of a reduced homology theory are the groups $\{h_n(S^0)\}$.

A homology theory is called *ordinary* (or *proper*) if it satisfies

$$h_n S^0 = 0 \text{ for } n \neq 0.$$

(This is the dimension axiom of Eilenberg–Steenrod.) Singular homology with coefficients in an abelian group A is an example of an ordinary theory.

It follows from a simple argument using the Atiyah-Hirzebruch spectral sequence that any ordinary reduced homology theory is isomorphic to reduced singular homology with coefficients in $A = h_0(S^0)$.

If (X, A) is a cofibration, then we saw in Chapter 7 that the mapping cone C_f is homotopy equivalent to X/A. Thus $h_nA \to h_nX \to h_n(X/A)$ is exact. Also in Chapter 7 we proved that the sequence

$$A \to X \to X/A \to SA \to SX \to S(X/A) \to \cdots$$

has each three term sequence a (homotopy) cofibration. Thus

$$h_nA \to h_nX \to h_n(X/A) \to h_n(SA) \to h_n(SX) \to \cdots$$

is exact. Applying the transformations e_n and using Axiom A3, we conclude that

$$\to h_nA \to h_nX \to h_n(X/A) \to h_{n-1}A \to h_{n-1}X \to \cdots$$

is exact. Thus to any reduced homology theory one obtains a long exact sequence associated to a cofibration.

Exercise 188. Let X be a based CW-complex which is the union of subcomplexes A and B, both of which contain the base point. Show that for any reduced homology theory h_* there is a Mayer–Vietoris long exact sequence

$$\cdots \to h_n(A \cap B) \to h_nA \oplus h_nB \to h_nX \to h_{n-1}(A \cap B) \to \cdots .$$

9.10.2. Unreduced homology theories. We will derive unreduced theories from reduced theories to emphasize that these are the same concept presented slightly differently.

Let CGH^2 denote the category of pairs (X, A) with $A \hookrightarrow X$ a cofibration. We allow the case when A is empty. Given a reduced homology theory $\{h_n, e_n\}$ define functors H_n on CGH^2 as follows (for this discussion, H_n *does not* denote ordinary singular homology!).

1. Let
$$H_n(X, A) = h_n(X_+/A_+) = \begin{cases} h_n(X/A) & \text{if } A \neq \phi, \\ h_n(X_+) & \text{if } A = \phi \end{cases}$$

2. Define the connecting homomorphism $\partial_n : H_n(X, A) \to H_{n-1}A$ to be the composite:

$$H_n(X, A) = h_n(X_+/A_+) \xrightarrow{\cong} h_n(C_i) \longrightarrow h_n(SA_+) \xrightarrow{\cong} h_{n-1}(A_+) = H_{n-1}A$$

where C_i is the mapping cone of the inclusion $i : A_+ \hookrightarrow X_+$, and $C_i \to SA_+$ is the quotient

$$C_i \to C_i/X_+ = SA_+.$$

Then $\{H_n, \partial_n\}$ satisfy the Eilenberg–Steenrod axioms:

A1. *(Homotopy)* If $f_0, f_1 : (X, A) \to (Y, B)$ are (freely) homotopic, then
$$H_n(f_0) = H_n(f_1) : H_n(X, A) \to H_n(Y, B).$$

A2. *(Exactness)* For a cofibration $i : A \hookrightarrow X$, let $j : (X, \phi) \hookrightarrow (X, A)$; then
$$\cdots \to H_{n+1}(X, A) \xrightarrow{\partial_{n+1}} H_n A \xrightarrow{H_n(i)} H_n X \xrightarrow{H_n(j)} H_n(X, A) \to \cdots$$
is exact.

A3. *(Excision)* Suppose that $X = A \cup B$, with A, B closed, and suppose that $(A, A \cap B)$ is a cofibration. Then
$$H_n(A, A \cap B) \to H_n(X, B)$$
is an isomorphism for all $n \in \mathbf{Z}$.

Exercise 189. Prove that these three properties hold using the axioms of a reduced theory.

If a reduced theory is additive and/or isotropic, the functors H_n likewise satisfy

A4. *(Additivity)* Let $X = \amalg_{j \in J} X_j,\ A \subset X,\ A_j = X_j \cap A$. Then
$$\bigoplus_{j \in J} H_n(X_j, A_j) \to H_n(X, A)$$
is an isomorphism for all $n \in \mathbf{Z}$.

A5. *(Isotropy)* If $f : X \to Y$ is a weak homotopy equivalence, then $H_n(f) : H_n X \to H_n Y$ is an isomorphism for all $n \in \mathbf{Z}$.

Notice that if the reduced theory is ordinary, then $H_n(\mathrm{pt}) = 0$ for $n \neq 0$.

One uses these properties to define an unreduced homology theory.

Definition 9.27. A collection of functors $\{H_n, \partial_n\}$ on CGH^2 is called an *(unreduced) homology theory* if it satisfies the three axioms A1, A2, and A3. It is called *additive* and/or *isotropic* if Axiom A4 and/or Axiom A5 hold. It is called *ordinary* or *proper* if $H_n(\mathrm{pt}) = 0$ for $n \neq 0$.

The *coefficients* of the unreduced homology theory are $\{H_n(\mathrm{pt})\}$.

One can go back and forth: an unreduced homology theory $\{H_n, \partial_n\}$ defines a reduced one by taking $h_n X = H_n(X, \{*\})$. The following theorem is proved in [**54**, Section XII.6].

Theorem 9.28. *These constructions set up a $1-1$ correspondence (up to natural isomorphism) between reduced homology theories on* CGH_* *and (unreduced) homology theories on* CGH^2*. Moreover the reduced theory is additive, isotropic, or ordinary if and only if the corresponding unreduced theory is.* □

The uniqueness theorem below has an easy inductive cell-by-cell proof in the case of finite CW-complexes, but requires a more delicate limiting argument for infinite CW-complexes.

Exercise 190. Show that $H_n(X \times S^1) \cong H_n X \oplus H_{n-1} X$ for any space X and any homology theory H_*.

Theorem 9.29 (Eilenberg–Steenrod uniqueness theorem)**.**

1. *Let $T : (H_n, \partial_n) \to (H'_n, \partial'_n)$ be a natural transformation of homology theories defined on the category of finite CW-pairs such that $T : H_*(\mathrm{pt}) \to H'_*(\mathrm{pt})$ is an isomorphism. Then $T : H_*(X, A) \to H'_*(X, A)$ is an isomorphism for all finite CW-pairs.*
2. *Let $T : (H_n, \partial_n) \to (H'_n, \partial'_n)$ be a natural transformation of additive homology theories defined on the category of CW-pairs where $T : H_*(\mathrm{pt}) \to H'_*(\mathrm{pt})$ is an isomorphism. Then $T : H_*(X, A) \to H'_*(X, A)$ is an isomorphism for all CW-pairs.*

9.10.3. Homology theories and spectra.

Theorem 9.30. *(Reduced) homology with coefficients in a spectrum* $\mathbf{K}$

$$\widetilde{H}_n(-; \mathbf{K}) : X \mapsto \operatorname*{colim}_{\ell \to \infty} \pi_{n+\ell}(X \wedge K_\ell)$$
$$H_n(-; \mathbf{K}) : (X, A) \mapsto \operatorname*{colim}_{\ell \to \infty} \pi_{n+\ell}((X_+/A_+) \wedge K_\ell)$$

is a (reduced) homology theory satisfying the additivity axiom.

One needs to prove the axioms A1, A2, A3, and A4. The homotopy axiom is of course obvious. The axiom A2 follows from the facts about the Puppe sequences we proved in Chapter 7 by passing to the limit. The suspension axiom holds almost effortlessly from the fact that the theory is defined by taking the direct limit (i.e. colimit) over suspension maps. The additivity axiom follows from the fact that the image of a sphere is compact and that a compact subspace of an infinite wedge is contained in a finite wedge.

A famous theorem of E. Brown (the Brown representation theorem) gives a converse of the above theorem. It leads to a shift in perspective on the functors of algebraic topology by prominently placing spectra as the source of homology theories. Here is a precise statement.

Theorem 9.31.

1. *Let $\{H_n, \partial_n\}$ be a homology theory. There there exists a spectrum $\mathbf{K}$ and a natural isomorphism $H_n(X, A) \cong H_n(X, A; \mathbf{K})$ for all finite CW-pairs.*

2. *Let $\{H_n, \partial_n\}$ be an additive homology theory. There there exists a spectrum $\mathbf{K}$ and a natural isomorphism $H_n(X,A) \cong H_n(X,A;\mathbf{K})$ for all CW-pairs.*

Note that the Brown representation theorem shows that for any homology theory there is a spectrum and hence an associated generalized cohomology theory.

Exercise 191. Give a definition of a map of spectra. Define maps of spectra $\mathbf{S} \to \mathbf{K}(\mathbf{Z})$ and $\mathbf{S} \to \mathbf{M}(\boldsymbol{\gamma})$ inducing the Hurewicz map $\pi_n^S X \to \widetilde{H}_n X$ and the map $\Omega_n^{\text{fr}} X \to \Omega_n^{\gamma} X$ from framed to $\boldsymbol{\gamma}$-bordism.

9.10.4. Generalized cohomology theories. The development of cohomology theories parallels that of homology theories following the principle of reversing arrows.

Exercise 192. Define reduced and unreduced *cohomology* theories.

There is one surprise however. In order for $H^n(\quad;\mathbf{K})$ to be an additive theory (which means the cohomology of a disjoint union is a direct *product*), one must require that $\mathbf{K}$ is a *loop spectrum,* also called an Ω-*spectrum*. By definition, a loop spectrum is a spectrum so that the adjoints

$$K_n \to \Omega K_{n+1}$$

of the structure maps k_n are homotopy equivalences. Conversely, the Brown representation theorem applied to an additive cohomology theory produces an Ω-spectrum. The Eilenberg–MacLane spectrum is an Ω-spectrum while the sphere spectrum or more generally bordism spectra are not.

An important example of a generalized cohomology theory whose spectrum is an Ω-spectrum is topological K-theory. It is the subject of one of the projects at the end of this chapter. Complex topological K-theory has a definition in terms of stable equivalence classes of complex vector bundles, but we instead indicate the definition in terms of a spectrum. Most proofs of the Bott periodicity theorem (Theorem 7.56, which states that $\pi_n U \cong \mathbf{Z}$ for n odd and $\pi_n U = 0$ for n even) actually prove a stronger result, that there is a homotopy equivalence

$$\mathbf{Z} \times BU \simeq \Omega^2(\mathbf{Z} \times BU).$$

This allows the definition of the complex K-theory spectrum with

$$K_n = \begin{cases} \mathbf{Z} \times BU & \text{if } n \text{ is even,} \\ \Omega(\mathbf{Z} \times BU) & \text{if } n \text{ is odd.} \end{cases} \tag{9.21}$$

The structure maps k_n

$$S(\mathbf{Z} \times BU) \to \Omega(\mathbf{Z} \times BU)$$
$$S\Omega(\mathbf{Z} \times BU) \to \mathbf{Z} \times BU$$

are given by the adjoints of the Bott periodicity homotopy equivalence and the identity map

$$\mathbf{Z} \times BU \to \Omega^2(\mathbf{Z} \times BU)$$
$$\Omega(\mathbf{Z} \times BU) \to \Omega(\mathbf{Z} \times BU).$$

Thus the complex K-theory spectrum is an Ω-spectrum. The corresponding cohomology theory is called complex K-theory and satisfies

$$K^n(X) = K^{n+2}(X) \quad \text{for all } n \in \mathbf{Z}.$$

In particular this is a nonconnective cohomology theory, where a *connective* cohomology theory is one that satisfies $H^n(X) = 0$ for all $n < n_0$. Ordinary homology, as well as bordism theories, are connective, since a manifold of negative dimension is empty.

A good reference for the basic results in the study of spectra (stable homotopy theory) is Adams' book [**2**].

9.11. Projects: Differential topology; K-theory; SW duality

9.11.1. Basic notions from differential topology. Define a smooth manifold and submanifold, the tangent bundle of a smooth manifold, a smooth map between manifolds and its differential, an isotopy, the Sard theorem, transversality, the tubular neighborhood theorem, and the decomposition

$$TM|_N = TN \oplus \nu(N \subset M),$$

where $N \subset M$ is a smooth submanifold, and show that if $f : M \to P$ is a smooth map transverse to a submanifold $Q \subset P$, with $N = f^{-1}Q$, then the differential of f induces a bundle map (i.e. a linear isomorphism in each fiber) $df : \nu(N \subset M) \to \nu(Q \subset P)$. A good reference is Hirsch's book [**22**].

9.11.2. Definition of topological K-theory. Define complex (topological) K-theory of a space in terms of vector bundles. Indicate why the spectrum for this theory is $\{K_n\}$ given in Equation (9.21). State the Bott periodicity theorem. Discuss vector bundles over spheres. Discuss real K-theory. References for this material are the books by Atiyah [**4**] and Husemoller [**23**].

9.11.3. Spanier-Whitehead duality. Spanier-Whitehead duality is a generalization of Alexander duality which gives a geometric method of going back and forth between a generalized homology theory and a generalized cohomology theory. Suppose that $X \subset S^{n+1}$ is a finite simplicial complex, and let $Y = S^{n+1} - X$, or better, $Y = S^{n+1} - U$ where U is some open simplicial neighborhood of X which deformation retracts to X. Recall that Alexander duality implies that

$$\widetilde{H}^p X \cong \widetilde{H}_{n-p} Y.$$

(See Theorem 4.32.) What this means is that the cohomology of X determines the homology of Y and vice versa.

The strategy is to make this work for generalized cohomology theories and any space X and to remove the dependence on the embedding. The best way to do this is to do it carefully using spectra. Look at Spanier's article [**44**]. There is a good sequence of exercises developing this material in [**45**, pages 462-463]. Another reference using the language of spectra is [**49**, page 321].

Here is a slightly low-tech outline. You should lecture on the following, providing details.

Given based spaces X and Y, let

$$\{X, Y\} = \operatorname*{colim}_{k\to\infty} [S^k \wedge X, S^k \wedge Y]_0.$$

Given a finite simplicial subcomplex $X \subset S^{n+1}$, let $D_n X \subset S^{n+1}$ be a finite simplicial subcomplex which is a deformation retract of $S^{n+1} - X$. Then $SD_n X$ is homotopy equivalent to $S^{n+2} - X$.

For k large enough, the homotopy type of the suspension $S^k D_n X$ depends only on X and $k + n$, and not on the choice of embedding into S^{n+1}. Moreover, for any spaces Y and Z

$$\{S^q Y, D_n X \wedge Z\} = \{S^{q-n} Y \wedge X, Z\}. \tag{9.22}$$

As an example, taking $Y = S^0$ and $Z = K(\mathbf{Z}, p+q-n)$, Equation (9.22) says that

$$\{S^q, D_n X \wedge K(\mathbf{Z}, p+q-n)\} = \{S^{q-n} \wedge X, K(\mathbf{Z}, p+q-n)\}. \tag{9.23}$$

Definition 9.24 says that the left side of Equation (9.23) is $\widetilde{H}_{n-p}(D_n X; \mathbf{K}(\mathbf{Z}))$. The right side is $\widetilde{H}^p(X; \mathbf{Z})$, using the fact that

$$[SA, K(\mathbf{Z}, k)] = [A, \Omega K(\mathbf{Z}, k)] = [A, K(\mathbf{Z}, k-1)].$$

What this means is that by combining Alexander duality, the result $H^q X = [X, K(\mathbf{Z}, q)]$ of obstruction theory, and Spanier-Whitehead duality (i.e. Equation (9.22)), the definition of homology with coefficients in the Eilenberg–MacLane spectrum given in Definition 9.24 coincides with the usual definition of (ordinary) homology (at least for finite simplicial complexes, but this works more generally).

This justifies Definition 9.24 of homology with coefficients in an arbitrary spectrum $\mathbf{K}$. It also gives a duality $\widetilde{H}_{n-p}(D_nX;\mathbf{K}) = \widetilde{H}^p(X;\mathbf{K})$, which could be either considered as a generalization of Alexander duality or as a further justification of the definition of (co)homology with coefficients in a spectrum.

Chapter 10

Spectral Sequences

Spectral sequences are powerful computational tools in topology. They also can give quick proofs of important theoretical results such as the Hurewicz theorem and the Freudenthal suspension theorem. Computing with spectral sequences is somewhat like computing integrals in calculus; it is helpful to have ingenuity and a supply of tricks, and even so, you may not arrive at the final solution to your problem. There are many spectral sequences which give different kinds of information. We will focus on one important spectral sequence, the *Leray-Serre-Atiyah-Hirzebruch spectral sequence* which takes as input a fibration over a CW-complex and a generalized homology or cohomology theory. This spectral sequence exhibits a complicated relationship between the generalized (co)homology of the total space and fiber and the ordinary (co)homology of the base. Many other spectral sequences can be derived from this one by judicious choice of fibration and generalized (co)homology theory.

Carefully setting up and proving the basic result requires very careful bookkeeping; the emphasis in these notes will be on applications and how to calculate. The project for this chapter is to outline the proof of the main theorem, Theorem 10.7.

10.1. Definition of a spectral sequence

Definition 10.1. A *spectral sequence* is a homological object of the following type:

One is given a sequence of chain complexes

$$(E^r, d^r) \text{ for } r = 1, 2, \dots$$

and isomorphisms

$$E^{r+1} \cong H(E^r, d^r) = \frac{\ker d^r : E^r \to E^r}{\operatorname{im} d^r : E^r \to E^r}.$$

The isomorphisms are fixed as part of the structure of the spectral sequence, so henceforth we will fudge the distinction between "≅" and "=" in the above context.

In this definition the term "chain complex" just means an abelian group (or R-module) with an endomorphism whose square is zero. In many important contexts, the spectral sequence has more structure: namely the chain complexes E^r are bigraded.

Definition 10.2. A *bigraded homology spectral sequence* is a spectral sequence such that each E^r has a direct sum decomposition

$$E^r = \bigoplus_{(p,q)\in \mathbf{Z}\oplus\mathbf{Z}} E^r_{p,q}$$

and the differential d^r has bidegree $(-r, r-1)$, that is,

$$d^r(E^r_{p,q}) \subset E^r_{p-r,q+r-1}$$

Write $d^r_{p,q}$ for the restriction of d^r to $E^r_{p,q}$, so that

$$E^{r+1}_{p,q} = \frac{\ker d^r_{p,q}}{\operatorname{im} d^r_{p+r,q-r+1}}.$$

A student first exposed to this plethora of notation may be intimidated; the important fact to keep in mind is that a bigrading decomposes a big object (E^r) into bite-sized pieces ($E^r_{p,q}$). Information about the $E^r_{p,q}$ for some pairs (p, q) gives information about $E^{r+1}_{p,q}$ for (probably fewer) pairs (p, q). But with luck one can derive valuable information. For example, from what has been said so far you should easily be able to see that if $E^r_{p,q} = 0$ for some fixed pair (p, q), then $E^{r+k}_{p,q} = 0$ for all $k \geq 0$. This simple observation can sometimes be used to derive highly nontrivial information.

One usually computes with a spectral sequence in the following way. A theorem will state that there exists a spectral sequence so that:

1. the modules $E^2_{p,q}$ (or $E^1_{p,q}$) can be identified with something known, and
2. Given p, q, there exists $r(p, q)$ so that $E^{r+1}_{p,q} = E^r_{p,q}$ for $r \geq r(p, q)$. Setting $E^\infty_{p,q} = E^r_{p,q}$ for any $r \geq r(p, q)$, then $E^\infty_{p,q}$ is related to something one wishes to compute.

It can also work the opposite way: E^∞ can be related to something known, and E^2 can be related to something we wish to compute. In either case, this gives a complicated relationship between two things. The relationship usually involves exact sequences. In favorable circumstances information can be derived by carefully analyzing this relationship.

A helpful way to organize the information contained in a bigraded homology spectal sequence is to think of a spectral sequence as a book, with E^r the rth page. Each page is a doubly indexed table, with (p,q) entry the module $E^r_{p,q}$. Every entry $E^r_{p,q}$ in the rth page has a differential exiting and entering it, namely $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ and $d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}$. Turning the page corresponds to passing to homology. In favorable situations, the (p,q) entry stabilizes, i.e. after a certain page the differentials exiting and entering $E^r_{p,q}$ are both zero, and hence turning the page no longer changes the the (p,q) entry. These stabilized entries form the entries of the E^∞ page.

As an example to see how this may be used, the Leray–Serre spectral sequence of a fibration implies that if $F \hookrightarrow E \to B$ is a fibration with B simply connected, then there is a spectral sequence with

$$E^2_{p,q} \cong H_p(B;\mathbf{Q}) \otimes H_q(F;\mathbf{Q})$$

and with

$$H_n(E;\mathbf{Q}) \cong \oplus_p E^\infty_{p,n-p}.$$

This establishes a relationship between the homology of the base, total space, and fiber of a fibration. Of course, the hard work when computing with this spectral sequence is in getting from E^2 to E^∞. But partial computations and results are often accessible. For example, we will show later (and the reader may wish to show as an exercise now) that if $\oplus_p H_p(B;\mathbf{Q})$ and $\oplus_q H_q(F;\mathbf{Q})$ are finite-dimensional, then so is $\oplus_n H_n(E;\mathbf{Q})$ and

$$\chi(B)\cdot\chi(F) = \chi(E).$$

Another example: if $H_p(B;\mathbf{Q}) \otimes H_{n-p}(F;\mathbf{Q}) = 0$ for all p, then $H_n(E;\mathbf{Q}) = 0$. This generalizes a similar fact which can be proven for the trivial fibration $B \times F \to B$ using the Künneth theorem.

The next few definitions will provide us with a language to describe the way that the parts of the spectral sequence fit together.

Definition 10.3. A *filtration of an R-module A* is an increasing sequence

$$0 \subset \cdots \subset F_{-1} \subset F_0 \subset F_1 \subset \cdots \subset F_p \subset \cdots \subset A$$

of submodules. A filtration is *convergent* if the union of the F_p's is A and their intersection is 0.

If $A = \oplus_n A_n$ is a graded module then we assume that the filtration preserves the grading, i.e. $F_p = \oplus_n (F_p \cap A_n)$. In this case we bigrade the filtration by setting

$$F_{p,q} = F_p \cap A_{p+q}.$$

We will mostly deal with filtrations that are *bounded below*, i.e. $F_s = 0$ for some s; or *bounded above*, i.e. $F_t = A$ for some t; or *bounded*, i.e. bounded above and bounded below. In this book, typically $F_{-1} = 0$.

Definition 10.4. Given a filtration $F = \{F_n\}$ of an R-module A, the *associated graded module* is the graded R-module denoted by $\mathrm{Gr}(A, F)$ and defined by

$$\mathrm{Gr}(A, F)_p = \frac{F_p}{F_{p-1}}.$$

We will usually just write $\mathrm{Gr}(A)$ when the filtration is clear from context.

In general, one is interested in the algebraic structure of A rather than $\mathrm{Gr}(A)$. Notice that $\mathrm{Gr}(A)$ contains some (but not necessarily all) information about A. For example, for a convergent filtration:

1. If $\mathrm{Gr}(A) = 0$, then $A = 0$.
2. If R is a field and A is a finite dimensional vector space, then each F_i is a subspace and $\mathrm{Gr}(A)$ and A have the same dimension. Thus in this case $\mathrm{Gr}(A)$ determines A up to isomorphism. This holds for more general R if each $\mathrm{Gr}(A)_n$ is free and the filtration is bounded above.
3. If $R = \mathbf{Z}$, then given a prime b, information about the b-primary part of $\mathrm{Gr}(A)$ gives information about the b-primary part of A; e.g. if $\mathrm{Gr}(A)_p$ has no b-torsion for all p, then A has no b-torsion for all p. However, the b-primary part of $\mathrm{Gr}(A)$ does not determine the b-primary part of A; e.g. if $\mathrm{Gr}(A)_0 = \mathbf{Z}$, $\mathrm{Gr}(A)_1 = \mathbf{Z}/2$, and $\mathrm{Gr}(A)_n = 0$ for $n \neq 0, 1$, it is impossible to determine whether $A \cong \mathbf{Z}$ or $A \cong \mathbf{Z} \oplus \mathbf{Z}/2$.

In short, knowing the quotients $\mathrm{Gr}(A)_p = F_p \big/ F_{p-1}$ determines A up to "extension questions," at least when the filtration is bounded.

Definition 10.5. Given a bigraded homology spectral sequence $(E^r_{p,q}, d^r)$ and a graded R-module A_*, we say *the spectral sequence converges to* A_* and write

$$E^r_{p,q} \Rightarrow A_{p+q}$$

if:

1. for each p, q, there exists an r_0 so that $d^r_{p,q}$ is zero for each $r \geq r_0$ (by Exercise 193 below this implies $E^r_{p,q}$ surjects to $E^{r+1}_{p,q}$ for $r \geq r_0$), and
2. there is a convergent filtration of A_*, so that for each n, the colimit $E^\infty_{p,n-p} = \operatorname*{colim}_{r\to\infty} E^r_{p,n-p}$ is isomorphic to the associated graded module $\mathrm{Gr}(A_n)_p$.

In many favorable situations (e.g. *first-quadrant spectral sequences* where $E^2_{p,q} = 0$ if $p < 0$ or $q < 0$) the convergence is stronger; namely for each pair (p, q) there exists an r_0 so that $E^r_{p,q} = E^\infty_{p,q}$ for all $r \geq r_0$.

Exercise 193. Fix $p, q \in \mathbf{Z} \oplus \mathbf{Z}$.

1. Show that if there exists $r_0(p, q)$ so that $d^r_{p,q} = 0$ for all $r \geq r_0(p, q)$, then there exists a surjection $E^r_{p,q} \to E^{r+1}_{p,q}$ for all $r \geq r_0(p, q)$.
2. Show that if $E^2_{p,q} = 0$ whenever $p < 0$, then there exists a number $r_0 = r_0(p, q)$ as in the first part.

An even stronger notion of convergence is the following. Suppose that there exists an r_0 so that for each (p, q) and all $r \geq r_0$, $E^r_{p,q} = E^\infty_{p,q}$. When this happens we say *the spectral sequence collapses at* E^{r_0}.

Theorems on spectral sequences usually take the form: "There exists a spectral sequence with $E^2_{p,q}$ (or $E^1_{p,q}$) some known object converging to A_*." This is an abbreviated way of saying that the E^∞-terms are, on the one hand, the colimits of the E^r-terms and, on the other, the graded pieces in the associated graded module $\mathrm{Gr}(A_*)$ to A_*.

10.2. The spectral sequence of a filtered complex

Suppose that (C, ∂) is a filtered chain complex of modules over a commutative ring. Thus one is given an increasing sequence of sub-chain complexes

$$\cdots \subset (F_pC, \partial) \subset (F_{p+1}C, \partial) \subset \cdots \subset (C, \partial).$$

We assume that the filtration is convergent, so that $\cup_p F_pC = C$ and $\cap_p F_pC = 0$. We further assume that it is bounded below, so that there exists a p_0 so that $F_pC = 0$ for all $p < p_0$. Taking the grading of C into account leads to the bigrading convention where the nth graded part of the subcomplex F_pC, F_pC_n, is denoted by $C_{p,n-p}$. Thus the first subscript corresponds to the filtration level, and the sum of the two subscripts corresponds to the grading index. Hence the subcomplex $(F_pC, \partial) \subset (C, \partial)$ is expressed in this notation as

$$\cdots \xrightarrow{\partial} C_{p,n-p} \xrightarrow{\partial} C_{p,n-1-p} \xrightarrow{\partial} \cdots$$

and the inclusions of the nth chain modules $F_pC_n \subset F_{p+1}C_n$ for increasing p are

$$\cdots \subset C_{p,n-p} \subset C_{p+1,n-p-1} \subset \cdots \subset C_n.$$

Notice that a filtration of a chain complex (C, ∂) gives, for each p, a quotient complex $(F_pC/F_{p-1}C, \partial)$. Moreover the short exact sequence

$$0 \to F_{p-1}C \to F_pC \to F_pC/F_{p-1}C \to 0$$

of chain complexes leads to a long exact sequence in homology, and in particular to the connecting homomorphism

$$\partial : H_n(F_pC/F_{p-1}C) \to H_{n-1}(F_{p-1}C).$$

The following theorem asserts that a filtered chain complex which is bounded below gives rise to a spectral sequence.

Theorem 10.6. *Suppose that (C, ∂) is a filtered chain complex which is convergent and bounded below. Then there exists a spectral sequence with*

$$E^1_{p,q} \cong H_{p+q}(F_pC/F_{p-1}C, \partial)$$

and $d^1 : E^1_{p,q} \to E^1_{p-1,q}$ the composite:

$$H_{p+q}(F_pC/F_{p-1}C) \xrightarrow{\partial} H_{p+q-1}(F_{p-1}C) \to H_{p+q-1}(F_{p-1}C/F_{p-2}C).$$

Moreover, this spectral sequence converges to the homology of (C, ∂) in the following sense. The homology of (C, ∂) is filtered by defining

$$F_pH_n(C, \partial) = \operatorname{im}\bigl(H_n(F_pC, \partial) \to H_n(C, \partial)\bigr).$$

With respect to this filtration, $E^r_{p,q}$ surjects to $E^{r+1}_{p,q}$ for r large enough, and

$$\operatorname*{colim}_{r\to\infty} E^r_{p,n-p} \cong \mathrm{Gr}(H_n(C, \partial))_p.$$

□

We outline the construction. More details can be found in e.g. [**45**] and [**54**].

Define, for each p, q and each $r \geq 1$,

$$Z^r_{p,q} = \ker\left(C_{p,q} \xrightarrow{\partial} C_{p,q-1} \to C_{p,q-1}/C_{p-r,q+r-1}\right)$$

and

$$Z^\infty_{p,q} = \ker\left(C_{p,q} \xrightarrow{\partial} C_{p,q-1}\right).$$

Explicitly, $Z^r_{p,q}$ is the submodule consisting of those elements of grading $p + q$ in the subchain complex $F_pC \subset C$ which are sent by the differential $\partial : F_pC \to F_pC$ into the subcomplex $F_{p-r}C$, and $Z^\infty_{p,q}$ consists of those elements of F_pC_{p+q} sent to zero.

Clearly, $Z^0_{p,q} = C_{p,q}$. Since the filtration is bounded below, $C_{p-r} = 0$ (and hence $C_{p-r,q+r-1} = 0$) for all large enough r, and so $Z^r_{p,q} = Z^\infty_{p,q}$ for all large enough r.

Solving the following simple but valuable exercise will bring the many indices into focus.

Exercise 194.

1. Show that $F_{p-1}C \subset F_pC$ implies that $Z^{r-1}_{p-1,q+1} \subset Z^r_{p,q}$.

2. Show that $\partial^2 = 0$ implies that $\partial Z^{r-1}_{p+r-1,q-r+2} \subset Z^r_{p,q}$.
3. Show that $\partial Z^r_{p,q} \subset Z^r_{p-r,q+r-1}$.

Now define

$$E^r_{p,q} = Z^r_{p,q}/\big(Z^{r-1}_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}\big).$$

To define $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, the third assertion of Exercise 194 shows that there is a well-defined composite

$$\tilde{d}^r : Z^r_{p,q} \xrightarrow{\partial} Z^r_{p-r,q+r-1} \to E^r_{p-r,q+r-1}.$$

Since

$$E^r_{p-r,q+r-1} = Z^r_{p-r,q+r-1}/\big(Z^{r-1}_{p-r-1,q+r} + \partial Z^{r-1}_{p-1,q+1}\big)$$

and $\partial^2 = 0$, it follows that $\tilde{d}^r$ descends to a well-defined differential

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

The fact that $H_*(E^r, d^r) = E^{r+1}$ is not hard, and amounts to careful bookkeeping. From the definition and the fact that $Z^0_{p,q} = C_{p,q}$ it is straightforward to check that

$$E^1_{p,q} \cong \frac{\ker\big(\partial : C_{p,q}/C_{p-1,q+1} \to C_{p,q-1}/C_{p-1,q}\big)}{\operatorname{im}\big(\partial : C_{p,q+1}/C_{p-1,q} \to C_{p,q}/C_{p-1,q+1}\big)}$$

from which the assertion about $E^1_{p,q}$ follows.

For all r large enough

$$E^r_{p,q} = Z^\infty_{p,q}/\big(Z^\infty_{p-1,q+1} + \partial Z^{r-1}_{p+r-1,q-r+2}\big).$$

Since $\partial Z^{r-1}_{p+r-1,q-r+2} \subset \partial Z^r_{p+r,q-r+1}$ by the first assertion of Exercise 194, $E^r_{p,q}$ surjects to

$$E^{r+1}_{p,q} = Z^\infty_{p,q}/\big(Z^\infty_{p-1,q+1} + \partial Z^r_{p+r,q-r+1}\big).$$

The fitration is convergent, and therefore the union $\cup_r \partial Z^r_{p+r,q-r+1}$ equals $\partial C \cap C_{p,q}$. Hence if one defines

$$E^\infty_{p,q} = Z^\infty_{p,q}/\big(Z^\infty_{p-1,q+1} + \partial C \cap C_{p,q}\big),$$

$\operatorname{colim}_{\to\infty} E^r_{p,q} \cong E^\infty_{p,q}$.

The assertion

$$E^\infty_{p,q} \cong \frac{\operatorname{im}\big(H_{p+q}(F_pC, \partial) \to H_{p+q}(C, \partial)\big)}{\operatorname{im}\big(H_{p+q}(F_{p-1}C, \partial) \to H_{p+q}(C, \partial)\big)}$$

is straightforward, if tedious, to check.

As an example, suppose X is a *filtered space*, that is, X is equipped with an increasing sequence of subspaces

$$\phi = X^{-1} \subset X^0 \subset \cdots X^p \subset X^{p+1} \subset \cdots \subset X$$

with $\cup_p X^p = X$. Assume further that any compact subset K of X is a subset of X^p for some p. For example, this holds if X is a CW-complex and X^p its p-skeleton, or, more generally, if $f : X \to Y$ is a continuous map with Y a CW-complex and $X^p = f^{-1}(Y^p)$ of X.

Since the n-simplex is compact, it follows that the singular chain complex $S_\bullet X$ of X has a convergent, bounded below filtration defined by

$$F_p(S_\bullet X) = \operatorname{im}\big(S_\bullet X^p \to S_\bullet X\big).$$

Exercise 195. Suppose that X is a CW-complex with its skeletal filtration. Filter the singular complex $S_\bullet X$ by $F_p(S_\bullet X)$ as above. The spectral sequence of Theorem 10.6 for this filtered chain complex has the properties:

1. (E^1, d^1) is the cellular chain complex for X (and hence E^2 is the cellular homology of X),
2. $d^r = 0$ for all $r \geq 2$, and so $E^2 = E^\infty$, (the spectral sequence collapses at the E^2 page) and
3. $\mathrm{Gr}(H_n(S_\bullet(X))_p$ is zero for $p \neq n$ and equals the singular homology $H_n X$ for $p = n$.
4. Conclude from Theorem 10.6 that the cellular and singular homology of X are isomorphic.

10.3. The Leray-Serre-Atiyah-Hirzebruch spectral sequence

Serre, based on earlier work of Leray, constructed a spectral sequence converging to $H_* E$, given a fibration

$$F \hookrightarrow E \xrightarrow{f} B.$$

Atiyah and Hirzebruch, based on earlier work of G. Whitehead, constructed a spectral sequence converging to $G_* B$ where G_* is an additive generalized homology theory and B is a CW-complex . The spectral sequence we present here is a combination of these spectral sequences and converges to $G_* E$ when G_* is an additive homology theory.

We may assume B is path-connected by restricting to path components, but we do not wish to assume B is simply connected. In order to deal with this case we will have to use local coefficients derived from the fibration. Theorem 7.15 shows that the homotopy lifting property gives rise to a homomorphism

$$\pi_1 B \to \{\text{Homotopy classes of homotopy equivalences } F \to F\}.$$

Applying the (homotopy) functor G_n, one obtains a representation

$$\pi_1 B \to \operatorname{Aut}(G_n F)$$

for each integer n. Thus for each n, G_nF has the structure of a $\mathbf{Z}[\pi_1 B]$ module, or, equivalently, one has a system of local coefficients over B with fiber G_nF. (Of course, if $\pi_1 B = 1$, then this is a trivial local coefficient system.) Taking (ordinary) homology with local coefficients, we can associate the group $H_p(B; G_qF)$ to each pair of integers p, q. Notice that $H_p(B; G_qF)$ is zero if $p < 0$.

Theorem 10.7. *Let $F \hookrightarrow E \xrightarrow{f} B$ be a fibration, with B a path-connected CW-complex. Let G_* be an additive homology theory. Then there exists a spectral sequence*

$$H_p(B; G_qF) \cong E^2_{p,q} \Rightarrow G_{p+q}E.$$

□

This spectral sequence is carefully constructed in [**54**], and we refer you there for a proof. It is based on variation of the construction of the previous section with respect to the filtration of G_*E given by

$$F_pG_*E = \mathrm{im}(G_*(f^{-1}(B^p)) \to G_*E$$

where $f : E \to B$ is the fibration and B^p denotes the p-skeleton of B. A key ingredient is the observation that $G_{p+q}((D^p, S^{p-1}) \times F) = G_qF$, which follows from the suspension isomorphism. This is applied over each cell of B, using the fact that the restriction of the fibration to a cell is fiber homotopically trivial, to establish a relationship between the various $G_{p+q}(f^{-1}(B^p))$ and the cellular chain complex of B with coefficients in G_*F.

Exercise 196. If G_* is an additive, isotropic homology theory, then the hypothesis that B is a CW-complex can be omitted. (Hint: for any space B there is a weak homotopy equivalence from a CW-complex to B.)

As a service to the reader, we will explicitly unravel the statement of Theorem 10.7. There exists

1. A (bounded below) filtration

   $$0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{p,n-p} \subset \cdots \subset G_nE$$

   of $G_nE = \cup_p F_{p,n-p}$ for each integer n.
2. A bigraded spectral sequence $(E^r_{*,*}, d^r)$ such that the differential d^r has bidegree $(-r, r-1)$ (i.e. $d^r(E^r_{p,q}) \subset E^r_{p-r,q+r-1}$), and so

   $$E^{r+1}_{p,q} = \frac{\ker\ d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\mathrm{im}\ d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$
3. Isomorphisms $E^2_{p,q} \cong H_p(B; G_qF)$ (local coefficients if $\pi_1(B)$ is nontrivial).

This spectral sequence converges to G_*E. That is, for each fixed p, q, there exists an r_0 so that

$$d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$$

is zero for all $r \geq r_0$ and so

$$E^{r+1}_{p,q} = E^r_{p,q}\big/d^r(E^r_{p+r,q-r+1})$$

for all $r \geq r_0$.

Define $E^\infty_{p,q} = \operatorname*{colim}_{r\to\infty} E^r_{p,q}$. There is an isomorphism

$$F_{p,q}\big/F_{p-1,q+1} \cong E^\infty_{p,q},$$

i.e.

$$\mathrm{Gr}(G_nE)_p \cong E^\infty_{p,n-p}$$

with respect to the filtration of G_nE.

In this spectral sequence, some filtrations of the groups G_nE are given, with the associated graded groups made up of the pieces $E^\infty_{p,n-p}$. So, for example, if $G_nE = 0$, then $E^\infty_{p,n-p} = 0$ for each $p \in \mathbf{Z}$.

As a first nontrivial example of computing with spectral sequences we consider the problem of computing the homology of the loop space of a sphere. Given $k > 1$, let $P = P_{x_0}S^k$ be the space of paths in S^k which start at $x_0 \in S^k$. As we saw in Chapter 7, evaluation at the end point defines a fibration $P \to S^k$ with fiber the loop space ΩS^k. Moreover, the path space P is contractible.

The spectral sequence for this fibration (using ordinary homology with integer coefficients for G_*) has $E^2_{p,q} = H_p(S^k; H_q(\Omega S^k))$. The coefficients are untwisted since $\pi_1 S^k = 0$. Therefore

$$E^2_{p,q} = \begin{cases} H_q(\Omega S^k) & \text{if } p = 0 \text{ or } p = k, \\ 0 & \text{otherwise.} \end{cases} \tag{10.1}$$

In particular this is a first-quadrant spectral sequence.

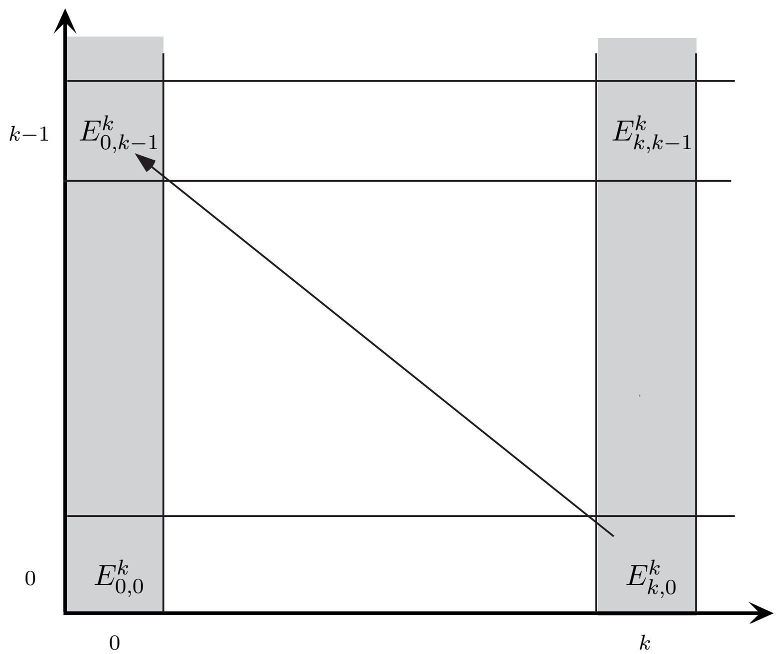
Since $H_nP = 0$ for all $n \neq 0$, the filtration of H_nP is trivial for $n > 0$ and so $E^\infty_{p,q} = 0$ if $p+q > 0$. Since this is a first-quadrant spectral sequence, $E^\infty_{p,q} = 0$ for all $(p,q) \neq (0,0)$, and, furthermore, given any $(p,q) \neq (0,0)$, $E^r_{p,q} = 0$ for some r large enough.

Now here's the cool part. Looking at the figure and keeping in mind the fact that the bidegree of d^r is $(-r, r-1)$, we see that all differentials $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$ either:

1. start or end at a zero group, or
2. $r = k$ and $(p,q) = (k,q)$ with $q \geq 0$, so that

$$d^k : E^k_{k,q} \to E^k_{0,q+k-1}.$$

The following picture shows the E^k-page and the differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$. The shaded columns contain the only possible nonzero entries, since $E^2_{p,q} = 0$ if $p \neq 0$ or k.



Hence

$$E^2_{p,q} = E^3_{p,q} = \cdots = E^k_{p,q}. \tag{10.2}$$

Thus if $(p, q) \neq (0, 0)$,

$$0 = E^\infty_{p,q} = E^{k+1}_{p,q} = \begin{cases} \ker d^k : E^k_{k,q} \to E^k_{0,q+k-1} & \text{if } (p,q) = (k,q), \\ \text{Coker } d^k : E^k_{k,q} \to E^k_{0,q+k-1} & \text{if } (p,q) = (0, q+k-1), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the spectral sequence collapses at E^{k+1} and $d^k : E^k_{k,q} \to E^k_{0,q+k-1}$ is an isomorphism whenever $q \neq 1 - k$. Using Equations (10.2) and (10.1) we can restate this as

$$H_q(\Omega S^k) \cong H_{q+k-1}(\Omega S^k).$$

Using induction, starting with $H_0(\Omega S^k) = \mathbf{Z}$ and $H_q(\Omega S^k) = 0$ for q negative, we conclude that

$$H_q(\Omega S^k) = \begin{cases} \mathbf{Z} & \text{if } q = a(k-1), a \geq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{10.3}$$

Exercise 197. If $S^k \hookrightarrow S^\ell \xrightarrow{f} S^m$ is a fibration, then $\ell = 2m - 1$ and $k = m - 1$. (In fact, it is a result of Adams that there are only such fibrations for $m = 1, 2, 4$ and 8.)

