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q-Poisson Bases and q-Poisson Curves

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ABSTRACT

We construct a new class of bases (q-Poisson bases) with one shape parameter based on q-integers. The q-Poisson bases have lots of good properties, including non-negativity, partition of unity, linear independence, which are suitable for modeling. Based on q-Poisson bases, we define q-Poisson curves, which have some properties similar to classical Poisson curves. We also present a degree elevation and de Casteljau algorithm for q-Poisson curve. The effect of the parameter q on q-Poisson curves is also studied. The introduction of the parameter q makes Poisson curves convenient and flexible for shape modeling.

INTRODUCTION

Last thirty years, q-calculus served as a bridge between mathematics and physics. Lots of experts concentrated on q-Hypergeometric series [1] and made a wide application of the Hypergeometric Series to Quantum Theory, Number Theory, Combinatorics, Statistical Mechanics and many other fields. After years of development, different kinds of q-special functions came up. On the other hand, discrete probability distributions play an important role in CAGD/CAD system and lots of blending functions are deduced from them. For example, the Bernstein-Bezier bases are formed from binomial distribution [2], the B-spline bases are formed from a simple stochastic distribution [3]. Morin and Goldman [4] extended the notion of Bezier curves to Poisson curves. In recent decades, people

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combined the q-calculus with discrete probability distributions to achieve a series of q-bases for modeling as q-bases have more degrees of freedom than normal bases. Lupas, first introduced a type of q-Bernstein operator in [5]. Then Phillips came up with the q-Bernstein operator in terms of q-difference [6]. After that, different kinds of q-operators were proposed (see [7] and [8]). In 2003, q-Bernstein operator [9] was used to construct q-Bezier curve due to its fine properties. In 2012, Simeonov and Goldman [10] defined q-B-splines (quantum B-splines), which were based on q-blossoming in [11].

In this paper we present a new operator called q-Poisson operator, which is an application of the q-integers in Poisson bases and curves. The paper is organized as follows. In Section 2, we review the basic knowledge of q-calculus, give the definition of q-Poisson bases and study some of their fundamental properties. Then we use the q-Poisson bases to construct the q-Poisson curves and discuss the properties of these curves in Section 3. After that, we talk about the effect of parameter q in shape control in Section 4. Finally, we conclude our paper in Section 4.

THE GENERATION OF q-POISSON BASES

In this section we first review some useful definitions about q-calculus [8], and then define the q-Poisson bases and discuss their properties.

Definition 1. Let q> 0, n∈N, q-integer ($[n]_q$) is defined as

$$[n]_q = \begin{cases} \dfrac{1-q^n}{1-q}, & q \neq 1 \\ n, & q = 1. \end{cases} \tag{1}$$

Definition 2. Let q> 0, n∈N, q-factorial ($[n]_q!$) is defined as

$$[n]_q! = \begin{cases} [1]_q[2]_q \cdots [n]_q, & n = 1,2,\cdots \\ 1, & n = 0. \end{cases} \tag{2}$$

Definition 3. Let q> 0, 0 ≤ k ≤ n, n, k∈N, the q-binomial coefficients is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}. \tag{3}$$

To simplify the notation, we use [n], [n]! and $\begin{bmatrix} n \\ k \end{bmatrix}$ to represent $[n]_q$, $[n]_q!$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ respectively in the rest part of the paper.

Definition 4. The q-analogue of (1 + x)n is the polynomial

$$(1+x)_q^n = \begin{cases} (1+x)(1+qx)\cdots(1+q^{n-1}x), & n=1,2,\cdots \\ 1, & n=0. \end{cases} \tag{4}$$

Definition 5. (q-Series expansions). For |x| < 1, |q| < 1,

$$\sum_{k=0}^{\infty} \frac{(1-a)_q^k}{(1-q)_q^k} x^k = \frac{(1-ax)_q^{\infty}}{(1-x)_q^{\infty}}. \tag{5}$$

Now we introduce two types of q-analogue of classical exponential function e^t :

$$\varepsilon_q(t) = \sum_{i=0}^{\infty} \frac{t^i}{[i]!} = \frac{1}{\left(1-(1-q)t\right)_q^{\infty}}, \quad |t| < \frac{1}{1-q},\ |q| < 1.$$

$$E_q(t) = \sum_{i=0}^{\infty} \frac{q^{i(i-1)/2}}{[i]!} t^i = \left(1+(1-q)t\right)_q^{\infty}, \quad t \in \Re,\ |q| < 1.$$

In the following part we define the degree *n* *q*-Poisson bases and discuss their properties. We take

$$b_k^n(t;q) = E_q(-[n]t)([n]t)^k / [k]!, \quad |t| < \frac{1}{1-q},\ |q| < 1 \tag{6}$$

as the degree n q-Poisson base. When q= 1 and n = 1, the base becomes $b_k^1(t;1) = e^{-t}t^k / [k]!$, which is the Poisson distribution function.