Returning to our general discussion, notice that $E^{r+1}_{p,q}$ and $E^{\infty}_{p,q}$ are *subquotients* of $E^r_{p,q}$; in particular, since $E^2_{p,q} \cong H_p(B; G_q F)$ we conclude the following fundamental fact.

Theorem 10.8. *The associated graded module to the filtration of $G_n E$ has graded summands which are subquotients of $H_p(B; G_{n-p} F)$.* □

This fact is the starting point for many spectral sequence calculations. For example,

Theorem 10.9. *If $H_p(B; G_{n-p} F) = 0$ for all p, then $G_n E = 0$.*

Proof. Since $E^2_{p,n-p} = 0$ for each p, it follows that $E^{\infty}_{p,n-p} = 0$ for each p and so $G_n E = 0$. □

10.4. The edge homomorphisms and the transgression

Before we turn to more involved applications, it is useful to know several facts about the Leray-Serre-Atiyah-Hirzebruch spectral sequence. These facts serve to identify certain homomorphisms which arise in the guts of the spectral sequence with natural maps induced by the inclusion of the fiber or the projection to the base in the fibration.

Lemma 10.10. *In the Leray-Serre-Atiyah-Hirzebruch spectral sequence there is a surjection*

$$E^2_{0,n} \to E^{\infty}_{0,n}$$

for all n.

Proof. Notice that

$$E^{r+1}_{0,n} = \frac{\ker d^r : E^r_{0,n} \to E^r_{-r,n+r-1}}{\text{im } d^r : E^r_{r,n-r+1} \to E^r_{0,n}} \quad \text{for } r > 1.$$

But, since $E^2_{p,q} = 0$ for $p < 0$, we must have $E^2_{-r,q} = 0$ for all q and so also its subquotient $E^r_{-r,q} = 0$ for all q.

Hence $E^r_{0,n} = \ker d^r : E^r_{0,n} \to E^r_{-r,n+r-1}$ and so

$$E^{r+1}_{0,n} = \frac{E^r_{0,n}}{\operatorname{im} d^r}.$$

Thus each $E^r_{0,n}$ surjects to $E^{r+1}_{0,n}$ and hence also to the colimit $E^\infty_{0,n}$. □

Proposition 6.8 says that if V is any local coefficient system over a path-connected space B, then

$$H_0(B;V) = V/\langle v - \alpha \cdot v \mid v \in V,\ \alpha \in \pi_1 B\rangle.$$

Applying this to $V = G_n F$, it follows that there is a surjection

$$G_n F \to H_0(B; G_n F) = E^2_{0,n}. \tag{10.4}$$

We can now use the spectral sequence to construct a homomorphism $G_*F \to G_*E$. Theorem 10.11 below asserts that this homomorphism is just the homomorphism induced by the inclusion of the fiber into the total space.

Since $F_{-1,n+1} = 0$,

$$E^\infty_{0,n} \cong F_{0,n}/F_{-1,n+1} = F_{0,n} \subset G_n E.$$

This inclusion can be precomposed with the surjections of Lemma 10.10 and Equation (10.4) to obtain a homomorphism (called an *edge homomorphism*)

$$G_n F \to H_0(B; G_n F) \cong E^2_{0,n} \to E^\infty_{0,n} \subset G_n E. \tag{10.5}$$

Theorem 10.11. *The edge homomorphism given by (10.5) equals the map $i_* : G_n F \to G_n E$ induced by the inclusion $i : F \hookrightarrow E$ by the homology theory G_*.* □

Another simple application of the spectral sequence is to compute oriented bordism groups of a space in low dimensions. We apply the Leray-Serre-Atiyah-Hirzebruch spectral sequence to the fibration $\mathrm{pt} \hookrightarrow X \xrightarrow{\mathrm{Id}} X$, and take $G_* = \Omega^{\mathbf{SO}}_*$, oriented bordism.

In this case the Leray-Serre-Atiyah-Hirzebruch spectral sequence says

$$H_p(X; \Omega^{\mathbf{SO}}_q(\mathrm{pt})) \Rightarrow \Omega^{\mathbf{SO}}_{p+q} X.$$

Notice that the coefficients are untwisted; this is because the fibration is trivial. Write $\Omega^{\mathbf{SO}}_n = \Omega^{\mathbf{SO}}_n(\mathrm{pt})$. Note that $\mathrm{pt} \hookrightarrow X$ is split by the constant map; hence the edge homomorphism $\Omega^{\mathbf{SO}}_n \to \Omega^{\mathbf{SO}}_n X$ is a split injection, so by Theorem 10.11, the differentials $d^r : E^r_{r,n-r+1} \to E^r_{0,n}$ whose targets are on the vertical edge of the first quadrant must be zero; i.e. every element of $E^2_{0,n}$ *survives* to $E^\infty_{0,n} = \Omega^{\mathbf{SO}}_n$.

Recall from Section **??** that $\Omega_q^{\mathbf{SO}} = 0$ for $q = 1, 2, 3$, and $\Omega_q^{\mathbf{SO}} = \mathbf{Z}$ for $q = 0$ and 4. Of course $\Omega_q^{\mathbf{SO}} = 0$ for $q < 0$. Thus for $n = p + q \leq 4$, the only (possibly) nonzero terms are $E^2_{n,0} \cong H_n X$ and $E^2_{0,4} \cong \Omega_4^{\mathbf{SO}}$. Hence $E^2_{p,n-p} \cong E^\infty_{p,n-p}$ for $n = 0, 1, 2, 3$, and 4. From the spectral sequence one concludes

$$\begin{aligned} \Omega_n^{\mathbf{SO}} X &\cong H_n X \qquad \text{for } n = 0, 1, 2, 3 \\ \Omega_4^{\mathbf{SO}} X &\cong \mathbf{Z} \oplus H_4 X. \end{aligned}$$

It can be shown that the map $\Omega_n^{\mathbf{SO}} X \to H_n X$ is a Hurewicz map which takes $f : M \to X$ to $f_*[M]$. In particular this implies that any homology class in $H_n X$ for $n = 0, 1, 2, 3$, and 4 is *represented* by a map from an oriented manifold to X. The morphism $\Omega_4^{\mathbf{SO}} X \to \mathbf{Z}$ is the map taking $f : M \to X$ to the signature of M.

We next identify another edge homomorphism which can be constructed in the same manner as (10.5). The analysis will be slightly more involved, and we will state it only in the case when G_* is ordinary homology with coefficients in an R-module (we suppress the coefficients).

In this context $E^2_{p,q} = H_p(B; H_q F) = 0$ for $q < 0$ or $p < 0$. So $E^*_{*,*}$ is a first-quadrant spectral sequence; i.e. $E^r_{p,q} = E^\infty_{p,q} = 0$ for $q < 0$ or $p < 0$.

This implies that the filtration of $H_n E$ has finite length

$$0 = F_{-1,n+1} \subset F_{0,n} \subset F_{1,n-1} \subset \cdots \subset F_{n,0} = H_n E$$

since

$$0 = E^\infty_{p,n-p} = F_{p,n-p}/F_{p-1,n-p+1}$$

if $p < 0$ or $n - p < 0$.

The second map in the short exact sequence

$$0 \to F_{n-1,1} \to F_{n,0} \to E^\infty_{n,0} \to 0$$

can thus be thought of as a homomorphism

$$H_n E \to E^\infty_{n,0}. \tag{10.6}$$

Lemma 10.12. *There is an inclusion*

$$E^\infty_{n,0} \subset E^2_{n,0}$$

for all n.

Proof. Since $E^r_{n+r,1-r} = 0$ for $r > 1$,

$$E^{r+1}_{n,0} = \frac{\ker\ d^r : E^r_{n,0} \to E^r_{n-r,r-1}}{\operatorname{im}\ d^r : E^r_{n+r,1-r} \to E^r_{n,0}} = \ker d^r : E^r_{n,0} \to E^r_{n-r,r-1}.$$

Thus

$$\cdots \subset E^{r+1}_{n,0} \subset E^r_{n,0} \subset E^{r-1}_{n,0} \subset \cdots$$

and hence
$$E^\infty_{n,0} = \bigcap_r E^r_{n,0} \subset E^2_{n,0}.$$
□

Note that the constant map from the fiber F to a point induces a homomorphism $H_n(B; H_0F) \to H_nB$. If F is path-connected, then the local coefficient system H_0F is trivial and $H_n(B; H_0F) = H_nB$ for all n.

Theorem 10.13. *The composite map (also called an edge homomorphism)*
$$H_nE = F_{n,0} \to E^\infty_{n,0} \subset E^2_{n,0} \cong H_n(B; H_0F) \to H_nB$$
is just the map induced on homology by the projection $f : E \to B$ of the fibration. □

The long differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ in the spectral sequence for a fibration (for ordinary homology) has an alternate geometric interpretation called the *transgression*. It is defined as follows. Suppose $f : E \to B$ is a fibration with fiber F. Fix $k > 0$. We assemble the homomorphism $f_* : H_k(E, F) \to H_k(B, b_0)$, the isomorphism $H_kB \cong H_k(B, b_0)$, and the connecting homomorphism $\partial : H_k(E, F) \to H_{k-1}F$ for the long exact sequence of the pair (E, F) to define a (not well-defined, multi–valued) function $\tau : H_kB$"$\to$"$H_{k-1}F$ as the "composite"
$$\tau : H_kB \cong H_k(B, b_0) \xleftarrow{f_*} H_k(E, F) \xrightarrow{\partial} H_{k-1}F.$$

To make this more precise, we take as the domain of τ the image of $f_* : H_k(E, F) \to H_k(B, b_0) \cong H_kB$, and as the range of τ the quotient of $H_{k-1}F$ by $\partial(\ker f_* : H_k(E, F) \to H_k(B, b_0))$. A simple diagram chase shows τ is well-defined with this choice of domain and range. Thus the transgression τ is an honest homomorphism from a subgroup of H_kB to a quotient group of $H_{k-1}F$.

Intuitively, the transgression is trying his/her best to imitate the connecting homomorphism in the long exact homotopy sequence for a fibration (see Corollary 7.49 and Theorem 10.16).

Assume for simplicity that F is path-connected, and consider the differential
$$d^k : E^k_{k,0} \to E^k_{0,k-1}$$
in the spectral sequence for this fibration (taking $G_* = H_* =$ ordinary homology). Its domain, $E^k_{k,0}$, is a subgroup of $E^2_{k,0} = H_k(B; H_0F) = H_kB$ because all differentials d^r into $E^r_{k,0}$ are zero for $r < k$ (this is a first-quadrant spectral sequence), and hence $E^k_{k,0}$ is just the intersection of the kernels of $d^r : E^r_{k,0} \to E^r_{k-r,r-1}$ for $r < k$.

Similarly the range $E^k_{0,k-1}$ of $d^k : E^k_{k,0} \to E^k_{0,k-1}$ is a quotient of $E^2_{0,k-1} = H_0(B; H_{k-1}F)$, which by Proposition 6.8 is just the quotient of $H_{k-1}F$ by the action of $\pi_1 B$.

We have shown that like the transgression, the differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ has domain identified with a subgroup of $H_k B$ and range a quotient of $H_{k-1}F$. The following theorem identifies the transgression and this differential.

Theorem 10.14 (transgression theorem)**.** *The differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ in the spectral sequence of the fibration $F \hookrightarrow E \xrightarrow{f} B$ coincides with the transgression*

$$\tau : \text{im } f_* \to H_{k-1}F/\partial(\ker f_*), \text{ where } f_* : H_k(E, F) \to H_k(B, b_0)$$

via the identifications of $E^2_{k,0}$ with $H_k B$ and $E^2_{0,k-1}$ with $H_{k-1}F$. □

The proofs of Theorems 10.11, 10.13, and 10.14 are not hard, but require an examination of the construction which gives the spectral sequence. We omit the proofs, but you should look them up when working through the project for this chapter.

In a principal $K(G, n)$-fibration, the k-invariant is the transgression of the fundamental class. What happens for a general principal fibration? Is the rational cohomology of a connected Lie group exterior, with generators that transgress to polynomial generators of the classifying space?

10.5. Applications of the homology spectral sequence

10.5.1. The five-term and Serre exact sequences.

Corollary 10.15 (five-term exact sequence)**.** *Suppose that $F \hookrightarrow E \xrightarrow{f} B$ is a fibration with B and F path-connected. Then there exists an exact sequence*

$$H_2 E \xrightarrow{f_*} H_2 B \xrightarrow{\tau} H_0(B; H_1 F) \to H_1 E \xrightarrow{f_*} H_1 B \to 0.$$

The composite of the surjection $H_1 F \to H_0(B; H_1 F)$ with the map $H_0(B; H_1 F) \to H_1 E$ in this exact sequence is the homomorphism induced by the inclusion $F \hookrightarrow E$, and τ is the transgression.

Proof. Take $G_* = H_*(-)$, ordinary homology, perhaps with coefficients. The corresponding first quadrant spectral sequence has

$$E^2_{p,q} \cong H_p(B; H_q F)$$

and converges to $H_* E$.

The local coefficient system $\pi_1 B \to \text{Aut}(H_0 F)$ is trivial since F is path-connected. Thus $E^2_{p,0} = H_p(B; H_0 F) = H_p B$.

The following facts either follow immediately from the statement of Theorem 10.7 or are easy to verify, using the bigrading of the differentials and the fact that the spectral sequence is a first–quadrant spectral sequence.

1. $H_1B \cong E^2_{1,0} = E^r_{1,0} = E^\infty_{1,0}$ for all $r \geq 2$.
2. $H_2B \cong E^2_{2,0}$.
3. $H_0(B; H_1F) = E^2_{0,1}$.
4. $E^\infty_{2,0} = E^r_{2,0} = E^3_{2,0} = \ker d^2 : E^2_{2,0} \to E^2_{0,1}$ for all $r \geq 3$.
5. $E^\infty_{0,1} = E^r_{0,1} = E^3_{0,1} = \operatorname{coker} d^2 : E^2_{2,0} \to E^2_{0,1}$ for all $r \geq 3$.

Exercise 198. Prove these five facts.

The last two facts give an exact sequence

$$0 \to E^\infty_{2,0} \to E^2_{2,0} \xrightarrow{d^2} E^2_{0,1} \to E^\infty_{0,1} \to 0$$

or, making the appropriate substitutions, the exact sequence

$$0 \to E^\infty_{2,0} \to H_2B \to H_0(B; H_1F) \to E^\infty_{0,1} \to 0. \tag{10.7}$$

Since the spectral sequence converges to H_*E, and the $E^\infty_{p,n-p}$ form the associated graded groups for H_nE, the two sequences

$$0 \to E^\infty_{0,1} \to H_1E \to E^\infty_{1,0} \to 0 \tag{10.8}$$

and

$$F_{1,1} \to H_2E \to E^\infty_{2,0} \to 0 \tag{10.9}$$

are exact.

Splicing the sequences (10.7), (10.8), and (10.9) together and using the first fact above, one obtains the exact sequence

$$F_{1,1} \to H_2E \to H_2B \to H_0(B; H_1F) \to H_1E \to H_1B \to 0.$$

In this sequence the homomorphism $H_iE \to H_iB$ is the edge homomorphism and hence is induced by the fibration $f : E \to B$. The map $H_0(B; H_1F) \to H_1E$ composes with $H_1F \to H_0(B; H_1F)$ to give the other edge homomorphism, induced by the inclusion of the fiber. The map $H_2B \to H_0(B; H_1F)$ is the transgression. These assertions follow by chasing definitions and using Theorems 10.11, 10.13 and 10.14. □

We have seen, beginning with our study of the Puppe sequences, that cofibrations give exact sequences in homology and fibrations give exact sequences in homotopy. One might say that a map is a "fibration or cofibration in some range" if there are partial long exact sequences. Corollary 10.15 implies that if π_1B acts trivially on H_1F, then *the fibration is a cofibration in* *a certain range*. A more general result whose proof is essentially identical to that of Corollary 10.15 is given in the following important theorem.

Theorem 10.16 (Serre exact sequence)**.** *Let $F \xrightarrow{i} E \xrightarrow{f} B$ be a fibration with B and F path-connected and with $\pi_1 B$ acting trivially on $H_* F$. Suppose $H_p B = 0$ for $0 < p < m$ and $H_q F = 0$ for $0 < q < n$. Then the sequence:*

$$H_{m+n-1}F \xrightarrow{i_*} H_{m+n-1}E \xrightarrow{f_*} H_{m+n-1}B \xrightarrow{\tau} H_{m+n-2}F \xrightarrow{i_*} \cdots$$
$$\cdots \xrightarrow{f_*} H_1 B \to 0.$$

is exact. □

Exercise 199. Prove Theorem 10.16.

To understand this result, suppose B is $(m-1)$-connected and F is $(n-1)$-connected. The long exact sequence for a fibration shows that E is $(\min(m,n)-1)$-connected, so that by the Hurewicz theorem, $H_q E = 0$ for $q < \min(m,n)$. So trivially the low-dimensional part of the Serre exact sequence is exact; indeed all groups are zero for $q < \min(m,n)$. The remarkable fact is that the sequence remains exact for all $\min\{m,n\} \le q < m+n$.

10.5.2. Euler characteristics and fibrations. Let k be a field. Recall that the Euler characteristic of a space Z is defined to be the alternating sum $\chi(Z) = \sum_n (-1)^n \beta_n(Z;k)$ of the Betti numbers $\beta_n(Z;k) = \dim_k(H_n(Z;k))$ whenever this sum is a finite sum of finite ranks. For finite CW-complexes it is equal to the alternating sum of the number of n-cells by the following standard exercise applied to the cellular chain complex.

Exercise 200. Let (C_*, ∂) be a chain complex over a field with $\oplus_i C_i$ finite-dimensional. Show that the alternating sum of the ranks of the C_i equals the alternating sum of the ranks of the homology groups $H_i(C_*, \partial)$.

Given a product space $E = B \times F$ with B and F finite CW-complexes, the Künneth theorem implies that the homology with field coefficients is a tensor product

$$H_*(E;k) \cong H_*(B;k) \otimes H_*(F;k)$$

from which it follows that the Euler characteristic is multiplicative

$$\chi(E) = \chi(B) \cdot \chi(F).$$

The following theorem extends this formula to the case when E is only a product locally, i.e. fiber bundles, and even to fibrations.

Notice that the homology itself need not be multiplicative for a nontrivial fibration. For example, consider the Hopf fibration $S^3 \hookrightarrow S^7 \to S^4$. The graded groups $H_*(S^7;k)$ and $H_*(S^3;k)\otimes H_*(S^4;k)$ are not isomorphic, even though the Euler characteristics multiply ($0 = 0\cdot 2$).

Theorem 10.17. *Let $p : E \to B$ be a fibration with fiber F, let k be a field, and suppose the action of $\pi_1 B$ on $H_*(F;k)$ is trivial. Assume that the Euler characteristics $\chi(B), \chi(F)$ are defined (e.g. if B, F are finite cell complexes). Then $\chi(E)$ is defined and*

$$\chi(E) = \chi(B)\cdot\chi(F).$$

Proof. Since k is a field and the action of $\pi_1 B$ on $H_*(F;k)$ is trivial,

$$H_p(B;H_q(F;k)) \cong H_p(B;k)\otimes_k H_q(F;k)$$

by the universal coefficient theorem. Theorem 10.7 with $G_* = H_*(-;k)$ implies that there exists a spectral sequence with

$$E^2_{p,q} \cong H_p(B;k)\otimes H_q(F;k).$$

By hypothesis, $E^2_{p,q}$ is finite-dimensional over k and is zero for all but finitely many pairs (p,q). This implies that the spectral sequence collapses at some stage and so $E^\infty_{p,q} = E^r_{p,q}$ for r large enough.

Define

$$E^r_n = \oplus_p E^r_{p,n-p}$$

for each n and $r \geq 2$ including $r = \infty$.

Then since the Euler characteristic of the tensor product of two graded vector spaces is the product of the Euler characteristics,

$$\chi(E^2_*) = \chi(B)\chi(F).$$

Notice that (E^r_*, d^r) is a (singly) graded chain complex with homology E^{r+1}_*. Exercise 200 shows that for any $r\geq 2$,

$$\chi(E^r_*) = \chi(H_*(E^r_*, d^r)) = \chi(E^{r+1}_*).$$

Since the spectral sequence collapses $\chi(E^2_*) = \chi(E^\infty_*)$.

Since we are working over a field, $H_n(E;k)$ is isomorphic to its associated graded vector space $\oplus_p E^\infty_{p,n-p} = E^\infty_n$. In particular $H_n(E;k)$ is finite-dimensional and $\dim H_n(E;k) = \dim E^\infty_n$.

Therefore,

$$\chi(B)\chi(F) = \chi(E^2_*) = \chi(E^\infty_*) = \chi(H_*(E;k)) = \chi(E).$$

□

10.5.3. The homology Gysin sequence.

Theorem 10.18. *Let R be a commutative ring. Suppose $F \hookrightarrow E \xrightarrow{f} B$ is a fibration, and suppose F is an R-homology n-sphere; i.e.*

$$H_i(F;R) \cong \begin{cases} R & \text{if } i = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that $\pi_1 B$ acts trivially on $H_n(F;R)$. Then there exists an exact sequence (R-coefficients):

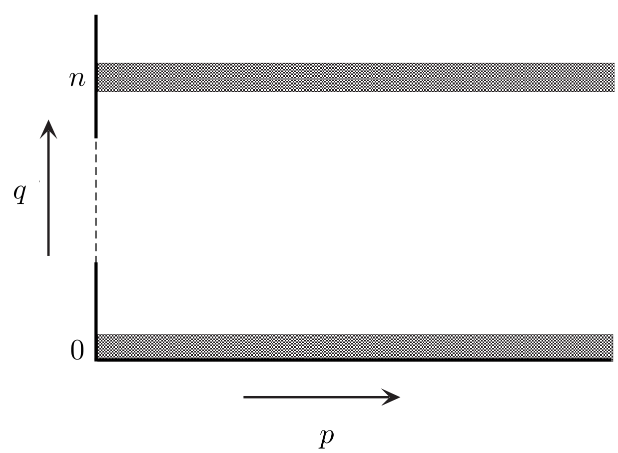
$$\cdots \to H_r E \xrightarrow{f_*} H_r B \to H_{r-n-1} B \to H_{r-1} E \xrightarrow{f_*} H_{r-1} B \to \cdots .$$

dont need R to be a ring.

Proof. The spectral sequence for the fibration (using ordinary homology with R-coefficients) has

$$E^2_{p,q} \cong H_p(B; H_q F) = \begin{cases} H_p(B;R) & \text{if } q = 0 \text{ or } n, \\ 0 & \text{otherwise.} \end{cases}$$

The following diagram shows the E^2-stage. The two shaded rows ($q = 0$ and $q = n$) are the only rows that might contain a nonzero $E^2_{p,q}$.



Thus the only possibly nonzero differentials are

$$d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n}.$$

It follows that

$$E^{n+1}_{p,q} \cong E^2_{p,q} \cong H_p(B; H_q F) = \begin{cases} H_p B & \text{if } q = 0 \text{ or } n, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$E^\infty_{p,q} \cong \begin{cases} 0 & \text{if } q \neq 0 \text{ or } n, \\ \ker\ d^{n+1} : E^{n+1}_{p,0} \to E^{n+1}_{p-n-1,n} & \text{if } q = 0, \\ \operatorname{coker} d^{n+1} : E^{n+1}_{p+n+1,0} \to E^{n+1}_{p,q} & \text{if } q = n. \end{cases} \tag{10.10}$$

The filtration of H_rE reduces to

$$0 \subset E^\infty_{r-n,n} \cong F_{r-n,n} \subset F_{r,0} = H_rE,$$

and so the sequences

$$0 \to E^\infty_{r-n,n} \to H_rE \to E^\infty_{r,0} \to 0$$

are exact for each r. Splicing these with the exact sequences

$$0 \to E^\infty_{p,0} \to E^{n+1}_{p,0} \xrightarrow{d^{n+1}} E^{n+1}_{p-n-1,n} \to E^\infty_{p-n-1,n} \to 0$$

(obtained from Equation (10.10)) gives the desired exact sequence

$$\cdots \to H_rE \xrightarrow{f_*} H_rB \to H_{r-n-1}B \to H_{r-1}E \to H_{r-1}B \to \cdots$$

with the map labelled f_* induced by $f : E \to B$ by Theorem 10.13. $\square$

Exercise 201. Derive the *Wang sequence*. If $F \hookrightarrow E \to S^n$ is a fibration over S^n, then there is an exact sequence

$$\cdots \to H_rF \to H_rE \to H_{r-n}F \to H_{r-1}F \to \cdots .$$

check n = 1

10.6. The cohomology spectral sequence

The examples in the previous section show that spectral sequences are a useful tool for establishing relationships between the homology groups of the three spaces forming a fibration. Much better information can often be obtained by using the ring structure on cohomology. We next introduce the cohomology spectral sequence and relate the ring structures on cohomology and the spectral sequence. The ring structure makes the cohomology spectral sequence a much more powerful computational tool than the homology spectral sequence.

Definition 10.19. A bigraded spectral sequence $(E_r^{p,q}, d_r)$ is called a *cohomology spectral sequence* if the differential d_r has bidegree $(r, 1-r)$.

Notice the change in placement of the indices in the cohomology spectral sequence. The contravariance of cohomology makes it necessary to change the notion of a filtration. There is a formal way to do this, namely by "lowering indices"; for example rewrite H^pX as $H_{-p}X$, rewrite F^p as F_{-p}, replace $E_r^{p,q}$ by $E^r_{-p,-q}$ and so forth. Unfortunately for this to work the notion of convergence of a spectral sequence has to be modified; with the definition we gave above the cohomology spectral sequence of a fibration will not converge. Rather than extending the formalism and making the notion of convergence technically more complicated, we will instead just make new definitions which apply in the cohomology setting.

Definition 10.20. A *(cohomology) filtration of an R-module A* is an increasing union

$$0 \subset \cdots \subset F^p \subset \cdots \subset F^2 \subset F^1 \subset F^0 \subset F^{-1} \subset \cdots \subset A$$

of submodules. A filtration is *convergent* if the union of the F_p's is A and their intersection is 0.

If $A = \oplus_n A^n$ is a graded module then we assume that the filtration preserves the grading, i.e. $F^p = \oplus_n (F^p \cap A^n)$. In this case we bigrade the filtration by setting

$$F^{p,q} = F^p \cap A^{p+q}.$$

Definition 10.21. Given a cohomology filtration $F = \{F^n\}$ of an R-module A, the *associated graded module* is the graded R-module denoted by $\mathrm{Gr}(A, F)$ and defined by

$$\mathrm{Gr}(A, F)^p = \frac{F^p}{F^{p+1}}.$$

Definition 10.22. Given a bigraded cohomology spectral sequence $(E_r^{p,q}, d_r)$ and a graded R-module A^*, we say *the spectral sequence converges to A^** and write

$$E_2^{p,q} \Rightarrow A^{p+q}$$

if:

1. for each (p, q) there exists an r_0 so that $d_r : E_r^{p-r,q+r-1} \to E_r^{p,q}$ is zero for all $r \geq r_0$; in particular there is an injection $E_{r+1}^{p,q} \hookrightarrow E_r^{p,q}$ for all $r \geq r_0$, and
2. there is a convergent filtration of A^* so that for each n the limit $E_\infty^{p,n-p} = \cap_{r \geq r_0} E_r^{p,n-p}$ is isomorphic to the associated graded module $\mathrm{Gr}(A^n)^p$.

Theorem 10.23. *Let $F \hookrightarrow E \xrightarrow{f} B$ be a fibration, with B a path-connected CW-complex. Let G^* be an additive cohomology theory. Assume either that B is a finite-dimensional CW-complex or else that there exists an N so that $G^q F = 0$ for all $q < N$. Notice that $\pi_1 B$ acts on $G^q F$ determining a local coefficient system.*

Then there exists a (cohomology) spectral sequence

$$H^p(B; G^q F) \cong E_2^{p,q} \Rightarrow G^{p+q} E.$$

□

There is a version of this theorem which applies to infinite CW-complexes; see [**54**].

Exercise 202. State and prove the cohomology versions of the Serre, Gysin, and Wang sequences. Construct the cohomology edge homomorphisms and the cohomology transgression and state the analogues of Theorems 10.11, 10.13, and 10.14.

Think through this exercise. Does it give that the map from the cohomology of the total space to that of the base (in the Gysin sequence) is a module map and "integration along the fiber"?

We turn now to an examination of the multiplicative properties of the cohomology spectral sequence. We start with a few definitions.

Definition 10.24. A *bigraded R-algebra* is a bigraded R-module $E^{*,*} = \oplus_{p,q} E^{p,q}$ equipped with a product satisfying $E^{p,q} \cdot E^{r,s} \subset E^{p+r,q+s}$, for which the underlying graded R-module $E^* = \oplus_n(\oplus_p E^{p,n-p})$ is a graded-commutative R-algebra (see Definition 4.1). In particular, if $a \in E^{p,q}$ and $b \in E^{r,s}$, then $ab = (-1)^{(p+q)(r+s)} ba$.

Lemma 10.25. *Let $F \hookrightarrow X \to B$ be a fibration and let H^* denote ordinary cohomology with coefficients in some commutative ring R. Assume that the action of $\pi_1(B)$ on $H^*(F)$ is trivial. Let $E_*^{*,*}$ denote the Leray-Serre cohomology spectral sequence of this fibration.*

Then the cup products in $H^*(B)$ and $H^*(F)$ give $E_2^{p,q}$ the structure of a bigraded R-algebra.

Proof. The cup product on H^*B induces a bilinear map

$$H^p(B; H^qF) \times H^r(B; H^sF) \to H^{p+r}(B; H^qF \otimes H^sF).$$

Composing with the coefficient homomorphism induced by the cup product on H^*F

$$H^qF \otimes H^sF \to H^{q+s}F$$

yielding the desired multiplication

(10.11)

$$E_2^{p,q} \otimes E_2^{r,s} = H^p(B; H^qF) \otimes H^r(B; H^sF) \to H^{p+r}(B; H^{q+s}F) = E_2^{p+r,q+s}.$$

□

In many contexts the map $E_2^{*,0} \otimes E_2^{0,*} \to E_2^{*,*}$ is an isomorphism. Theorem 3.35 can be quite useful in this regard. For example, if R is a field and B and F are simply connected finite CW-complexes, then the map is an isomorphism.

Definition 10.26. Suppose that $E^* = \oplus_p E^p$ is a commutative graded R-algebra (Definition 4.1) and $d : E^* \to E^*$ an endomorphism of the underlying R-module of degree 1. Then d is called a *derivation* provided

$$d(a \cdot b) = (da) \cdot b + (-1)^p a \cdot db \quad \text{for } a \in E^p.$$

If, in addition, $d^2 = 0$, then the pair (E^*, d) is called a *differential graded R-algebra.*

Exercise 203. Suppose that (E^*, d) is a differential graded R-algebra. Show that the product on E^* well-defines a (graded) commutative product on cohomology $H^*(E^*, d)$.

Theorem 10.27. *Let $F \hookrightarrow E \xrightarrow{f} B$ be a fibration, with B a path-connected CW-complex. Let H^* denote ordinary cohomology with coefficients in some commutative ring R. Assume that the action of $\pi_1(B)$ on $H^*(F)$ is trivial.*

Then the (Leray-Serre) cohomology spectral sequence of the fibration is a spectral sequence of differential graded R-algebras. More precisely,

1. *The differential $d_2 : E_2^{*,*} \to E_2^{*,*}$ is a derivation of the product (10.11), and hence induces a product on $E_3 = H(E_2, d_2)$. Inductively, $d_{t-1} : E_{t-1}^{*,*} \to E_{t-1}^{*,*}$ is a derivation, inducing products:*

$$E_t^{p,q} \times E_t^{r,s} \to E_t^{p+r,q+s}$$

for $t = 3, 4, \cdots$ which, since $E_\infty^{p,q} = E_t^{p,q}$ for t large enough, determines a product $E_\infty^{p,q} \times E_\infty^{r,s} \to E_\infty^{p+r,q+s}$.

2. *The cup product on the total space X, $\cup : H^*X \times H^*X \to H^*X$, is filtration preserving; i.e. the diagram*

$$\begin{tikzcd}
F^{p,q} \times F^{r,s} \arrow[r] \arrow[d] & F^{p+r,q+s} \arrow[d] \\
H^{p+q}X \times H^{r+s}X \arrow[r, "\cup"] & H^{p+q+r+s}X
\end{tikzcd}$$

commutes, and so also induces (Exercise 204) a product

$$E_\infty^{p,q} \times E_\infty^{r,s} \to E_\infty^{p+r,q+s}$$

on $E_\infty^{p,q} = \mathrm{Gr}(H^{p+q}X)^p = F^{p,q}/F^{p+1,q-1}$.

These two products on $E_\infty^{*,*}$ coincide. □

Exercise 204. Show that a filtration-preserving multiplication on a filtered algebra induces a multiplication on the associated graded algebra.

10.7. Applications of the cohomology spectral sequence

As a first example, we show how to compute the complex K-theory of complex projective space $\mathbf{C}P^k$ (see Section 9.10.4 and the project for Chapter 9). The computation of complex K-theory was the original motivation for Atiyah-Hirzebruch to set up their spectral sequence. Complex K-theory is a cohomology theory satisfying $K^nX = K^{n+2}X$, and its coefficients are given by

$$K^{2n}(\mathrm{pt}) = \pi_0(\mathbf{Z} \times BU) = \mathbf{Z}$$

and

$$K^{2n+1}(\mathrm{pt}) = \pi_1(\mathbf{Z} \times BU) = 0.$$

Theorem 10.23, applied to the trivial fibration

$$\mathrm{pt} \hookrightarrow \mathbf{C}P^k \xrightarrow{\mathrm{Id}} \mathbf{C}P^k,$$

says there exists a cohomology spectral sequence $E_r^{p,q}$ satisfying

$$H^p(\mathbf{C}P^k; K^q(\mathrm{pt})) \cong E_2^{p,q} \Rightarrow K^{p+q}(\mathbf{C}P^k).$$

The coefficients are untwisted since the fibration is trivial. Since

$$H^p(\mathbf{C}P^k) = \begin{cases} \mathbf{Z} & \text{if } p \text{ is even, } 0 \leq p \leq 2k \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$E_2^{p,q} = \begin{cases} \mathbf{Z} & \text{if } p \text{ and } q \text{ are even, } 0 \leq p \leq 2k \\ 0 & \text{otherwise.} \end{cases}$$

This checkerboard pattern forces every differential to be zero, since one of the integers $(r, 1-r)$ must be odd! Notice, by the way, that this is not a first-quadrant spectral sequence since the K-theory of a point is nonzero in positive and negative dimensions.

Therefore $E_2^{p,q} = E_\infty^{p,q}$ and the associated graded group to $K^n(\mathbf{C}P^k)$, $\oplus_p E_\infty^{p,n-p}$, is a direct sum of $k+1$ copies of $\mathbf{Z}$, one for each pair (p,q) so that $p+q=n$, both p and q are even, and $0 \leq p \leq 2k$. Inducting down the filtration we see that $K^n(\mathbf{C}P^k)$ has no torsion and hence is isomorphic to its associated graded group. Therefore

$$K^n(\mathbf{C}P^k) = \begin{cases} \mathbf{Z}^{k+1} & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

We next turn to calculations with the cohomology spectral sequence which make use of its multiplicative properties.

Proposition 10.28. *The rational cohomology ring of $K(\mathbf{Z}, n)$ is a polynomial ring on one generator if n is even and a truncated polynomial ring on one generator (in fact an exterior algebra on one generator) if n is odd:*

$$H^*(K(\mathbf{Z}, n); \mathbf{Q}) = \begin{cases} \mathbf{Q}[\iota_n] & \text{if } n \text{ is even,} \\ \mathbf{Q}[\iota_n]/\iota_n^2 & \text{if } n \text{ is odd} \end{cases}$$

where $\deg(\iota_n) = n$.

Proof. We induct on n. For $n = 1$, $K(\mathbf{Z}, 1) = S^1$ which has cohomology ring $\mathbf{Z}[\iota_1]/\iota_1^2$.

Suppose the theorem is true for $k < n$. Consider the Leray–Serre spectral sequence for the path space fibration $K(\mathbf{Z}, n-1) \hookrightarrow P \to K(\mathbf{Z}, n)$ for cohomology with rational coefficients. Then

$$E_2^{p,q} = H^p(K(\mathbf{Z}, n); \mathbf{Q}) \otimes_{\mathbf{Q}} H^q(K(\mathbf{Z}, n-1); \mathbf{Q}) \Rightarrow H^{p+q}(P; \mathbf{Q}).$$

Since $H^{p+q}(P; \mathbf{Q}) = 0$ for $(p, q) \neq (0, 0)$, the differential

$$d_n : E_n^{0,n-1} \to E_n^{n,0}$$

must be an isomorphism. Since $E_n^{0,n-1} = H^{n-1}(K(\mathbf{Z}, n-1); \mathbf{Q}) \cong \mathbf{Q}$, generated by ι_{n-1}, and $E_n^{n,0} = E_2^{n,0} = H^n(K(\mathbf{Z}, n); \mathbf{Q}) \cong \mathbf{Q}$, generated by ι_n, it follows that $d_n(\iota_{n-1})$ is a nonzero multiple of ι_n. By rescaling the generator ι_n by a rational number, assume inductively that $d_n(\iota_{n-1}) = \iota_n$.

Consider the cases n even and n odd separately. If n is even, then since $H^q(K(\mathbf{Z}, n-1); \mathbf{Q}) = 0$ unless $q = 0$ or $n-1$, $E_2^{p,q} = 0$ unless $q = 0$ or $n-1$. This implies that $0 = E_\infty^{p,q} = E_{n+1}^{p,q}$ for $(p, q) \neq (0, 0)$ and the derivation property of d_n says that $d_n(\iota_{n-1}\iota_n^r) = \iota_n^{r+1}$ which, by induction on r, is nonzero. It follows easily from $0 = E_\infty^{p,q} = E_{n+1}^{p,q}$ for $(p, q) \neq (0, 0)$ that $H^p(K(\mathbf{Z}, n); \mathbf{Q}) = 0$ if p is not a multiple of n and is isomorphic to $\mathbf{Q}$ for $p = nr$. Since ι_n^r is nonzero it generates $H^{nr}(K(\mathbf{Z}, n); \mathbf{Q}) \cong \mathbf{Q}$, and so $H^*(K(\mathbf{Z}, n); \mathbf{Q})$ is a polynomial ring on ι_n as required.

If n is odd, the derivation property of d_n implies that

$$d_n(\iota_{n-1}^2) = d_n(\iota_{n-1})\iota_{n-1} + (-1)^{n-1}\iota_{n-1}d_n(\iota_{n-1}) = 2\iota_{n-1}\iota_n.$$

Hence $d_n : E_n^{0,2n-2} \to E_2^{n,n-1}$ is an isomorphism. More generally by induction one sees that $d_n(\iota_{n-1}^r) = r\iota_n\iota_{n-1}^{r-1}$, so that $d_n : E_n^{0,r(n-1)} \to E_2^{n,(r-1)(n-1)}$ is an isomorphism. It is then easy to see that the spectral sequence collapses at E_{n+1}, and hence $H^p(K(\mathbf{Z}, n); \mathbf{Q}) = \mathbf{Q}$ for $p = 0$ or n and zero otherwise. □

We will show how to use Theorem 10.27 to compute $\pi_4 S^3$. This famous theorem was first proven by G. Whitehead and Rohlin (independently). The argument is effortless using spectral sequences.

Theorem 10.29. $\pi_4 S^3 = \mathbf{Z}/2$.

Proof. Since $\mathbf{Z} = H^3(S^3) = [S^3, K(\mathbf{Z},3)]$, choose a map $f : S^3 \to K(\mathbf{Z},3)$ representing the generator. For example, $K(\mathbf{Z},3)$ can be obtained by adding 5-cells, 6-cells, etc., to S^3 inductively to kill all the higher homotopy groups of S^3, and then f can be taken to be the inclusion. The Hurewicz theorem implies that $f_* : \pi_3 S^3 \to \pi_3(K(\mathbf{Z},3))$ is an isomorphism.

Pull back the fibration

$$K(\mathbf{Z},2) \to * \to K(\mathbf{Z},3)$$

(this is shorthand for $\Omega K(\mathbf{Z},3) \hookrightarrow P \to K(\mathbf{Z},3)$ where P is the contractible path space) via f to get a fibration

$$K(\mathbf{Z},2) \to X \to S^3. \tag{10.12}$$

Alternatively, let X be the homotopy fiber of f; i.e. $X \to S^3 \to K(\mathbf{Z},3)$ is a fibration up to homotopy. Then $\Omega K(\mathbf{Z},3) \simeq K(\mathbf{Z},2)$ is the homotopy fiber of $X \to S^3$ by Theorem 7.45. (We will use the fibration (10.12) again in Chapter 11.)

From the long exact sequence of homotopy groups for a fibration we see that $\partial : \pi_3 S^3 \to \pi_2(K(\mathbf{Z},2))$ is an isomorphism. Hence

$$\pi_k X = \begin{cases} 0 & \text{if } k \le 3, \\ \pi_k S^3 & \text{if } k > 3. \end{cases}$$

In particular, $H_4 X = \pi_4 X = \pi_4 S^3$. We will try to compute $H_4 X$ using a spectral sequence.

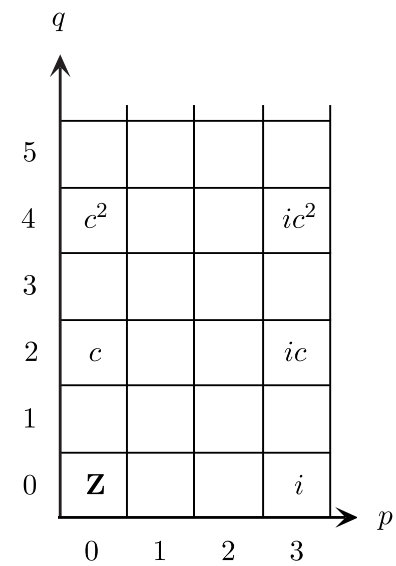
Consider the cohomology spectral sequence for the fibration (10.12). Then $E_2^{p,q} = H^p(S^3; H^q(K(\mathbf{Z},2)))$. Recall that $K(\mathbf{Z},2)$ is the infinite complex projective space $\mathbf{C}P^\infty$ whose cohomology algebra is the 1-variable polynomial ring $H^*(K(\mathbf{Z},2)) = \mathbf{Z}[c]$ where $\deg(c) = 2$.

Exercise 205. Give another proof of the fact that $H^*(K(\mathbf{Z},2)) = \mathbf{Z}[c]$ using the spectral sequence for the path space fibration

$$K(\mathbf{Z},1) \to * \to K(\mathbf{Z},2)$$

and the identification of $K(\mathbf{Z},1)$ with S^1. (Hint: the argument is contained in the proof of Proposition 10.28.)

Let $i \in H^3 S^3$ denote the generator. Then the E_2-stage in the spectral sequence is indicated in the following diagram. The labels mean that the groups in question are infinite cyclic with the indicated generators. The empty entries are zero. The entries in this table are computed using Lemma 10.25.



Since $H^2X = 0 = H^3X$ it follows that $d_3 c = i$. Therefore,

$$d_3 c^2 = ic + ci = 2ci.$$

This implies that $\mathbf{Z}/2 \cong E_4^{3,2} = E_\infty^{3,2} \cong H^5X$ and $0 = E_4^{0,4} = E_\infty^{0,4} = H^4X$. The universal coefficient theorem implies that $H_4X = \mathbf{Z}/2$. We conclude that $\mathbf{Z}/2 \cong \pi_4 X = \pi_4 S^3$, as desired. □

Corollary 10.30. *$\pi_{n+1}S^n = \mathbf{Z}/2$ for all $n \geq 3$. In particular, $\pi_1^S = \mathbf{Z}/2$.*

Proof. This is an immediate consequence of the Freudenthal suspension theorem (Theorem 9.7). □

Corollary 10.31. *$\pi_4 S^2 = \mathbf{Z}/2$.*

Proof. Apply the long exact sequence of homotopy groups to the Hopf fibration $S^1 \hookrightarrow S^3 \to S^2$. □

Exercise 206. Show that $\pi_n(\mathbf{C}P^\infty \vee \mathbf{C}P^\infty) \cong \pi_n S^3$ for $n > 2$. (Hint: Use the cohomology spectral sequence to show hofiber($\mathbf{C}P^\infty \vee \mathbf{C}P^\infty \to \mathbf{C}P^\infty \times \mathbf{C}P^\infty$) has the homology of S^3. Then argue that the hofiber is simply connected and use the Hurewicz Theorem and the relative Hurewicz Theorem to show that the homotopy groups of the hofiber and the 3-sphere are isomorphic.)

The reader should think about the strategy used to make these computations. On the one hand fibrations were used to relate homotopy groups of various spaces; on the other spectral sequences are used to compute homology groups. The Hurewicz theorem is then used to conclude that a homology group computation in fact gives a homotopy group computation.

10.8. Homology of groups

Definition 10.32. Let G be a group. Define the *cohomology of G with* $\mathbf{Z}$ *coefficients* by

$$H^k(G;\mathbf{Z}) = H^k(K(G,1);\mathbf{Z}).$$

Similarly define the *homology of G*

$$H_k(G;\mathbf{Z}) = H_k(K(G,1);\mathbf{Z}).$$

More generally define the homology and cohomology of G with coefficients in any R-module A to be the corresponding homology or cohomology of $K(G,1)$.

Corollary 8.25 implies that the homology and cohomology of a group are well-defined. Moreover, the assignment $G \mapsto K(G,1)$ is functorial and takes short exact sequences to fibrations. (The functoriality can be interpreted in two different ways. For every group one associates a homotopy type of spaces, and a group homomorphism leads to a homotopy class of maps between the spaces. Alternatively, one can construct an honest functor from the category of groups to the category of spaces by giving a specific model of $K(G,1)$ related to the bar resolution in homological algebra.)

Groups are very mysterious nonabelian things and thus are hard to study. The homology of groups gives abelian invariants and has been very useful in group theory as well as topology.

It follows that to understand the homology of groups related by exact sequences amounts to understanding the homology of a fibration, for which, as we have seen, spectral sequences are a good tool.

It is easy to see that $K(A \times B,1) = K(A,1) \times K(B,1)$, and so the Künneth theorem can be used to compute the cohomology of products of groups. Therefore the following result is all that is needed to obtain a complete computation of the cohomology of finitely generated abelian groups.

Theorem 10.33. *The cohomology of* $\mathbf{Z}/n$ *is given by*

$$H^q(\mathbf{Z}/n;\mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } q = 0, \\ 0 & \text{if } q \text{ is odd, and} \\ \mathbf{Z}/n & \text{if } q > 0 \text{ is even.} \end{cases}$$

Proof. The exact sequence $0 \to \mathbf{Z} \xrightarrow{\times n} \mathbf{Z} \to \mathbf{Z}/n \to 0$ induces a fibration sequence

$$K(\mathbf{Z},2) \to K(\mathbf{Z},2) \to K(\mathbf{Z}/n,2)$$

(see Proposition 8.27). By looping this fibration twice (i.e. taking iterated homotopy fibers twice; see Theorem 7.45) we obtain the fibration

$$K(\mathbf{Z},1) \to K(\mathbf{Z}/n,1) \to K(\mathbf{Z},2).$$

The fiber $K(\mathbf{Z},1)$ is a circle.

Consider the spectral sequence for this fibration. The base is simply connected so there is no twisting in the coefficients. Notice that

$$E_2^{p,q} = H^p(K(\mathbf{Z},2);H^qS^1) = \begin{cases} 0 & \text{if } q>1, \text{ and} \\ H^p(K(\mathbf{Z},2);\mathbf{Z}) & \text{if } q=0 \text{ or } 1. \end{cases}$$

Using Lemma 10.25, the E_2-stage is given by the following table, with the empty entries equal to 0 and the others infinite cyclic with the indicated generators (where i is the generator of H^1S^1).

q \ p	0	1	2	3	4	5	6
1	i		ic		ic^2		ic^3
0	1		c		c^2		c^3

Of course $d^2(i) = kc$ for some integer k, and the question is: what might k be? We can find out by "peeking at the answer". Since $E_\infty^{0,2} = 0 = E_\infty^{1,1}$, we see that $H^2(K(\mathbf{Z}/n,1)) = E_\infty^{2,0} \cong \mathbf{Z}/k$. Since $\pi_1(K(\mathbf{Z}/n,1)) = \mathbf{Z}/n$, by the universal coefficient theorem, we see that H^2 must be $\mathbf{Z}/n$ and hence $k = \pm n$. (Neat, huh?)

Let $\overline{c}$ be the image of c in $E_3^{2,0}$. Here is a picture of the E^3-stage.

q \ p	0	1	2	3	4	5	6
1							
0	1		$\overline{c}$		$\overline{c}^2$		$\overline{c}^3$

From this we see that the spectral sequence collapses at E^3, and that as graded rings $E_\infty^{*,0} \cong H^*(K(\mathbf{Z}/n,1))$. This not only completes the proof of the theorem, but also computes the cohomology ring

$$H^*(K(\mathbf{Z}/n,1)) = \mathbf{Z}[\bar{c}] \ /\langle n\bar{c}\rangle.$$

□

Also, we can get the homology from the cohomology by using the universal coefficient theorem:

$$H_q(\mathbf{Z}/n) = \begin{cases} \mathbf{Z} & \text{if } q = 0, \\ \mathbf{Z}/n & \text{if } q \text{ is odd, and} \\ 0 & \text{if } q > 0 \text{ is even.} \end{cases}$$

In applications, it is important to know the mod p-cohomology ring (which is the mod p-cohomology ring on an infinite-dimensional lens space). By the Künneth theorem (which implies that, if we use field coefficients, $H^*(X \times Y) \cong H^*X \otimes H^*Y$, it suffices to consider the case where n is a prime power. Let $\mathbf{F}_p$ denote the field $\mathbf{Z}/p\mathbf{Z}$ for a prime p.

Exercise 207. Show that $H^*(\mathbf{Z}/2;\mathbf{F}_2) \cong \mathbf{F}_2[a]$ where a has degree one, and if $p^k \neq 2$, $H^*(\mathbf{Z}/p^k;\mathbf{F}_p) \cong \Lambda(a) \otimes \mathbf{F}_p[b]$, where a has degree one and b has degree 2. Here $\Lambda(a)$ is the 2-dimensional graded algebra over $\mathbf{F}_p$ with $\Lambda(a)^0 \cong \mathbf{F}_p$ with generator 1, and $\Lambda(a)^1 \cong \mathbf{F}_p$ with generator a. (Hint: use $\mathbf{R}P^\infty = K(\mathbf{Z}/2,1)$ and $a \cdot a = -a \cdot a$ for $a \in H^1$.)

Exercise 208. Compute $H^p(K(\mathbf{Z}/2,n);\mathbf{Z}/2)$ for as many p and n as you can. (Hint: try induction on n, using the fibration

$$K(\mathbf{Z}/2,n) \to * \to K(\mathbf{Z}/2,n+1).)$$

10.9. Homology of covering spaces

Suppose that $f : \tilde{X} \to X$ is a regular cover of a path-connected space X. Letting $G = \pi_1 X/f_*(\pi_1\tilde{X})$, $f : \tilde{X} \to X$ is a principal G-bundle (with G discrete). Thus $G \hookrightarrow \tilde{X} \to X$ is pulled back from the universal G-bundle $G \hookrightarrow EG \to BG$ (see Theorem 9.15). In other words, there is a diagram

$$\begin{tikzcd} G \arrow[r, "="] \arrow[d] & G \arrow[d] \\ \tilde{X} \arrow[r] \arrow[d, "f"'] & EG \arrow[d] \\ X \arrow[r, "h"] & BG \end{tikzcd}$$

It follows that the sequence

$$\tilde{X} \to X \to BG$$

is a fibration (up to homotopy). (One way to see this is to consider the Borel fibration $\tilde{X} \hookrightarrow \tilde{X} \times_G EG \to BG$. Since G acts freely on $\tilde{X}$, there is another fibration $EG \hookrightarrow \tilde{X} \times_G EG \to \tilde{X}/G$. Since EG is contractible we see that the total space of the Borel fibration is homotopy equivalent to X.) Since G is discrete, $BG = K(G,1)$. Applying the homology (or cohomology) spectral sequence to this fibration immediately gives the following *spectral sequence of a covering space* (we use the notation $H_*G = H_*(K(G,1))$).

Theorem 10.34. *Given a regular cover $f : \tilde{X} \to X$ with group of covering automorphisms $G = \pi_1 X / f_*(\pi_1 \tilde{X})$, there is a homology spectral sequence*

$$H_p(G; H_q\tilde{X}) \cong E^2_{p,q} \Rightarrow H_{p+q}X$$

and a cohomology spectral sequence

$$H^p(G; H^q\tilde{X}) \cong E_2^{p,q} \Rightarrow H^{p+q}X.$$

The twisting of the coefficients is just the one induced by the action of G on $\tilde{X}$ by covering transformations. □

Applying the five-term exact sequence (Corollary 10.15) in this context gives the very useful exact sequence

$$H_2X \to H_2G \to H_0(G; H_1\tilde{X}) \to H_1X \to H_1G \to 0.$$

Exercise 209. Use the spectral sequence of the universal cover to show that for a path-connected space X the sequence

$$\pi_2 X \xrightarrow{\rho} H_2X \to H_2(\pi_1 X) \to 0$$

is exact, where ρ denotes the Hurewicz map.

As an application we examine the problem of determining which finite groups G can act freely on S^k. Equivalently, what are the fundamental groups of manifolds covered by the k-sphere? First note that if $g : S^k \to S^k$ is a fixed-point free map, then g is homotopic to the antipodal map (can you remember how to prove this?), and so is orientation-preserving if k is odd and orientation-reversing if k is even. Thus if k is even, the composite of any two nontrivial elements of G must be trivial, from which it follows that G has 1 or 2 elements. We shall henceforth assume k is odd, and hence that G acts by orientation-preserving fixed-point free homeomorphisms.

Thus the cohomology spectral sequence for the cover has

$$E_2^{p,q} = \begin{cases} H^p(G; H^q S^k) = H^pG & \text{if } q = 0 \text{ or } q = k, \\ 0 & \text{otherwise} \end{cases}$$

and converges to $H^{p+q}(S^k/G)$. This implies that the only possible nonzero differentials are

$$d_{k+1} : E^{p,k}_{k+1} \to E^{p-k-1,0}_{k+1}$$

and that the spectral sequence collapses at E_{k+2}.

Notice that S^k/G is a compact manifold of dimension k, and in particular $H^n(S^k/G) = 0$ for $n > k$. This forces $E^{p,q}_\infty = 0$ whenever $p + q > k$. Hence the differentials $d_{k+1} : E^{p,k}_{k+1} \to E^{p+k+1,0}_{k+1}$ are isomorphisms for $p \geq 1$, and since these are the only possible nonzero differentials we have

$$E^{p,k}_{k+1} = E^{p,k}_2 \cong H^pG \text{ and } E^{p+k+1,0}_{k+1} = E^{p+k+1,0}_2 \cong H^{p+k+1}G$$

so that $H^pG \cong H^{p+k+1}G$ for $p \geq 1$.

Thus G has *periodic cohomology* with *period* $k + 1$. Any subgroup of G also acts freely on S^k by restricting the action. This implies the following theorem.

Theorem 10.35. *If the finite group G acts freely on an odd-dimensional sphere S^k, then every subgroup of G has periodic cohomology of period $k+1$.* □

As an application, first note the group $\mathbf{Z}/p \times \mathbf{Z}/p$ does not have periodic cohomology; this can be checked using the Künneth theorem. We conclude that any finite group acting freely on a sphere cannot contain a subgroup isomorphic to $\mathbf{Z}/p \times \mathbf{Z}/p$.

10.10. Relative spectral sequences

In studying maps of fibrations, it is useful to have relative versions of the homology and cohomology spectral sequence theorems. There are two relative versions, one involving a subspace of the base and one involving a subspace of the fiber.

Theorem 10.36. *Let $F \hookrightarrow X \xrightarrow{f} B$ be a fibration with B a CW-complex. Let $A \subset B$ be a subcomplex. Let $Y = p^{-1}A$. Suppose G_* (resp. G^*) is a generalized homology (resp. cohomology) theory.*

1. *There is a homology spectral sequence with*

$$H_p(B, A; G_qF) \cong E^2_{p,q} \Rightarrow G_{p+q}(X, Y).$$

2. *If B is finite-dimensional or if there exists an N so that $G^qF = 0$ for all $q < N$, there is a cohomology spectral sequence with*

$$H^p(B, A; G^qF) \cong E^{p,q}_2 \Rightarrow G^{p+q}(X, Y).$$

□

Theorem 10.37. *Let $F \hookrightarrow X \xrightarrow{f} B$ be a fibration with B a CW-complex. Let $X_0 \subset X$ so that $f|_{X_0} : X_0 \to B$ is a fibration with fiber F_0. Suppose G_* (resp. G^*) is a generalized homology (resp. cohomology) theory.*

1. *There is a homology spectral sequence with*
$$H_p(B; G_q(F, F_0)) \cong E^2_{p,q} \Rightarrow G_{p+q}(X, X_0).$$
2. *If B is finite-dimensional or if there exists an N so that $G^q(F, F_0) = 0$ for all $q < N$, there is a cohomology spectral sequence with*
$$H^p(B; G^q(F, F_0)) \cong E_2^{p,q} \Rightarrow G^{p+q}(X, X_0).$$

□

The multiplicative properties of relative cohomology spectral sequences are similar to those in the absolute case.

More generally, one can generalize to spectral sequences the cup product $H^p(X) \times H^q(X, A) \to H^{p+q}(X, A)$ of Corollary 4.30 as follows. In the following theorem, $f : X \to B$ denotes a fibration over a CW complex with fiber F and $X_0 \subset X$ a subset satisfying $f|_{X_0} : X_0 \to B$ is a fibration with fiber F_0. Also, H^* denotes ordinary homology with coefficients in some commutative ring. Assume that $\pi_1(B)$ acts trivially on $H^*(F)$ and $H^*(F, F_0)$.

In this situation, one has two relative cup products

$$H^p(X) \times H^q(X, X_0) \to H^{p+q}(X, X_0) \tag{10.13}$$

and

$$H^q(F) \times H^s(F, F_0) \to H^{q+s}(F, F_0).$$

This latter product can be viewed as a bilinear pairing on coefficients, inducing a cup product

$$H^p(B; H^q(F)) \times H^r(B; H^q(F, F_0)) \to H^{p+r}(B; H^{q+s}(F, F_0)) \tag{10.14}$$

as in (the proof of) Lemma 10.25.

Assume either that B is finite-dimensional, or that $H^k(F)$ and $H^k(F, F_0)$ vanish for k large enough.

Theorem 10.38. *Let $\bar{E}_*^{*,*}$ be the cohomology Leray-Serre spectral sequence of the fibration $F \hookrightarrow X \to B$ and $E_*^{*,*}$ be the cohomology Leray-Serre spectral sequence of the relative fibration $(F, F_0) \hookrightarrow (X, X_0) \to B$ as in Theorem 10.37, so that*
$$\bar{E}_2^{p,q} \cong H^p(B; H^q(F)) \Rightarrow H^{p+q}(X)$$
$$E_2^{p,q} \cong H^p(B; H^q(F, F_0)) \Rightarrow H^{p+q}(X, X_0).$$

The cup product of Equation (10.13) induces a cup product on associated graded modules:

$$\bar{E}_\infty^{p,q} \times E_\infty^{r,s} \to E_\infty^{p+r,q+s} \tag{10.15}$$

The cup product $\bar{E}_2^{p,q} \times E_2^{r,s} \to E_2^{p+r,q+s}$ *given in Equation (10.14) induces products*

$$\bar{E}_k^{p,q} \times E_k^{r,s} \to E_k^{p+r,q+s} \tag{10.16}$$

for $k = 2, 3, \cdots, \infty$.

The product (10.15) coincides with the product (10.16) when $k = \infty$.

10.11. Projects: Construction of the spectral sequence

10.11.1. Construction of the spectral sequence. Give (or outline) the construction of the Leray-Serre-Atiyah-Hirzebruch spectral sequence and prove the main theorem, Theorem 10.7. References include [**54**, Sect. XIII.5] and [**45**, Ch. 9] (only for ordinary homology).

Chapter 11

Further Applications of Spectral Sequences

11.1. Serre classes of abelian groups

Definition 11.1. A *Serre class of abelian groups* is a nonempty collection $\mathcal{C}$ of abelian groups satisfying:

1. If $0 \to A \to B \to C \to 0$ is a short exact sequence, then $B \in \mathcal{C}$ if and only if $A, C \in \mathcal{C}$.

Moreover, there are additional axioms which can be useful:

2A. If $A, B \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ and $\mathrm{Tor}(A, B) \in \mathcal{C}$.

2B. If $A \in \mathcal{C}$, then $A \otimes B \in \mathcal{C}$ for *any* abelian group B.

3. If $A \in \mathcal{C}$, then $H_n(A; \mathbf{Z}) = H_n(K(A, 1); \mathbf{Z})$ is in $\mathcal{C}$ for every $n > 0$.

Exercise 210. Prove that Axiom 2B implies Axiom 2A. (Hint: show that $\mathrm{Tor}(A, B) \subset A \otimes F$ for some F.)

There are many examples of Serre classes, including the trivial class, the class of all abelian groups, the class of torsion abelian groups, torsion abelian groups such that no element is p^r-torsion for a fixed prime p, the class of finite abelian groups, and the class of abelian p-groups. You should think about which of the axioms these classes satisfy.

It suffices for our exposition to consider the following two examples.

1. The class $\mathcal{C}_{FG}$ of finitely generated abelian groups. Axioms 1 and 2A clearly hold (see Exercise 51 and the remark preceding it). Note, however, that $\mathbf{Z} \in \mathcal{C}_{FG}$, but $\mathbf{Z} \otimes \mathbf{Q}$ is not in $\mathcal{C}_{FG}$, so 2B does not

hold. Axiom 3 follows from Theorem 10.33, the Künneth theorem, and the fact that $K(\mathbf{Z}, 1) = S^1$.

2. Let P denote a subset of the set of all prime numbers. Let $\mathcal{C}_P$ denote the class of torsion abelian groups A so that no element of A has order a positive power of p for $p \in P$. Thus for example, if P is empty, then $\mathcal{C}_P$ is the class of all torsion abelian groups. If P denotes all primes, then $\mathcal{C}_P$ is the class containing only the trivial group. If P consists of the single prime p, then we use the notation $\mathcal{C}_p$ for $\mathcal{C}_P$.

We will show that the class $\mathcal{C}_P$ satisfies Axioms 1, 2B, and 3. First some terminology: given a prime p, the *p-primary subgroup* of an abelian group consists of the subgroup of those elements whose order is a power of p. Thus $\mathcal{C}_P$ consists of those torsion abelian groups whose p-primary subgroup is trivial for any $p \in P$.

Lemma 11.2. *The class $\mathcal{C}_P$ satisfies Axioms 1, 2B, and 3.*

Proof. Say that an integer $n \neq 0$ is *prime to P* if p does not divide n for all $p \in P$. Then an abelian group A is in $\mathcal{C}_P$ if and only if for all $a \in A$, there is an n prime to P so that $na = 0$.

We first prove Axiom 1. Let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be an exact sequence of abelian groups. If $B \in \mathcal{C}_P$, then for $a \in A$, there is an n prime to P so that $n\alpha(a) = 0$. Hence $na = 0$, and so $A \in \mathcal{C}_P$. If $B \in \mathcal{C}_P$, then for $c \in C$, choose $b \in \beta^{-1}\{c\}$ and n prime to P so that $nb = 0$. Then $nc = n\beta(b) = 0$ and hence $C \in \mathcal{C}_P$. Conversely assume $A, C \in \mathcal{C}_P$. Then for $b \in B$, there exists an n prime to P so that $n\beta(b) = 0$. By exactness $nb = \alpha(a)$ for some a. Choose m prime to P so that $ma = 0$. Then $mnb = m\alpha(a) = 0$, so $B \in \mathcal{C}_P$.

Next comes Axiom 2B. Suppose that $A \in \mathcal{C}_P$ and let B be an arbitrary abelian group. Pick an element $t = \sum_i a_i \otimes b_i \in A \otimes B$. Since $A \in \mathcal{C}_P$, we can find integers n_i prime to P so that $n_i a_i = 0$. Let $n = \prod n_i$; this is prime to P and $nt = 0$. Thus $A \otimes B \in \mathcal{C}_P$.

We turn to the proof of Axiom 3. Let $A \in \mathcal{C}_P$. Suppose first that A is finitely generated. Then A is isomorphic to the finite direct sum of cyclic groups $A \cong \oplus_i \mathbf{Z}/p_i^{r_i}$ where $p_i \notin P$. Using Theorem 10.33, the Künneth theorem, and induction, it follows that $H_n(K(A, 1); \mathbf{Z})$ is a finitely generated torsion abelian group with trivial p-primary subgroup for any $p \in P$.

Next let $A \in \mathcal{C}_P$ be arbitrary and pick an $\alpha \in H_n(K(A, 1); \mathbf{Z})$. Choose a cycle z representing the homology class α. Since z is a finite sum of singular simplices there is a finite subcomplex $X \subset K(A, 1)$ containing the image of every singular simplex in z. Therefore α lies in the image of $H_n X \to$

$H_n(K(A,1))$. Let $A' \subset A$ denote the finitely generated subgroup $\mathrm{im}(\pi_1 X \to \pi_1(K(A,1)))$. Since $\pi_1(K(A,1)) = A \in \mathcal{C}_P$, the subgroup A' is also in $\mathcal{C}_P$. The space $K(A',1)$ can be constructed by adding k-cells to X for $k \geq 2$, and since $A' \to A$ is injective the inclusion $X \subset K(A,1)$ can be extended to give the commutative diagram

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & K(A,1) \\ & K(A',1) \arrow[ur] & \end{tikzcd}$$

Thus $\alpha \in \mathrm{im}(H_n(K(A',1)) \to H_n(K(A,1)))$, and since A' is finitely generated, α is torsion with order relatively prime to p for $p \in P$. Thus $H_n(K(A,1)) \in \mathcal{C}_P$. □

Definition 11.3. Given a Serre class $\mathcal{C}$, a homomorphism $\varphi : A \to B$ between two abelian groups is called:

1. a $\mathcal{C}$-*monomorphism* if $\ker \varphi \in \mathcal{C}$,
2. a $\mathcal{C}$-*epimorphism* if $\operatorname{coker} \varphi \in \mathcal{C}$, and
3. a $\mathcal{C}$-*isomorphism* if $\ker \varphi \in \mathcal{C}$ and $\operatorname{coker} \varphi \in \mathcal{C}$.

Two abelian groups A and B are called $\mathcal{C}$-*isomorphic* if there exists an abelian group C and two $\mathcal{C}$-isomorphisms $f : C \to A$ and $g : C \to B$.

Lemma 11.4. *Let $\alpha : A \to B$ and $\beta : B \to C$ be homomorphisms of abelian groups. If two of the three maps α, β, and $\beta \circ \alpha$ are $\mathcal{C}$-isomorphisms, then so is the third.*

Proof. This follows from the exact sequence

$$0 \to \ker \alpha \to \ker \beta \circ \alpha \xrightarrow{\alpha} \ker \beta \to \operatorname{coker} \alpha \xrightarrow{\beta} \operatorname{coker} \beta \circ \alpha \to \operatorname{coker} \beta \to 0.$$

□

Lemma 11.5. *$\mathcal{C}$-isomorphism is an equivalence relation.*

Proof. Reflexivity and symmetry are clear. For transitivity, suppose that A and B are $\mathcal{C}$-isomorphic, and B and C are $\mathcal{C}$-isomorphic. Choose $\mathcal{C}$-isomorphisms $\alpha : H \to A$, $\beta : H \to B$, $\gamma : K \to B$, and $\delta : K \to C$. Let $L = \{(h,k) \in H \oplus K \mid \beta(h) = \gamma(k)\}$ and let $\pi_H : L \to H, \pi_K : L \to K$ be the two projections.

The kernel of π_H is $\{(0,k) \in L\} \cong \ker \gamma \in \mathcal{C}$. Since $\alpha \circ \pi_H = \gamma \circ \pi_K$, α induces an injective morphism $\bar{\alpha} : \operatorname{coker} \pi_H \to \operatorname{coker} \gamma \in \mathcal{C}$. Hence $\operatorname{coker} \pi_H \in \mathcal{C}$. Thus π_H is a $\mathcal{C}$-isomorphism. The same argument shows that

π_K is a $\mathcal{C}$-isomorphism. Lemma 11.4 then implies that both $\alpha \circ \pi_H : L \to A$ and $\delta \circ \pi_K : L \to C$ are $\mathcal{C}$-isomorphisms. □

We will sometimes write $A \cong B \bmod \mathcal{C}$ to indicate that A and B are $\mathcal{C}$-isomorphic.

Exercise 211. Let $\mathcal{C}_\phi$ be the class of torsion abelian groups. Show that $A \cong B \bmod \mathcal{C}_\phi$ if and only if $A \otimes \mathbf{Q} \cong B \otimes \mathbf{Q}$.

Exercise 212. Prove the five-lemma "mod $\mathcal{C}$".

The Hurewicz theorem has the following extremely useful generalization.

Theorem 11.6.

1. (mod $\mathcal{C}$ Hurewicz theorem) *Let X be 1-connected, and suppose $\mathcal{C}$ satisfies Axioms 1, 2A, and 3.*
   (a) *If $\pi_i X \in \mathcal{C}$ for all $i < n$, then $H_i X \in \mathcal{C}$ for all $0 < i < n$ and the Hurewicz map $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.*
   (b) *If $H_i X \in \mathcal{C}$ for all $0 < i < n$, then $\pi_i X \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.*
2. (mod $\mathcal{C}$ relative Hurewicz theorem) *Suppose $A \subset X$, A and X are 1-connected, and $\pi_2(X, A) = 0$. Suppose $\mathcal{C}$ satisfies Axioms 1, 2B, and 3.*
   (a) *If $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$, then $H_i(X, A) \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is a $\mathcal{C}$-isomorphism.*
   (b) *If $H_i(X, A) \in \mathcal{C}$ for all $i < n$, then $\pi_i(X, A) \in \mathcal{C}$ for all $i < n$ and the Hurewicz map $\pi_n(X, A) \to H_n(X, A)$ is a $\mathcal{C}$-isomorphism.*

Actually, as you can easily check, the part (b)'s above follow from the part (a)'s. We will give a proof of the theorem using spectral sequences and the fact that $\pi_1 Y \to H_1 Y$ is an isomorphism when the fundamental group is abelian. By taking $\mathcal{C}$ to be the class consisting of the trivial group, we obtain proofs of the classical Hurewicz and relative Hurewicz theorems. The proof we give simplifies a bit in the classical case. A proof of the classical case without the use of spectral sequences was a project in Chapter 7.

The mod $\mathcal{C}$ relative Hurewicz theorem implies the mod $\mathcal{C}$ Whitehead theorem.

Theorem 11.7 (mod $\mathcal{C}$ Whitehead theorem). *Let $f : A \to X$, where A, X are 1-connected, and suppose $f : \pi_2 A \to \pi_2 X$ is an epimorphism. Let $\mathcal{C}$ satisfy Axioms 1, 2B, and 3. Then the following two statements are equivalent.*

1. *$f_* : \pi_i A \to \pi_i X$ is an $\mathcal{C}$-isomorphism for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.*
2. *$f_* : H_i A \to H_i X$ is a $\mathcal{C}$-isomorphism for $i < n$ and a $\mathcal{C}$-epimorphism for $i = n$.*

Exercise 213. Show that Theorem 11.7 follows from Theorem 11.6.

Since the homology groups of a finite CW-complex are all in $\mathcal{C}_{FG}$, the mod $\mathcal{C}$ Hurewicz theorem has the following important consequence.

Corollary 11.8. *If X is a simply connected finite CW-complex, then all the homotopy groups of X are finitely generated. More generally a simply connected space has finitely generated homology groups in every dimension if and only if it has finitely generated homotopy groups in each dimension.*

This sounds great, but we warn you that the only simply connected finite CW-complexes for which all homotopy groups have been computed are contractible.

The hypothesis in the corollary that X be simply connected is necessary, as the following exercise shows.

Exercise 214. Prove that $\pi_2(S^1 \vee S^2)$ is not finitely generated.

Exercise 215. Show that Corollary 11.8 holds more generally when $\pi_1 X$ is finite.

We turn now to the proof of the Hurewicz theorem.

Proof of Theorem 11.6. Here is the idea of the proof. For a space X, consider the path fibration

$$\Omega X \to PX \xrightarrow{f} X.$$

There is a commutative diagram

$$\begin{tikzcd}
\pi_n(X, x_0) \arrow[d, "\rho"'] & \pi_n(PX, \Omega X) \arrow[l, "f_*"', "\cong"] \arrow[r, "\partial", "\cong"'] \arrow[d, "\rho"'] & \pi_{n-1}(\Omega X) \arrow[d, "\rho"'] \\
H_n(X, x_0) & H_n(PX, \Omega X) \arrow[l, "f_*"'] \arrow[r, "\partial", "\cong"'] & H_{n-1}(\Omega X)
\end{tikzcd} \tag{11.1}$$

where the vertical maps are Hurewicz maps. The boundary maps ∂ are isomorphisms since PX is contractible. The top f_* is an isomorphism since f is a fibration (see Lemma 7.59). With the mod $\mathcal{C}$-connectivity hypothesis, a spectral sequence argument given below shows that the bottom f_* is a $\mathcal{C}$-isomorphism. Inductively, the right-hand ρ is a $\mathcal{C}$-isomorphism, so thereby the left-hand ρ is a $\mathcal{C}$-isomorphism. There are three difficulties with this outline. We have to get the induction started; we have to make the mod $\mathcal{C}$

spectral sequence argument; and we have to deal with the fact that if $\pi_2 X \neq 0$, then ΩX is not simply connected, so, strictly speaking, the inductive hypothesis does not apply.

We will use the following lemma, which shows why Serre classes are tailor-made to be used with spectral sequences.

Lemma 11.9. *Let $(E^r_{*,*}, d^r)$ be a first quadrant spectral sequence. Let $\mathcal{C}$ denote a class of abelian groups.*

1. *For any bigraded spectral sequence, if $\mathcal{C}$ satisfies Axiom 1, $E^n_{p,q} \in \mathcal{C}$ for some p, q implies that $E^r_{p,q} \in \mathcal{C}$ for all $r \geq n$.*
2. *Let $F \hookrightarrow E \xrightarrow{f} B$ be a fibration over a simply connected base space. If $\mathcal{C}$ satisfies Axioms 1 and 2A, if $H_p B \in \mathcal{C}$ for $0 < p < n$, and if $H_q F \in \mathcal{C}$ for $0 < q < n-1$, then $f_* : H_i(E, F) \to H_i(B, b_0)$ is a $\mathcal{C}$-isomorphism for $i \leq n$.*

Proof. Part 2 is connected with the transgression theorem

1. A subgroup or a quotient group of a group in $\mathcal{C}$ is in $\mathcal{C}$ by Axiom 1. Thus a subquotient of a group in $\mathcal{C}$ is in $\mathcal{C}$. Since $E^r_{p,q}$ is a subquotient of $E^n_{p,q}$, the first statement follows.

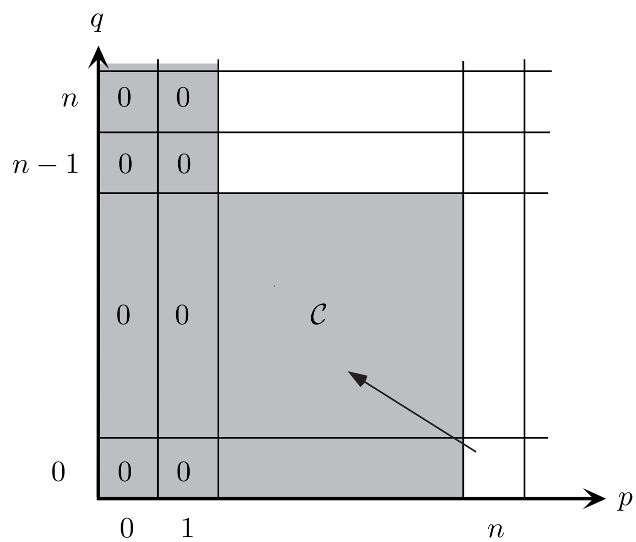
2. Consider the spectral sequence of the relative fibration (Theorem 10.36)

$$F \to (E, F) \to (B, b_0).$$

The E^2-term is

$$E^2_{p,q} \cong H_p(B, b_0; H_q F) \cong (H_p(B, b_0) \otimes H_q F) \oplus \operatorname{Tor}(H_{p-1}(B, b_0), H_q F),$$

and the spectral sequence converges to $H_*(E, F)$. Thus $E^2_{p,q} \in \mathcal{C}$ and hence $E^\infty_{p,q} \in \mathcal{C}$ when $p = 0, 1$ or when $1 < p < n$ and $0 < q < n-1$ (see the shaded area in the picture below).



The picture gives a convincing argument that

$$H_i(E,F) \cong H_i(B,b_0) \mod \mathcal{C} \quad \text{for } i \le n,$$

but here is a precise one.

The spectral sequence gives a filtration

$$0 = F_{-1,i+1} \subset F_{0,i} \subset F_{1,i-1} \subset \cdots \subset F_{i-1,1} \subset F_{i,0} = H_i(E,F)$$

with

$$E^\infty_{p,i-p} = F_{p,i-p}/F_{p-1,i-p+1}.$$

It follows by induction on p that $F_{p,i-p} \in \mathcal{C}$ for $p < i$, and hence that

$$H_i(E,F) \to E^\infty_{i,0} \quad \text{is a } \mathcal{C}\text{-isomorphism.} \tag{11.2}$$

On the other hand, for $r \ge 2$, the exact sequence

$$0 \to E^{r+1}_{i,0} \to E^r_{i,0} \xrightarrow{d^r} E^r_{i-r,r-1}$$

with the range of d^r in $\mathcal{C}$ shows that $E^{r+1}_{i,0} \to E^r_{i,0}$ is a $\mathcal{C}$-isomorphism. Since the composite of $\mathcal{C}$-isomorphisms is a $\mathcal{C}$-isomorphism, it follows by induction that

$$E^\infty_{i,0} \to E^2_{i,0} \quad \text{is a } \mathcal{C}\text{-isomorphism.} \tag{11.3}$$

Therefore, the composite of (11.2) and (11.3) is a $\mathcal{C}$-isomorphism. But this composite is identified with the edge homomorphism (see Theorem 10.13)

$$f_* : H_i(E,F) \to H_i(B,b_0).$$

□

We return to the proof of Part 1(a). The proof is by induction on n. For $n = 1$, $\pi_1 X = 0 = H_1 X$. For $n = 2$, $\pi_1 X = 0 = H_1 X$, and so Lemma 11.9 shows that $f_* : H_2(PX, \Omega X) \to H_2(X, x_0)$ is an isomorphism (use the class consisting only of the trivial group). Hence diagram (11.1), the fact that $\pi_1(\Omega X)$ is abelian since ΩX is an H-space, and Theorem 1.9 (the π_1-version of the Hurewicz theorem) show that $\rho : \pi_2 X \to H_2 X$ is an isomorphism for any simply connected X.

Now suppose $n > 2$ and inductively assume for simply connected spaces Y with $\pi_i Y \in \mathcal{C}$ for $i < n - 1$ that $H_i Y \in \mathcal{C}$ for $0 < i < n - 1$ and that the Hurewicz map $\rho : \pi_{n-1} Y \to H_{n-1} Y$ is a $\mathcal{C}$-isomorphism. Let X be a simply connected space so that $\pi_i X \in \mathcal{C}$ for $i < n$. There will be two cases: where $\pi_2 X = 0$ and where $\pi_2 X \neq 0$.

In the first case $\pi_1(\Omega X) = \pi_2 X = 0$, and $\pi_i(\Omega X) = \pi_{i+1} X$, so we can apply the inductive hypothesis to ΩX and conclude that the right-hand $\rho : \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X)$ in diagram (11.1) is a $\mathcal{C}$-isomorphism and that $H_i(\Omega X) \in \mathcal{C}$ for $i < n - 1$. Then Lemma 11.9 applied to the path fibration

$$\Omega X \to PX \xrightarrow{f} X$$

shows that the lower f_* in diagram (11.1) is a $\mathcal{C}$-isomorphism. Then Lemma 11.4 applied repeatedly to diagram (11.1) shows that $\pi_n X \to H_{n-1} X$ is a $\mathcal{C}$-isomorphism as desired.

Now suppose we are in the case where $\pi_2 X \neq 0$. By hypothesis, $\pi_2 X \in \mathcal{C}$. There is a map $f : X \to K(\pi_2 X, 2)$ inducing the identity on π_2. Let $X_2 \to X$ be the homotopy fiber of this map. Now turn this map into a fibration (see Theorem 7.45) to obtain the fibration

$$K(\pi_2 X, 1) \to X_2 \to X.$$

Note by Axiom 3, $H_i(K(\pi_2 X, 1)) \in \mathcal{C}$ for $i > 0$. This has two consequences, first that

$$H_i(X_2) \to H_i(X_2, K(\pi_2 X, 1))$$

is a $\mathcal{C}$-isomorphism for $i > 0$, and second that Lemma 11.9 applies and so

$$H_i(X_2, K(\pi_2 X, 1)) \to H_i(X, x_0)$$

is a $\mathcal{C}$-isomorphism for $0 < i \leq n$. Thus the composite of these two maps $H_i(X_2) \to H_i X$ is a $\mathcal{C}$-isomorphism for $0 < i \leq n$. Summarizing,

1. $H_i X_2 \in \mathcal{C}$ for all $0 < i < n$,
2. $H_n X_2 \cong H_n X \bmod \mathcal{C}$,
3. $\pi_i X_2 = \pi_i X$ for all $i > 2$,
4. $\pi_1 X_2 = 0 = \pi_2 X_2$.

Thus $\rho : \pi_n X_2 \to H_n X_2$ is a $\mathcal{C}$-isomorphism and $H_i X_2 \in \mathcal{C}$ for $0 < i < n$ by the $\pi_2 = 0$ case; hence the same is true for X. This completes the proof of Part 1(a) of the mod $\mathcal{C}$ Hurewicz theorem.

Part 1(b) follows formally from Part 1(a). Indeed, let X be simply connected and suppose $H_i X \in \mathcal{C}$ for $0 < i < n$. Then use induction on i to show $\rho : \pi_i X \to H_i X$ is a $\mathcal{C}$-isomorphism for $1 < i \leq n$.

We now show how to deduce the relative Hurewicz theorem from the absolute theorem. We assume that X and A are simply connected, nonempty, and that $\pi_2(X, A) = 0$. The diagram

$$\begin{tikzcd} \pi_2 A \arrow[r] \arrow[d, "\rho"'] & \pi_2 X \arrow[d, "\rho"'] \\ H_2 A \arrow[r] & H_2 X \end{tikzcd}$$

commutes, with the vertical maps Hurewicz isomorphisms and the horizontal maps induced by inclusion. Since the top horizontal map is surjective, so is the bottom one, and it follows from the long exact sequence in homology that $H_2(X, A) = 0$.

Now suppose $n > 2$ and inductively assume that for any simply connected pair $B \subset Y$ with $\pi_2(Y, B) = 0$ and $\pi_i(Y, B) \in \mathcal{C}$ for $i < n - 1$, that $H_i(Y, B) \in \mathcal{C}$ and that the Hurewicz map $\rho : \pi_i(Y, B) \to H_i(Y, B)$ is a $\mathcal{C}$-isomorphism for $i < n - 1$.

Let (X, A) be a pair of simply connected spaces with $\pi_k(X, A) \in \mathcal{C}$ for $k < n$. Then by induction $H_k(X, A) \in \mathcal{C}$ for $k < n$. We must show that $\rho : \pi_n(X, A) \to H_n(X, A)$ is a $\mathcal{C}$-isomorphism.

Let $f : PX \to X$ denote the path space fibration, and let $L = L(X, A) = f^{-1}A$. Thus we have a relative fibration

$$\Omega X \to (PX, L) \to (X, A).$$

Recall from page 204 in Chapter 7 that $\pi_{k-1} L \cong \pi_k(X, A)$ for all k. The Leray–Serre spectral sequence for this fibration has

$$E^2_{p,q} = H_p(X, A; H_q(\Omega X))$$

and converges to $H_{p+q}(PX, L)$. The coefficients are untwisted since X and A are simply connected.

We have

$$H_p(X, A; H_q(\Omega X)) = H_p(X, A) \otimes H_q(\Omega X) \oplus \mathrm{Tor}(H_{p-1}(X, A), H_q(\Omega X)) \in \mathcal{C}$$

for $p < n$. This follows from the fact that $H_p(X, A) \in \mathcal{C}$ for $p < n$ and Axiom 2B. (This is where we need Axiom 2B, which is stronger than 2A.)

Therefore (see the picture on page 334 again):

1. all differentials out of $E^r_{n,0}$ have range in $\mathcal{C}$, and so $H_n(X,A) \cong E^r_{n,0} \cong E^\infty_{n,0} \bmod \mathcal{C}$ for all r, and
2. $E^\infty_{p,n-p} \in \mathcal{C}$ for $p > 0$, and so $H_n(PX,L) \cong E^\infty_{n,0} \bmod \mathcal{C}$.

Arguing as above we have:

$$\begin{aligned} H_n(PX,L) &\cong E^\infty_{n,0} \bmod \mathcal{C} \\ &\cong H_n(X,A) \bmod \mathcal{C}. \end{aligned} \tag{11.4}$$

This $\mathcal{C}$-isomorphism is induced by the edge homomorphism and hence coincides with the homomorphism induced by $f:(PX,L)\to(X,A)$.

The diagram

$$\begin{tikzcd} \pi_n(X,A) \arrow[d, "\rho"'] & \pi_n(PX,L) \arrow[l, "f_*"'] \arrow[r, "\partial", "\cong"'] \arrow[d, "\rho"'] & \pi_{n-1}L \arrow[d, "\rho"'] \\ H_n(X,A) & H_n(PX,L) \arrow[l, "f_*"'] \arrow[r, "\partial", "\cong"'] & H_{n-1}L \end{tikzcd}$$

commutes, with the two right horizontal arrows isomorphisms by the long exact sequence of the pair in homology and homotopy groups and the fact that PX is contractible.

The top left horizontal arrow is an isomorphism since $f: PX \to X$ is a fibration (see Lemma 7.59). Since $\pi_{k-1}L = \pi_k(X,A)$, $\pi_1 L = 0$ and $\pi_k L \in \mathcal{C}$ for all $k < n-1$. The absolute Hurewicz theorem implies that $\rho : \pi_{n-1}L \to H_{n-1}L$ is a $\mathcal{C}$-isomorphism.

Finally the bottom left horizontal map is a $\mathcal{C}$-isomorphism by (11.4). Moving around the diagram shows that the Hurewicz map $\rho : \pi_n(X,A) \to H_n(X,A)$ is a $\mathcal{C}$-isomorphism. This proves Part 2(a) of Theorem 11.6. As before, Part 2(b) follows from Part 2(a). □

11.2. Homotopy groups of spheres

In this section we will use the machinery of spectral sequences and Serre classes to obtain more nontrivial information about the elusive homotopy groups of spheres. An immediate consequence of Corollary 11.8 is the following.

Corollary 11.10. *The homotopy groups of spheres $\pi_k S^n$ are finitely generated abelian groups.* □

Here is a result which follows easily from Serre mod $\mathcal{C}$ theory.

Theorem 11.11. *If n is odd, $\pi_m S^n$ is finite for $m \neq n$.*

Proof. If $n = 1$, then $\pi_m S^1 = \pi_m \mathbf{R} = 0$ for $m \neq 1$. If $n > 1$ is odd, then a map $f : S^n \to K(\mathbf{Z}, n)$ inducing an isomorphism on π_n, induces an isomorphism on homology with rational coefficients by Proposition 10.28. Taking the Serre class $\mathcal{C}_\phi$ as in Exercise 211, we see f_* induces a $\mathcal{C}_\phi$-isomorphism on homology, and hence, by the mod $\mathcal{C}$ Whitehead theorem, also a $\mathcal{C}_\phi$-isomorphism on homotopy. Thus for all m, the kernel and cokernel of

$$f_* : \pi_m S^n \to \pi_m(K(\mathbf{Z}, n))$$

are torsion groups. However, the homotopy groups of spheres are finitely generated. The result follows.

$\square$

Corollary 11.12. *The stable homotopy groups of spheres π_n^S are finite for $n > 0$.* $\square$

Exercise 216. Prove that if n is even, then $\pi_k S^n$ is finite except for $k = n$ and $k = 2n - 1$, and that $\pi_{2n-1} S^n$ is the direct sum of $\mathbf{Z}$ and a finite abelian group. (Hint: Let $S^{n-1} \to T \to S^n$ be the unit tangent bundle of S^n for n even. Show that $\pi_n T \to \pi_n S^n$ is not onto by, for example, showing that a lift of $\mathrm{Id} : S^n \to S^n$ leads to a nonzero vector field on S^n, and hence a homotopy from the identity to the antipodal map. Conclude that $\pi_n T \to \pi_n S^n$ is the zero map. By looking at the transgression, deduce that $H_k T$ is finite except for $H_0 T = H_{2n-1} T = \mathbf{Z}$. Find a map inducing an isomorphism $H_k T \cong H_k S^{2n-1}$ mod $\mathcal{C}_\phi$. Then $\pi_k T \cong \pi_k S^{2n-1}$ mod $\mathcal{C}_\phi$ via the Whitehead theorem. Then apply Theorem 11.11.)

For the next sequence of results, let $K(\mathbf{Z}, 2) \to X \to S^3$ be the fibration from (10.12). Thus X is the homotopy fiber of the map $S^3 \to K(\mathbf{Z}, 3)$ inducing an isomorphism on π_3. It follows that $\pi_n X = 0$ for $n \leq 3$ and $\pi_n X = \pi_n S^3$ for $n > 3$.

Lemma 11.13.

$$H_k X = \begin{cases} 0 & \text{if } k \text{ is odd,} \\ \mathbf{Z} & \text{if } k = 0, \text{ and} \\ \mathbf{Z}/n & \text{if } k = 2n. \end{cases}$$

Proof. The integral cohomology ring of $K(\mathbf{Z}, 2) = \mathbf{C}P^\infty$ is a polynomial ring $H^*(K(\mathbf{Z}, 2); \mathbf{Z}) \cong \mathbf{Z}[c]$, where $\deg\, c = 2$ (see Exercise 205).

Consider the cohomology spectral sequence for the fibration (10.12). This has

$$E_2 = H^*(S^3) \otimes \mathbf{Z}[c].$$

More precisely,

$$\begin{aligned} E_2^{p,q} &= H^p(S^3; H^q(K(\mathbf{Z},2))) \\ &= \begin{cases} 0 & \text{if } p = 1,2 \text{ or } p > 3, \text{ or if } q \text{ is odd,} \\ \mathbf{Z}\cdot c^k & \text{if } p = 0 \text{ and } q = 2k \text{ is even, and} \\ \mathbf{Z}\cdot \iota c^k & \text{if } p = 3 \text{ and } q = 2k \text{ is even,} \end{cases} \end{aligned}$$

where $\iota \in H^3(S^3)$ denotes the generator, using Theorem 3.35 (the universal coefficient theorem for cohomology). Notice also that since all the differentials d_2 are zero, $E_2 = E_3$.

Since $H^2X = 0 = H^3X$, the differential $d_3 : E_3^{0,2} \to E_3^{3,0}$ must be an isomorphism, and so $d_3c = \iota$ (after perhaps replacing ι by $-\iota$). Thus

$$d(c^2) = (dc)\cdot c + c \cdot dc = \iota \cdot c + c \cdot \iota = 2\iota \cdot c.$$

More generally, one shows by an easy induction argument that

$$d(c^n) = n\iota \cdot c^{n-1}.$$

All other differentials in the spectral sequence are zero since either their domain or range is zero. Therefore $E_\infty^{3,2n-2} = \mathbf{Z}/n$ and hence $H^{2n+1}X = \mathbf{Z}/n$ if $n \geq 1$. The universal coefficient theorem now implies that $H_{2n}X = \mathbf{Z}/n$ for $n \geq 1$. □

Corollary 11.14. *If p is a prime, the p-primary component of $\pi_i S^3$ is zero if $3 < i < 2p$, and is $\mathbf{Z}/p$ if $i = 2p$.*

Proof. We use the class $\mathcal{C}_p$. As before, let X be the space from the fibration (10.12). Lemma 11.13 implies that $H_iX \in \mathcal{C}_p$ for $0 < i < 2p$. Using the mod $\mathcal{C}$ Hurewicz theorem, we conclude that $\pi_iX \in \mathcal{C}_p$ for $0 < i < 2p$, and $\mathbf{Z}/p = H_{2p}X \cong \pi_{2p}X$ mod $\mathcal{C}_p$. This implies that the p-primary part of $\pi_{2p}X$ is $\mathbf{Z}/p$. The corollary now follows from the fact that $\pi_iX = \pi_iS^3$ for $i \neq 3$. □

With a bit more work one can show that the p-primary component of π_n^S is trivial for $n < 2p-3$ and equals $\mathbf{Z}/p$ for $n = 2p-3$ (see [**45**]). Take a look at the table on page 271 to verify this in low dimensions. So for example, π_2^S has trivial p-primary part for $p > 2$ and $\pi_3^S = \mathbf{Z}/3 \oplus$ (2-primary subgroup).

We turn now to the computation of π_2^S.

In Theorem 10.31 we computed that $\pi_4S^2 = \mathbf{Z}/2$. Consider the suspension map $s : S^2 \to \Omega S^3$, i.e. the adjoint of the identification $S(S^2) = S^3$. Then the Freudenthal suspension theorem implies that s induces an isomorphism $\mathbf{Z} \cong \pi_2(S^2) \xrightarrow{s_*} \pi_2(\Omega S^3) \cong \pi_3(S^3)$ and a surjection $\mathbf{Z} \cong \pi_3(S^2) \xrightarrow{s_*} \pi_3(\Omega S^3) \cong \pi_4S^3 \cong \mathbf{Z}/2$.

Let F be the homotopy fiber of s. The long exact sequence of homotopy groups of the fibration

$$F \to S^2 \xrightarrow{s} \Omega S^3 \tag{11.5}$$

shows that $\pi_1 F = \pi_2 F = 0$. Thus by the Hurewicz theorem $\pi_3 F = H_3 F$. The spectral sequence for the fibration (11.5) shows that the transgression $\tau : H_3 F \to H_4(\Omega S^3)$ is an isomorphism. In Chapter 10 we computed that $\mathbf{Z} \cong H_4(\Omega S^3)$ (see Equation (10.3)), and hence $\mathbf{Z} \cong H_3 F = \pi_3 F$.

The long exact sequence in homotopy groups for (11.5) is

$$\cdots \to \pi_4 S^2 \xrightarrow{s_*} \pi_4(\Omega S^3) \to \pi_3 F \to \pi_3 S^2 \to \pi_4 S^3 \to 0.$$

From the Hopf fibration $S^1 \to S^3 \to S^2$ we know that $\pi_3 S^2 = \mathbf{Z}$, and from Theorem 10.29 we know $\pi_4 S^3 = \mathbf{Z}/2$. Since $\pi_3 F = \mathbf{Z}$ it follows from this exact sequence that the suspension map $s_* : \pi_4 S^2 \to \pi_4(\Omega S^3) = \pi_5 S^3$ is onto. Therefore, $\pi_5 S^3$ is either 0 or $\mathbf{Z}/2$. We will show that $\pi_5 S^3 = \mathbf{Z}/2$.

Consider once again our friend the space X of the fibration (10.12). Since $\pi_4 X = \pi_4 S^3 = \mathbf{Z}/2$, let $f : X \to K(\mathbf{Z}/2, 4)$ be a map inducing an isomorphism on π_4 and let Y denote the homotopy fiber of f. Since Y is 4-connected, $H_5(Y; \mathbf{Z}) = \pi_5 Y = \pi_5 X = \pi_5 S^3$. Since $\pi_5 S^3$ is either 0 or $\mathbf{Z}/2$, the universal coefficient theorem implies that $\pi_5 S^3 = H^5(Y; \mathbf{Z}/2)$.

In the spectral sequence in $\mathbf{Z}/2$-cohomology for the fibration $Y \to X \to K(\mathbf{Z}/2, 4)$, the differential

$$H^5(Y; \mathbf{Z}/2) = E_2^{0,5} = E_6^{0,5} \xrightarrow{d_6} E_6^{6,0} = E_2^{6,0} = H^6(K(\mathbf{Z}/2, 4); \mathbf{Z}/2)$$

is surjective. This follows from the fact that Y is 4-connected, $K(\mathbf{Z}/2, 4)$ is 3-connected, and from Lemma 11.13 which implies that $H^6(X; \mathbf{Z}/2) = 0$ (you should check this fact).

We will show in Section 11.5 below (Equation (**??**)) that

$$H^6(K(\mathbf{Z}/2, 4); \mathbf{Z}/2) = \mathbf{Z}/2.$$

Hence $H^5(F; \mathbf{Z}/2)$ surjects to $\mathbf{Z}/2$ and therefore equals $\mathbf{Z}/2$. Thus we have computed $\pi_5 S^3 = \mathbf{Z}/2$.

The homotopy exact sequence for the Hopf fibration $S^3 \to S^7 \to S^4$ shows that $\pi_2^S = \pi_6 S^4 \cong \pi_5 S^3$. In particular this shows that the sequence of suspension homomorphisms

$$\pi_2 S^0 \to \pi_3 S^1 \to \pi_4 S^2 \to \pi_5 S^3 \to \pi_6 S^4 = \pi_2^S$$

is

$$0 \to 0 \to \mathbf{Z}/2 \xrightarrow{\cong} \mathbf{Z}/2 \xrightarrow{\cong} \mathbf{Z}/2.$$

The long exact sequence of homotopy groups for the Hopf fibration $S^1 \to S^3 \to S^2$ shows that $\pi_5 S^2 = \pi_5 S^3$, and so $\pi_5 S^2 = \mathbf{Z}/2$ also.

11.3. Suspension, looping, and the transgression

In this section, H_k and H^k denote ordinary homology and cohomology with some fixed (untwisted) coefficients, and X, Y denote simply connected and path-connected based spaces.

Identify the reduced cone CX as a quotient of $[0,1]\times X$ with the inclusion $X \subset CX$ corresponding to $x \mapsto (1, x)$. Take the reduced suspension SX to be CX/X, and let $c : CX \to SX$ denote the quotient map. Denote by $E : H_{k-1}X \to H_k(SX)$ the suspension isomorphism on homology, defined as the composite of isomorphisms

$$E : H_{k-1}X \xleftarrow{\partial} H_k(CX, X) \xrightarrow{c_*} H_k(SX, *) \cong H_k(SX).$$

The purpose of this section is to prove the Freudenthal suspension theorem (Theorem 9.7) and to develop material for a dual result about stable cohomology operations given in the next section.

Consider the two fundamental maps

$$s : Y \to \Omega SY,\ y \mapsto (t \mapsto (t, y))$$

and

$$\ell : S\Omega X \to X,\ (t, \alpha) \mapsto \alpha(t).$$

We will relate these maps to the transgression for the path space fibration.

Consider the path space fibration over SY:

$$\Omega SY \to PSY \xrightarrow{e} SY$$

where e evaluates a path at its end point. The transgression in homology for this fibration (for $k > 0$) is the "composite" (with domain a submodule of $H_k(SY) = H_k(SY, *)$ and range a quotient module of $H_{k-1}(\Omega SY)$):

$$\tau : H_k(SY, *) \supset \mathrm{Im}(e_*) \xleftarrow{e_*} H_k(PSY, \Omega SY) \xrightarrow{\partial} H_{k-1}(\Omega SY)/\partial(\ker e_*).$$

By the transgression theorem (Theorem 10.14), τ agrees with the differential $d^k : E^k_{k,0} \to E^k_{0,k-1}$ in the spectral sequence for this fibration.

Theorem 11.15. *The domain of the transgression τ for the path space fibration over SY is all of $H_k(SY)$. Moreover, the homomorphism $s_* : H_{k-1}Y \to H_{k-1}(\Omega SY)$ induced by s is a lift of $\tau \circ E$, where $E : H_{k-1}Y \cong H_k(SY)$ denotes the suspension isomorphism.*

Proof. Consider the map $f : CY \to PSY$ defined by

$$f(t, y) = (r \mapsto (rt, y)).$$

Then $e\circ f : CY \to SY$ is just the map $(t, y) \mapsto (t, y)$, i.e. the natural collapse map c. Moreover, the restriction of f to $Y = \{1\} \times Y \subset CY$ is the map $y \mapsto (r \mapsto (r, y))$; this is exactly the map s, with image in ΩSY. Hence f induces a map of pairs $f : (CY, Y) \to (PSY, \Omega SY)$ whose restriction to the subspaces is s. Thus f induces a map between the long exact sequences of these pairs; since PSY and CY are contractible every third term vanishes and so we obtain commuting diagrams with the horizontal arrows isomorphisms:

$$\begin{tikzcd} H_k(CY,Y) \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{k-1}Y \arrow[d, "s_*"] \\ H_k(PSY,\Omega SY) \arrow[r, "\partial"] & H_{k-1}(\Omega SY) \end{tikzcd}$$

Since $e \circ f = c$, the diagram

$$\begin{tikzcd} H_k(CY,Y) \arrow[rr, "f_*"] \arrow[dr, "c_*"'] & & H_k(PSY,\Omega SY) \arrow[dl, "e_*"] \\ & H_k(SY,*) & \end{tikzcd}$$

commutes, with c_* an isomorphism. It follows that e_* is onto, so that the domain of τ is all of $H_k(SY)$.

In the commuting diagram:

$$\begin{tikzcd} H_k(SY,*) \arrow[d, "\mathrm{Id}"'] & H_k(CY,Y) \arrow[l, "c_*"'] \arrow[r, "\partial"] \arrow[d, "f_*"'] & H_{k-1}Y \arrow[d, "s_*"'] & \\ H_k(SY,*) & H_k(PSY,\Omega SY) \arrow[l, "e_*"] \arrow[r, "\partial"'] & H_{k-1}(\Omega SY) \arrow[r] & H_{k-1}(\Omega SY)/\partial(\ker e_*) \end{tikzcd}$$

The top horizontal line from right to left equals E, and the bottom horizontal line from left to right equals τ. It follows that $s_* : H_{k-1}Y \to H_{k-1}(\Omega SY)$ is a lift of $\tau \circ E : H_{k-1}Y \to H_{k-1}(\Omega SY)/\partial(\ker e_*)$. $\square$

As an application of Theorem 11.15, we prove the Freudenthal suspension theorem (Theorem 9.7). Theorem 7.42 implies that for any based spaces X and Y, the adjoint function

$$A : [X, \Omega Y]_0 \to [SX, Y]_0, \quad A(f)(t,x) = f(x)(t)$$

is a natural bijection.

The maps $s : Y \to \Omega SY$ and A compose to induce the suspension map

$$S : [X, Y]_0 \xrightarrow{s_*} [X, \Omega SY]_0 \overset{A}{\cong} [SX, SY]_0, \quad S(f)(t,x) = (t, f(x)),$$

which takes a function to the induced function on suspensions. In particular, taking $X = S^k$ yields the suspension homomorphism on homotopy groups

$$S : \pi_k Y \to \pi_{k+1}(SY).$$

The Freudenthal suspension theorem asserts that if Y is $(n-1)$-connected, S is an isomorphism for $k < 2n-1$ and an epimorphism for $k = 2n-1$.

Proof of Theorem 9.7. Suppose that Y is an $(n-1)$-connected space, with $n > 1$. Using the Hurewicz theorem and the suspension isomorphism $E : \widetilde{H}_k Y \cong \widetilde{H}_{k+1}(SY)$ (with $\mathbf{Z}$ coefficients), we see that SY is n-connected. Since $\pi_k(\Omega SY) = \pi_{k+1}(SY)$, ΩSY is $(n-1)$-connected, and hence its homology vanishes in dimensions less than n by the Hurewicz theorem.

The Serre exact sequence (Theorem 10.16) for the fibration

$$\Omega SY \to PSY \xrightarrow{e} SY$$

implies that the sequence

$$H_{2n}(\Omega SY) \to H_{2n}(PSY) \xrightarrow{e_*} H_{2n}(SY) \xrightarrow{\tau} H_{2n-1}(\Omega SY) \to \cdots$$

is exact. Since the path space PSY is contractible it follows that the transgression $\tau : H_k(SY) \to H_{k-1}(\Omega SY)$ is an isomorphism for all $k \leq 2n$. It follows from Theorem 11.15 that $s_* : H_{k-1}Y \to H_{k-1}(\Omega SY)$ is an isomorphism for all $k \leq 2n$. Hence the relative homology groups $H_k(\Omega SY, Y)$ vanish for $k \leq 2n-1$. From the relative Hurewicz theorem (Theorem 11.6) we conclude that $\pi_k(\Omega SY, Y) = 0$ for $k \leq 2n-1$ and so $s_* : \pi_k(Y) \to \pi_k(\Omega SY)$ is an isomorphism for $k < 2n-1$ and an epimorphism for $k = 2n-1$. Since the composite of $s_* : \pi_k Y \to \pi_k(\Omega SY)$ and the adjoint isomorphism $A : \pi_k(\Omega SY) \to \pi_{k+1}(SY)$ is the suspension homomorphism $S : \pi_k Y \to \pi_{k+1}(SY)$, the Freudenthal suspension theorem follows. $\square$

We turn our attention to the map $\ell : S\Omega X \to X$. Our main use for ℓ will be in studying cohomology operations, so we relate it to the cohomology transgression. For any space Y, the cohomology suspension isomorphism $E^* : H^k(SY) \to H^{k-1}Y$ is defined as the composite of the isomorphisms

$$H^k(SY) = H^k(SY, *) \xrightarrow{c^*} H^k(CY, Y) \xleftarrow{\delta} H^{k-1}Y.$$

Taking $Y = \Omega X$, the composite of $E^* : H^k(S\Omega X) \to H^{k-1}(\Omega X)$ and the homomorphism $\ell^* : H^k(X) \to H^k(S\Omega X)$ induced by ℓ is a homomorphism

$$E^* \circ \ell^* : H^k X \to H^{k-1}(\Omega X).$$

On the other hand, the cohomology transgression for the path space fibration over X,

$$\Omega X \to PX \xrightarrow{e} X,$$

is the "composite" of the homomorphisms

$$H^{k-1}(\Omega X) \xrightarrow{\delta} H^k(PX, \Omega X) \xleftarrow{e^*} H^k(X, *) = H^k X,$$

i.e.

$$\tau^* : \delta^{-1}(\text{im } e^*) \to H^k(X)/\ker e^*, \text{ where } e^* : H^k(X,*) \to H^k(E,F).$$

The transgression theorem implies that τ^* coincides with the differential $d_k : E_k^{0,k-1} \to E_k^{k,0}$ in the spectral sequence for this fibration via the identifications $E_2^{0,k-1} \cong H^{k-1}(\Omega X)$ and $E_2^{0,k} \cong H^k X$.

The following theorem asserts that τ^* and $E^* \circ \ell^*$ are essentially inverses.

Theorem 11.16. *The domain $\delta^{-1}(\text{Im } e^*)$ of τ^* is equal to the image of $E^* \circ \ell^*$. The transgression induces an isomorphism $\tau^* : H^{k-1}(\Omega X) \to H^k X$ if and only if ℓ^* is an isomorphism, in which case $E^* \circ \ell^*$ is the inverse of τ^*.*

Proof. This time we use the map $g : C\Omega X \to PX$ defined by

$$(t,\alpha) \mapsto (s \mapsto \alpha(st)).$$

Thus $g(0,\alpha)$ and $g(t,\text{const}_*)$ are both the constant path at $*$, so that this indeed gives a well-defined map on the reduced cone. Moreover, $g(1,\alpha) = \alpha$, and so g defines a map of pairs $g : (C\Omega X, \Omega X) \to (PX, \Omega X)$ which restricts to the identity map on ΩX. Since both $C\Omega X$ and PX are contractible it follows that the maps between the cohomology long exact sequences of these pairs reduce to a commuting triangle of isomorphisms

$$\begin{tikzcd} & H^{k-1}(\Omega X) \arrow[dl, "\delta"'] \arrow[dr, "\delta"] & \\ H^k(PX,\Omega X) \arrow[rr, "g^*"'] & & H^k(C\Omega X,\Omega X) \end{tikzcd}$$

The composite $e \circ g : C\Omega X \to X$ is the map $(t,\alpha) \mapsto \alpha(t)$. Since ΩX is the fiber of the fibration e, $e \circ g$ factors through the suspension $S\Omega X$, and in fact the diagram

$$\begin{tikzcd} C\Omega X \arrow[rr, "e\circ g"] \arrow[dr, "c"'] & & X \\ & S\Omega X \arrow[ur, "\ell"'] & \end{tikzcd}$$

commutes. This gives the commuting diagram:

$$\begin{tikzcd} H^{k-1}(\Omega X) \arrow[r, "\delta"] & H^k(C\Omega X,\Omega X) & H^k(S\Omega X,*) \arrow[l, "c^*"'] & \\ H^{k-1}(\Omega X) \arrow[u, "\mathrm{Id}"] \arrow[r, "\delta"'] & H^k(PX,\Omega X) \arrow[u, "g_*"] & H^k(X,*) \arrow[l, "e^*"] \arrow[u, "\ell^*"] \arrow[r] & H^k(X)/(\ker e^*) \end{tikzcd}$$

The top horizontal row from right to left is the isomorphism E^*, and the bottom row from left to right defines τ^*. Since the maps labelled "δ" are isomorphisms, it follows that $\delta^{-1}(\mathrm{Im}\ e^*)$ equals the image of $E^* \circ \ell^*$. Since E^* is an isomorphism, the domain of τ^* is all of $H^{k-1}(\Omega X)$ if and only if e^* is surjective, which happens if and only if ℓ^* is surjective. Moreover, $\ker e^* = 0$ if and only if $\ker \ell^* = 0$. Hence $\tau^* : H^{k-1}(\Omega X) \to H^k X$ is an isomorphism if and only if $\ell^* : H^k X \to H^k(S\Omega X)$ is an isomorphism, and in this case $E^* \circ \ell^* = (\tau^*)^{-1}$. □

11.4. Cohomology operations

We have seen that the cohomology of a space with coefficients in a ring has a natural ring structure. Cohomology operations are a further refinement of the structure of the cohomology of a space. We have already come across cohomology operations in Chapter 8 (see Definition 8.23 and Exercise 160).

11.4.1. Definition and simple examples. We recall the definition.

Definition 11.17. If A, C are abelian groups, a *cohomology operation of type* $(n, A; q, C)$ is a natural transformation of functors

$$\theta : H^n(-; A) \to H^q(-; C).$$

The set of all cohomology operations of type $(n, A; q, C)$ is denoted by $O(n, A; q, C)$.

The following are some standard examples.

Coefficient homomorphisms. If $h : A \to C$ is a homomorphism, then h induces homomorphisms

$$h_* : H^n(X; A) \to H^n(X; C)$$

for all n; these are natural, so h defines an operation h_* of type $(n, A; n, C)$ for any n.

Bockstein homomorphisms. If

$$0 \to A \to B \to C \to 0 \tag{11.6}$$

is a short exact sequence of abelian groups, then $0 \to \mathrm{Hom}(C_\bullet X, A) \to \mathrm{Hom}(C_\bullet X, B) \to \mathrm{Hom}(C_\bullet X, C) \to 0$ is exact, where $C_\bullet X$ denotes the singular or cellular chain complex of X. Thus one obtains a long exact sequence in cohomology

$$\cdots \to H^k(X; A) \to H^k(X; B) \to H^k(X; C) \to H^{k+1}(X; A) \to \cdots .$$

The connecting homomorphisms

$$\beta_k : H^k(X; C) \to H^{k+1}(X; A)$$

are called the *Bockstein operators* associated to the short exact sequence (11.6). For each k this construction defines a cohomology operation β_k of type $(k, C; k+1, A)$.

Squaring. If R is a ring, let $\theta_n : H^n(X; R) \to H^{2n}(X; R)$ be the map

$$x \mapsto x \cup x.$$

Then θ_n is a natural transformation since $f^*(x \cup x) = f^*(x) \cup f^*(x)$, and hence a cohomology operation of type $(n, R; 2n, R)$.

Remark. At this point we would like to avoid using the symbol "$\cup$" for the product in the cohomology ring of a space and will use juxtaposition to indicate multiplication whenever it is convenient.

Notice that θ_n is *not a homomorphism*, since $(x+y)^2 \neq x^2 + y^2$ in general. In fact the definition of a cohomology operation does not require it to be a homomorphism.

Main example. Let A, C be abelian groups, and let

$$u \in H^q(K(A, n); C).$$

For CW-complexes Theorem 8.20 says that

$$[X, K(A, n)] \cong H^n(X; A), \text{ via } \big(f : X \to K(A, n)\big) \mapsto f^*(\iota),$$

where $\iota \in H^n(K(A, n); A)$ is the fundamental class of $K(A, n)$ (see Definition 8.19).

Thus, $u \in H^q(K(A, n); C)$ defines a map (up to homotopy)

$$f_u : K(A, n) \to K(C, q),$$

and hence u defines a cohomology operation θ_u as the composite

$$H^n(X; A) = [X; K(A, n)] \xrightarrow{(f_u)_*} [X, K(C, q)] = H^q(X; C).$$

So $u \in H^q(K(A, n); C)$ defines the operation θ_u of type $(n, A; q, C)$.

In Exercise 160 you showed that the correspondence $u \mapsto \theta_u$ gave a bijection between $H^q(K(A, n); C)$ and $O(n, A; q, C)$; the inverse map is $\theta \mapsto \theta(\iota)$.

11.4.2. Stable operations.

Definition 11.18. Given a cohomology operation $\theta \in O(n, A; q, C)$, the *suspension of* θ, $\sigma^*(\theta)$, is the operation of type $(n-1, A; q-1, C)$ defined by requiring the following diagram to commute:

$$\begin{tikzcd} H^{n-1}(X;A) \arrow[r, "\sigma^*(\theta)"] & H^{q-1}(X;C) \\ H^n(SX;A) \arrow[u, "\cong"] \arrow[r, "\theta"] & H^q(SX;C) \arrow[u, "\cong"'] \end{tikzcd}$$

where the vertical maps are the suspension isomorphisms in cohomology, $E^* : H^n(SX;A) \cong H^{n-1}(X;A)$ and $E^* : H^q(SX;C) \cong H^{q-1}(X;C)$.

Thus to any cohomology operation θ we can associate the sequence $\sigma^*(\theta), \sigma^*(\sigma^*(\theta)), \cdots$. This motivates the following definition.

Definition 11.19. A *stable cohomology operation of degree* r *and type* (A,C) is a sequence of operations $\theta = \{\theta_n\}$ where

$$\theta_n \in O(n, A; n+r, C) \quad \text{and} \quad \sigma^*(\theta_n) = \theta_{n-1}.$$

Thus a stable operation of degree r and type (A,C) is the same thing as an element in the limit of the sequence

$$\cdots \xrightarrow{\sigma^*} H^{n+r}(K(A,n);C) \xrightarrow{\sigma^*} H^{n+r-1}(K(A,n-1);C) \xrightarrow{\sigma^*} \cdots \xrightarrow{\sigma^*} H^{r+1}(K(A,1);C).$$

Denote by $A^r(A;C)$ the set of all stable cohomology operations $\theta = \{\theta_n\}$ of degree r and type (A,C). Hence

$$A^r(A;C) = \varprojlim_n H^{n+r}(K(A,n);C).$$

To decide whether a cohomology operation $\theta \in O(n,A;q,C)$ forms a component of a stable cohomology operation, at the very least we need to know whether $\theta = \sigma^*(\theta')$ for some θ'. This is possible if θ is transgressive, as we now explain.

Start with $\theta \in O(n,A;q,C) = [K(A,n), K(C,q)]$. There are homotopy equivalences $h_{n-1} : K(A,n-1) \to \Omega K(A,n)$ and $h_{q-1} : K(C,q-1) \to \Omega K(C,q)$ whose adjoints are the structure maps for the Eilenberg-Maclane spectra of A and C. This gives rise to a commutative diagram for any space X:

$$\begin{tikzcd} {[X, K(A,n-1)]_0} \arrow[r, "h_{n-1}"] \arrow[d] & {[X, \Omega K(A,n)]_0} \arrow[r, "A"] \arrow[d, "(\Omega\theta)_*"'] & {[SX, K(A,n)]_0} \arrow[d, "\theta_*"'] \\ {[X, K(C,q-1)]_0} \arrow[r, "h_{q-1}"] & {[X, \Omega K(C,q)]_0} \arrow[r, "A"] & {[SX, K(C,q)]_0} \end{tikzcd}$$

With respect to the identification $[X, K(A,n-1)]_0 \cong H^{n-1}(X;A)$ and $[SX, K(A,n)]_0 \cong H^n(SX;A)$, the top horizontal row from right to left is the cohomology suspension isomorphism E^* by Proposition 9.22 (whose proof is clearly valid for any coefficients). Similarly the bottom horizontal row from right to left equals $E^* : H^q(SX; B) \cong H^{q-1}(X; B)$. Thus the unlabelled vertical map equals $\sigma^*(\theta)$.

To simplify notation, assume that the structure maps h_{n-1}, h_{q-1} are the identity (and hence $A^{-1} = E^*$). The previous paragraph can be summarized by saying that the map

$$\sigma^* : O(n, A; q, C) \to O(n-1, A; q-1, C)$$

is given by "looping"; i.e.

$$\sigma^* = \Omega : [K(A,n), K(C,q)]_0 \to [\Omega K(A,n), \Omega K(C,q)]_0, \ (\Omega\theta)(\alpha)(t) = \theta(\alpha(t)).$$

Therefore, the composite of σ^* with the isomorphism given by the adjoint

$$A : [\Omega K(A,n), \Omega K(C,q)]_0 \cong [S\Omega K(A,n), K(C,q)]_0$$

is ℓ^*, the map induced by $\ell : S\Omega K(A,n) \to K(A,n)$ of Section 11.3. We conclude that $\sigma^* = A^{-1} \circ \ell^* = E^* \circ \ell^*$.

Theorem 11.16 implies that $\theta' \in H^{q-1}(K(A, n-1); C)$ is in the image of $\sigma^* = E^* \circ \ell^*$ if and only if θ' is *transgressive*, i.e. in the domain of the cohomology transgression τ^* for the path space fibration over $K(A,n)$

$$K(A, n-1) \to P \xrightarrow{e} K(A,n). \tag{11.7}$$

Moreover, in the special case when τ^* induces an isomorphism, then Theorem 11.16 implies that $(\tau^*)^{-1} = E^* \circ \ell^* = \sigma^*$. Thus we have proved the following corollary.

Corollary 11.20. *If the class $\theta' \in O(n-1, A; q-1, C)$ is transgressive, then there exists a class $\theta \in O(n, A; q, C)$ so that $\sigma^*(\theta) = \theta'$. In particular if the transgression is an isomorphism*

$$\tau^* : H^{q-1}(K(A, n-1); C) \to H^q(K(A,n); C),$$

then $\theta' = \sigma^*(\tau^*(\theta'))$. □

If one starts with a cohomology operation $\theta \in O(n, A; q, C)$, then the sequence $\theta, \sigma^*(\theta), \sigma^*(\sigma^*(\theta)), \cdots$ can be extended to the left to give a stable operation provided θ, $\tau^*(\theta)$, $\tau^*(\tau^*(\theta))$, etc., are transgressive. The following theorem shows that this is possible for n large enough.

Theorem 11.21. *If $n \geq 2$, the transgression for the fibration (11.7) induces isomorphisms*

$$\tau : H^{q-1}(K(A, n-1); C) \to H^q(K(A,n); C)$$

for $2n \geq q+2$.

Proof. This is an immediate consequence of the cohomology version of the Serre exact sequence (Theorem 10.16). For the convenience of the reader we give the details.

Consider the Leray–Serre cohomology spectral sequence with C coefficients for the fibration (11.7). Since $E_2^{p,q} = H^p(K(A,n); H^q(K(A,n-1))$, the $E_2^{p,q}$ terms vanish if $1 \leq p \leq n-1$ or if $1 \leq q \leq n-2$.

This implies that if $2n \geq q+2$,

$$H^{q-1}(K(A,n-1)) \cong E_2^{0,q-1} = E_q^{0,q-1}$$

and

$$H^q(K(A,n)) \cong E_2^{q,0} = E_q^{q,0}.$$

Since the total space is contractible, $E_\infty^{p,q} = 0$ if $p+q \neq 0$. Hence the differential $d_q : E_q^{0,q-1} \to E_q^{q,0}$ is an isomorphism. Theorem 10.14 states that this differential coincides with the transgression, and so we conclude that the transgression $\tau : H^{q-1}(K(A,n-1)) \to H^q(K(A,n))$ is an isomorphism for $2n \geq q+2$. □

Thus Theorem 11.21 implies that

$$A^r(A;C) = \varprojlim_n H^{n+r}(K(A,n);C) = H^{2r+1}(K(A,r+1);C) \tag{11.8}$$

and so a class $\theta \in H^{2r+1}(K(A,r+1);C)$ defines the stable operation

$$\cdots, (\tau^*)^2(\theta), \tau^*(\theta), \theta, \sigma^*(\theta), (\sigma^*)^2(\theta), \cdots, (\sigma^*)^{r+1}(\theta).$$

Exercise 217. Show that the composition of two stable cohomology operations is a stable cohomology operation.

The proof of the following proposition is easy and is left to the reader.

Proposition 11.22. *If G is an abelian group, then the sum and composition give $\mathcal{A}(G) = \underset{r}{\oplus} A^r(G,G)$ the structure of a graded, associative ring with unit.* □

Exercise 218. Prove Proposition 11.22.

Exercise 219. Show that if θ is a stable cohomology operation, then θ is a homomorphism. (Hint: if $f, g \in H^n(X;A) = [X, K(A,n)]_0 = [X, \Omega K(A,n-1)]_0$, then the group structure is given by taking composition of loops, which is preserved by σ^*.)

An interesting consequence of Proposition 11.22 and Exercise 219 is that for any space X, the cohomology $H^*(X;G)$ has the structure of a module over $\mathcal{A}(G)$. This additional structure is functorial.

Definition 11.23. Take $G = \mathbf{Z}/p$, p a prime. Then $\mathcal{A}_p = \mathcal{A}(\mathbf{Z}/p)$ is called the *mod p Steenrod algebra.* It is a graded algebra over $\mathbf{Z}/p$.

Thus the $\mathbf{Z}/p$ cohomology algebra of a space is a module over the mod p Steenrod algebra.

Exercise 220. Given two spectra $\mathbf{K}$ and $\mathbf{K}'$, define what a map of degree r from $\mathbf{K}$ to $\mathbf{K}'$ is, and what a homotopy of such maps is. Then show that taking $\mathbf{K}(A)$ (resp. $\mathbf{K}(B)$) to be the Eilenberg–MacLane spectrum for the abelian group A (resp. B),

$$\mathcal{A}^*(A, B) = [\mathbf{K}(A), \mathbf{K}(B)]_*.$$

Can you define stable cohomology operations for arbitrary generalized cohomology theories?

11.5. The mod 2 Steenrod algebra

In this section we explore briefly the mod 2 Steenrod algebra $\mathcal{A}_2$. A systematic exposition of this important algebra can be found in many homotopy theory texts. The standard reference is [**37**]. The complete structure of $\mathcal{A}_2$ is described in Theorem 11.24, most of whose proof we will omit. In this section, H^* denotes ordinary cohomology with coefficients in $\mathbf{Z}/2$.

Construct an associative algebra as follows. Let V be the graded $\mathbf{Z}/2$ vector space with basis the (infinite) set of symbols

$$\{Sq^0, Sq^1, \cdots\}.$$

The vector space is graded by assigning the basis vector Sq^i the grading i. Then, let $T(V)$ be the tensor algebra of V. Thus

$$T(V) = \mathbf{Z}/2 \oplus V \oplus (V \otimes V) \oplus \cdots$$

with multiplication given by tensor product.

Theorem 11.24. *Let $I \subset T(V)$ be the two-sided homogeneous ideal generated by:*

1. $1 + Sq^0$ *and*
2. (Adem relations)

$$Sq^a \otimes Sq^b + \sum_{c=0}^{[a/2]} \binom{b-c-1}{a-2c} Sq^{a+b-c} \otimes Sq^c$$

for all $0 < a < 2b$.

Then $\mathcal{A}_2$ is isomorphic to $T(V)/I$. The identification takes the Sq^i to stable operations satisfying:

a. $Sq^i(x) = 0$ *if* $x \in H^{i-p}X, p > 0$.
b. $Sq^i(x) = x^2$ *if* $x \in H^iX$.
c. Sq^1 *is the Bockstein associated to the short exact sequence*

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0.$$

d. (Cartan formula)

$$Sq^i(xy) = \sum_j Sq^j x \; Sq^{i-j} y.$$

□

In Theorem 11.24, the Sq^i should be interpreted as a stable operations in the sense that $Sq^i = \{Sq^i_{(n)}\}$ where $Sq^i_{(n)} : H^nX \to H^{n+i}X$ and $\sigma^*(Sq^i_{(n)}) = Sq^i_{(n-1)}$.

Exercise 219 says that each component θ_n of a stable operation is a group homomorphism. Thus the Sq^i are additive; i.e. $Sq^i_{(n)} : H^nX \to H^{n+i}X$ is a group homomorphism for all n. The operation Sq^i is not a ring homomorphism; this is clear from the Cartan formula. For example,

$$Sq^1(ab) = Sq^1 a \cdot b + a \cdot Sq^1 b,$$

so Sq^1 is a derivation (Definition 10.26). However, if we define the *total square* by the formula

$$Sq = \sum_{i=0}^{\infty} Sq^i$$

(on each element $x \in H^p(X)$ the sum is finite; $Sq(x) = x + Sq^1x + \cdots + Sq^px$ since $Sq^{p+k}x = 0$ for $k > 0$) then the Cartan formula simplifies to

$$Sq(xy) = Sq(x)Sq(y).$$

What this says is that the Sq^i are the *homogeneous components* of a ring endomorphism Sq of the cohomology algebra $H^*(X; \mathbf{Z}/2)$.

There are several ways of constructing the Sq^i and verifying their properties. We will not prove Theorem 11.24 in general, but instead will construct the operations Sq^i and focus on some special cases, taking the point of view that computing $\mathcal{A}_2^r$ is the same, using Equation (11.8), as computing the cohomology $H^{2r+1}(K(\mathbf{Z}/2, r+1); \mathbf{Z}/2)$.

To streamline notation, for the remainder of this section, we denote $K(\mathbf{Z}/2, n)$ by K_n and, when no chance of confusion is possible, H^* will denote ordinary (singular) homology with coefficients in $\mathbf{Z}/2$.

Let $\iota_n \in H^n(K_n)$ denote the fundamental class, corresponding to the identity function via the identification $H^n(K_n) = [K_n, K_n]$. Then let

$$y_0 = \iota_n^2 \in H^{2n}(K_n).$$

The identification $H^{2n}(K_n) = O(n, \mathbf{Z}/2; 2n, \mathbf{Z}/2)$ takes y_0 to the cohomology operation

$$H^n(-) \to H^{2n}(-), \ a \mapsto a^2.$$

We leave to the reader the following easy spectral sequence exercise.

Exercise 221.

1. In the spectral sequence for the path space fibration

$$K_n \to * \to K_{n+1}$$

the differential d_{n+1} takes ι_n to ι_{n+1}. Thus $d_{n+1}(y_0) = 2\iota_n i_{n+1} = 0$. Conclude that y_0 is transgressive.

2. For $k \geq 1$, in the spectral sequence for the path space fibration

$$K_{n+k} \to * \to K_{n+k+1},$$

show that

$$E_{2n+k+1}^{0,2n+k} = E_2^{0,2n+k} \cong H^{2n+k}(K_{n+k}),$$

$$E_{2n+k+1}^{2n+k+1,0} = E_2^{2n+k+1,0} = H^{2n+k+1}(K_{n+k+1}),$$

and that the differential

$$d_{2n+k+1} : E_{2n+k+1}^{0,2n+k} \to E_{2n+k+1}^{2n+k+1,0}$$

is an isomorphism, and hence by (the cohomology version of) Theorem 10.14,

$$\tau^* : H^{2n+k}(K_{n+k}) \to H^{2n+k+1}(K_{n+k+1})$$

is an isomorphism.

From Exercise 221 and Corollary 11.20 it follows that defining

$$y_1 = d_{2n+1}(y_0) = \tau^*(y_0)$$

and then defining y_k inductively by

$$y_k = \tau^*(y_{k-1}) \in H^{2n+k}(K_{n+k}),$$

the sequence $y_0, y_1, y_2, \cdots$ satisfies $\sigma^*(y_k) = y_{k-1}$ for $k \geq 1$. Taking $X = K_n$ in Exercise 222, and referring to Definition 11.18, one sees that $\sigma^*(y_0) = 0$, so that we may extend this sequence to the left by taking $y_k = 0$ for $k < 0$.

Exercise 222. Show that for any space X, the map

$$H^n(SX) \to H^{2n}(SX)$$

given by $x \mapsto x^2$ is zero for $n > 0$. (Hint: consider the cup product $H^n(CX, X) \times H^n(CX) \to H^{2n}(CX, X)$ of Corollary 4.30.)

Since $\tau^*(y_k) = y_{k-1}$, the sequence $\{y_k\}$ defines a stable cohomology operation $Sq^n = \{Sq^n_{(r)}\}$ with y_k corresponding to $Sq^n_{(n+k)}$.

This completes the construction of the Steenrod operations Sq^n. This approach does not reveal much about their properties beyond showing that $Sq^0 = 1$ (since $\iota_0^2 = 1 = \iota_0$) and that $Sq^n(x) = x^2$ if $x \in H^n(X)$. Showing that the Sq^n generate the Steenrod algebra $\mathcal{A}_2$, establishing the Adem relations, and proving the rest of Theorem 11.24 are more involved and require a more detailed analysis of the cohomology of the Eilenberg–MacLane spaces. We will content ourselves with proving part (c) of Theorem 11.24, identifying Sq^1 with the Bockstein operator.

Lemma 11.25. *The stable operation Sq^1 is equal to the Bockstein associated to the exact sequence*

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0.$$

Proof. Fix k, and consider the coefficient long exact sequence in cohomology of $K_k = K(\mathbf{Z}/2, k)$ associated to the short exact sequence

$$0 \to \mathbf{Z}/2 \to \mathbf{Z}/4 \to \mathbf{Z}/2 \to 0.$$

This is the sequence

$$\cdots \to H^{k-1}(K_k; \mathbf{Z}/2) \to H^k(K_k; \mathbf{Z}/2) \to$$
$$H^k(K_k; \mathbf{Z}/4) \to H^k(K_k; \mathbf{Z}/2) \xrightarrow{\beta} H^{k+1}(K_k; \mathbf{Z}/2) \to \cdots.$$

The Hurewicz and universal coefficient theorems show that

$$H^{k-1}(K_k; \mathbf{Z}/2) = 0.$$

Also,

$$H^k(K_k; \mathbf{Z}/2) = \mathrm{Hom}(H_k(K_k), \mathbf{Z}/2) = \mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}/2) = \mathbf{Z}/2$$

and

$$H^k(K_k; \mathbf{Z}/4) = \mathrm{Hom}(H_k(K_k); \mathbf{Z}/4) = \mathrm{Hom}(\mathbf{Z}/2, \mathbf{Z}/4) = \mathbf{Z}/2.$$

It follows that β is an injection.

Induction using the spectral sequences for the fibrations

$$K_{n-1} \to * \to K_n$$

and the fact that the cellular chain complex for $K(\mathbf{Z}/2, 1) = \mathbf{R}P^\infty$ is

$$\cdots \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0$$

show that

$$\begin{aligned} H_{k+1}(K_k;\mathbf{Z}) &\cong H_k(K_{k-1};\mathbf{Z}) \\ \cdots &\cong H_2(K_1;\mathbf{Z}) \\ &= 0. \end{aligned}$$

Thus

$$\begin{aligned} H^{k+1}(K_k;\mathbf{Z}/2) &= \mathrm{Hom}(H_{k+1}(K_k,\mathbf{Z}),\mathbf{Z}/2)) \oplus \mathrm{Ext}\,(H_k(K;\mathbf{Z}),\mathbf{Z}/2) \\ &= 0 \oplus \mathbf{Z}/2, \end{aligned}$$

so that $\beta : H^k(K_k;\mathbf{Z}/2) = \mathbf{Z}/2 \to H^{k+1}(K_k;\mathbf{Z}/2) = \mathbf{Z}/2$ is injective, hence an isomorphism.

By definition, the cohomology operation β corresponds to the nonzero element $\beta(i_k)$ in $H^{k+1}(K_k;\mathbf{Z}/2) = \mathbf{Z}/2$.

But, by construction, $(\sigma^*)^{k-1}(Sq^1_{(k)}) = Sq^1_{(1)}$ which is the cup square $\iota_1^2 \in H^2(K_1)$. This is nonzero, since the cohomology of $K_1 = \mathbf{R}P^\infty$ is the polynomial ring generated by ι_1. Thus $Sq^1_{(k)}$ is nonzero and hence must equal β. $\square$

Here is an interesting application of the Steenrod squares to the homotopy groups of spheres. Consider the Hopf fibration $S^3 \to S^7 \xrightarrow{h} S^4$. Using h as an attaching map for an 8-cell to S^4, we obtain the quaternionic projective plane $X = \mathbf{H}P^2$. This has $\mathbf{Z}/2$-cohomology $\mathbf{Z}/2$ in dimensions $0, 4$ and 8. Poincaré duality implies that the intersection form on fourth cohomology is nondegenerate. Therefore, (using $\mathbf{Z}/2$-coefficients) if $x \in H^4(X)$, $x^2 = Sq^4(x) \in H^8(X) = \mathbf{Z}/2$ is nonzero.

If we use the suspension $Sh : S^8 \to S^5$ to attach a 9-cell to S^5, we obtain the suspension SX (prove this). We will show that Sh is not nullhomotopic, and hence the suspension homomorphism $\pi_7 S^4 \to \pi_8 S^5 = \pi_3^S$ (which is onto by the Freudenthal suspension theorem) is nonzero.

Let $y \in H^5(SX)$ denote the nonzero element. Suppose to the contrary that Sh is nullhomotopic. Then SX is homotopy equivalent to the wedge $S^5 \vee S^9$. In particular the map $Sq^4 : H^5(SX) \to H^9(SX)$ is trivial, since if y is the nonzero element of $H^5(S^5 \vee S^9)$, then y is pulled back from $H^5(S^5)$ via the projection $S^5 \vee S^9 \to S^5$, but $H^9(S^5) = 0$ and so by naturality $Sq^4(y) = 0$.

But, since Sq^4 is a stable operation, the diagram

$$\begin{tikzcd} H^4X \arrow[r, "Sq^4"] & H^8X \\ H^5(SX) \arrow[u, "\cong"] \arrow[r, "Sq^4"] & H^9(SX) \arrow[u, "\cong"] \end{tikzcd}$$

commutes, and so $Sq^4(y) \neq 0$.

Thus $Sh : S^8 \to S^5$ is non-nullhomotopic, and so π_3^S is nonzero.

A similar argument, using the Hopf fibration $S^7 \to S^{15} \to S^8$ solves the following exercise.

Exercise 223. Show that the suspension

$$\pi_{15}S^8 \to \pi_{16}S^9 = \pi_7^S$$

is nontrivial on the homotopy class of the Hopf map $S^{15} \to S^8$.

11.6. The Thom isomorphism theorem

The Thom isomorphism theorem is a generalization of the fact that suspension induces $H^nB \cong \widetilde{H}^n(B_+) \cong H^{k+n}(S^k(B_+))$, where the + means add on a disjoint base point. Roughly speaking, the Thom isomorphism theorem says that the suspension isomorphism continues to hold when one "twists" the suspension construction. More precisely, the k-fold suspension $S^k(B_+)$ can be considered as the "half-smash" quotient $(B \times D^k)/(B \times S^{k-1})$. One generalizes this by replacing the space $B \times D^k$ by the disk bundle of a vector bundle over B and replacing $B \times S^{k-1}$ by the corresponding sphere bundle.

Let $\gamma : X \to B$ be an Euclidean vector bundle of rank k, with unit disk bundle $D(\gamma) \to B$ and unit sphere bundle $S(\gamma) \to B$. Denote the fibers of these bundles over $b \in B$ by $D(\gamma)_b$ and $S(\gamma)_b$, so that $(D(\gamma)_b, S(\gamma)_b) \cong (D^k, S^{k-1})$.

The inclusion $(D(\gamma)_b, S(\gamma)_b) \subset (D(\gamma), S(\gamma))$ induces a restriction homomorphism (for any choice of coefficients C)

(11.9)

$$H^m(D(\gamma), S(\gamma); C) \to H^m(D(\gamma)_b, S(\gamma)_b; C) \cong H^m(D^k, S^{k-1}; C) \cong \begin{cases} C & m = k, \\ 0 & m \neq k. \end{cases}$$

We will be concerned only with the two fundamental cases of $C = \mathbf{Z}/2$ and $C = \mathbf{Z}$.

In preparation for the statement of the Thom isomorphism theorem, notice that there is a cup product (with any ring coefficients)

(11.10) $$H^p(B) \times H^q(D(\gamma), S(\gamma)) \xrightarrow{\cup} H^{p+q}(D(\gamma), S(\gamma))$$

obtained by pre-composing the cup product

$$H^p(D(\gamma)) \times H^q(D(\gamma), S(\gamma)) \to H^{p+q}(D(\gamma), S(\gamma))$$

(see Corollary 4.30) with the isomorphism $\gamma^* : H^p(B) \to H^p(X)$ induced by the bundle projection (and homotopy equivalence) $\gamma : X \to B$.

Theorem 11.26 (Thom isomorphism theorem)**.** *Given a rank k Euclidean vector bundle $\gamma : X \to B$ with unit disk and unit sphere bundles $D(\gamma) \to B$ and $S(\gamma) \to B$,*

1. *There exists a unique class $u \in H^k(D(\gamma), S(\gamma); \mathbf{Z}/2)$ so that for each $b \in B$, the restriction to the fiber over b,*

   $$H^k(D(\gamma), S(\gamma); \mathbf{Z}/2) \to H^k(D(\gamma)_b, S(\gamma)_b; \mathbf{Z}/2) \cong \mathbf{Z}/2$$

   (see Equation (11.9)) takes u to the unique nonzero element. This class u has the property that the homomorphism

   $$\Phi : H^p(B; \mathbf{Z}/2) \to H^{k+p}(D(\gamma), S(\gamma); \mathbf{Z}/2), \quad x \mapsto \gamma^*(x) \cup u$$

   (using the cup product (11.10)) is an isomorphism for all n.
2. *If the vector bundle $X \to B$ is orientable, then there exists a class $\tilde{u} \in H^k(D(\gamma), S(\gamma); \mathbf{Z})$ so that for each $b \in B$ the restriction to the fiber over b,*

   $$H^k(D(\gamma), S(\gamma); \mathbf{Z}) \to H^k(D(\gamma)_b, S(\gamma)_b; \mathbf{Z}) \cong \mathbf{Z}$$

   takes $\tilde{u}$ to a generator. A choice of orientation of $\gamma : X \to B$ uniquely specifies $\tilde{u}$ and conversely a choice of $\tilde{u}$ determines an orientation of $\gamma : X \to B$. For any such $\tilde{u}$, the homomorphism

   $$\tilde{\Phi} : H^p(B; \mathbf{Z}) \to H^{k+p}(D(\gamma), S(\gamma); \mathbf{Z}), \quad x \mapsto \gamma^*(x) \cup \tilde{u}$$

   is an isomorphism for all n. Moreover the coefficient homomorphism $H^k(D(\gamma), S(\gamma); \mathbf{Z}) \to H^k(D(\gamma), S(\gamma); \mathbf{Z}/2)$ takes $\tilde{u}$ to u.

The classes u and $\tilde{u}$ are natural with respect to pulling back vector bundles: if $f : B' \to B$ is a continuous map, and $\gamma' = f^*(\gamma)$ the pulled back bundle, then (with the obvious notation) $u' = f^*(u)$ and (with the pulled back orientation) $\tilde{u}' = f^*(\tilde{u})$.

Definition 11.27. The cohomology class $u \in H^k(D(\gamma), S(\gamma), \mathbf{Z}/2)$ (resp. $\tilde{u} \in H^k(D(\gamma), S(\gamma), \mathbf{Z})$) whose existence is assured by Theorem 11.26 is called the *Thom class for* the rank k vector bundle $\gamma : X \to B$. If we wish to emphasize the bundle, we will denote its Thom class by u_γ (resp. $\tilde{u}_\gamma$).

Proof of Theorem 11.26. Assume B is path connected. We prove the two cases ($\mathbf{Z}$ and $\mathbf{Z}/2$) simultaneously.

Consider the spectral sequence of the relative fibration:

$$E_2^{p,q} \cong H^p(B; H^q(D^k, S^{k-1})) \Rightarrow H^*(D(\gamma), S(\gamma))$$

with $\mathbf{Z}/2$ or $\mathbf{Z}$ coefficients.

Using the universal coefficient theorem (and, for the orientable case, the fact that the action of $\pi_1(B)$ on cohomology of the fiber $H^k(D^k, S^{k-1}; \mathbf{Z})$ is trivial, so that the coefficients are untwisted),

$$E_2^{p,q} = H^p(B; H^q(D^k, S^{k-1})) \cong \begin{cases} H^p(B) & \text{if } q = k, \\ 0 & \text{if } q \neq k. \end{cases}$$

It follows that all differentials in the spectral sequence are zero and therefore

$$E_2^{p,q} = E_\infty^{p,q} = \begin{cases} H^{p+k}(D(\gamma), S(\gamma)) & \text{if } q = k, \\ 0 & \text{if } q \neq k. \end{cases}$$

Hence $H^pB \cong H^{p+k}(D(\gamma), S(\gamma))$.

Let u (resp. $\tilde{u}$) generate $E_\infty^{0,k} = H^k(D(\gamma), S(\gamma)) \cong H^0B$, which is isomorphic to $\mathbf{Z}/2$ (resp. $\mathbf{Z}$). Clearly u is unique (resp. $\tilde{u}$ is unique up to sign).

The edge homomorphism $H^k(D(\gamma), S(\gamma)) \to H^k(D^k, S^{k-1})$, which equals the map induced by the inclusion $(D^k, S^{k-1})_b \subset (D(\gamma), S(\gamma))$ (see Theorem 10.13 and Exercise 202) is a composite of isomorphisms

$$H^k(D(\gamma), S(\gamma)) \cong E_\infty^{0,k} \cong E_2^{0,k} \cong H^0(B; H^k(D^k, S^{k-1})) \cong H^k(D^k, S^{k-1}).$$

Therefore, u (resp. $\tilde{u}$) restricts to a generator in each fiber. In particular $\tilde{u}$ orients the vector bundle $\gamma : X \to B$.

Naturality of the classes u and $\tilde{u}$ follows from the naturality properties of the spectral sequences.

The fact that $\tilde{u}$ is sent to u by the coefficient homomorphism

$$H^k(D(\gamma), S(\gamma); \mathbf{Z}) \to H^k(D(\gamma), S(\gamma); \mathbf{Z}/2)$$

follows from uniqueness of the Thom class and commutativity of the diagram

$$\begin{tikzcd} H^k(D(\gamma), S(\gamma); \mathbf{Z}) \arrow[r] \arrow[d] & H^k(D(\gamma), S(\gamma); \mathbf{Z}/2) \arrow[d] \\ H^k(D^k, S^{k-1}; \mathbf{Z}) \arrow[r] & H^k(D^k, S^{k-1}; \mathbf{Z}/2) \end{tikzcd}$$

where the vertical arrows are induced by the inclusions and the horizontal arrows by the coefficient homomorphism $\mathbf{Z} \to \mathbf{Z}/2$.

It remains to show that the isomorphism $H^pB \cong H^{p+k}(D(\gamma), S(\gamma))$ is given by taking the cup product with u (resp. $\tilde{u}$). We invoke Theorem 10.38 which asserts that spectral sequence $\bar{E}_*^{*,*}$ of the fibration $D^k \to D(\gamma) \to B$,

$$\bar{E}_2^{p,q} \cong H^p(B; H^q(D^k)) \Rightarrow H^*(D(\gamma)),$$

pairs with $E_*^{*;*}$. Since $H^q(D^k) = 0$ for $q \neq 0$,

$$\bar{E}_2^{p,q} \cong \begin{cases} H^p(B) & \text{if } q = 0 \\ 0 & \text{if } q \neq 0 \end{cases}$$

so that all differentials are zero, and

$$H^p(B) \cong H^p(B; H^0(D^k)) \cong \bar{E}_2^{p,0} \cong \bar{E}_\infty^{p,0} \cong H^p(D(\gamma)).$$

This composite is the edge homomorphism, hence coincides with the induced morphism $\gamma^* : H^p(B) \to H^p(D(\gamma))$, reflecting the fact that $\gamma : D(\gamma) \to B$ is a homotopy equivalence. Moreover, Theorem 10.38 implies that the diagram, with all vertical arrows isomorphims,

$$\begin{tikzcd}
H^p(B) \times H^0(B) \arrow[r, "\cup"] \arrow[d] & H^p(B) \arrow[d] \\
H^p(B; H^0(D^k)) \times H^0(B; H^k(D^k, S^{k-1})) \arrow[r, "\cup"] \arrow[d] & H^p(B; H^k(D^k, S^{k-1})) \arrow[d] \\
\bar{E}_2^{p,0} \times E_2^{0,k} \arrow[r] \arrow[d] & E_2^{p,k} \arrow[d] \\
\bar{E}_\infty^{p,0} \times E_\infty^{0,k} \arrow[r] \arrow[d] & E_\infty^{p,k} \arrow[d] \\
H^p(D(\gamma)) \times H^k(D(\gamma), S(\gamma)) \arrow[r, "\cup"] & H^{p+k}(D(\gamma), S(\gamma))
\end{tikzcd}$$

commutes.

By construction, the composite of the left vertical isomorphisms takes $(x, 1) \in H^p(B) \times H^0(B)$ to $(\gamma^*(x), u)$ (resp. $(\gamma^*(x), \tilde{u})$). Since $x \cup 1 = x$, the outer square in the diagram above gives a commutative diagram

$$\begin{tikzcd}
H^p(B) \arrow[r, "\cup 1", "\cong"'] \arrow[d, "\gamma^*", "\cong"'] & H^p(B) \arrow[d, "\cong"] \\
H^p(D(\gamma)) \arrow[r, "\cup u"] & H^{p+k}(D(\gamma), S(\gamma))
\end{tikzcd}$$

showing that $x \mapsto \gamma^*(x) \cup u$ is an isomorphism, and similarly for $\tilde{u}$. □

Recall that the Thom space $T(\gamma)$ of a vector bundle $\gamma : X \to B$ with metric is defined as $D(\gamma)/S(\gamma)$ (Definition 9.3). The collapse map defines, via excision, an isomorphism (with any coefficients)

$$H^m(T(\gamma), p) \xrightarrow{\cong} H^m(D(\gamma), S(\gamma).)$$

Hence the Thom isomorphism theorem asserts the existence of a class $u \in H^k(T(\gamma))$ giving an isomorphism

$$\cup u : H^p B \xrightarrow{\cong} \widetilde{H}^{k+p}(T(\gamma)),$$

with $\mathbf{Z}/2$-coefficients (in general) or $\mathbf{Z}$-coefficients (in the orientable case).

Observe that if $X_0 \subset X$ denotes the complement of the zero section $B \to X$, then the inclusion $(D(\gamma), S(\gamma)) \subset (X, X_0)$ is a homotopy equivalence of pairs, and so the Thom isomorphism theorem can also be stated as giving an isomorphism

$$H^p(B) \xrightarrow{\cong} H^{k+p}(X, X_0).$$

The Thom isomorphism theorem has a homology counterpart.

Exercise 224. Show that for a rank k Euclidean vector bundle, there is an isomorphism $H_{n+k}(D(\gamma), S(\gamma); \mathbf{Z}/2) \to H_k(B; \mathbf{Z}/2)$ and that for an orientable vector bundle, there is an isomorphism $H_{n+k}(D(\gamma), S(\gamma); \mathbf{Z}) \to H_k(B; \mathbf{Z})$.

In fact one can show that the isomorphisms are given by cap product with the Thom class.

11.7. Intersection theory

We could weave the following threads: Poincare duality, Spanier-Whitehead duality, Thom isomorphism (for normal bundle), Poincare-Lefschetz duality, the Milnor-Spanier theorem

One very useful consequence of the Thom isomorphism theorem is the identification of intersection numbers with cup products in manifolds. For simplicity we will discuss only the case of smooth compact manifolds, but everything we say holds in greater generality (with trickier proofs). In this section all homology and cohomology is taken with $\mathbf{Z}$ coefficients.

As orientations play an important role in intersection theory, we refer the reader to the discussion of the various approaches to orientations discussed in Section 9.7.3. In particular, Exercise 185 identifies two notions of orientation for a smooth, compact connected n-manifold M: an orientation of its tangent bundle TM, and a choice of generator $[M, \partial M] \in H_n(M, \partial M; \mathbf{Z}) \cong \mathbf{Z}$. The homology class $[M, \partial M]$ is called the fundamental class of M.

Suppose that V and W are oriented subspaces of an oriented vector space Z and that $\dim(V) + \dim(W) = \dim(Z)$. Suppose further that V and W are transverse, i.e. $V \cap W = 0$. Then the *intersection number* of V and W is the number in $\{\pm 1\}$ defined to be the sign of the determinant of the change of basis matrix from the (ordered) basis $\{\mathbf{b}_V, \mathbf{b}_W\}$ to $\mathbf{b}_Z$, where

$\mathbf{b}_V, \mathbf{b}_W$, and $\mathbf{b}_Z$ denote bases in the given equivalence classes. Reversing the order of V and W changes the intersection number by $(-1)^{\dim(V)\dim(W)}$.

Let A and B be smooth, compact, connected, oriented submanifolds of dimensions a and b of a smooth, compact, connected oriented manifold M of dimension m. (A smooth manifold is oriented if its tangent bundle is oriented.) Assume that A is properly embedded; i.e. the boundary of A is embedded in the boundary of M. Assume also that the boundary of B is empty and that B is contained in the interior of M. Finally assume that A and B are transverse. This means that at each point $p \in A \cap B$, the tangent subspaces T_pA and T_pB span T_pM.

Definition 11.28. Suppose that $a + b = m$. Then since A and B are transverse and compact, their intersection consists of a finite number of points. Because A, B, and M are oriented, we can define $\varepsilon_p \in \{\pm 1\}$ for each intersection point $p \in A \cap B$ to be the intersection number (as above) of the oriented subspaces T_pA and T_pB in T_pM. Then define the *intersection number of A and B* to be the integer

$$A \cdot B = \sum_{p \in A \cap B} \varepsilon_p.$$

Notice that $A \cdot B = (-1)^{ab} B \cdot A$.

Since A and B are oriented, they have fundamental classes $[A, \partial A] \in H_a(A, \partial A)$, $[B] \in H_b B$. Let $e_A : A \hookrightarrow M$ and $e_B : B \hookrightarrow M$ denote the inclusions. Then $e_A[A, \partial A] \in H_a(M, \partial M)$ and $e_B[B] \in H_b M$.

Theorem 11.29. *Let $\alpha \in H^b M$ be the Poincaré dual to $e_A[A, \partial A]$, and $\beta \in H^a(M, \partial M)$ the Poincaré dual to $e_B[B]$, i.e.*

$$\alpha \cap [M, \partial M] = e_A[A, \partial A] \quad \textit{and} \quad \beta \cap [M, \partial M] = e_B[B].$$

Then

$$A \cdot B = \langle \alpha \cup \beta, [M, \partial M] \rangle \tag{11.11}$$

where $\langle\ ,\ \rangle$ denotes the Kronecker pairing.

Theorem 11.29 justifies the terminology "intersection pairing" for the cup product

$$H^b M \times H^a(M, \partial M) \xrightarrow{\cup} H^m(M, \partial M) \xrightarrow{\cong} \mathbf{Z}$$

(see Section 4.7.2). Moreover, it implies that the intersection number $A \cdot B$ depends only on the homology classes $e_A[A, \partial A]$ and $e_B[B]$. In particular, given any not necessarily transverse submanifolds A and B as above, the transversality theorems imply that B can be isotoped to be transverse to

A. This preserves the class $e_B[B]$, and so the resulting intersection number $A \cdot B$ is independent of the choice of the isotopy.

With more work one can define $x \cdot y$ for any classes $x \in H_a(M, \partial M)$ and $y \in H_b M$, or even on the chain level for a simplicial complex and its dual cell complex (by thinking of a simplex as a submanifold). Theorem 11.29 is true in this greater generality. Alternatively, this approach can be reversed to give a proof of Poincaré duality and a definition of cup products in terms of intersections.

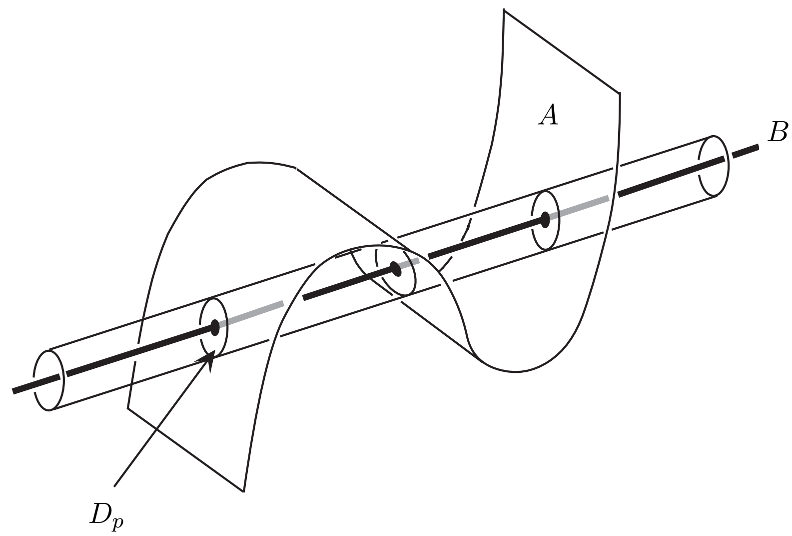
There is also a mod 2 version of Theorem 11.29 in which orientation issues do not play a role; one defines $A \cdot_2 B$ to be the reduction modulo 2 of number of intersection points of A and B when A and B are transverse. The mod 2 version holds in greater generality since none of the manifolds are required to be be orientable. To help you digest the following argument, you might first consider the mod 2 case, thereby avoiding sign and orientation issues which complicate the proof.

Proof of Theorem 11.29. Let $\nu_B : E \to B$ denote the normal bundle of $B \subset M$. The tubular neighborhood theorem implies that E can be embedded as a neighborhood of B in M, with B itself corresponding to the zero section. Endow E with a Euclidean metric.

It is a straightforward consequence of the fact that A and B are transverse that for $\varepsilon > 0$ small enough, the disk bundle $D(\nu_B)$ of vectors of E of length less than or equal to ε intersects A in a finite number of disks D_p, one for each p in $A \cap B$, with each D_p isotopic rel boundary in $(D(\nu_B), \partial D(\nu_B))$ to a fiber of the disk bundle $D(\nu_B) \to B$. Using the isotopy extension theorem (and maybe making ε smaller if necessary), we may assume A intersects $D(\nu_B)$ exactly in a union of fibers, one for each point $p \in A \cap B$. In other words, after an isotopy supported in E which fixes each $p \in A \cap B$,

$$A \cap D(\nu_B) = \bigcup_{p \in A \cap B} D(\nu_B)_p.$$

For convenience we simplify notation by setting $D = D(\nu_B)$. Thus the boundary ∂D is the ε-sphere bundle of E. The submanifold A intersects D in a union of disks D_p, one for each $p \in A \cap B$. The situation is illustrated in the following figure.



The manifold D is orientable and oriented. Indeed, since D has codimension zero, $TD = TM|_D$, and so the orientation of the tangent bundle of M restricts to an orientation of the tangent bundle of D.

The normal bundle $\nu_B : E \to B$ is an orientable vector bundle. One way to see this is to use the Whitney sum decomposition

$$E \oplus TB \cong TM|_B.$$

The fact that TB and TM are orientable implies that ν_B is orientable. Orient ν_B so that the intersection number of a fiber E_x of E with the zero section B equals 1:

$$E_x \cdot B = 1.$$

Exercise 225. Show that this condition uniquely specifies an orientation of the normal bundle E.

This orients the fibers $D_x \subset E_x$ and therefore, for each $x \in B$ one has a preferred generator $[D_x, \partial D_x]$ of $H_a(D_x, \partial D_x)$.

The Thom isomorphism theorem says that there is a unique Thom class $\tilde{u} \in H^a(D, \partial D)$ so that $\cup \tilde{u} : H^k(B) \to H^{k+a}(D, \partial D)$ is an isomorphism for all k, and so that the restriction of $\tilde{u}$ to the fiber D_x satisfies $\tilde{u}|_{D_x} = [D_x, \partial D_x]^*$, i.e. $\langle \tilde{u}, [D_x, \partial D_x] \rangle = 1$.

Now $[B]^* \cup \tilde{u}$ generates $H^m(D, \partial D)$ and so equals $[D, \partial D]^*$ up to sign. The sign is equal to $(-1)^{ab}$. To see this one can use naturality of the Thom class and work over a small open set in B diffeomorphic to a ball of dimension a.

Exercise 226. Prove that $\tilde{u} \cup [B]^* = [D, \partial D]^*$ (and hence, by Theorem **??** $[B]^* \cup \tilde{u} = (-1)^{ab}[D, \partial D]^*$) by pulling $E \to B$ back over a small neighborhood $U \subset B$ contained in an oriented chart for B.

Thus, using Exercises 66, **??**, and Proposition 4.28,

$$\begin{aligned}(-1)^{ab} &= \langle [B]^* \cup \tilde{u}, [D, \partial D] \rangle \\ &= ([B]^* \cup \tilde{u}) \cap [D, \partial D] \\ &= [B]^* \cap (\tilde{u} \cap [D, \partial D]) \\ &= \langle [B]^*, \tilde{u} \cap [D, \partial D] \rangle\end{aligned}$$

and so, since $H^b D \cong \mathbf{Z}$ and $H_b D \cong \mathbf{Z}$,

$$\tilde{u} \cap [D, \partial D] = (-1)^{ab}[B].$$

In other words, $(-1)^{ab}\tilde{u}$ is the Poincaré dual to $[B]$ in D.

The inclusion $i_1 : (D, \partial D) \hookrightarrow (M, M - \mathrm{Int}(D))$ induces excision isomorphisms in homology and cohomology. Hence $H_n(M, M - \mathrm{Int}(D))$ is isomorphic to $\mathbf{Z}$. The inclusion $i_2 : (M, \partial M) \subset (M, M - \mathrm{Int}(D))$ is not an excision, but induces an isomorphism

$$H_n(M, \partial M) \to H_n(M, M - \mathrm{Int}(D))$$

since both groups are isomorphic to $\mathbf{Z}$ and the inclusions of both pairs to $(M, M - x)$ are excisions. Because the orientations were chosen compatibly,

$$i_1[D, \partial D] = [M, M - \mathrm{Int}(D)] = i_2[M, \partial M].$$

It follows from naturality of the cap product

$$H^p(X, Y) \times H_q(X, Y) \xrightarrow{\cap} H_{q-p}X$$

for any pair (X, Y) that the diagram

$$\begin{tikzcd} H^p(D, \partial D) \arrow[r, "{\cap[D,\partial D]}"] & H_{n-p}D \arrow[d, "i_1"] \\ H^p(M, M - \mathrm{Int}(D)) \arrow[u, "i_1"] \arrow[d, "i_2"] \arrow[r, "{\cap i_1[D,\partial D]}"] & H_{n-p}M \\ H^p(M, \partial M) \arrow[r, "{\cap[M,\partial M]}"] & H_{n-p}M \arrow[u, "\mathrm{Id}"] \end{tikzcd}$$

commutes.

Denote by j^* the composite $i_2 \circ (i_1)^{-1} : H^p(D, \partial D) \to H^p(M, \partial M)$. The diagram above shows that if $x \in H^p(D, \partial D)$, $i_1(x \cap [D, \partial D]) = j^*(x) \cap$

$[M, \partial M]$. Taking $x = \tilde{u}$ and using the notation $[B]$ for the image of the fundamental class of B in either D or M, we conclude that

$$j^*(\tilde{u}) \cap [M, \partial M] = i_1(\tilde{u} \cap [D, \partial D]) = (-1)^{ab}[B].$$

In particular

$$\beta = (-1)^{ab} j^*(\tilde{u}).$$

We can think of the homomorphism j^* as being induced by the quotient map $j : M/\partial M \to D/\partial D$. Thus we have a corresponding homomorphism $j_* : H_p(M, \partial M) \to H_*(D, \partial D)$. Using the notation $[A, \partial A]$ for the fundamental class of A in $H_a(M, \partial M)$, we see the class $j_*[A, \partial A]$ is represented by the union of fibers D_p, one for each $p \in A \cap B$, but oriented according to the local intersection number of A and B at p. Precisely:

$$j_*[A, \partial A] = \sum_{p \in A \cap B} \varepsilon_p [D_p, \partial D_p],$$

where $\varepsilon_p = 1$ or -1 according to whether or not the two orientations of $D_p \subset (A \cap D)$ given by

1. restricting the orientation of A to D_p, and
2. the orientation of D_p as a fiber of the normal disk bundle D

agree. This is because the map $j : M/\partial M \to D/\partial D$ takes $A/\partial A$ to

$$\cup_p (D_p/\partial D_p) = \vee_p (D_p/\partial D_p).$$

By definition,

$$\sum_p \varepsilon_p = A \cdot B.$$

We now compute:

$$\begin{aligned}
\langle \alpha \cup \beta, [M, \partial M] \rangle &= \langle \alpha \cup (-1)^{ab} j^*(\tilde{u}), [M, \partial M] \rangle \\
&= \langle j^*(\tilde{u}) \cup \alpha, [M, \partial M] \rangle \\
&= \big(j^*(\tilde{u}) \cup \alpha\big) \cap [M, \partial M] \\
&= j^*(\tilde{u}) \cap \big(\alpha \cap [M, \partial M]\big) \\
&= j^*(\tilde{u}) \cap [A, \partial A] \\
&= \langle j^*(\tilde{u}), [A, \partial A] \rangle \\
&= \langle \tilde{u}, j_*[A, \partial A] \rangle \\
&= \sum_{p \in A \cap B} \varepsilon_p \langle \tilde{u}, [D_p, \partial D_p] \rangle \\
&= A \cdot B.
\end{aligned}$$

□

Exercise 227. State (and prove) the mod 2 version of Theorem 11.29.

During the proof of Theorem 11.29 we also proved the following.

Corollary 11.30. *Let $e : B \subset M$ be an embedding of a smooth, closed, oriented manifold in a compact, oriented manifold. Let D denote a closed tubular neighborhood of B in M, with Thom class $\tilde{u} \in H^{n-b}(D, \partial D)$, and let $j : M/\partial M \to D/\partial D$ denote the collapse map. Then $j^*(\tilde{u})$ is the Poincaré dual to $e_*([B])$ (up to sign).* □

The sign ambiguity in Corollary 11.30 comes from the fact that there are two possible choices of Thom classes $\tilde{u}$; during the proof of Corollary 11.30 we made a specific choice by requiring that $\langle \tilde{u}, [D_x, \partial D_x] \rangle = D_x \cdot B$.

We describe the usual way that a geometric topologists think of the Poincaré dual $\beta \in H^a(M, \partial M)$ to a cycle represented by a submanifold $B \subset M$. Given a cycle $x \in H_a(M, \partial M)$ represented by an oriented submanifold $(A, \partial A) \subset (M, \partial M)$, the class β is represented by the cochain whose value on x is given by the formula

$$\langle \beta, x \rangle = B \cdot A.$$

In brief, "the Poincaré dual β to B is given by intersecting with B."

To see why this is true, we compute:

$$\begin{aligned} B \cdot A &= (-1)^{ab} A \cdot B \\ &= (-1)^{ab} \langle \alpha \cup \beta, [M, \partial M] \rangle \\ &= (-1)^{ab} (\alpha \cup \beta) \cap [M, \partial M] \\ &= (\beta \cup \alpha) \cap [M, \partial M] \\ &= \beta \cap (\alpha \cap [M, \partial M]) \\ &= \beta \cap [A, \partial A] \\ &= \langle \beta, [A, \partial A] \rangle. \end{aligned}$$

Exercise 228. Show that if A and B are closed submanifolds of S^n intersecting transversally in finitely many points, then they intersect in an even number of points.

Exercise 229. Let M be a closed manifold and $f : M \to M$ a smooth map. Let $\Delta \subset M \times M$ be the diagonal and

$$G(f) = \{(m, f(m))\} \subset M \times M$$

the graph of f. Show that if $\Delta \cdot G(f)$ is nonzero, then any map homotopic to f has a fixed point. Can you show that $\Delta \cdot G(\mathrm{Id})$ equals the Euler characteristic of M or, more generally, that $\Delta \cdot G(f)$ equals the Lefschetz number of f?

Exercise 230. Think about how to modify the proof of Theorem 11.29 to handle the situation when A and B are only immersed instead of embedded.

A more ambitious exercise is the following, which says that the intersection of submanifolds is identified with the cup product even when the dimensions are not complementary.

Exercise 231. The transversality theorems show that if A and B are transverse closed submanifolds of a closed manifold M with $a+b>m$, then the intersection $A\cap B$ is an oriented, closed submanifold of dimension $m-a-b$. Prove that the Poincaré dual of $[A\cap B]$ is the class $\alpha\cup\beta$. Use the fact that $A\cap D$ is the pull-back of the disk bundle $D\to B$ over $A\cap B$, use naturality of the Thom class, and apply Corollary 11.30. (You might try the $\mathbf{Z}/2$ version first, to avoid orientation issues.)

11.8. Stiefel–Whitney classes

Recall from Section 9.6 that $u_k : E_{O(k)} \to BO(k)$ denotes the universal vector bundle over $BO(k)$, $T(u_k)$ denotes its Thom space. Proposition 9.18 shows that the map $s_k : BO(k) \to BO(k+1)$ induced by the inclusion of matrices

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

is covered by a bundle map $E_{O(k)} \oplus \underline{\mathbb{R}} \to E_{O(k+1)}$, inducing $s_k : ST(u_k) \to T(u_{k+1})$.

To avoid notational conflict, we denote by $\tau_k \in \widetilde{H}^k(T(u_k))$ the Thom class for the universal bundle $u_k : E_{O(k)} \to BO(k)$. Hence the Thom isomorphism theorem implies that the cup product

$$\Phi_k = \cup\tau_k : H^n(BO(k);\mathbf{Z}/2) \to \widetilde{H}^{n+k}(T(u_k);\mathbf{Z}/2)$$

is an isomorphism for all n.

Proposition 11.31. *The diagram*

$$\begin{tikzcd} H^n(BO(k)) \arrow[r, "\Phi_k"] \arrow[d] & H^{n+k}(T(u_k)) \arrow[d] \\ H^n(BO(k-1)) \arrow[r, "\Phi_{k-1}"] & H^{n+k-1}(T(u_{k-1})) \end{tikzcd}$$

commutes, where the left vertical map is induced by the inclusion

$$BO(k-1) \to BO(k)$$

and the right vertical map is induced by the composite

$$H^{n+k}(T(u_k)) \xrightarrow{s_{k-1}^*} H^{n+k}(ST(u_{k-1})) \xrightarrow{\cong} H^{n-k-1}(T(u_{k-1}))$$

with the second map the suspension isomorphism.

Proof. Notice that $T(\xi \oplus \underline{\mathbb{R}}) = ST(\xi)$ (Exercise 171). Restricting to a fiber corresponds to the suspension $S(D^k, S^{k-1}) = (D^{k+1}, S^k)$. It follows by naturality of the suspension isomorphism that the Thom class for $\xi \oplus \underline{\mathbb{R}}$ is the suspension of the Thom class for ξ.

If

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d] & E' \arrow[d] \\ X \arrow[r, "f"] & X' \end{tikzcd}$$

is a map of $\mathbf{R}^k$-vector bundles which is an isomorphism on each fiber (equivalently, $E \cong f^*(E')$), then the Thom class pulls back, $u_E = \tilde{f}^*(u_{E'})$; this follows again by uniqueness of the Thom class and by restricting to fibers.

The corollary now follows from these observations and the fact that $s_{k-1}^*(u_k) \cong u_{k-1} \oplus \underline{\mathbb{R}}$ (Proposition 9.18). $\square$

Proposition 11.32. *The homomorphism*

$$H^n(BO(k); \mathbf{Z}/2) \to H^n(BO(k-1); \mathbf{Z}/2)$$

induced by the natural map $BO(k-1) \to BO(k)$ is an isomorphism for $n < k-1$.

Proof. Consider the fibration

$$O(k-1) \hookrightarrow O(k) \to S^{k-1}$$

taking a matrix in $O(k)$ to its last column. This deloops twice (see Theorem 7.45) to give a fibration

$$S^{k-1} \to BO(k-1) \to BO(k).$$

The spectral sequence for this fibration (or the Gysin sequence) shows that the sequence

$$\cdots \to H^{n-k}(BO(k)) \to H^n(BO(k)) \to H^n(BO(k-1)) \to H^{n-k+1}(BO(k)) \to \cdots$$

is exact. Thus if $n - k + 1 < 0$, $H^n(BO(k)) \to H^n(BO(k-1))$ is an isomorphism. $\square$

The Stiefel–Whitney classes can now be defined using the Steenrod operations and the Thom isomorphism theorem.

Definition 11.33. Define the *nth (universal) Stiefel–Whitney class* to be

$$w_n = \Phi_k^{-1}\big(Sq^n(\tau_k)\big) \in H^n(BO(k); \mathbf{Z}/2),$$

where $\tau_k \in H^k(T(u_k); \mathbf{Z}/2)$ denotes the Thom class of the universal bundle $u_k : E_{O(k)} \to BO(k)$, and

$$\Phi_k = \cup\tau_k : H^n(BO(k); \mathbf{Z}/2) \to H^{k+n}(T(u_k)); \mathbf{Z}/2)$$

denotes the Thom isomorphism.

Proposition 11.31, Proposition 11.32, and naturality of the Steenrod operations imply

Proposition 11.34. *The restriction*

$$H^n(BO(k); \mathbf{Z}/2) \to H^n(BO(k-r); \mathbf{Z}/2)$$

takes w_n for $BO(k)$ to w_n for $BO(k-r)$. □

Hence the notation w_n is unambiguous. Notice that if $n > k$, then $Sq^n(\tau_k) = 0$, and so $w_n = 0$ in $H^n(BO(k); \mathbf{Z}/2)$ for $n > k$.

Definition 11.35. The *nth Stiefel–Whitney class* of an $\mathbf{R}^k$-vector bundle $\xi : E \to B$ is the class

$$w_n(\xi) = f_\xi^*(w_n) \in H^n(B; \mathbf{Z}/2)$$

where $f_\xi : B \to BO(k)$ denotes the classifying map for ξ (see Theorem 9.15).

It follows immediately from their definition that the Stiefel–Whitney classes are natural with respect to pulling back bundles. In particular, if ξ and ξ' are isomorphic bundles over B, then $w_n(\xi) = w_n(\xi')$ for all n. Moreover, since the w_n are compatible with respect to the maps $BO(k) \to BO(k+1)$,

$$w_n(\xi \oplus \underline{\mathbb{R}}) = w_n(\xi).$$

In other words, the Stiefel–Whitney classes are invariants of the stable equivalence class of a vector bundle.

Exercise 232. Show that a vector bundle ξ is orientable if and only if $w_1(\xi) = 0$. (Hint: first relate Sq^1 to the Bockstein associated to the exact sequence $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \to \mathbf{Z}/2 \to 0$.)

The Cartan formula (see Theorem 11.24) easily implies the following theorem.

Theorem 11.36. *The Stiefel–Whitney classes of a Whitney sum of vector bundles satisfy*

$$w_k(\xi \oplus \xi') = \sum_n w_n(\xi) \cup w_{k-n}(\xi').$$

□

Exercise 233. Suppose that ξ and ξ' are vector bundles over a finite-dimensional CW-complex so that $\xi \oplus \xi'$ is trivial (i.e. ξ and ξ' are stable inverses. For example, take ξ to be the tangent bundle of a smooth compact manifold and ξ' its normal bundle for some embedding in S^n.) Use Theorem 11.36 to prove that

$$w_1(\xi') = w_1(\xi), w_2(\xi') = w_1(\xi)^2 + w_2(\xi), w_3(\xi') = w_1(\xi)^3 + w_3(\xi),$$

and, in general, that

$$w_n(\xi') = \sum_{i_1+2i_2+\cdots+ki_k=n} \frac{(i_1+\cdots+i_k)!}{i_1!\cdots i_k!} w_1(\xi)^{i_1}\cdots w_k(\xi)^{i_k}.$$

The Stiefel–Whitney classes generate the cohomology ring of $BO(k)$, as the following theorem shows.

Theorem 11.37. *The $\mathbf{Z}/2$-cohomology ring of $BO(k)$ is a polynomial ring on the Stiefel–Whitney classes of degree less than or equal to k:*

$$H^*(BO(k);\mathbf{Z}/2) = \mathbf{Z}/2[w_1, w_2, \cdots, w_k]$$

where $w_i \in H^i(BO(k);\mathbf{Z}/2)$ denotes the ith Stiefel–Whitney class.

Proof. (Use $\mathbf{Z}/2$ coefficients.) First we show that $w_k \in H^k(BO(n))$ is nonzero if $k \leq n$. To see this it suffices by naturality to find one $\mathbf{R}^n$-bundle with w_k nonzero. Let $\mathbf{R}^1 \hookrightarrow E \to S^1$ denote the "Möbius band" bundle over S^1, i.e. the bundle with clutching function $S^0 \to O(1)$ the non-constant map. This bundle has $w_1 \neq 0$ (for example, it is not orientable). Thus w_1 is nonzero in $H^1(BO(1))$, and since the restrictions

$$H^k(BO(n)) \to H^k(BO(n-1))$$

preserve the w_i by Proposition 11.34, w_1 is nonzero in $H^1(BO(n))$ for all $n \geq 1$.

Since $BO(1) = K(\mathbf{Z}/2, 1) = \mathbf{R}P^\infty$ and $w_1 \neq 0$, $H^*(BO(1)) = \mathbf{Z}/2[w_1]$. Let $\xi : E \to BO(1)$ be any bundle with $w_1(\xi) = w_1$. Then Theorem 11.36 (and induction) shows that

$$w_k(\underbrace{\xi \oplus \cdots \oplus \xi}_{k \text{ times}}) = w_1(\xi)^k,$$

which is nonzero in $H^k(BO(1))$. Therefore $w_k \in H^k(BO(n))$ is nonzero for all $n \geq k$.

We prove the theorem by induction. The case $n = 1$ is contained in the previous paragraph. Let $i : BO(n-1) \to BO(n)$ denote the inclusion. The induced map $i^* : H^*(BO(n)) \to H^*(BO(n-1))$ is surjective since by induction $H^*(BO(n-1))$ is generated by the w_i for $i \leq n-1$, and these are in the image of i^*.

The fiber of $i : BO(n-1) \to BO(n)$ is S^{n-1}; in fact the fibration obtained by taking an orthogonal matrix to its last column

$$O(n-1) \hookrightarrow O(n) \to S^{n-1}$$

deloops twice to give the fibration

$$S^{n-1} \to BO(n-1) \to BO(n).$$

Consider the cohomology spectral sequence for this fibration. It has $E_2^{p,q} = H^p(BO(n)) \otimes H^q(S^{n-1})$ which is zero if $q \neq 0$ or $n-1$. Hence

$$E_k^{p,q} = \begin{cases} E_2^{p,q} = H^p(BO(n)) \otimes H^q(S^{n-1}) & \text{if } k \leq n, \\ E_\infty^{p,q} & \text{if } k > n. \end{cases}$$

This leads to the exact sequence (this is just the Gysin sequence in cohomology)

$$\cdots \to H^{k-1}(BO(n)) \to H^{k-1}(BO(n-1)) \to$$

$$H^{k-n}(BO(n)) \otimes H^{n-1}(S^{n-1}) \xrightarrow{d_n} H^k(BO(n)) \to H^k(BO(n-1)) \to \cdots$$

which reduces to short exact sequences
(11.12)

$$0 \to H^{k-n}(BO(n)) \otimes H^{n-1}(S^{n-1}) \xrightarrow{d_n} H^k(BO(n)) \to H^k(BO(n-1)) \to 0$$

since $H^*(BO(n)) \to H^*(BO(n-1))$ is onto. The map labelled d_n in (11.12) is the differential $d_n : E_n^{q,n-1} \to E_n^{q+n,0}$.

Taking $k = n$ in the sequence (11.12), we obtain

$$0 \to H^{n-1}(S^{n-1}) \xrightarrow{d_n} H^n(BO(n)) \to H^n(BO(n-1)) \to 0.$$

Since $H^{n-1}(S^{n-1}) = \mathbf{Z}/2$, generated by the fundamental class $[S^{n-1}]^*$, and since $w_n \in H^n(BO(n))$ is nonzero and in the kernel of the restriction $H^n(BO(n)) \to H^n(BO(n-1))$, it follows that $d_n([S^{n-1}]^*) = w_n$.

Applying the sequence (11.12), the fact that $d_n(\alpha \otimes [S^{n-1}]^*) = \alpha \cup w_n$ for $\alpha \in H^{k-1}(BO(n))$, and induction completes the proof, since this sequence shows that any element in $H^k(BO(n))$ can be written uniquely as a sum of classes of the form

$$w_1^{i_1} \cdots w_{n-1}^{i_{n-1}} \text{ with } i_1 + 2i_2 + \cdots + (n-1)i_{n-1} = k$$

and classes of the form

$$d_n([S^{n-1}]^*)\alpha = w_n\alpha$$

for some $\alpha \in H^{k-n}(BO(n))$. $\square$

Exercise 234. Show that if $\xi : E \to B$ is an $\mathbf{R}^k$-vector bundle, then $w_k(\xi)$ is the image of the Thom class under the composite

$$H^k(E, E_0; \mathbf{Z}/2) \xrightarrow{i^*} H^k(E; \mathbf{Z}/2) \xrightarrow{z^*} H^k(B; \mathbf{Z}/2)$$

where $z : B \to E$ denotes the zero section.

Stiefel-Whitney classes are special cases of *characteristic classes*, defined as classes in the cohomology of the classifying space BG for any Lie group G. See Theorem 11.40 for the case of Chern classes of unitary groups, $G = U(n)$.

An important example of a characteristic class is the Euler class $e \in H^k(BSO(k); \mathbf{Z})$ of an oriented $\mathbf{R}^k$-vector bundle. Concretely, given an oriented $\mathbf{R}^k$-vector bundle $\xi : E \to B$ with oriented Thom class $\tilde{u}(\xi) \in H^k(T(\xi); \mathbf{Z})$, the class

$$e(\xi) = z^*(i^*(\tilde{u}(\xi))) \in H^k(B; \mathbf{Z})$$

is called the Euler class of ξ. Compare this with the definition we gave of the Euler class in Section 8.11. The Euler class is not a stable class, for example $e(TS^2) = 2[S^2]^* \in H^2(S^2; \mathbf{Z})$, but $e(TS^2 \oplus \underline{\mathbb{R}}) = e(T\mathbf{R}^3|_{S^2}) = 0$. It follows from Exercise 234 that for any oriented $\mathbf{R}^k$-vector bundle ξ, the mod 2 reduction of $e(\xi)$ equals $w_k(\xi)$.

11.9. Localization

Given a subset P of the set of prime numbers, let $\mathbf{Z}_{(P)}$ denote the integers localized at P. This is the subring of the rationals consisting of all fractions whose denominator is relatively prime to each prime in P:

$$\mathbf{Z}_{(P)} = \{\tfrac{r}{s} \mid (r, s) = 1 \text{ and } (s, p) = 1 \text{ for each prime } p \in P\}.$$

Thus

$$\mathbf{Z} \subset \mathbf{Z}_{(P)} \subset \mathbf{Q}.$$

If P consists of a single prime p, we write $\mathbf{Z}_{(P)} = \mathbf{Z}_{(p)}$.

Definition 11.38. Given a set P of prime numbers, an abelian group A is called P-*local* if the homomorphism

$$A \to A \otimes_{\mathbf{Z}} \mathbf{Z}_{(P)}, \quad a \mapsto a \otimes 1$$

is an isomorphism.

If p is a prime and $r > 0$,

$$\mathbf{Z}/p^r \otimes \mathbf{Z}_{(P)} = \begin{cases} \mathbf{Z}/p^r & \text{if } p \in P, \\ 0 & \text{if } p \notin P. \end{cases} \tag{11.13}$$

More generally, if $A \in \mathcal{C}_P$, then $A \otimes \mathbf{Z}_{(P)} = 0$. This is because if $a \in A$, choose $r > 0$ relatively prime to each $p \in P$ so that $ra = 0$. Then r is invertible in $\mathbf{Z}_{(P)}$, and so for each $z \in \mathbf{Z}_{(P)}$,

$$a \otimes z = a \otimes \tfrac{zr}{r} = ra \otimes \tfrac{z}{r} = 0.$$

Since $\mathbf{Z}_{(P)}$ is torsion free is a localization, it is flat as an abelian group (see Theorem 3.40); i.e. the functor $- \otimes_{\mathbf{Z}} \mathbf{Z}_{(P)}$ is exact. In particular, if $f : A \to B$ is a $\mathcal{C}_P$ isomorphism, then tensoring the exact sequence

$$0 \to \ker f \to A \xrightarrow{f} B \to \operatorname{coker} f \to 0$$

with $\mathbf{Z}_{(P)}$ and using the fact that

$$\ker f \otimes \mathbf{Z}_{(P)} = 0 = \operatorname{coker} f \otimes \mathbf{Z}_{(P)},$$

we conclude that

$$f \otimes 1 : A \otimes \mathbf{Z}_{(P)} \to B \otimes \mathbf{Z}_{(P)}$$

is an isomorphism. This implies that if A and B are $\mathcal{C}_P$-isomorphic, then $A \otimes \mathbf{Z}_{(P)}$ is isomorphic to $B \otimes \mathbf{Z}_{(P)}$. Conversely, suppose that A and B are finitely generated abelian groups so that $A \otimes \mathbf{Z}_{(P)}$ is isomorphic to $B \otimes \mathbf{Z}_{(P)}$. Then A and B have the same rank and their p-primary subgroups are isomorphic for $p \in P$. Thus there is a $\mathcal{C}_P$-isomorphism from A to B.

The (relative) Hurewicz theorem mod $\mathcal{C}$ implies the following result, when applied to $\mathcal{C}_P$.

Theorem 11.39. *Let A, X be spaces such that H_iA and H_iX are finitely generated for each i, such that $\pi_1 A = \pi_1 X = 0$.*

Let $f : A \to X$ be a map with $\pi_2(X, A) = 0$. Then the statements:

1. *$f_* : H_i(A; \mathbf{Z}_{(P)}) \to H_i(X; \mathbf{Z}_{(P)})$ is an isomorphism for $i < n$ and an epimorphism for $i = n$,*
2. *$H_i(X, A; \mathbf{Z}_{(P)}) = 0$ for $i \leq n$,*
3. *$H_i(X, A; \mathbf{Z}) \in \mathcal{C}_P$ for $i \leq n$,*
4. *$\pi_i(X, A) \in \mathcal{C}_P$ for $i \leq n$,*
5. *$\pi_i A \to \pi_i X$ is a $\mathcal{C}_P$-isomorphism for $i < n$ and a $\mathcal{C}_P$-epimorphism for $i = n$,*
6. *$\pi_i(A) \otimes \mathbf{Z}_{(P)} \to \pi_i(X) \otimes \mathbf{Z}_{(P)}$ is an isomorphism for $i < n$ and an epimorphism for $i = n$*

are equivalent and imply that if $i < n$, then $\pi_i A$ and $\pi_i X$ have equal rank and isomorphic p-primary components for each $p \in P$.

Proof. Since $\mathbf{Z}_{(P)}$ is flat, the universal coefficient theorem (Corollary 3.34) implies that $H_k(Y; \mathbf{Z}_{(P)}) = H_k(Y; \mathbf{Z}) \otimes \mathbf{Z}_{(P)}$ for any space Y and any k. Since X and A have finitely generated $\mathbf{Z}$-homology it follows from the discussion preceding this theorem that the second and third assertions are equivalent. The long exact sequence in homology and homotopy for a pair and the relative Hurewicz theorem mod $\mathcal{C}_P$ imply that (1) through (6) are equivalent.

The Hurewicz theorem mod $\mathcal{C}_{FG}$ implies that $\pi_i A$ and $\pi_i X$ are finitely generated for each i. Thus (6) and Equation (11.13) imply that $\pi_i A$ and $\pi_i X$ have isomorphic p-primary components and equal rank for $i < n$. $\square$

An application of the universal coefficient theorem shows that a map $f : A \to X$ induces a $\mathbf{Z}_{(P)}$-homology isomorphism in all degrees if and only if it induces a $\mathbf{Z}_{(P)}$-cohomology isomorphism in all degrees.

Theorem 11.39 can be used to construct a functor (called the *localization of a space at* P)

$$L_{(P)} : \left\{ \begin{array}{l} \text{simply connected spaces with} \\ \text{finitely generated homology} \end{array} \right\} \to \{ \text{ simply connected spaces } \}$$

so that:

1. there exists a natural transformation from the identity functor to $L_{(P)}, \Phi : \mathrm{Id} \to L_{(P)}$,
2. for each X, $\Phi : X \to L_{(P)}(X)$ induces an isomorphism in $\mathbf{Z}_{(P)}$-homology, and
3. $H_i(L_{(P)}(X); \mathbf{Z}_{(P)}) = H_i(L_{(P)}(X); \mathbf{Z})$ for $i > 0$.

We write

$$L_{(P)}(X) = X_{(P)}.$$

The space $X_{(P)}$ is a good enough approximation to X to compute the p-primary part of its homotopy groups for $p \in P$; i.e. the p-primary part of $\pi_n X$ is isomorphic to the p-primary part of $\pi_n(X_{(P)})$ for $p \in P$ and the q-primary part of $\pi_n(X_{(P)}) = 0$ for $q \notin P$. In this manner one can study the algebraic topology of spaces one prime at a time, by taking $P = \{p\}$, and also the *rational homotopy* of a space, by taking P empty.

Such a functor $L_{(P)} : X \mapsto X_{(P)}$ exists and can be constructed by first constructing it for an Eilenberg–MacLane space $K(\pi, n)$ and then using a Postnikov decomposition of an arbitrary space into $K(\pi, n)s$.

We outline how to construct the localization functor $L_{(P)}$. For $K(\pi, n)$ with π a finitely generated abelian group, one just replaces π by $\pi \otimes \mathbf{Z}_{(P)}$. The natural map $\pi \to \pi \otimes \mathbf{Z}_{(P)}$ defines a (homotopy class of) map $K(\pi, n) \to K(\pi \otimes \mathbf{Z}_{(P)}, n)$. Thus we define

$$K(\pi, n)_{(P)} = K(\pi \otimes \mathbf{Z}_{(P)}, n).$$

For a general space one constructs $X_{(P)}$ inductively by assembling the pieces of its Postnikov tower, pulling back its k-invariants using the $\mathbf{Z}_{(P)}$

cohomology isomorphisms. Thus, if X has Postnikov system

$$\big(\pi_n = \pi_n(X), X_n, p_n : X_n \to X_{n-1}, k^n \in H^n(X_{n-1}; \pi_{n-1})\big),$$

then first define $(X_2)_{(P)} = K(\pi_2 \otimes \mathbf{Z}_{(P)}, 2)$. Since $X_2 = K(\pi_2, 2)$, the homomorphism $\pi_2 \to \pi_2 \otimes \mathbf{Z}_{(P)}$ induces a map $X_2 \to (X_2)_{(P)}$. The fibration $p_3 : X_3 \to X_2$ is obtained by pulling back the path space fibration $K(\pi_3, 3) \to * \to K(\pi_3, 4)$ via $k^4 \in H^4(X_2; \pi_3) = [X_2, K(\pi_3, 4)]$. Since the map $X_2 \to (X_2)_{(P)}$ induces an isomorphism

$$H^4((X_2)_{(P)}; \pi_3 \otimes \mathbf{Z}_{(P)}) \to H^4(X_2; \pi_3 \otimes \mathbf{Z}_{(P)}) \tag{11.14}$$

(using the universal coefficient theorem), it follows that there is a unique $k^4_{(P)} \in H^4((X_2)_{(P)}; \pi_3 \otimes \mathbf{Z}_{(P)})$ so that the image of $k^4_{(P)}$ via the homomorphism of Equation (11.14) coincides with the image of k^4 under the coefficient homomorphism

$$H^4(X_2; \pi_3) \to H^4(X_2; \pi_3 \otimes \mathbf{Z}_{(P)}).$$

Inductively, if $(X_k)_{(P)}$ and fibrations $(X_k)_{(P)} \to (X_{k-1})_{(P)}$ with fiber $K(\pi_k \otimes \mathbf{Z}_{(P)}, k)$ classified by $k^{k+1}_{(P)} \in H^{k+1}((X_{k-1})_{(P)}; \pi_k \otimes \mathbf{Z}_{(P)})$ have been defined for $k \leq n$, define $k^{n+2}_{(P)} \in H^{n+2}((X_n)_{(P)}; \pi_{n+1} \otimes \mathbf{Z}_{(P)})$ to be the image of the $(n+2)$-nd Postnikov invariant of X, k^{n+2}, under the composite

$$H^{n+2}(X_n; \pi_{n+1}) \to H^{n+2}(X_n; \pi_{n+1} \otimes \mathbf{Z}_{(P)}) \cong H^{n+2}((X_n)_{(P)}; \pi_{n+1} \otimes \mathbf{Z}_{(P)}).$$

Then take $(X_{n+1})_{(P)}$ to be total space in the fibration pulled back from the path space fibration $K(\pi_{n+1} \otimes \mathbf{Z}_{(P)}, n+1) \to * \to K(\pi_{n+1} \otimes \mathbf{Z}_{(P)}, n+2)$ using $k^{n+2}_{(P)} \in H^{n+2}((X_n)_{(P)}; \pi_{n+1} \otimes \mathbf{Z}_{(P)}) = [(X_n)_{(P)}, K(\pi_{n+1} \otimes \mathbf{Z}_{(P)}, n+2)]$.

Notice that the construction also gives a map $X_n \to (X_n)_{(P)}$ inducing the homomorphisms $\pi_k X = \pi_k X_n \to \pi_k(X_n) \otimes \mathbf{Z}_{(P)} = \pi_k((X_n)_{(P)})$ for all $k \leq n$. Thus if $X_{(P)}$ denotes the space determined by the Postnikov system $(X_n)_{(P)}$ with k-invariants $k^{n+1}_{(P)}$, there is a map $X \to X_{(P)}$ (this gives the natural transformation Φ) so that the induced map $\pi_n(X) \to \pi_n(X_{(P)})$ coincides with

$$\pi_n X \to \pi_n(X) \otimes \mathbf{Z}_{(P)}, \quad a \mapsto a \otimes 1.$$

From Theorem 11.39 we conclude that $X \to X_{(P)}$ induces an isomorphism on homology with $\mathbf{Z}_{(P)}$ coefficients (and so also on cohomology with $\mathbf{Z}_{(P)}$ coefficients). The facts that localization is functorial and that $X \to X_{(P)}$ defines a natural transformation $\Phi : \mathrm{Id} \to L_{(P)}$ can be proven by carrying out the construction we gave in a systematic fashion.

Here are some examples with $P = \phi$ to show you why localization is useful. The space $X_{(\phi)}$ is usually denoted by $X_{(0)}$ and is called the *rationalization of X*.

From Proposition 10.28 it follows that if n is odd, the map $S^n \to K(\mathbf{Z}, n)$ generating $H^n(S^n)$ induces an isomorphism on rational cohomology, and hence a homotopy equivalence $S^n_{(0)} \to K(\mathbf{Q}, n) = K(\mathbf{Z}, n)_{(0)}$. Therefore

$$\pi_k(S^n) \otimes \mathbf{Q} = \pi_k(K(\mathbf{Q}, n)) = 0 \text{ for } q \neq n.$$

This implies that $\pi_k S^n$ is finite for $k \neq n$.

For n even, $S^n \to K(\mathbf{Q}, n)$ induces an isomorphism in rational homology through dimensions $2n-1$. Hence $\pi_k S^n$ is finite for $k \leq 2n-1, k \neq n$. We can do better by taking E to be the homotopy fiber of the map $K(\mathbf{Q}, n) \to K(\mathbf{Q}, 2n)$ representing $\iota_n^2 \in H^{2n}(K(\mathbf{Q}, n); \mathbf{Q})$. The map $S^n \to K(\mathbf{Q}, n)$ lifts to E since $H^{2n}(S^n; \mathbf{Q}) = 0$. The long exact sequence in homotopy shows that

$$\pi_k E = \begin{cases} \mathbf{Q} & \text{if } k = n, 2n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Again, a simple application of the Leray–Serre spectral sequence for the fibration $K(\mathbf{Q}, 2n-1) \to E \to K(\mathbf{Q}, n)$ and Proposition 10.28 shows that $H^*(E; \mathbf{Q}) = H^*(S^n; \mathbf{Q})$; the isomorphism is induced by the map $S^n \to E$. Thus $S^n_{(0)} = E$ and so $\pi_k(S^n) \otimes \mathbf{Q} = \pi_k E$. This shows that $\pi_k S^n$ is finite for $k \neq n, 2n-1$ and that the rank of $\pi_k S^n$ is 1 for $k = n$ or $2n-1$.

These two calculations were obtained in Theorem 11.11 and Exercise 216 by similar arguments; the point is that the argument using localization is conceptually much simpler since calculating with the Leray-Serre spectral sequence using rational coefficients is easier than using integer coefficients; for example $E_2^{p,q} = H^p(B) \otimes H^q(F)$. Moreover, the rational cohomology of $K(\mathbf{Q}, n)$ is simple, and so constructing rational Postnikov systems which do what we want is a more manageable problem than constructing an arbitrary Postnikov system.

As a new example, consider the space $\mathbf{C}P^n$. The (rational) cohomology ring of $\mathbf{C}P^n$ is a truncated polynomial ring, and the cohomology of $\mathbf{C}P^\infty$ is a polynomial ring. The inclusion $\mathbf{C}P^n \to \mathbf{C}P^\infty = K(\mathbf{Z}, 2)$ induces isomorphisms on (rational) cohomology through dimension $2n$. Let $c \in H^2(K(\mathbf{Q}, 2); \mathbf{Q})$ denote a generator. Think of $c^{n+1} \in H^{2n+2}(K(\mathbf{Q}, 2); \mathbf{Q})$ as a map $c^{n+1} : K(\mathbf{Q}, 2) \to K(\mathbf{Q}, 2n+2)$ and let E be its homotopy fiber. The map $\mathbf{C}P^n \to K(\mathbf{Q}, 2)$ lifts to E since $H^{2n+2}(\mathbf{C}P^n; \mathbf{Q}) = 0$. The spectral sequence for the fibration $K(\mathbf{Q}, 2n+1) \to E \to K(\mathbf{Q}, 2)$ and the calculation of Proposition 10.28 shows that $\mathbf{C}P^n \to E$ induces an isomorphism on rational cohomology.

Exercise 235. Prove this to see how easy it is.

Using the long exact sequence in homotopy, we conclude that

$$\pi_k(\mathbf{C}P^n) \otimes \mathbf{Q} = \pi_k(E) = \begin{cases} \mathbf{Q} & \text{if } k = 2, 2n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathbf{C}P^n$ is a finite complex this shows that $\pi_k(\mathbf{C}P^n)$ is finite for $k \neq 2, 2n+1$ and has rank 1 for $k = 2$ and $k = 2n+1$.

Another application is to Chern classes and Bott periodicity for the unitary group. First, we have the following complex analogue of Theorem 11.37.

Theorem 11.40. *Let $BU(n)$ denote the classifying space for $U(n)$. Then the cohomology ring of $BU(n)$ is a polynomial ring:*

$$H^*(BU(n); \mathbf{Z}) = \mathbf{Z}[c_1, c_2, \cdots, c_n]$$

where the generators c_k have degree $2k$. The inclusion $U(n-1) \to U(n)$ induces a map $H^*(BU(n)) \to H^*(BU(n-1))$ which preserves the c_k. □

Exercise 236. Prove Theorem 11.40 using induction and the multiplicative properties of the Leray–Serre spectral sequence for the fibration

$$S^{2n-1} \to BU(n-1) \to BU(n)$$

obtained by delooping the fibration $U(n-1) \to U(n) \to S^{2n-1}$ twice. You may use the proof of Theorem 11.37 as a guide, but the argument in this case is much simpler.

The class $c_k \in H^{2k}(BU(n); \mathbf{Z})$ is called the kth *Chern class*. Since isomorphism classes of $\mathbf{C}^n$-vector bundles are classified by homotopy classes of maps to $BU(n)$, the Chern classes determine characteristic classes of complex vector bundles. By construction, c_k is a stable class; i.e. if E is a complex vector bundle and $\underline{\mathbb{C}}$ denotes the trivial 1-dimensional complex vector bundle, then $c_k(E \oplus \underline{\mathbb{C}})$ is sent to $c_k(E)$ by the map $H^k(BU(n+1)) \to H^k(BU(n))$.

Now consider the map $BU(n) \to \prod_{k=1}^n K(\mathbf{Q}, 2k)$ given by the product of the Chern classes, thinking of $c_k \in H^{2k}(BU(n); \mathbf{Q}) = [BU(n); K(\mathbf{Q}, 2k)]$. By the Künneth theorem and Proposition 10.28 this map induces an isomorphism on rational cohomology. Therefore the rationalization of $BU(n)$ is $\prod_{k=1}^n K(\mathbf{Q}, 2k)$. Since $\Omega(X \times Y) = \Omega X \times \Omega Y$, $\Omega Z = \text{pt}$ if Z is discrete, and $\Omega K(G, n) \simeq K(G, n-1)$, we see that

$$\Omega^2 BU(n)_{(0)} = \Omega^2(\prod_{k=1}^n K(\mathbf{Q}, 2k)) \simeq \prod_{k=0}^{n-1} K(\mathbf{Q}, 2k) \simeq \mathbf{Q} \times BU(n-1)_{(0)}.$$

In particular, letting n go to infinity we obtain a proof of the rational form of Bott periodicity:

$$\Omega^2(\mathbf{Q} \times BU_{(0)}) \simeq \mathbf{Q} \times BU_{(0)}.$$

11.10. Construction of bordism invariants

We finish this chapter with some comments on Thom's computation of the unoriented bordism groups. An *invariant* of unoriented bordism is a homomorphism $w : \Omega_n^{\mathbf{O}} \to G$ for some abelian group G. Since $2M = \partial(M \times I)$, every bordism class in the group $\Omega_n^{\mathbf{O}}$ has order 2. Thus to construct bordism invariants one might as well restrict to constructing homomorphisms $w : \Omega_n^{\mathbf{O}} \to \mathbf{Z}/2$. Thom computed $\Omega_n^{\mathbf{O}}$ in this fashion for all n in his famous 1954 paper [**50**], using Stiefel-Whitney classes to construct bordism invariants $\Omega_n^{\mathbf{O}} \to \mathbf{Z}/2$, called the *Stiefel-Whitney numbers.*

We will outline some of the ingredients in Thom's arguments.

Proposition 11.41. *Let*

$$w = w_1^{i_1} \cdots w_n^{i_n} \in H^n(BO(n)),$$

so $i_1 + 2i_2 + \cdots + ni_n = n$. If M is a smooth n-manifold with tangent bundle TM, then the number

$$\langle w(TM), [M] \rangle \in \mathbf{Z}/2$$

is a bordism invariant.

Proof. Since the expression $\langle w(TM), [M] \rangle$ is additive with respect to the sum in the bordism group (disjoint union), it suffices to show that if M is null-bordant, i.e. $M = \partial W$, then $\langle w(TM), [M] \rangle = 0$.

The tangent bundle of W and M are related by $TM \oplus \underline{\mathbb{R}} = TW|_M$. Hence if $i : M \subset W$ denotes the inclusion,

$$\begin{aligned} \langle w(TM), [M] \rangle &= \langle w(TM \oplus \underline{\mathbb{R}}), [M] \rangle \\ &= \langle w(i^*(TW)), [M] \rangle \\ &= \langle i^*(w(TW)), [M] \rangle \\ &= i^*(w(TW)) \cap [M] \\ &= w(TW) \cap i_*[M] = 0, \end{aligned}$$

since in the sequence $H_{n+1}(W, M) \xrightarrow{\partial} H_n M \xrightarrow{i_*} H_n W$ the map labelled ∂ takes the generator $[W, M]$ to $[M]$. □

Definition 11.42. A *partition* of the positive integer n is an n–tuple of nonnegative integers $(i_1, \cdots, i_n)$ so that $i_1 + 2i_2 + \cdots + ni_n = n$. Denote by P_n the set of all partitions of n.

Given a closed manifold M and a partition $\alpha = (i_1, \cdots, i_n)$ of n, the number

$$w_\alpha(M) = \langle w_1^{i_1} \cdots w_n^{i_n}(TM), [M] \rangle \in \mathbf{Z}/2$$

is called the *Stiefel–Whitney number* associated to the partition α. Proposition 11.41 shows that $w_\alpha(M)$ depends only on the bordism class of M.

Thom's theorem is the following. It is particularly simple to state in terms of the graded ring structure on $\Omega_*^{\mathbf{O}}$ induced by taking the cartesian product of manifolds, i.e. $[M]\cdot[N] = [M \times N] \in \Omega_{m+n}^{\mathbf{O}}$ for $[M] \in \Omega_m^{\mathbf{O}}$ and $[N] \in \Omega_n^{\mathbf{O}}$.

Theorem 11.43. *The map taking a manifold to its Stiefel–Whitney numbers induces an monomorphism*

$$\bigoplus_{\alpha \in P_n} w_\alpha : \Omega_n^{\mathbf{O}} \to \bigoplus_{\alpha \in P_n} \mathbf{Z}/2.$$

In other words, two closed manifolds are bordant if and only if they have the same Stiefel-Whitney numbers.

Moreover, $\Omega_*^{\mathbf{O}}$ is a polynomial ring over $\mathbf{Z}/2$ on generators $x_k \in \Omega_k^{\mathbf{O}}$, one for each nonnegative integer k not of the form $2^m - 1$. Thus $\Omega_n^{\mathbf{O}}$ is a $\mathbf{Z}/2$ vector space of rank the number of partitions in P_n of the form $(i_1, \cdots, i_n)$ satisfying $i_k = 0$ when $k = 2^j - 1$. □

Although the Stiefel-Whitney numbers and the generators of $\Omega_*^{\mathbf{O}}$ are both indexed by partitions, the relationship between them is not the obvious one. For example, the first few ring generators of $\Omega_*^{\mathbf{O}}$ are $x_2, x_4, x_5, x_6, x_8, \cdots$. Thus $\Omega_5^{\mathbf{O}}$ has rank one, generated by x_5. But every 5-manifold M has $w_5(M) = 0$, and it turns out that any manifold M representing x_5 has $w_2(M)w_3(M) \neq 0$. See [**48**] for more details.

Thom proves this theorem by a method analogous to the example of $BU_{(0)}$ we gave in the previous section. First, Thom finds sufficiently many examples of manifolds with the appropriate Stiefel–Whitney numbers, and then he uses these to define a map from the Thom spectrum to a product of Eilenberg-MacLane spectra $\mathbf{K}(\mathbf{Z}/2)$. He shows this map induces an isomorphism on homology, using the Thom isomorphism to compute the cohomology of the Thom spectrum as a module over the mod 2 Steenrod algebra. The Whitehead theorem then implies that the map is a homotopy equivalence, and so the Stiefel–Whitney numbers classify bordism.

11.11. Projects: Unstable homotopy theory

Blakers-Massey?

11.11.1. Unstable homotopy theory. Unstable homotopy theory is significantly harder than the stable theory, essentially because $\pi_n(X, A) \not\cong \pi_n(X/A)$. There are nevertheless some useful results; you should lecture on some or all of these.

Since S^m has a cell structure with only one 0-cell and one m-cell, the product $S^k \times S^n$ has a cell structure with 4 cells, a 0-cell $e^0 \times e^0$, a k-cell $e^k \times e^0$, an n-cell $e^0 \times e^n$, and a $(k+n)$-cell $e^k \times e^n$. Removing the top cell leaves the wedge

$$S^k \times S^n - (e^k \times e^n) = S^k \vee S^n.$$

Let $a : S^{k+n-1} \to S^k \vee S^n$ denote the attaching map for the (top) $(k+n)$-cell of $S^k \times S^n$.

The map a can be used to construct interesting elements in $\pi_n X$.

Definition 11.44. Given $f \in \pi_k X$ and $g \in \pi_n X$, define the *Whitehead product* $[f, g] \in \pi_{k+n-1} X$ to be the (homotopy class of) the composite

$$S^{k+n-1} \xrightarrow{a} S^k \vee S^n \xrightarrow{f \vee g} X.$$

For example, if $k = n = 1$, the attaching map for the 2-cell of a torus represents the commutator of the two generators, and hence the Whitehead product $\pi_1 X \times \pi_1 X \to \pi_1 X$ takes a pair of loops to their commutator. Since $\pi_2(X)$ is abelian, the suspension map $s_* : \pi_1 X \to \pi_1(\Omega SX) = \pi_2 X$ takes any commutator to zero. More generally show (or look up) the following fact.

Proposition 11.45. *The suspension of the attaching map for the top cell of $S^k \times S^n$,*

$$Sa : S(S^{k+n-1}) = S^{k+n} \to S(S^k \vee S^n),$$

is nullhomotopic. Hence $s_*[f, g] = 0$ for any $f \in \pi_k X, g \in \pi_n X$; i.e. the Whitehead product $[f, g]$ is in the kernel of the suspension homomorphism $\pi_{n+k-1} X \to \pi_{n+k} SX$. □

Thus Whitehead products produce decidedly unstable elements in $\pi_m X$. The map $s : X \to \Omega SX$ (defined in Section 11.3) induces the suspension homomorphism $s_* : \pi_\ell(X) \to \pi_\ell(\Omega SX) = \pi_{\ell+1}(SX)$. It can be studied in a large range (the "metastable range") by using the EHP sequence:

Theorem 11.46. *If X is an $(n-1)$-connected space, there is an exact sequence*

$$\begin{gathered}\pi_{3n-2} X \xrightarrow{s_*} \pi_{3n-1}(SX) \to \pi_{3n-1}(SX \wedge X) \to \pi_{3n-3} X \to \cdots \\ \cdots \to \pi_k X \to \pi_{k+1}(SX) \to \pi_{k+1}(SX \wedge X) \to \cdots .\end{gathered}$$

□

The map s_* is sometimes denoted "E" in the literature (from the German word "<u>E</u>inhängung" for suspension), the map $\pi_{k+1}(SX) \to \pi_{k+1}(SX \wedge X)$ is usually denoted by "H" since it generalizes the <u>H</u>opf invariant, and the map

$\pi_{k+1}(SX \wedge X) \to \pi_{k-1}X$ is usually denoted "P" since its image is generated by Whitehead Products. Hence the name EHP sequence.

Thus, in the range $k \leq 3n-2$, the EHP sequence gives some control over what the kernel and cokernel of the suspension map on homotopy groups are.

An important special case of the EHP sequence is obtained by setting $X = S^n$. The sequence is

$$\pi_{3n-2}S^n \to \pi_{3n-1}S^{n+1} \to \pi_{3n-1}S^{2n+1} \to \pi_{3n-3}S^n \to \cdots . \tag{11.15}$$

A proof of Theorem 11.46 can be found in [**54**] (although it is hard to reconstruct the argument since it is explained as a consequence of a more general result of James). The proof that (11.15) is exact as well as the material below on the Hopf invariant can be found in [**45**, Section 9.3].

After substituting $\pi_{2n+1}S^{2n+1} = \mathbf{Z}$ a part of the sequence (11.15) can be written:

$$\cdots \to \pi_{2n}S^n \to \pi_{2n+1}S^{n+1} \xrightarrow{H} \mathbf{Z} \to \pi_{2n-1}S^n \to \cdots .$$

If n is even, Theorem 11.11 implies that the map H is zero and therefore $\pi_{2n}S^n \to \pi_{2n+1}S^{n+1}$ is onto. The Freudenthal suspension theorem implies that $\pi_{2n+1}S^{n+1} \to \pi_{2n+2}S^{n+2} = \pi_n^S$ is onto, and so we conclude that for n even, $\pi_{2n}S^n \to \pi_n^S$ is onto.

If n is odd, then Exercise 216 shows that $\pi_{2n+1}S^{n+1}/\text{torsion}= \mathbf{Z}$. Since $\pi_{2n}S^n$ is finite the map $H : \pi_{2n+1}S^{n+1} \to \mathbf{Z}$ is nonzero; it is called the *Hopf invariant.*

The famous "Hopf invariant one" problem, solved by J. F. Adams [**3**], asserts that H is onto only for $n+1 = 1, 2, 4,$ and 8, and in fact the Hopf fibrations are the only maps which have Hopf invariant one. The Whitehead product $[i, i] \in \pi_{2n-1}S^{n+1}$, where $i \in \pi_{n+1}S^{n+1}$ denotes the generator, has Hopf invariant 2.

There are several other definitions of the Hopf invariant for a map $f : S^{2n+1} \to S^{n+1}$, and you should lecture on some of these. Here are two.

1. Assume that f is smooth (this can always be arranged by a small homotopy) and let $x_0, x_1 \in S^{n+1}$ be two regular values of f. Let $M_0 = f^{-1}\{x_0\}$ and $M_1 = f^{-1}\{x_1\}$. Then $H(f) = \mathrm{lk}(M_0, M_1)$, where "lk" denotes the linking number.
2. Let X be the CW-complex obtained by attaching a $(2n+2)$-cell to S^{n+1} using f as the attaching map. Then $H^q X = \mathbf{Z}$ for $q = 0, n+1$, and $2n+2$, and is zero otherwise. Let $e_{n+1} \in H^{n+1}X$ and $e_{2n+2} \in H^{2n+2}X$ denote the generators. Then $(e_{n+1})^2 = H(f)e_{2n+2}$.

Chapter 12

Simple-Homotopy Theory

Two basic references for the material in this chapter are Cohen's book [**10**] and Milnor's article [**34**].

12.1. Introduction

Whitehead's theorem (Theorem 8.33) says that a map $f : X \to Y$ between CW-complexes is a homotopy equivalence if $f_* : \pi_n X \to \pi_n Y$ is an isomorphism for all n. Thus homotopy groups and hence the tools of homotopy theory give important information about the homotopy type of a space. However, important questions in geometric topology center around distinguishing homeomorphism types within a class of homotopy equivalent manifolds, to which the methods we have studied so far do not directly apply.

For example, suppose W is a compact manifold with two boundary components: $\partial W = M_0 \amalg M_1$. Suppose the inclusion $M_0 \hookrightarrow W$ is a homotopy equivalence. Is W homeomorphic to $M_0 \times [0,1]$, and, in particular, is M_0 homeomorphic to M_1? The answer to this question is provided by the *s-cobordism theorem* (see Section 12.7) which states that there exists a functor

$$\text{Wh}: \{\text{Groups}\} \to \{\text{Abelian groups}\}$$

so that in the situation described above, an element $\tau(W, M_0) \in \text{Wh}(\pi_1 W)$ is defined and vanishes if W is homeomorphic to $M_0 \times [0,1]$. Conversely, if the dimension of M_0 is greater than 4 and $\tau(W, M_0) = 0$, then W is homeomorphic to $M_0 \times [0,1]$.

Of course the point of this theorem is that the functor Wh has a functorial, geometric, and somewhat computable definition. (One could stupidly define $\mathrm{Wh}(\pi) = \mathbf{Z}/2$ for all groups π and define $\tau(W, M_0)$ to be 0 or 1 according to whether or not W is homeomorphic to a product.)

In this chapter we will give the complete definition of the Whitehead group $\mathrm{Wh}(\pi)$ and the Whitehead torsion τ.

Exercise 237. Use the fact that $\mathrm{Wh}(1) = 0$ to prove the generalized Poincaré conjecture for $n > 5$: a closed manifold Σ which has the homotopy type of S^n is homeomorphic to S^n. (Hint: Remove two open disks; it can be shown that the complement has the homotopy type of a CW-complex. Assume this and apply the s-cobordism theorem.)

Another collection of examples is provided by lens spaces. We will use obstruction theory to give a homotopy classification of 3-dimensional lens spaces, and then use the machinery of simple-homotopy theory to prove the following theorem.

Theorem 12.1. *The 3-dimensional lens spaces $L(p, q)$ and $L(p', q')$ are homotopy equivalent if and only if $p = p'$ and there exists an integer b so that $q \equiv \pm b^2 q' \mod p$.*

Moreover, $L(p, q)$ is homeomorphic to $L(p, q')$ if and only if $p = p'$ and $q \equiv \pm(q')^{\pm 1} \mod p$.

In particular $L(7, 1)$ and $L(7, 2)$ are closed three-manifolds which have the same homotopy type but are not homeomorphic. Whitehead torsion must be a subtle and powerful invariant to make such a distinction.

J.H.C. Whitehead developed the theory of *simple-homotopy equivalence*, a refinement of homotopy equivalence for finite CW-complexes which takes into account the cell structure. It was proven by Chapman that homeomorphic finite CW-complexes are simple-homotopy equivalent. Hence simple-homotopy theory provides a weapon by which to attack homeomorphism problems that are impervious to the homotopy theoretic machinery developed in the previous chapters.

Suppose X, Y are finite CW-complexes. Then a cellular map $f : X \to Y$ is a homotopy equivalence if and only if the mapping cylinder of f deformation retracts to X via a cellular map.

Exercise 238. Prove this using obstruction theory.

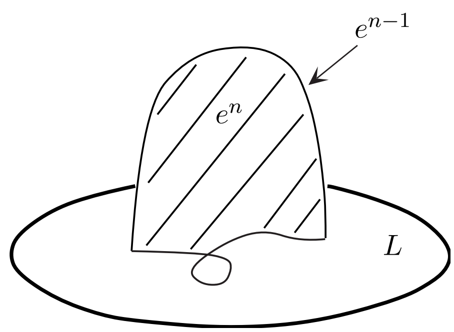
The geometric approach of simple-homotopy theory is to investigate when a pair (K, L) which admits a deformation retract of K to L admits a particular "simple" type of deformation.

In this chapter a *finite CW-pair* (K, L) will mean a CW-complex K with finitely many cells and a subcomplex $L \subset K$. Thus L is a closed subspace of K which is a union of cells. (See Definition 1.18 for the definition of a CW-complex.) We will also use infinite CW-pairs, but these will always be of the form $(\tilde{K}, \tilde{L})$ where (K, L) is a finite CW-pair, $\tilde{K} \to K$ a covering space, and $\tilde{L}$ the inverse image of L in $\tilde{K}$.

Definition 12.2. If (K, L) is a finite CW-pair, we say K *collapses to* L *by an elementary collapse*, denoted $K \overset{e}{\searrow} L$, if the following two conditions hold.

1. $K = L \cup e^{n-1} \cup e^n$ where the cells e^{n-1} and e^n are not in L.
2. Write $\partial D^n = S^{n-1} = D^{n-1}_+ \cup_{S^{n-2}} D^{n-1}_-$. Then there exists a characteristic map $\chi : D^n \to K$ for e^n so that
   (a) $\chi_{|D^{n-1}_+} : D^{n-1}_+ \to K$ is a characteristic map for e^{n-1}, and
   (b) $\chi(D^{n-1}_-) \subset L$.

Thus K is obtained by gluing D^n to L along a map $D^{n-1}_- \longrightarrow L$, where $D^{n-1}_- \subset \partial D^n \subset D^n$.



K

Note K can be viewed as the mapping cylinder of a map $D^{n-1}_- \to L$. Thus L is a deformation retract of K.

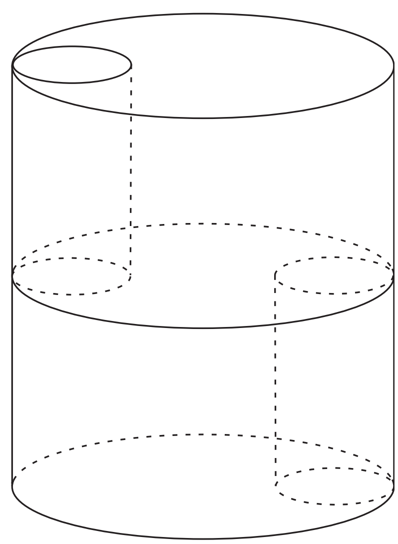
Definition 12.3.

1. One says that K *collapses to* L, or L *expands to* K, if there are subcomplexes $K = K_0 \supset K_1 \supset \cdots \supset K_n = L$ so that $K_0 \overset{e}{\searrow} K_1 \overset{e}{\searrow} \cdots \overset{e}{\searrow} K_n$. Write $K \searrow L$ or $L \nearrow K$.
2. A map $f : K \to L$ is called a *simple-homotopy equivalence* if there exists a finite sequence of CW-complexes $K = K_0, K_1, \ldots, K_n = L$ so that f is homotopic to a composite $K_0 \to K_1 \to K_2 \to \cdots \to K_n$ where each map $K_i \to K_{i+1}$ is either the inclusion map of an expansion, the deformation retraction of a collapse, or a cellular homeomorphism.

Exercise 239. Prove that simple-homotopy equivalence is an equivalence relation.

We now give two examples concerned with collapsing. Suppose L is a finite simplicial subcomplex of a triangulated open subset of Euclidean space. Then the *regular neighborhood* $K = N(L)$ is the union of all simplices whose closure intersects L. This is an analogue of a normal bundle, but L does not have to be a manifold. It is not difficult to see that $K \searrow L$.

The second example is where K is the "house with two rooms" pictured below. Here K is a 2-dimensional CW-complex. To get to the large room on the lower floor, you must enter the house from the top through the small cylinder on the left. Similarly, one enters the upper room via the small right cylinder. Then it is not difficult to see that K is simple-homotopy equivalent to a point, but that K does not collapse to a point; i.e. some expansions are needed.



12.2. Invertible matrices and $K_1(R)$

In this section we will define the Whitehead group and in the next section define torsion. Since there are two sections of algebra coming up, we will give you some geometric motivation to help you through.

It will turn out that the question of whether a homotopy equivalence is simple can be understood in the following way. Assume $f : L \to K$ is a cellular inclusion, and a homotopy equivalence. Then if $\pi = \pi_1 L = \pi_1 K$, the relative chain complex $C_\bullet(\tilde{K}, \tilde{L})$ (where $\tilde{K},\ \tilde{L}$ denote universal covers) is a *free and acyclic* $\mathbf{Z}\pi$*-chain complex* and has a $\mathbf{Z}\pi$*-basis* labeled by the cells of

$K - L$. In the context of simple-homotopy theory it is traditional to define an acyclic complex to be a complex $C_\bullet = (C_*, d)$ satisfying $H_n(C_*, d) = 0$ for *all* integers n, and we adopt this terminology in this chapter. Compare Definition 3.21 with Section 12.3 below.

Elementary collapses and changing base points change the cellular chain complex $C_\bullet(\tilde{K}, \tilde{L})$, and so one wants to classify acyclic, based chain complexes over $\mathbf{Z}\pi$ up to some equivalence relation, so that $C_\bullet(\tilde{K}, \tilde{L})$ is equivalent to 0 if and only if $K \hookrightarrow L$ is a simple-homotopy equivalence. The main result will be that the chain complex $C_\bullet(\tilde{K}, \tilde{L})$ determines an element $\tau(K, L) \in \mathrm{Wh}(\pi)$ which vanishes if and only if the map f is a simple-homotopy equivalence.

Once the machinery is set up, other useful applications will follow from considering rings R more general than the integral group ring $\mathbf{Z}\pi$. For example, if $\mathbf{Z}\pi \to R$ is a ring homomorphism, it may be easier to work with the chain complex $R \otimes_{\mathbf{Z}\pi} C_\bullet$ than to work directly with $C_\bullet$. This is especially true if R is a commutative ring or, even better, a field.

The simplest acyclic, based chain complexes are of the form:

$$0 \to C_n \xrightarrow{\partial} C_{n-1} \to 0.$$

Since this complex is based, ∂ is given by a matrix, which is invertible since (C_*, ∂) is acyclic.

Motivated by the previous discussion, we study invertible matrices over a (not necessarily commutative) ring R. We assume that all our rings are rings with 1. Unfortunately, two bizarre phenomena can arise when considering free modules over a ring R.

- It may be the case that $R^m \cong R^n$ with $m \neq n$.
- It may be the case that $M \oplus R^m \cong R^n$, but that M itself is not free. In this case we say the module M is *stably free* but not free.

Fortunately, the first problem does not occur for group rings, because there is a homomorphism $\varepsilon : \mathbf{Z}\pi \to \mathbf{Z}$, $\sum a_g g \mapsto \sum a_g$. Thus $(\mathbf{Z}\pi)^m \cong (\mathbf{Z}\pi)^n$ implies

$$\mathbf{Z}^m = \mathbf{Z} \otimes_{\mathbf{Z}\pi} (\mathbf{Z}\pi)^m \cong \mathbf{Z} \otimes_{\mathbf{Z}\pi} (\mathbf{Z}\pi)^n = \mathbf{Z}^n$$

and so $m = n$. Henceforth

We assume all rings have the property that $R^m \cong R^n$ implies $m = n$.

Thus we exclude rings like the endomorphism ring of an infinite-dimensional vector space.

The second pathology does occur for certain group rings, so we cannot assume it away. It will be a thorn in our side in the next section, but we will deal with it.

Definition 12.4. Denote by $GL(n, R)$ the group of all $n \times n$ matrices over R which have a two-sided inverse. An inclusion $GL(n, R) \hookrightarrow GL(n+1, R)$ is defined by

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $GL(R) = \bigcup_n GL(n, R)$. Think of $GL(R)$ as the group of all invertible infinite matrices which are "eventually" the identity. We will always identify an invertible $n \times n$ matrix with its image in $GL(R)$. In particular, if A and B are invertible matrices, their product in $GL(R)$ makes sense even if their sizes are different.

We next define an important subgroup of $GL(R)$, the subgroup generated by elementary matrices.

Definition 12.5. $E(R) \subset GL(R)$ is the subgroup generated by the *elementary matrices*, i.e. the matrices of the form:

$$I + rE_{ij} \quad (i \neq j)$$

where I is the identity matrix, E_{ij} is the matrix with 1 in the ij spot and 0's elsewhere, and $r \in R$.

The effect of multiplying a matrix A by the elementary matrix $I + rE_{ij}$ on the right is the column operation which replaces the jth column of A by the sum of the jth column of A and r times the ith column of A. Multiplying A on the left by an elementary matrix performs the corresponding row operation.

Recall that the *commutator subgroup* of a group G is the subgroup $[G, G]$ generated by all *commutators* $ghg^{-1}h^{-1}$ where $g, h \in G$. This is the smallest normal subgroup of G such that the corresponding quotient group is abelian.

Lemma 12.6 (Whitehead lemma)**.** *The group generated by elementary matrices equals the commutator subgroup of* $GL(R)$

$$E(R) = [GL(R), GL(R)].$$

Proof. First, $(I + rE_{ij})^{-1} = I - rE_{ij}$, and

$$E_{ij}E_{k\ell} = \begin{cases} 0 & \text{if } j \neq k \\ E_{i\ell} & \text{if } j = k. \end{cases}$$

Thus if i, j, k are *distinct*,

$$I + rE_{ik} = (I + rE_{ij})(I + E_{jk})(I + rE_{ij})^{-1}(I + E_{jk})^{-1}.$$

So any $n \times n$ elementary matrix with $n \geq 3$ can be expressed as a commutator. Hence $E(R) \subset [GL(R), GL(R)]$.

The opposite inclusion follows from the matrix identities

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & I \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}$$

$$\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A^{-1} & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I - A & I \end{pmatrix}$$

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} = \prod_{i=1}^{m} \prod_{j=1}^{n} (I + x_{ij} E_{i,j+m})$$

which are valid for $A \in GL(m, R)$, $B \in GL(n, R)$, and $X = (x_{ij})$ an $m \times n$ matrix. The identities show that any commutator in $GL(n, R)$ can be expressed as a product of elementary matrices in $GL(2n, R)$. All three identities are easily checked; the last two are motivated by the elementary row operations one would do to transform $\begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}$ and $\begin{pmatrix} I & X \\ 0 & I \end{pmatrix}$ to $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$.

□

So $E(R)$ is a normal subgroup of $GL(R)$ with abelian quotient.

Definition 12.7.

$$K_1(R) = GL(R)/E(R).$$

The quotient homomorphism $GL(R) \to K_1(R)$, $A \mapsto [A]$ should be thought of as a generalized determinant function.

Exercise 240.

1. For a commutative ring R, there is a well-defined map $K_1(R) \to R^\times$, $[A] \mapsto \det$ A, which is a split epimorphism. Here $R^\times$ is the group of units of R, where a unit is an element of R with a two-sided multiplicative inverse.
2. For a field F, show that $K_1(F) \cong F^\times = F - \{0\}$.
3. Show that $K_1(\mathbf{Z}) = \{[(\pm 1)]\} \cong \mathbf{Z}/2$.

Exercise 241. Show that K_1 is a *functor* from the category of rings with 1 to the category of abelian groups.

In fact, for every $n \in \mathbf{Z}$, there is a functor K_n, with the various K_n's intertwined by Künneth theorems. Composing K_n with the functor taking a group π to its integral group ring $\mathbf{Z}\pi$ defines a functor $\pi \mapsto K_n(\mathbf{Z}\pi)$ from the category of groups to the category of abelian groups.

The following equalities in $K_1(R)$ are useful in computations and applications. They are reminiscent of properties of determinants.

Theorem 12.8.

1. *Let $A \in GL(R, m)$, $B \in GL(R, n)$, X be an $m \times n$ matrix, and Y an $n \times m$ matrix. Then*
$$\left[\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}\right] = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] = \left[\begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix}\right] \in K_1(R),$$
$$\left[\begin{pmatrix} A & 0 \\ Y & B \end{pmatrix}\right] = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right] = \left[\begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix}\right] \in K_1(R).$$

2. *Let $P \in GL(n, R)$ be the permutation matrix obtained by permuting the columns of the identity matrix using the permutation $\sigma \in S_n$. Let* $\operatorname{sign}(\sigma) \in \{\pm 1\}$ *be the sign of the permutation. Then*
$$[P] = [\operatorname{sign}(\sigma)] \in K_1(R).$$

Proof. Note
$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix} = \begin{pmatrix} I & XB^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$
and the middle matrix is in $E(R)$ as in the proof of the Whitehead lemma. Likewise
$$\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix}.$$
The last matrix is in $E(R)$ as in the proof of the Whitehead lemma. The first equation in Part 1 above follows. The proof of the second equation is similar.

For Part 2, note
$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$
The last three matrices are in $E(R)$, so we see the assertion is true for 2×2 matrices. For a general 2-cycle σ, the same method shows that P is equivalent to a diagonal matrix with 1's down the diagonal except for a single -1. By Part 1, this is equivalent to the 1×1 matrix (-1).

Every permutation is a product of 2-cycles so the result follows. □

Theorem 12.8 shows that the group operation in the abelian group $K_1(R) = GL(R)/E(R)$ can be thought of either as matrix multiplication
$$([A], [B]) \mapsto [AB]$$
or as *block sum*
$$([A], [B]) \mapsto \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right].$$

The group operation in $K_1(R)$ will be written additively. Hence

$$[A] + [B] = [AB] = \left[\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}\right].$$

Definition 12.9. Define the *reduced K-group*

$$\widetilde{K}_1(R) = K_1(R)/[(-1)].$$

Using this group will allow us to use unordered bases for free modules.

Exercise 242. Let $i : \mathbf{Z} \to R$ be the unique ring map from the integers to R taking the 1 of $\mathbf{Z}$ to the 1 of R. Show that $\widetilde{K}_1(R)$ is the cokernel of the induced map $K_1(\mathbf{Z}) \to K_1(R)$.

Now we switch to group rings. For a group ring, the map $i : \mathbf{Z} \to \mathbf{Z}\pi$ is split by the augmentation map $\varepsilon : \mathbf{Z}\pi \to \mathbf{Z}$, $\sum a_g g \mapsto \sum a_g$. Hence $K_1(\mathbf{Z}\pi) = \mathbf{Z}/2 \oplus \widetilde{K}_1(\mathbf{Z}\pi)$.

If X is a CW-complex with fundamental group π and universal cover $\tilde{X}$, then $C_\bullet(\tilde{X})$ is a free $\mathbf{Z}\pi$-chain complex with generators corresponding to the cells of X. However, the generators are not uniquely determined by the cells; in addition one must choose an orientation and a lift of the cell to the cover. In other words, generators are determined only up to a multiple $\pm g$ where $g \in \pi$. This helps motivate the definition of the Whitehead group.

Definition 12.10. Let E_π be the subgroup of $GL(\mathbf{Z}\pi)$ generated by $E(\mathbf{Z}\pi)$ and 1×1 matrices $(\pm g)$, where $g \in \pi$. Then the *Whitehead group of* π is

$$\mathrm{Wh}(\pi) = GL(\mathbf{Z}\pi)/E_\pi = K_1(\mathbf{Z}\pi)/\{[(\pm g)] : g \in \pi\}.$$

The elements that we mod out by are represented by matrices of the form

$$\begin{pmatrix} \pm g & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \text{ for } g \in \pi.$$

The Whitehead group is a functor from groups to abelian groups. There is a short exact sequence of abelian groups

$$0 \to \{\pm 1\} \times \pi^{ab} \to K_1(\mathbf{Z}\pi) \to \mathrm{Wh}(\pi) \to 0,$$

where $\pi^{ab} = \pi/[\pi, \pi]$. The reason for injectivity is that the composite of the maps

$$\{\pm 1\} \times \pi^{ab} \to K_1(\mathbf{Z}\pi) \to K_1(\mathbf{Z}[\pi^{ab}]) \xrightarrow{\det} \mathbf{Z}[\pi^{ab}]^\times$$

is the inclusion.

The elements of the subgroup $\pm\pi = \{\pm g : g \in \pi\}$ of $(\mathbf{Z}\pi)^\times$ are called the *trivial units* of $\mathbf{Z}\pi$. The ring $\mathbf{Z}\pi$ might contain other units, depending on what π is. To some extent the existence of nontrivial units is measured by the nontriviality of the Whitehead group, but the only precise statement in this direction is that if π is abelian and $\mathbf{Z}\pi$ contains nontrivial units, then the Whitehead group $\mathrm{Wh}(\pi)$ is nontrivial. This uses the fact that the determinant map $K_1(R) \to R^\times$ is a split epimorphism for a commutative ring.

Here are three interesting examples.

1. Let $\mathbf{Z}/5$ have generator t. Then in $\mathbf{Z}[\mathbf{Z}/5]$,
$$(1-t+t^2)(t+t^2-t^4) = 1.$$
Thus $1-t+t^2$ is a nontrivial unit and the Whitehead group is nontrivial. It can be shown that $\mathrm{Wh}(\mathbf{Z}/5)$ is infinite cyclic with this unit as generator.
2. It is easy to see that $\mathbf{Z}[\mathbf{Z}]$ has only trivial units (exercise!). It can be shown that $\mathrm{Wh}(\mathbf{Z}) = 0$.
3. This next example due to Whitehead is a nontrivial unit which represents the zero element of $\mathrm{Wh}(\pi)$.

   Let $\pi = \langle x, y \mid y^2 = 1\rangle = \mathbf{Z} * \mathbf{Z}/2$. Let $a = 1-y$ and $b = x(1+y)$ in $\mathbf{Z}\pi$. Notice $1-ab$ is a nontrivial unit, since $(1-ab)(1+ab) = 1$. However we will show $[(1-ab)]$ is zero in the Whitehead group. It can be shown that $\mathrm{Wh}(\pi) = 0$.

   Note that $(1-y)(x(1+y)) = x + xy - yx - yxy \neq 0$ and also $(x(1+y))(1-y) = x(1-y^2) = 0$. So $ab \neq 0$ and $ba = 0$.

   Then
$$\begin{aligned}\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}^{-1} &= \begin{pmatrix}1 & 0\\ b & 1\end{pmatrix}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ -b & 1\end{pmatrix}\begin{pmatrix}1 & -a\\ 0 & 1\end{pmatrix}\\ &= \begin{pmatrix}1-ab & 0\\ 0 & 1\end{pmatrix}.\end{aligned}$$

   Thus one must *stabilize* (i.e. include the 1×1 matrices into the 2×2 matrices) before $1-ab$ becomes "trivial", i.e. becomes a product of elementary matrices.

The actual computation of Whitehead groups can be a difficult business, involving number theory in the case of finite groups and geometry in the case of infinite groups. We mention a result and a conjecture. The result, due to Bass-Milnor-Serre, is that $\mathrm{Wh}(\mathbf{Z}/n)$ is a free abelian group of rank $[n/2] + 1 - d(n)$ where $d(n)$ is the number of positive divisors of n [**5**]. The conjecture (proven in many cases) is that $\mathrm{Wh}(\pi) = 0$ when π is a torsion-free group.

The next lemma will enable us to remove the dependence on base points when we move to a geometric context. In particular, it shows that the assignment $X \to \mathrm{Wh}(\pi_1 X)$ gives a well-defined functor from the category of path-connected spaces to the category of abelian groups.

Lemma 12.11. *If $f : \pi \to \pi$ is the inner automorphism given by $f(g) = xgx^{-1}$ for some $x \in \pi$, then the induced map on Whitehead groups $f_* : \mathrm{Wh}(\pi) \to \mathrm{Wh}(\pi)$ is the identity map.*

Proof. The automorphism f induces $f : \mathbf{Z}\pi \to \mathbf{Z}\pi$ by the formula

$$f(\sum n_g g) = \sum n_g x g x^{-1} = x(\sum n_g g)x^{-1},$$

which in turn induces a group automorphism $f : GL(n, \mathbf{Z}\pi) \to GL(n, \mathbf{Z}\pi)$ by $A \mapsto (xI)A(x^{-1}I)$. Hence,

$$\begin{aligned} f_*[A] &= [xI \cdot A \cdot x^{-1}I] \\ &= [xI \cdot x^{-1}I \cdot A] \qquad\qquad \text{since } \mathrm{Wh}(\pi) \text{ is abelian} \\ &= [A]. \end{aligned}$$

□

This is reminiscent of the fact that an inner automorphism of π induces the identity on $H_*\pi$, pointing out an analogy between the two functors from groups to abelian groups.

We conclude this section with a remark about matrices over noncommutative rings. If $f : M \to M'$ is an isomorphism of R-modules and if M and M' have bases $\{b_1, \ldots, b_n\}$ and $\{b'_1, \ldots, b'_n\}$ respectively, we wish to define $[f] \in K_1(R)$ to be $[F]$, where F is a matrix representative of f. There are several ways to define a matrix for f, and the result depends on whether we are working with right or left R-modules.

For our main application, the modules we take are the cellular n-chains on the universal cover of a CW-complex X. These are right $\mathbf{Z}\pi$-modules. For that reason we consider right R-modules and define the *matrix* of a map $f : M \to M'$ of right based R-modules to be (f_{ij}) where

$$f(b_i) = \sum_{j=1}^{n} b'_j f_{ij}.$$

With this definition, assigning a matrix to a map of right based R-modules

$$(-) : \mathrm{Hom}_R(M, M') \to M_n(R)$$

is a homomorphism; i.e. $(f + g) = (f) + (g)$ and $(fg) = (f)(g)$ and taking the equivalence class defines a *homomorphism*

$$\mathrm{Iso}_R(M, M') \to K_1(R), \ f \mapsto [(f_{ij})].$$

We will write $[f] = [(f_{ij})] \in K_1(R)$.

12.3. Torsion for chain complexes

We next make the transition from matrices to acyclic, based chain complexes. A *based* R-module is a free, finite-dimensional R-module with a specified basis. A chain complex $C_\bullet$ over a ring R is *bounded* if there exists an N so that $C_n = 0$ for $|n| > N$, *bounded below* if there exists an N so that $C_n = 0$ for $n < N$, *based* if each C_n is based and $C_\bullet$ is bounded, *free* if each C_n is free, *projective* if each C_n is projective, *finite* if $\oplus C_n$ is finitely generated, and *acyclic* if the homology of $C_\bullet$ is zero. We will often write C instead of $C_\bullet$. As above we assume that the ring R has the property that $R^m \cong R^n$ implies that $m = n$. For example, a group ring $\mathbf{Z}\pi$ has this property since it maps epimorphically to $\mathbf{Z}$.

Let $\widetilde{K}_1(R) = K_1(R)/[(-1)]$ where $(-1) \in GL(1, R)$. An isomorphism $f : M' \to M$ of based R-modules determines an element $[f] \in \widetilde{K}_1(R)$. (The reason that we use $\widetilde{K}_1$ rather than K_1 is that it is both messy and unnecessary for us to fuss with ordered bases.)

We wish to generalize $[f]$ in two ways. First, we wish to replace M and M' by chain complexes. Given a chain isomorphism $f : C' \to C$ between based chain complexes, define the *torsion of* f by

$$\tau(f) = \sum (-1)^n [f_n : C'_n \to C_n] \in \widetilde{K}_1(R).$$

The second way we will generalize $[f]$ is to consider $f : M' \to M$ as an acyclic, based chain complex

$$\cdots \to 0 \to M' \xrightarrow{f} M \to 0 \to \cdots .$$

Then we will have $[f] = \pm\tau(f)$.

The following theorem gives an axiomatic characterization of the torsion $\tau(C)$ of an acyclic, based chain complex. Its proof will be an easy consequence of Theorem 12.14 discussed below.

Theorem 12.12. *Let $\mathcal{C}$ be the class of acyclic, based chain complexes over R. Then there is a unique map $\mathcal{C} \to \widetilde{K}_1(R), C \mapsto \tau(C)$ satisfying the following axioms:*

1. *If $f : C \to C'$ is a chain isomorphism, then $\tau(f) = \tau(C') - \tau(C)$.*
2. *$\tau(C \oplus C') = \tau(C) + \tau(C')$.*
3. *$\tau(0 \to C_n \xrightarrow{f} C_{n-1} \to 0) = (-1)^{n-1}[f]$.*

Definition 12.13. For an R-module M and an integer n, define the *elementary chain complex* $E(M, n)$

$$E(M,n)_i = \begin{cases} 0 & \text{if } i \neq n, n-1 \\ M & \text{if } i = n, n-1, \end{cases}$$

and with all differentials zero except $\partial_n = \mathrm{Id} : E(M,n)_n \to E(M,n)_{n-1}$. A *simple chain complex* is a finite direct sum of elementary chain complexes of the form $E(R^{k_n}, n)$.

For example, if K collapses to L by an elementary collapse, then $C(\tilde{K}, \tilde{L})$ is an elementary chain complex with $M = R = \mathbf{Z}\pi$. If K collapses to L, then $C(\tilde{K}, \tilde{L})$ is a simple chain complex.

Note that a simple chain complex is an acyclic, based complex. It is of the shape pictured below.

$$\begin{array}{ccccccccccc} & R^{k_n} & \xrightarrow{\mathrm{Id}} & R^{k_n} & & R^{k_{n-2}} & \xrightarrow{\mathrm{Id}} & R^{k_{n-2}} & \\ & \oplus & & \oplus & & \oplus & & \oplus & \\ \xrightarrow{\mathrm{Id}} & R^{k_{n+1}} & & R^{k_{n-1}} & \xrightarrow{\mathrm{Id}} & R^{k_{n-1}} & & R^{k_{n-3}} & \xrightarrow{\mathrm{Id}} \end{array}$$

Theorem 12.14.

1. *Let $f : C \to C'$ be a chain isomorphism between simple chain complexes. Then $\tau(f) = 0 \in \widetilde{K}_1(R)$.*
2. *Let C be a finite, free, acyclic chain complex. There are simple chain complexes E and F and a chain isomorphism $f : E \to C \oplus F$.*

Corollary 12.15. *Let C be an acyclic, based chain complex. If E, F, E', F' are simple chain complexes and if $f : E \to C \oplus F$ and $g : E' \to C \oplus F'$ are chain isomorphisms, then $\tau(f) = \tau(g)$.*

Proof. Consider the three chain isomorphisms

$$\begin{aligned} \tilde{f} &= f \oplus \mathrm{Id}_{F'} : E \oplus F' \to C \oplus F \oplus F' \\ \tilde{p} &= \mathrm{Id}_C \oplus s : C \oplus F \oplus F' \to C \oplus F' \oplus F \\ \tilde{g} &= g \oplus \mathrm{Id}_F : E' \oplus F \to C \oplus F' \oplus F \end{aligned}$$

where $s : F \oplus F' \to F' \oplus F$ is the obvious switch map. We then have

$$\begin{aligned} 0 &= \tau(\tilde{g}^{-1} \circ \tilde{p} \circ \tilde{f}) && \text{by Theorem 12.14, Part 1} \\ &= \tau(\tilde{g}^{-1}) + \tau(\tilde{p}) + \tau(\tilde{f}) \\ &= -\tau(g) + \tau(f) && \text{Theorem 12.8, Part 2 shows that } \tau(\tilde{p}) = 0. \end{aligned}$$

□

We can now use the previous theorem and corollary to define torsion.

Definition 12.16. Let C be an acyclic, based complex. Define the *torsion of C* by

$$\tau(C) = \tau(f : E \to C \oplus F)$$

where E, F are simple chain complexes and f is a chain isomorphism.

Exercise 243. Prove Theorem 12.12 assuming Theorem 12.14 and Corollary 12.15.

What remains is to prove Theorem 12.14. We strongly advise you to put down this book and prove the theorem by yourself, assuming (at first) that all stably free modules are free.

Welcome back! The proof that we give for Part 1 is the same as you found, but the proof we will give for Part 2 uses the fundamental lemma of homological algebra and is much slicker and less illuminating than the inductive proof you figured out.

We separate out Part 1 as a lemma.

Lemma 12.17. *Let $f : C \to C'$ be a chain isomorphism between simple chain complexes. Then $\tau(f) = 0 \in \widetilde{K}_1(R)$.*

Proof. Write

$$C_n = E(R^{k_n}, n)_n \oplus E(R^{k_{n+1}}, n+1)_n = C'_n.$$

It is easy to see using the fact that f is a chain map that the block matrix form of $f_n : C_n \to C'_n$ is

$$\begin{pmatrix} A_n & 0 \\ B_n & A_{n+1} \end{pmatrix}.$$

Then

$$\begin{aligned} \tau(f) &= \sum (-1)^n \left[\begin{pmatrix} A_n & 0 \\ B_n & A_{n+1} \end{pmatrix} \right] && \text{definition of } \tau(f) \\ &= \sum (-1)^n ([A_n] + [A_{n+1}]) && \text{by Theorem 12.8} \\ &= 0. \end{aligned}$$

□

Before we prove Part 2 we need some preliminaries.

Definition 12.18. A *chain contraction* $s : C \to C$ for a chain complex C is a sequence of maps $s_n : C_n \to C_{n+1}$ satisfying $\partial_{n+1} s_n + s_{n-1} \partial_n = \mathrm{Id}_{C_n}$.

A chain contraction is a chain homotopy between the identity map and the zero map. If C has a chain contraction, then $H_*(C) = 0$.

Proposition 12.19. *Let $s : C \to C$ be a chain contraction. Let $B_n = \partial(C_{n+1}) \subset C_n$.*

1. $C_n = B_n \oplus s(B_{n-1})$.
2. $\partial : s(B_n) \to B_n$ *is an isomorphism with inverse* $s : B_n \to s(B_n)$.
3. C *is isomorphic to the direct sum of chain complexes* $\oplus_n E(B_n, n+1)$.

Proof. Consider the short exact sequence

$$0 \to B_n \hookrightarrow C_n \xrightarrow{\partial} B_{n-1} \to 0.$$

The formula $x = \partial s(x) + s\partial(x)$ is valid for all x. So if $x \in B_{n-1}$, then $\partial(x) = 0$ and hence $\partial s(x) = x - s\partial(x) = x$. Therefore, this short exact sequence is split by $s_{n-1} : B_{n-1} \to C_n$, proving the first assertion. Similarly, if $y \in B_n$, $\partial s(y) = y$ and $s\partial s(y) = s(y)$, which implies the second assertion.

The map

$$\begin{aligned} C_n = B_n \oplus s(B_{n-1}) &\to B_n \oplus B_{n-1} = E(B_n, n+1)_n \oplus E(B_{n-1}, n)_n \\ a \oplus b &\mapsto a \oplus \partial b \end{aligned}$$

is an isomorphism by the second assertion. It is easy to check that this is a chain map.

□

Lemma 12.20.

1. *If C is projective, acyclic, and bounded below, then C has a chain contraction.*
2. *If C is finite, free, and acyclic, the modules $s(B_n)$ are stably free for all n.*

Proof.

1. By reindexing if necessary, assume that $C_n = 0$ for n negative. Then by the fundamental lemma of homological algebra (Theorem 3.22), the identity and the zero map are chain maps from C to C inducing the same map on H_0, hence are chain homotopic.
2. By Proposition 12.19, $C_n \cong B_n \oplus B_{n-1}$. Using induction on n, one sees B_n is stably free for all n.

□

Proof of Theorem 12.14. We have already proven Part 1.

Let C be a finite, free, acyclic chain complex. Then there is a chain contraction $s : C \to C$ by Lemma 12.20, and hence C is chain isomorphic to $\oplus_n E(B_n, n+1)$ by Proposition 12.19. Now for every n there is a finitely generated free module F_n so that $B_n \oplus F_n$ is free. Give it a finite basis. Then $C \oplus (\oplus_n E(F_n, n+1))$ is chain isomorphic to the simple complex $\oplus_n E(B_n \oplus F_n, n+1)$. □

Exercise 244. Show that for an acyclic, based chain complex C with a chain contraction s, $s + \partial : C_{\text{odd}} \to C_{\text{even}}$ is an isomorphism and $[s+\partial] = \tau(C) \in \widetilde{K}_1(R)$. Here $C_{\text{odd}} = \oplus C_{2i+1}$ and $C_{\text{even}} = \oplus C_{2i}$. This is called "wrapping up" the chain complex and is the approach to torsion used in [**10**].

An isomorphism of based R-modules $f : M \to M'$ determines an element $[f] \in \widetilde{K}_1(R)$. We generalized this in two ways: to $\tau(f)$ for a chain isomorphism between based chain complexes, and to $\tau(C)$ for an acyclic, based chain complex. We wish to generalize further and define $\tau(f)$ for $f : C \to C'$ a chain homotopy equivalence between based complexes. We need some useful constructs from homological algebra.

Definition 12.21. Let $f : C \to C'$ be a chain map between chain complexes. Define the *algebraic mapping cone of* f to be the chain complex $C(f)$ where

$$C(f)_n = C_{n-1} \oplus C'_n$$
$$\partial = \begin{pmatrix} -\partial & 0 \\ f & \partial' \end{pmatrix} : C(f)_n \to C(f)_{n-1}.$$

Define the *algebraic mapping cylinder of* f to be the chain complex $M(f)$ where

$$M(f)_n = C_{n-1} \oplus C_n \oplus C'_n$$
$$\partial = \begin{pmatrix} -\partial & 0 & 0 \\ -\mathrm{Id} & \partial & 0 \\ f & 0 & \partial' \end{pmatrix} : M(f)_n \to M(f)_{n-1}.$$

For a chain complex C, define the *cone on* C

$$\mathrm{Cone}(C) = C(\mathrm{Id} : C \to C),$$

the *cylinder on* C

$$\mathrm{Cyl}(C) = M(\mathrm{Id} : C \to C),$$

and the *suspension of* C, which is the chain complex SC where $(SC)_n = C_{n-1}$ and $\partial_{SC}(x) = -\partial_C(x)$. Note $H_n(SC) = H_{n-1}C$.

If the chain complexes involved are based, then $C(f)$, $M(f)$, $\mathrm{Cone}(C)$, $\mathrm{Cyl}(C)$, and SC have obvious bases.

All of these constructions are interrelated. There are short exact sequences of chain complexes

$$0 \to C' \to C(f) \to SC \to 0 \tag{12.1}$$

$$0 \to C \to M(f) \to C(f) \to 0 \tag{12.2}$$

$$0 \to C' \to M(f) \to \mathrm{Cone}(C) \to 0. \tag{12.3}$$

Here is some geometric motivation. If $f : X \to Y$ is a cellular map between CW-complexes, and $f_\bullet : C_\bullet(X) \to C_\bullet(Y)$ is the associated cellular chain map, then the mapping cone $C(f)$, the mapping cylinder $M(f)$, the reduced cone CX, and the reduced suspension SX all have CW-structures. One can make the following identifications:

$$\begin{aligned} C(f_\bullet) &= C_\bullet(C(f), \mathrm{pt}) \\ M(f_\bullet) &= C_\bullet(M(f)) \\ \mathrm{Cone}(C_\bullet(X)) &= C_\bullet(CX, \mathrm{pt}) \\ \mathrm{Cyl}(C_\bullet(X)) &= C_\bullet(I \times X) \\ S(C_\bullet(X)) &= C_\bullet(SX, \mathrm{pt}). \end{aligned}$$

The exact sequence (12.1) gives a long exact sequence in homology

$$\cdots \to H_nC \to H_nC' \to H_n(C(f)) \to H_{n-1}C \to \cdots$$

and one can check easily that the map $H_nC \to H_nC'$ is just the map induced by f. In particular, if f induces an isomorphism in homology, then $C(f)$ is acyclic.

Definition 12.22. A chain map $f : C \to C'$ is a *quasi-isomorphism* if it induces an isomorphism on homology. If f is a *quasi-isomorphism* between finite, based chain complexes, then $C(f)$ is a finite, acyclic, based chain complex. Define

$$\tau(f) = \tau(C(f)).$$

Exercise 245. If $f : C \to C'$ is a chain isomorphism of finite, based complexes, we unfortunately have two different definitions of the torsion $\tau(f)$: as $\sum(-1)^n[f_n : C_n \to C'_n]$ and as $\tau(C(f))$. Show that they coincide.

A quasi-isomorphism is often called a weak homotopy equivalence. The justification for this term is given by the following algebraic analogue of Whitehead's theorem.

Exercise 246. If $f : C \to C'$ is a *quasi-isomorphism* between projective chain complexes which are bounded below, then $C(f)$ has a chain contraction and f is a chain equivalence; i.e. there is a chain map $g : C' \to C$ so that $f \circ g$ and $g \circ f$ are chain homotopic to identity maps.

Theorem 12.23. *Let*

$$0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$$

be a short exact sequence of acyclic, based chain complexes. Assume the bases are compatible, which means that for every n, the basis of C_n is of the form

$$\{i(b'_1), i(b'_2), \ldots, i(b'_{i_j}), c''_1, c''_2, \ldots, c''_{i_k}\} \subset C_n$$

where

$$\{b'_1, b'_2, \ldots, b'_{i_j}\} \subset C'_n$$

is the given basis for C'_n and

$$\{p(c''_1), p(c''_2), \ldots, p(c''_{i_k})\} \subset C''_n$$

is the given basis for C''_n. Then $\tau(C) = \tau(C') + \tau(C'')$.

Lemma 12.24. *Let*

$$0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$$

be a short exact sequence of free chain complexes which are bounded below. If i is a weak homotopy equivalence, then the sequence splits. Hence $C \cong C' \oplus C''$.

Proof. Following [**10**, pg. 48], we construct a chain map $t : C'' \to C$ which splits p. Since C'' is free we can find a sequence of homomorphisms $\sigma_k : C''_k \to C_k$ which split $C_k \to C''_k$ as R-modules. Write $\sigma : C'' \to C$ for the sum of the σ_k. We turn σ into a chain map t as follows.

Since i is a *quasi-isomorphism*, the long exact sequence in homology shows that C'' is acyclic. Let $\delta'' : C'' \to C''$ be a chain contraction for C'', guaranteed to exist by Lemma 12.20. Let $t = \partial\sigma\delta'' + \sigma\delta''\partial''$. Note that $\partial t = \partial\sigma\delta''\partial'' = t\partial''$, so t is a chain map. Also, $pt = p\partial\sigma\delta'' + \delta''\partial'' = \partial'' p\sigma\delta'' + \delta''\partial'' = \partial''\delta'' + \delta''\partial'' = \mathrm{Id}_{C''}$, so t is a chain map which splits p. □

Proof of Theorem 12.23. Let $f : C \to C' \oplus C''$ be the chain isomorphism given by Lemma 12.24. Since f is a chain isomorphism,

$$\begin{aligned} \tau(f) &= \tau(C' \oplus C'') - \tau(C) \\ &= \tau(C') + \tau(C'') - \tau(C). \end{aligned}$$

On the other hand, since the bases are compatible, then the matrix of f_n is $\begin{pmatrix} \mathrm{Id} & * \\ 0 & \mathrm{Id} \end{pmatrix}$ in block matrix form after partitioning each basis into its $'$ part and its $''$ part. But such a matrix is trivial in $\widetilde{K}_1(R)$ by Theorem 12.8. □

Corollary 12.25. *If $f : C \to C'$ is a chain map of acyclic, based chain complexes, then $\tau(f) = \tau(C') - \tau(C)$.*

This follows from Theorem 12.23 and the short exact sequence (12.1).

We wish to prove homotopy invariance and additivity of torsion. The next lemma is a key ingredient.

Lemma 12.26. *If C is a based chain complex, then* Cone(C) *is an acyclic, based chain complex with trivial torsion.*

Proof. Cone(C) is finite with an obvious basis; we will show Cone(C) is acyclic by induction on the total rank $\sum \dim C_i$. If the total rank is 1, both assertions are clear.

Suppose C is a based chain complex of total rank $n > 1$. Let b be a basis element of C of minimal degree. Let $C' = \{\cdots \to 0 \to Rb \to 0 \cdots \to\}$ be the corresponding subcomplex of C and let $C'' = C/C'$ be the quotient complex. There is a short exact sequence of finite complexes

$$0 \to C' \to C \to C'' \to 0$$

with compatible bases. It is easy to see there is a short exact sequence of finite complexes

$$0 \to \mathrm{Cone}(C') \to \mathrm{Cone}(C) \to \mathrm{Cone}(C'') \to 0$$

with compatible bases. By induction, Cone(C') and Cone(C'') are acyclic with trivial torsion. By the long exact sequence in homology, Cone(C) is acyclic, and by Theorem 12.23, Cone(C) has trivial torsion. $\square$

Theorem 12.27. *Let C and D be based chain complexes and let $f, g : C \to D$ be weak homotopy equivalences which are chain homotopic. In symbols,*

$$f \simeq g : C \xrightarrow{\sim} D.$$

Then $\tau(f) = \tau(g)$.

Proof. Let $s = \{s_n : C_n \to D_{n+1}\}$ be a chain homotopy from f to g satisfying $s\partial + \partial s = f - g$. Then $F = \begin{pmatrix} s & f & g \end{pmatrix} : \mathrm{Cyl}(C) \to D$ is a chain map. There is a short exact sequence

$$0 \to C(f) \to C(F) \to S(\mathrm{Cone}(C)) \to 0 \tag{12.4}$$

of chain complexes. For an acyclic, based complex B, $\tau(S(B)) = -\tau(B)$. Hence $\tau(f) = \tau(F)$ by Lemma 12.26 and Theorem 12.23. Likewise $\tau(g) = \tau(F)$. $\square$

Exercise 247. Use Definition 12.21 to construct the sequences (12.4), (12.1), (12.2), and (12.3) and verify that that they are short exact sequences of chain complexes.

Exercise 248. Let X and Y be finite complexes and f and g be cellular homotopy equivalences from X to Y which are homotopic. Show that the mapping cylinder of f is simple-homotopy equivalent to the mapping cylinder to g.

Finally, there is an additivity property of torsion.

Theorem 12.28. *Let $f : C \to C'$ and $g : C' \to C''$ be quasi-isomorphisms between based chain complexes. Then*

$$\tau(g \circ f) = \tau(f) + \tau(g).$$

The idea is to convert g to an inclusion and analyze what happens. To this end we need a definition and a lemma.

Definition 12.29. A chain map $f : C \to C'$ between based complexes is a *based injection* if for all n, the map $f : C_n \to C'_n$ is an injection and f applied to the basis of C_n is a subset of the basis of C'_n.

Lemma 12.30.

1. *Let $g : C' \to C''$ be a based injection which is a quasi-isomorphism. Then $\tau(g) = \tau(C''/C')$.*
2. *Let $f : C \to C'$ and $g : C' \to C''$ be quasi-isomorphisms between based chain complexes. If g is a based injection, then*

$$\tau(g \circ f) = \tau(f) + \tau(C''/C').$$

Proof. We prove Part 2 first. There is a based injection

$$\begin{pmatrix} \mathrm{Id} & 0 \\ 0 & g_n \end{pmatrix} : C(f)_n = C_{n-1} \oplus C'_n \to C(g \circ f)_n = C_{n-1} \oplus C''_n.$$

This is part of a short exact sequence of chain complexes

$$0 \to C(f) \to C(g \circ f) \to C''/C' \to 0$$

with compatible bases. The result follows by Theorem 12.23. Part 1 is a special case of Part 2 taking $f = \mathrm{Id}$ and applying Theorem 12.23 and Lemma 12.26. □

Proof of Theorem 12.28. As advertised, we convert g to an inclusion and consider chain maps

$$C \xrightarrow{g \circ f} C'' \hookrightarrow M(g).$$

Then

$$
\begin{aligned}
\tau(C \to M(g)) &= \tau(g \circ f) + \tau(M(g)/C'') && \text{by Lemma 12.30}\\
&= \tau(g \circ f) + \tau(\mathrm{Cone}(g)) && \text{by (12.3)}\\
&= \tau(g \circ f) && \text{by Lemma 12.26.}
\end{aligned}
$$

Finally, we consider the chain maps

$$C \xrightarrow{f} C' \hookrightarrow M(g),$$

and see

$$
\begin{aligned}
\tau(g \circ f) &= \tau(C \to M(g)) && \text{we just proved this}\\
&= \tau(f) + \tau(M(g)/C') && \text{by Lemma 12.30}\\
&= \tau(f) + \tau(C(g)) && \text{by (12.2)}\\
&= \tau(f) + \tau(g).
\end{aligned}
$$

□

Perhaps the homotopy invariance and additivity of torsion are analogous to the homotopy invariance and functoriality of homology.

12.4. Whitehead torsion for CW-complexes

Let K be a finite CW-complex. Assume that K is connected. Let $x_0 \in K$ and let $\pi = \pi_1(K, x_0)$. We identify π with the group of covering transformations of the universal cover $\tilde{K} \to K$ in the usual way. We have seen (in Chapter 6) that $C_\bullet(\tilde{K})$ is a free $\mathbf{Z}\pi$-chain complex. A basis of this chain complex is obtained by choosing a lift $\tilde{e} \subset \tilde{K}$ for each cell e of K and choosing an orientation of e or, equivalently, $\tilde{e}$. The set of lifts of cells of K with the chosen orientations defines a basis over $\mathbf{Z}\pi$ for the free $\mathbf{Z}\pi$-chain complex $C_\bullet(\tilde{K})$.

Now suppose that $f : K \to L$ is a homotopy equivalence of finite CW-complexes. We can homotop f to a cellular map $g : K \to L$, which in turn defines a *quasi-isomorphism*of based $\mathbf{Z}\pi$-chain complexes

$$g_\bullet : C_\bullet(\tilde{K}) \to C_\bullet(\tilde{L}).$$

Hence we have all the data needed to define torsion as in the previous section. Define

$$\tau(f) = \tau(g_\bullet) \in \mathrm{Wh}(\pi) = \widetilde{K}_1(\mathbf{Z}\pi)/\pm\pi.$$

The main geometric result of simple-homotopy theory is the following.

Theorem 12.31. *Let $f : K \to L$ be a homotopy equivalence of finite CW-complexes. Define the torsion $\tau(f)$ as above.*

1. *The torsion $\tau(f)$ is well-defined, independent of choice of orientations, lifts, base point x_0, identification of π with the group of covering transformations, and cellular approximation g.*
2. *If f is a simple-homotopy equivalence, then $\tau(f) = 0$.*
3. *If $\tau(f) = 0$, then f is a simple-homotopy equivalence.*

Proof. We give complete proofs of Part 1 and 2, but only the vaguest of sketches for the proof of Part 3.

Changing the lift and orientation of a cell replaces $\tilde{e}$ by $\pm\gamma\tilde{e}$ for some $\gamma \in \pi$. Thus the torsion changes by the change of basis matrix

$$\left[\begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & \pm\gamma & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}\right] = 0 \in \mathrm{Wh}(\pi).$$

The choice of base point and the identification of π with the group of covering transformations are dealt with by Lemma 12.11, which says that conjugation in π induces the identity map on $\mathrm{Wh}(\pi)$. Independence of the choice of cellular approximation follows from Theorem 12.27, the homotopy invariance of torsion.

Next we need to show that if $f : K \to L$ is a simple-homotopy equivalence, then $\tau(f) = 0$. Now f is a simple-homotopy equivalence if f is homotopic to a composite $K_0 \to K_1 \to K_2 \to \cdots \to K_n$ where each map $K_i \to K_{i+1}$ is either the inclusion map of an elementary expansion, the deformation retraction of an elementary collapse, or a cellular homeomorphism. We analyze the pieces. A cellular homeomorphism clearly has trivial torsion. If $i : A \hookrightarrow B$ is the inclusion map of an elementary collapse, then $\tau(i) = \tau(C(i_\bullet : C_\bullet\tilde{A} \hookrightarrow C_\bullet\tilde{B}))$ which is $\tau(C_\bullet(\tilde{B}, \tilde{A}))$ by Lemma 12.30. But $C_\bullet(\tilde{B}, \tilde{A})$ is an elementary chain complex, so has trivial torsion. Finally, if $d : B \to A$ is the associated deformation retract, then

$$\begin{aligned} \tau(d) &= \tau(d_\bullet : C_\bullet\tilde{B} \to C_\bullet\tilde{A}) \\ &= \tau(d_\bullet \circ i_\bullet) - \tau(i_\bullet) && \text{additivity of torsion} \\ &= \tau(\mathrm{Id}_{C_\bullet(\tilde{A})}) - \tau(i_\bullet) && \text{homotopy invariance of torsion} \\ &= 0. \end{aligned}$$

The composite $K_0 \to K_1 \to K_2 \to \cdots \to K_n$ must have trivial torsion since all the pieces do. Thus $\tau(f) = 0$.

For the proof of Part 3, suppose that $f : K \to L$ is a cellular map between finite complexes with trivial torsion. Then f factors as $K \hookrightarrow M(f) \to L$. The second map is a collapse map and hence is a simple-homotopy equivalence and has trivial torsion. By the additivity of torsion, $\tau(K \hookrightarrow M(f)) = 0$, and it suffices to show that this map is a simple-homotopy equivalence.

Recycling our notation, we will assume K is a subcomplex of L and that the torsion of the inclusion map is trivial. The first step in showing that $K \hookrightarrow L$ is a simple-homotopy equivalence is *cell-trading* [**10**, 7.3]. If e is a cell of $L - K$ of minimal dimension (say k), one constructs a simple-homotopy equivalence $L \to L'$ rel K so that L' has one fewer k-cell than L, one more $(k+2)$-cell, and for $i \neq k, k+2,$ the number of i-cells of L and of L' is the same. By a simple-homotopy equivalence $h : L \to L'$ rel K, we mean that K is a subcomplex of both L and L' and that h restricted to K is the identity. By continuing to trade, one reduces to proving that an inclusion $i : K \hookrightarrow L$ with trivial torsion is a simple-homotopy equivalence when the cells of $L - K$ all are in two adjoining dimensions, say n and $n+1$. Then the chain complex $C_\bullet(\tilde{L}, \tilde{K})$ is described by a matrix! We can stabilize the matrix, if desired, by making expansions. Since the torsion is zero, we may assume that the matrix is a product of elementary matrices. There is a technique called *cell-sliding* [**10**, 8.3] (changing the attaching map of an $(n+1)$-cell) which gives a simple-homotopy equivalence $L \to L'$ rel K so that the matrix in the chain complex is replaced by the matrix multiplied by an elementary matrix. Thus one reduces to the case where $K \hookrightarrow L$ has the chain complex

$$C_\bullet(\tilde{L}, \tilde{K}) = \{\cdots \to 0 \to \mathbf{Z}[\pi]^m \xrightarrow{\text{Id}} \mathbf{Z}[\pi]^m \to 0 \to \cdots\}$$

in which case there is one last technique, *cell-cancellation* [**10**, 8.2], which says that $K \hookrightarrow L$ is a simple-homotopy equivalence. $\square$

Two finite CW-complexes are said to have the same *simple-homotopy type* if there is a simple-homotopy equivalence between them. Homeomorphic CW-complexes could have drastically different CW-structures. Do they have the same simple-homotopy type? A fundamental theorem of Chapman implies that simple-homotopy type is a homeomorphism invariant:

Theorem 12.32 (Chapman [**9**]). *If $f : X \to Y$ is a homeomorphism between finite CW-complexes, then f is a simple-homotopy equivalence.* $\square$

It follows that the torsion of a homotopy equivalence $f : X \to Y$ between finite CW-complexes depends only on the underlying topological spaces.

How does simple-homotopy theory apply to manifolds? Typically, a smooth manifold is given the structure of a simplicial complex (and hence a CW-complex) by constructing a triangulation. This triangulation is unique up to subdivision, and it is not difficult to show [**34**] that the identity map between a complex and a subdivision has trivial torsion. Thus compact smooth manifolds have a well-defined simple-homotopy type. A deep theorem of Kirby-Siebenmann shows that this theory also applies for topological manifolds (Hausdorff spaces which are locally Euclidean). They show that a compact topological manifold has a canonical simple-homotopy type.

In the next section we will define Reidemeister torsion for certain finite CW-complexes. This will be an interesting and computable invariant of simple-homotopy type. If two manifolds have different Reidemeister torsions, by Chapman's theorem they cannot be homeomorphic.

12.5. Reidemeister torsion

Suppose that (C, ∂) is a finite, based (with basis $\{b_i\}$) chain complex, not necessarily acyclic, over a ring S. Let $f : S \to R$ be a ring homomorphism. Then $(C \otimes_S R, \partial \otimes \mathrm{Id}_R)$ is a finite, based (with basis $\{b_i \otimes 1\}$) chain complex over R.

If (C, ∂) is acyclic with chain contraction s, then $(C \otimes_S R, \partial \otimes \mathrm{Id}_R)$ is acyclic since it has the chain contraction $s \otimes \mathrm{Id}_R$. The torsions are related by

$$\tau(C \otimes_S R) = f_* \tau(C)$$

where $f_* : \widetilde{K}_1(S) \to \widetilde{K}_1(R)$ is the induced homomorphism. However, in many interesting cases it may happen that $C \otimes_S R$ is acyclic although C is not, so that $\tau(C \otimes_S R)$ may be defined even though $\tau(C)$ is not.

Moreover, if R is a commutative ring, or better yet, a field, then the determinant defines a homomorphism (an isomorphism if R is a field) $\det : \widetilde{K}_1(R) \to R^\times / \pm 1$ (see Exercise 240).

Definition 12.33. Let C be a based chain complex over a ring S, and $f : S \to R$ be a ring homomorphism to a commutative ring. Suppose that $C \otimes_S R$ is acyclic. Then

$$\Delta_R(C) = \det(\tau(C \otimes_S R)) \in R^\times / \pm 1$$

is called the *Reidemeister torsion of C with respect to $f : S \to R$*. Since $\Delta_R(C)$ is a unit in R we use multiplicative notation for Reidemeister torsion.

Let X be a CW-complex. Let R be some ring, which for our purposes may be taken to be commutative. Suppose

$$\rho : \mathbf{Z}\pi_1 X \to R$$

is a ring homomorphism. The chain complex

$$C_\bullet(\tilde{X})\otimes_{\mathbf{Z}\pi_1 X} R$$

was used in Chapter 6 to define homology with twisted coefficients $H_*(X;R)$.

A choice of lifts and orientations of cells makes this a based (right) R-complex. If it happens to be acyclic, then the torsion $\tau \in \widetilde{K}_1(R)$ is defined, and so we can take the Reidemeister torsion

$$\Delta_R(X) = \Delta_R(C_\bullet X) \in R^\times / \pm 1$$

to be the determinant of τ.

The following exercise shows how to remove the dependence of Δ_R on the choice of lifts, orientation, and ordering of cells.

Exercise 249. Let $G \subset R^\times$ be the subgroup generated by -1 and $\det(\rho(\gamma))$ for $\gamma \in \pi_1 X$, where $\rho : \mathbf{Z}\pi_1 X \to R$ is the given ring homomorphism. Show that the Reidemeister torsion $\Delta_R(X)$, taken in $R^\times/G$, is well-defined, independent of the choice of lifts, orientation, or ordering of cells.

We usually abuse notation and consider Reidemeister torsion as an element of $R^\times$, and omit mentioning that it is only well-defined up to multiplication by an element of G.

The disadvantage of Reidemeister torsion is that it requires a map ρ so that $C_\bullet(\tilde{X}) \otimes_{\mathbf{Z}\pi} S$ is acyclic. Perhaps for a given X, no useful map exists. The advantage is that when ρ does exist, the Reidemeister torsion gives an invariant of the space. The following proposition is a corollary of Corollary 12.25.

Proposition 12.34. *Let $f : X \to Y$ be a homotopy equivalence between finite, connected CW-complexes. Let $\pi = \pi_1 X = \pi_1 Y$. Suppose $\rho : \mathbf{Z}\pi \to R$ is a ring homomorphism to a commutative ring so that $C_\bullet(\tilde{X}) \otimes_{\mathbf{Z}\pi} R$ and $C_\bullet(\tilde{Y}) \otimes_{\mathbf{Z}\pi} R$ are acyclic. Then*

$$\rho_*\tau(f) = \Delta_R(Y)/\Delta_R(X) \in R^\times/G.$$

Therefore Reidemeister torsion is an invariant capable of distinguishing simple-homotopy type from homotopy type and of showing that two homotopy equivalent spaces are not homeomorphic. In the next section we will apply this idea to lens spaces.

Reidemeister torsion can also lead to interesting invariants, for example, the Alexander polynomial of a knot. Let $K \subset S^3$ be a knot, that is, a smooth submanifold of S^3 diffeomorphic to S^1. Let $X = S^3 - N(K)$, where $N(K)$ is an open tubular neighborhood of K. Alexander duality implies that $H_1X \cong \mathbf{Z}$, and so every knot has a canonical (up to multiplication by ± 1) homomorphism $a : \pi_1 X \to H_1 X \to \mathbf{Z}$.

Let $R = \mathbf{Q}(t)$, the field of rational functions. Then define $\rho : \mathbf{Z}\pi_1 X \to \mathbf{Q}(t)$ by

$$\rho(\sum a_\gamma \gamma) = \sum a_\gamma t^{a(\gamma)}.$$

It turns out (see the exercise below) that the chain complex

$$C = C_\bullet(\tilde{X}) \otimes_{\mathbf{Z}\pi_1 X} \mathbf{Q}(t)$$

is acyclic. Its Reidemeister torsion $\Delta_{\mathbf{Q}(t)}(X)$ is a nonzero rational function and is well-defined up to sign and powers of t. Moreover $\Delta_{\mathbf{Q}(t)}(X)$ is always of the form

$$\Delta_{\mathbf{Q}(t)}(X) = (t-1)/\Delta_K(t)$$

for some polynomial $\Delta_K(t) \in \mathbf{Z}[t]$. This polynomial is called the *Alexander polynomial of* K. It is a useful invariant for distinguishing isotopy classes of knots.

Exercise 250. Let $\mu \subset X$ be a *meridian* of the knot K, that is, the boundary circle of a small embedded disk in S^3 intersecting K transversely in one point. Show that the inclusion $\mu \hookrightarrow S^3$ induces an isomorphism on integral homology. Conclude that $H_*(X, \mu; \mathbf{Z}) = 0$ and so the cellular chain complex $C_\bullet(X, \mu)$ is acyclic.

Let $s : C_\bullet(X, \mu) \to C_\bullet(X, \mu)$ be a chain contraction. Lift s to a map

$$\tilde{s} : C_\bullet(\tilde{X}, \tilde{\mu}) \otimes_{\mathbf{Z}\pi_1 X} \mathbf{Q}(t) \to C_\bullet(\tilde{X}, \tilde{\mu}) \otimes_{\mathbf{Z}\pi_1 X} \mathbf{Q}(t)$$

and show that $\tilde{s}\partial + \partial\tilde{s}$ is a chain isomorphism. Conclude that the chain complex $C_\bullet(\tilde{X}, \tilde{\mu}) \otimes_{\mathbf{Z}\pi_1(X)} \mathbf{Q}(t)$ is acyclic. The long exact sequence for the pair $(\tilde{X}, \tilde{\mu})$ then shows that $C_\bullet(\tilde{X}) \otimes_{\mathbf{Z}\pi_1 X} \mathbf{Q}(t)$ is acyclic.

12.6. Torsion and lens spaces

In this section we will use Reidemeister torsion to classify 3-dimensional lens spaces up to simple-homotopy and prove Theorem 12.1. The homotopy classification differs from the simple-homotopy classification, but the simple-homotopy classification is the same as the homeomorphism classification, the diffeomorphism classification, and the isometry classification.

Let (p, q) be a pair of relatively prime integers, with $p > 0$. For convenience let r denote an inverse for q mod p, so $rq \equiv 1$ mod p. There are many descriptions of $L(p, q)$. We will give a description which makes the cell structure on its universal cover easy to see.

Let $X = S^1 \times D^2$ be the solid torus, which we parameterize as the subset $\{(z_1, z_2) \mid |z_1| = 1, |z_2| \leq 1\} \subset \mathbf{C}^2$. The quotient of X by the equivalence relation $(z_1, z_2) \sim (z_1', z_2)$ if $|z_2| = 1$ is the 3-sphere S^3. In fact the map $f : X \to S^3$ defined by

$$f(z_1, z_2) = (z_1\sqrt{1 - |z_2|^2}, z_2)$$

defines a homeomorphism from $X/\sim$ to $S^3 \subset \mathbf{C}^2$.

Write $\zeta = e^{2\pi i/p}$ and let $\mathbf{Z}/p$ act on X by

$$(z_1, z_2) \cdot g = (z_1\zeta, z_2\zeta^q)$$

where $\mathbf{Z}/p$ is written multiplicatively as $\langle g \mid g^p = 1\rangle$ (we use a right action to be consistent with the previous sections). This defines the free $\mathbf{Z}/p$-action on $S^3 = X/\sim$

$$(w_1, w_2) \cdot g = (w_1\zeta, w_2\zeta^q).$$

The map $f : X \to S^3$ is an equivariant map. By definition the quotient space $S^3/(\mathbf{Z}/p)$ is the lens space $L(p, q)$.

We use the description $X/\sim$ to construct an equivariant cell structure on S^3. Let $\tilde{e}_0$ be the image under f of the point $(1, 1) \in X$. We take the 0-cells of S^3 to be the p points $\tilde{e}_0, \tilde{e}_0 g, \cdots, \tilde{e}_0 g^{p-1}$. Notice that

$$\tilde{e}_0 g^n = f((1, 1)g^n) = f(\zeta^n, \zeta^{qn}) = f(1, \zeta^{qn}).$$

Next, let $\tilde{e}_1$ be the image under f of the arc $\{(1, e^{i\theta}) \mid 0 \leq \theta \leq \frac{2\pi}{p}\}$, and take as 1-cells the translates $\tilde{e}_1, \tilde{e}_1 g, \cdots, \tilde{e}_1 g^{p-1}$. Then

$$\begin{aligned} \partial\tilde{e}_1 &= f(1, \zeta) - f(1, 1) \\ &= f(1, \zeta^{qr}) - f(1, 1) \\ &= \tilde{e}_0(g^r - 1), \end{aligned} \tag{12.5}$$

since $qr \equiv 1 \bmod p$.

Let $\tilde{e}_2$ be $f(1 \times D^2)$. We take as 2-cells the translates of $\tilde{e}_2$ by g, $\tilde{e}_2, \tilde{e}_2 g, \cdots, \tilde{e}_2 g^{p-1}$. Then

$$\begin{aligned} \partial\tilde{e}_2 &= f(\{(1, e^{i\theta}) \mid 0 \leq \theta \leq 2\pi\}) \\ &= \tilde{e}_1 + \tilde{e}_1 g + \cdots + \tilde{e}_1 g^{p-1} \\ &= \tilde{e}_1(1 + g + \cdots + g^{p-1}). \end{aligned} \tag{12.6}$$

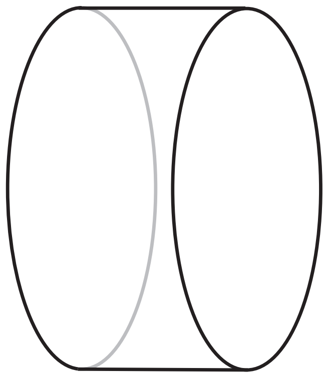
For the 3-cells, consider the solid cylinder

$$[0, \tfrac{2\pi}{p}] \times D^2 \cong \{e^{i\theta} \mid \theta \in [0, \tfrac{2\pi}{p}]\} \times D^2 \subset X.$$

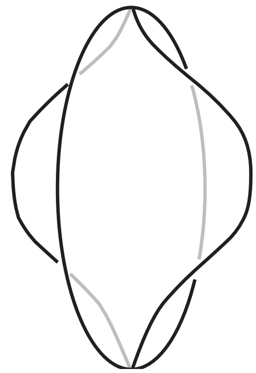
This is homeomorphic to a closed 3-ball. Its image in S^3

$$f([0, \tfrac{2\pi}{p}] \times D^2) = ([0, \tfrac{2\pi}{p}] \times D^2)/\sim$$

is a "lens", also homeomorphic to a closed 3-ball.



$[0, \frac{2\pi}{p}] \times D^2$



$f([0, \frac{2\pi}{p}] \times D^2)$

We let $\tilde{e}_3$ be the 3-cell $f([0, \frac{2\pi}{p}] \times D^2)$. Then

$$
\begin{aligned}
\partial \tilde{e}_3 &= f(\{\zeta\} \times D^2) - f(1 \times D^2) \\
&= f(\{\zeta\} \times (D^2 \zeta^q)) - f(1 \times D^2) \\
&= f((\{1\} \times D^2)\zeta) - f(1 \times D^2) \\
&= \tilde{e}_2(g-1).
\end{aligned} \tag{12.7}
$$

Thus we have described a $\mathbf{Z}/p$-equivariant cell structure for S^3; this defines a cell structure on $L(p,q)$ with one 0-cell e_0, one 1-cell e_1, one 2-cell e_2, and one 3-cell e_3. We calculated the $\mathbf{Z}[\pi]$-chain complex for the universal cover $\widetilde{L(p,q)} = S^3$ of $L(p,q)$ to be

$$0 \longrightarrow \mathbf{Z}\pi(\tilde{e}_3) \xrightarrow{\partial_3} \mathbf{Z}\pi(\tilde{e}_2) \xrightarrow{\partial_2} \mathbf{Z}\pi(\tilde{e}_1) \xrightarrow{\partial_1} \mathbf{Z}\pi(\tilde{e}_0) \longrightarrow 0 \tag{12.8}$$

with $\partial_3 = g - 1$, $\partial_2 = 1 + g + g^2 + \cdots + g^{p-1}$, and $\partial_1 = g^r - 1$.

Notice that the map from the lens to $L(p,q)$ is a quotient map which is a homeomorphism on its interior. Therefore $L(p,q)$ can be described as the identification space of the lens, where the left 2-disk in the boundary of the lens is identified with the right 2-disk by a $\frac{2\pi q}{p}$-twist.

Exercise 251. Draw a picture of this cell structure on S^3 for $p = 5$ and $q = 2$ by thinking of S^3 as $\mathbf{R}^3 \cup \{\infty\}$. Label the i-cells $\tilde{e}^i, g\tilde{e}^i, \cdots, g^4\tilde{e}^i$.

Exercise 252. Show that $L(2,1)$ is $\mathbf{R}P^3$, real projective 3-space. Show that $L(p,q)$ is the union of two solid tori.

Since $\pi_1(L(p,q)) = \mathbf{Z}/p$, if $L(p,q)$ is homotopy equivalent to $L(p',q')$, then $p = p'$. The following theorem gives the homotopy classification of 3-dimensional lens spaces.

Theorem 12.35. *Suppose $f : L(p,q) \to L(p,q')$ takes g to $(g')^a$, where g, g' are the generators of $\pi_1(L(p,q)), \pi_1(L(p,q'))$ as above. Assume $(a,p) = 1$, so that f induces an isomorphism on fundamental groups. Then*

1. $q \deg(f) \equiv q'a^2 \bmod p$.
2. *f is a homotopy equivalence if and only if* $\deg(f) = \pm 1$.

Moreover, if there exists an integer a so that $a^2 q' \equiv \pm q \bmod p$, then there is a homotopy equivalence $f : L(p,q) \to L(p,q')$ whose induced map on fundamental groups takes g to $(g')^a$.

Proof. Let $L = L(p,q),\ L' = L(p,q')$. First note that $f_* : \pi_1 L \to \pi_1 L'$ is an isomorphism since $(a,p) = 1$ implies $(g')^a$ generates $\mathbf{Z}/p = \pi_1 L'$. Using the cellular approximation theorem and the homotopy extension property, we may assume that f is cellular. Denote the cells of L' by e'.

Recall that the group of covering transformations of a universal cover $p : \tilde{X} \to X$ is identified with the fundamental group of X by taking the covering transformation $h : \tilde{X} \to \tilde{X}$ to the loop $p(\alpha)$, where α is a path in $\tilde{X}$ from $\tilde{x}_0$ to $h(\tilde{x}_0)$. Since $\tilde{e}_1$ is a path in S^3 from $\tilde{e}_1$ to $\tilde{e}_1 g^r$ by (12.5), e_1 (which is a loop since there is only one 0-cell in L) represents g^r in $\pi_1 L$. Similarly e_1' represents $(g')^{r'}$ in $\pi_1 L'$.

Because f is cellular and takes g to $(g')^a$ on fundamental groups, it follows that the loop $f(e_1)$ represents $(g')^{ar} = (g')^{r'q'ar}$, and so the chain map $f_\bullet : C_1 L \to C_1 L'$ takes e_1 to $(q'ar)e_1'$.

Lift f to $\tilde{f} : \tilde{L} \to \tilde{L}'$, so that $\tilde{f}(\tilde{e}_0) = \tilde{e}_0'$. Then since $f(e_1)$ wraps $q'ar$ times around e_1', $f(\tilde{e}_1)$ lifts to a sum of $q'ar$ translates of $\tilde{e}_1'$. Precisely,

$$
\begin{aligned}
\tilde{f}(\tilde{e}_1) &= \tilde{e}_1' + \tilde{e}_1'(g')^{r'} + \cdots + \tilde{e}_1'(g')^{r'(q'ar-1)} \\
&= \tilde{e}_1'(1 + (g')^{r'} + \cdots + (g')^{r'(q'ar-1)}).
\end{aligned} \tag{12.9}
$$

To avoid being confused by isomorphic rings, write

$$\Lambda = \mathbf{Z}[\mathbf{Z}/p] = \mathbf{Z}[t]/(t^p - 1).$$

Identify $\mathbf{Z}[\pi_1 L]$ with Λ using the isomorphism determined by $g \mapsto t^a$, and identify $\mathbf{Z}[\pi_1 L']$ with Λ via $g' \mapsto t$. With these identifications, the equivariant chain complexes of $\widetilde{L(p,q)}$ and $\widetilde{L(p,q')}$ and the chain map between them are given by the diagram

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r, "\partial_3"] \arrow[d, "f_3"] & \Lambda \arrow[r, "\partial_2"] \arrow[d, "f_2"] & \Lambda \arrow[r, "\partial_1"] \arrow[d, "f_1"] & \Lambda \arrow[r] \arrow[d, "f_0"] & 0 \\
0 \arrow[r] & \Lambda \arrow[r, "\partial'_3"'] & \Lambda \arrow[r, "\partial'_2"'] & \Lambda \arrow[r, "\partial'_1"'] & \Lambda \arrow[r] & 0
\end{tikzcd}$$

where the differentials in two chain complexes are given by multiplication by an element in Λ as follows:

$$\begin{aligned}
\partial_3 &= t^a - 1 \\
\partial_2 &= 1 + t^a + \cdots + (t^a)^{p-1} = 1 + t + \cdots + t^{p-1} \\
\partial_1 &= t^{ar} - 1 \\
\partial'_3 &= t - 1 \\
\partial'_2 &= 1 + t + \cdots + t^{p-1} \\
\partial'_1 &= t^{r'} - 1.
\end{aligned}$$

These equations follow from Equations (12.5), (12.6), and (12.7) and the identifications of Λ with $\mathbf{Z}[\pi_1 L]$ and $\mathbf{Z}[\pi_1 L']$.

Since $\tilde{f}(\tilde{e}_0) = \tilde{e}'_0$ and f takes g to $(g')^a$, it follows that $f_0 = \mathrm{Id}$. From Equation (12.9) we conclude that

$$f_1 = 1 + t^{r'} + \cdots + t^{r'(q'ar-1)}.$$

Now $f_1\partial_2 = \partial'_2 f_2$, i.e. $(1+t+\cdots+t^{p-1})f_1 = (1+t+\cdots+t^{p-1})f_2$. This implies that

$$f_2 = f_1 + \xi(1-t) \text{ for some } \xi \in \Lambda. \tag{12.10}$$

Similarly we have that $f_2\partial_3 = \partial'_3 f_3$; i.e. $f_2(t^a - 1) = (t-1)f_3$, and therefore $(t-1)(t^{a-1} + \cdots + 1)f_2 = (t-1)f_3$. Hence

$$f_3 = (t^{a-1} + \cdots + 1)f_2 + \beta(1 + t + \cdots + t^{p-1}) \text{ for some } \beta \in \Lambda. \tag{12.11}$$

Let $\varepsilon : \Lambda \to \mathbf{Z}$ be the augmentation map defined by map $\sum n_i t^i \mapsto \sum n_i$. Then $H_3\tilde{L} = \ker\ \partial_3 = \operatorname{span}(1 + t + \cdots + t^{p-1}) \cong \mathbf{Z}$. The isomorphism is given by

$$(1 + t + \cdots + t^{p-1}) \cdot \alpha \mapsto \varepsilon(\alpha).$$

(Check that this indeed gives an isomorphism. Facts like this come from the identity $(1 + t + \cdots + t^{p-1})(1-t) = 0$ in $\mathbf{Z}[\mathbf{Z}/p]$.)

Similarly

$$H_3\tilde{L}' \cong \ker \partial'_3 = \operatorname{span}(1 + t + \cdots + t^{p-1}) \cong \mathbf{Z}.$$

Thus, $\deg \tilde{f} = n$ if and only if $f_3(1+t+\cdots+t^{p-1}) = n(1+t+\cdots+t^{p-1})$. Now

$$f_3(1+t+\cdots+t^{p-1}) = \varepsilon(f_3)(1+t+\cdots+t^{p-1})$$

and, using the computations above,

$$\begin{aligned}
\varepsilon(f_3) &= \varepsilon((t^{a-1}+t+\cdots+1)f_2 + \beta \cdot (1+t+\cdots+t^{p-1})) \\
&= a\varepsilon(f_2) + \varepsilon(\beta)\cdot p \\
&= a\varepsilon(f_1 + \xi(1-t)) + \varepsilon(\beta)\cdot p \\
&= a\varepsilon(f_1) + \varepsilon(\beta)\cdot p \\
&= a \cdot q'ar + \varepsilon(\beta)\cdot p.
\end{aligned}$$

In these equations, ξ and β are the elements defined in Equations (12.10) and (12.11). Thus the degree of $\tilde{f}$ equals $a^2 \cdot q'r + \varepsilon(\beta)\cdot p$, and in particular the degree of $\tilde{f}$ is congruent to $a^2q'r$ mod p.

The covers $\tilde{L} \to L$, $\tilde{L}' \to L'$ both have degree p. Since the degree multiplies under composition of maps between oriented manifolds, it follows that $p \deg f = p \deg \tilde{f}$, and so $\deg f = \deg \tilde{f} = a^2q'r + \varepsilon(\beta)p$, proving the second assertion.

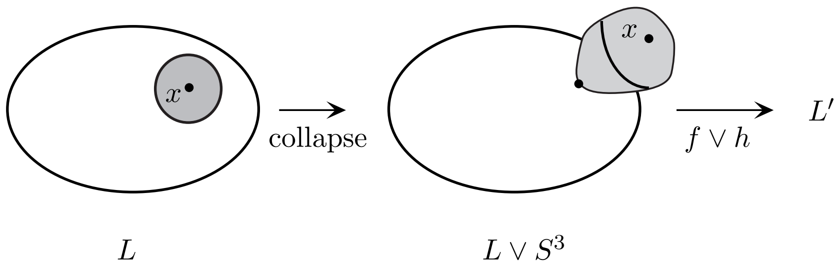
If $\deg f = \deg \tilde{f} = \pm 1$, then since $\tilde{L} \cong S^3 \cong \tilde{L}'$, the map $\tilde{f} : S^3 \to S^3$ is a homotopy equivalence. Thus $f : L \to L'$ induces an isomorphism on all homotopy groups and is therefore a homotopy equivalence by Theorem 7.76.

It remains to prove the last assertion of Theorem 12.35. Define a map on the 1 skeleton $f^{(1)} : L^{(1)} \to L'$ so that the induced map on fundamental groups takes g to $(g')^a$. Since $((g')^a)^p = 1$, $f^{(1)}$ extends over the 2 skeleton. The obstruction to extending over the 3-skeleton lies in $H^3(L;\pi_2 L')$. Notice that $\pi_1 L'$ acts trivially on $\pi_k L'$ for all k, since the covering transformations $S^3 \to S^3$ have degree 1 and so are homotopic to the identity. Thus the results of obstruction theory (Chapter 8) apply in this situation.

Since $\pi_2 L' = \pi_2 S^3 = 0$, $H^3(L;\pi_2 L') = 0$, and so this obstruction vanishes. Hence we can extend over the 3-skeleton; since this does not alter the map on the 1-skeleton, we obtain a map $f : L \to L'$ so that $f_*(g) = (g')^a$.

If $a^2q'r \equiv \pm 1$ mod p, then $\tilde{f}$ has a degree ± 1 mod p. We assert that f can be modified so that the resulting lift $\tilde{f}$ is replaced by another equivariant map $\tilde{f}'$ such that $\deg \tilde{f}' = \deg \tilde{f} \pm p$. This is a formal consequence of the technique used to extend f over the 3-skeleton in obstruction theory; we outline the construction in this specific case.

Let $x \in L$, and redefine f on a neighborhood of x as indicated in the following figure.



Here $h : S^3 \to L'$ is a degree $\pm p$ map (e.g. take h to be the universal covering). Denote by f' this composition of the collapsing map $L \to L \vee S^3$ and $f \vee h$. There are many ways to see that the degree of f' equals the $\deg(f) \pm p$. Notice that since f is only modified on a 3-cell, f and f' induce the same map on fundamental groups.

If $a^2 q' r \equiv \pm 1 \bmod p$, then repeating this modification as needed we can arrange that $\deg f = \pm 1$, and so f is a homotopy equivalence. This completes the proof of Theorem 12.35. □

Exercise 253.

1. Show that $L(5,1)$ and $L(5,2)$ have the same homotopy and homology groups, but are not homotopy equivalent.
2. Show that $L(7,1)$ and $L(7,2)$ are homotopy equivalent. Show that any homotopy equivalence is orientation preserving, i.e. has degree 1. Is this true for any pair of homotopy equivalent lens spaces?

Having completed the homotopy classification, we turn now to the simple-homotopy classification. This is accomplished using Reidemeister torsion. The chain complex of the universal cover of a lens space $L = L(p,q)$ is not acyclic, since it has the homology of S^3. We will tensor with $\mathbf{C}$ to turn them into acyclic complexes and compute the corresponding Reidemeister torsion. Thus we need a ring map $\mathbf{Z}[\mathbf{Z}/p] \to \mathbf{C}$.

Let $\zeta = e^{2\pi i/p} \in \mathbf{C}$. Let $\rho : \mathbf{Z}[\mathbf{Z}/p] \to \mathbf{C}$ be the ring homomorphism defined by

$$h(t) = \zeta.$$

Note that $h(1 + t + \cdots + t^{p-1}) = 0$, and that if $p \not| a$, then $\zeta^a \neq 1$.

In this notation we assumed $\mathbf{Z}/p$ had the generator t. Let g be the generator of $\pi_1 L$, and choose an isomorphism $\pi_1 L \cong \mathbf{Z}/p$ so that g corresponds to t^a where $(a,p) = 1$. Let $D_\bullet = C_\bullet(\tilde{L}) \otimes_{\pi_1 L} \mathbf{C}$. This is a based complex over the complex numbers with basis of the form $e \otimes 1$ where e is an (oriented) cell of L. Since L has one cell in each dimension $d = 0, 1, 2, 3$,

$$D_n = \begin{cases} \mathbf{C} & \text{if } n = 0, 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, from the chain complex (12.8) one easily sees that

$$D_\bullet = \{\ 0 \longrightarrow \mathbf{C} \xrightarrow{\zeta^a - 1} \mathbf{C} \xrightarrow{0} \mathbf{C} \xrightarrow{\zeta^{ar}-1} \mathbf{C} \longrightarrow 0\ \}$$

and hence $D_\bullet$ is acyclic.

The following diagram exhibits a chain isomorphism of $D_\bullet$ with an elementary complex

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{C} \arrow[r, "\zeta^a-1"] \arrow[d, "\zeta^a-1"] & \mathbf{C} \arrow[r, "0"] \arrow[d, "\mathrm{Id}"] & \mathbf{C} \arrow[r, "\zeta^{ar}-1"] \arrow[d, "\zeta^{ar}-1"] & \mathbf{C} \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbf{C} \arrow[r, "\mathrm{Id}"] & \mathbf{C} \arrow[r, "0"] & \mathbf{C} \arrow[r, "\mathrm{Id}"] & \mathbf{C} \arrow[r] & 0 \end{tikzcd}$$

and so the Reidemeister torsion is

$$\Delta_{\mathbf{C}}(L) = (\zeta^a - 1)(\zeta^{ar} - 1)$$

(recall that we are using multiplicative notation). Notice that the Reidemeister torsion takes its values in $\mathbf{C}^\times / \pm\{1, \zeta, \cdots, \zeta^{p-1}\}$.

Now suppose $f : L(p,q) \to L(p,q')$ is a homotopy equivalence which takes g to $(g')^a$ for some a so that $(a,p) = 1$. Then if we choose the isomorphism $\pi_1 L' \cong \mathbf{Z}/p$ so that g' corresponds to t, we have

$$\Delta_{\mathbf{C}}(L') = (\zeta - 1)(\zeta^{r'} - 1).$$

If f is a simple-homotopy equivalence, $\tau(f) = 0$; by Proposition 12.34 $\Delta_{\mathbf{C}}(L)$ and $\Delta_{\mathbf{C}}(L')$ are equal in the quotient $\mathbf{C}^\times / \pm\{1, \zeta, \cdots, \zeta^{p-1}\}$. We summarize this conclusion (and the conclusion of Theorem 12.35 for convenience) in the following proposition.

Proposition 12.36. *If $f : L(p,q) \to L(p,q')$ is a simple-homotopy equivalence which takes the generator $g \in \pi_1 L$ to $(g')^a \in \pi_1 L'$, then*

1. $a^2 q' \equiv \pm q \bmod p$.
2. *For each p^{th} root of unity $\zeta \neq 1$, there exists an $s \in \mathbf{Z}/p$ so that*

$$(\zeta^a - 1)(\zeta^{ar} - 1) = \pm\zeta^s(\zeta - 1)(\zeta^{r'} - 1)$$

where r and r' are determined by the equations $rq \equiv 1 \bmod p$ and $r'q' \equiv 1 \bmod p$.

□

Example of L(7,1) and L(7,2). Let $L(7,1) = L$ and $L(7,2) = L'$. Then the equation $a^2 = \pm 2 \bmod 7$ has only the solutions $a = 3,\ a = 4$. We have $r = 1$ and $r' = 4$.

Exercise 253 shows that L and L' are homotopy equivalent. Suppose that L and L' were simple-homotopy equivalent.

1. If $a = 3$, then for each seventh root of unity ζ there exists an $s \in \mathbf{Z}$ with
$$(\zeta^3 - 1)^2 = \pm\zeta^s(\zeta - 1)(\zeta^4 - 1).$$
This implies that
$$|\zeta^3 - 1|^2 = |\zeta - 1||\zeta^4 - 1|.$$
Note that $|\zeta^3 - 1| = |\zeta^4 - 1|$. But we leave it to you as an exercise to show $|\zeta^b - 1| = 2\sin(b\pi/p)$, and we will allow the use of a calculator to show $|\zeta - 1| \neq |\zeta^3 - 1|$.
2. If $a = 4$, the equation reads:
$$(\zeta^4 - 1)^2 = \pm\zeta^s(\zeta - 1)(\zeta^4 - 1).$$
For similar reasons as in the first case this is impossible.

Thus $L(7,1)$ is homotopy equivalent but not simple-homotopy equivalent to $L(7,2)$. Theorem 12.32 implies that $L(7,1)$ and $L(7,2)$ are not homeomorphic.

We will now give the simple-homotopy classification for 3-dimensional lens spaces. The proof will rely on a number-theoretic result about roots of unity.

Theorem 12.37. *If $f : L(p,q) \to L(p,q')$ is a simple-homotopy equivalence, with $f_*(g) = (g')^a$, then $L(p,q)$ is homeomorphic to $L(p,q')$ and either*
$$a = \pm 1 \text{ and } q \equiv \pm q' \bmod p$$
or
$$a = \pm q \text{ and } q \equiv \pm (q')^{-1} \bmod p.$$
In particular, $q \equiv \pm (q')^{\pm 1} \bmod p$.

Proof. Suppose that $f : L(p,q) \to L(p,q')$ is a simple-homotopy equivalence. Proposition 12.36 shows that for any p^{th} root of unity $\zeta \neq 1$ there exists an s with
$$(\zeta^a - 1)(\zeta^{ar} - 1) = \pm\zeta^s(\zeta - 1)(\zeta^{r'} - 1).$$

Note that $|\zeta^s|^2 = 1$ and for any x, $|\zeta^x - 1|^2 = (\zeta^x - 1)(\zeta^{-x} - 1)$. Thus
$$1 = (\zeta^a - 1)(\zeta^{-a} - 1)(\zeta^{ar} - 1)(\zeta^{-ar} - 1)\left[(\zeta - 1)(\zeta^{-1} - 1)(\zeta^{r'} - 1)(\zeta^{-r'} - 1)\right]^{-1}.$$

For each j so that $0 < j < p$ and $(j,p) = 1$ define

$$m_j = \#\{x \in \{a, -a, ar, -ar\} \mid x \equiv j \bmod p\}$$

and

$$n_j = \#\{x \in \{1, -1, r', -r'\} \mid x \equiv j \bmod p\}.$$

Then clearly $\sum_j m_j = 4 = \sum_j n_j$.

Let $a_j = m_j - n_j$. Then

(a) $\sum a_j = 0$.

(b) $a_j = m_j - n_j = m_{p-j} - n_{p-j} = a_{p-j}$.

(c) $\prod_j (\zeta^j - 1)^{a_j} = 1$.

A theorem of Franz (for a proof see [**11**]) says that if a_j is a sequence of integers so that (a), (b), and (c) hold for all p^{th} roots of unity $\zeta \neq 1$, then $a_j = 0$ for all j.

Thus $m_j = n_j$ for each j. It follows that either

1. $a \equiv \varepsilon_1$ and $ar \equiv \varepsilon_2 r' \bmod p$ for some $\varepsilon_i \in \{\pm 1\}$, or
2. $a \equiv \varepsilon_1 r'$ and $ar \equiv \varepsilon_2 \bmod p$ for some $\varepsilon_i \in \{\pm 1\}$.

The second part of the theorem follows from this and the facts that $rq \equiv 1 \bmod p$ and $r'q' \equiv 1 \bmod p$.

The homeomorphism $h : S^3 \to S^3$ taking (w_1, w_2) to $(w_1, \bar{w}_2)$ is equivariant with respect to the actions $(w_1, w_2)g = (w_1\zeta, w_2\zeta^q)$ and $(w_1, w_2)g = (w_1\zeta, w_2\zeta^{-q})$. This implies that $L(p,q)$ and $L(p,-q)$ are homeomorphic.

The homeomorphism $k : S^3 \to S^3$ taking (w_1, w_2) to (w_2, w_1) is equivariant with respect to the actions $(w_1, w_2)g = (w_1\zeta, w_2\zeta^q)$ and $(w_1, w_2)g = (w_1\zeta^q, w_2\zeta)$. The quotient space of the $\mathbf{Z}/p$ action $(w_1, w_2)g = (w_1\zeta^q, w_2\zeta)$ is the same as the quotient space of the $\mathbf{Z}/p$ action $(w_1, w_2)g = (w_1\zeta, w_2\zeta^r)$, since p and q are relatively prime and $(w_1\zeta^q, w_2\zeta) = (w_1\zeta^q, w_2(\zeta^q)^r)$. This implies that $L(p,q)$ and $L(p,r)$ are homeomorphic (where $r = q^{-1} \bmod p$). Thus if $L(p,q)$ and $L(p,q')$ are simple-homotopy equivalent, they are homeomorphic. □

Proof of Theorem 12.1. The homotopy classification was obtained in Theorem 12.35. Theorem 12.37 shows that if $q' = \pm q^{\pm 1}$, then $L(p,q)$ and $L(p,q')$ are homeomorphic.

If $q' \neq \pm q^{\pm 1}$, then $L(p,q)$ and $L(p,q')$ are not simple-homotopy equivalent, and so by Chapman's theorem (Theorem 12.32) $L(p,q)$ and $L(p,q')$ are not homeomorphic. □

12.7. The s-cobordism theorem

Finally, we end this chapter with the statement of the s-cobordism theorem, a fundamental result of geometric topology.

Theorem 12.38 (s-cobordism theorem). *Let W be a smooth (respectively piecewise-linear, topological) compact manifold of dimension 6 or more whose boundary consists of two path components M_0 and M_1. Suppose that the inclusions $M_0 \hookrightarrow W$ and $M_1 \hookrightarrow W$ are homotopy equivalences. Let $\tau(W, M_0) \in \mathrm{Wh}(\pi)$ denote the Whitehead torsion of the acyclic, based $\mathbf{Z}[\pi_1 W]$–complex $C_\bullet(\tilde{W}, \tilde{M_0})$.*

Then W is diffeomorphic (respectively PL-homeomorphic, homeomorphic) to $M_0 \times [0,1]$ if and only if $\tau(W, M_0) = 0$ vanishes. □

A good exposition of the proof in the smooth case is given in [**24**] and in the PL-case in [**42**]. The topological case is much harder and is based on the breakthroughs of Kirby and Siebenmann [**25**] for topological manifolds. The theorem is false in the smooth case if W has dimension 5 by results of Donaldson [**13**], and is true in the topological case in dimension 5 for many fundamental groups (e.g. $\pi_1 W$ finite) by work of Freedman-Quinn [**15**].

The method of proof of the s-cobordism theorem is to develop *handlebody structures* on manifolds. A handlebody structure is an enhanced analogue of a CW-decomposition. Provided the dimension of the manifold is high enough, then handles can be manipulated in a manner similar to the way cells are manipulated in the proof of Theorem 12.31, and the proof of the s-cobordism theorem proceeds using handle-trading, handle-sliding, and handle-cancellation.

12.8. Projects: Handlebody theory and torsion for manifolds

12.8.1. Handlebody theory and torsion for manifolds. Discuss handlebody theory for smooth (or PL) manifolds and use it to indicate how torsion can be useful in the study of diffeomorphism (or PL homeomorphism) problems for manifolds. In particular, discuss how handlebody structures relate Theorems 12.31 and 12.38.

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