Theorem 1. The q-Poisson bases have the following properties:

1. *Non-negativity*: $b_k^n(t;q) \geq 0, \quad k = 0,1,2,\ldots;$
2. Partition of unity: $\sum_{k=0}^{\infty} b_k^n(t;q) = 1;$
3. End-point property: $b_k^n(0;q) = \begin{cases} 1, & k=0, \\ 0, & k \neq 0, \end{cases}$ and $\lim_{t \to (1/1-q^n)^-} b_k^n(t;q) = 0$;
4. Linear independence: $\sum_k c_k b_k^n(t;q) = 0 \Leftrightarrow c_k = 0 \quad \text{for} \quad \forall k \cdot$

Proof. Properties 1, 3 and 4 are obvious so here we only give the proof of property 2.

$$\sum_{k=0}^{\infty} b_k^n(t;q) = \sum_{k}^{\infty} E_q(-[n]t) \frac{1}{k!}([n]t)^k = E_q(-[n]t)\varepsilon_q([n]t) = 1.$$

Theorem 2. (The property of degree elevation) A degree n q-Poisson base can be expressed as a combination of a series of degree n+1 q-Poisson bases

$$b_k^n(t;q) = \left(\frac{[n]}{[n+1]}\right)^k \sum_{t=k}^{\infty} \begin{bmatrix} l \\ k \end{bmatrix} \left(1 - \frac{[n]}{[n+1]}\right)_q^{l-k} b_l^{n+1}(t;q) \cdot \tag{7}$$

Proof. According to the definition of q-Poisson base (6), we have

$b_k^n(t;q) = E_q(-[n]t)([n]t)^k 1/[k]!$ and $b_{k+m}^{n+1}(t;q) = E_q(-[n+1]t)([n+1]t)^k 1/[k+m]!$

Comparing these two bases, we derive

$$\frac{b_{k+m}^{n+1}(t;q)}{b_k^n(t;q)} = \frac{E_q(-[n+1]t)([n+1]t)^k 1/[k+m]!}{E_q(-[n]t)([n]t)^k 1/[k]!}.$$

By using $\varepsilon_q(t)E_q(-t) = 1$, Equation (3.14) in [8], and simply calculations, we get (7).

Theorem 3. (The property of degree reduction) A degree n base $b_k^n(t;q)$ can be represented by combination of a series of degree n-1 bases, thus

$$b_k^n(t;q) = \left(\frac{[n]}{[n-1]}\right)^k \sum_{t=k}^{\infty} \begin{bmatrix} l \\ k \end{bmatrix} \left(1 - \frac{[n]}{[n-1]}\right)_q^{l-k} b_l^{n-1}(t;q) \cdot \tag{8}$$

Proof. According to the definition of q-Poisson base (6), we have

$b_k^n(t;q) = E_q(-[n]t)([n]t)^k 1/[k]!$ and $b_{k+m}^{n\text{-}1}(t;q) = E_q(-[n\text{-}1]t)([n\text{-}1]t)^k 1/[k+m]!$

By direct calculations and using the method similar to Theorem 2, we obtain Equation (8).

q-POISSON CURVES

In this section we use q-Poisson bases to construct q-Poisson curves and study their properties.

Definition 6. We define the degree n q-Poisson curve as

$$\boldsymbol{p}(t;q) = \sum_{k=0}^{\infty} \boldsymbol{P}_k b_k^n(t;q), \quad 0 \le t \le \frac{1}{1-q^n},\ |q| < 1, \tag{9}$$

Where $\boldsymbol{P_k} \in \Re^3$ (k $\ge$ 0, k$\in$N) are the control points and $b_k^n(t;q)$ are the q-Poisson bases defined by (6). Joining up the control points $\boldsymbol{P_k}$ in sequence, we obtain a polygon, which is called the control polygon of q-Poisson curve.

Theorem 4. From the definition, we can derive some basic properties of q-Poisson curves:

1. Geometric and affine invariance.
2. The q-Poisson curve lies inside the convex hull of its control polygon.
3. The end-point interpolation property: p(0; q) = P0.
4. Reducibility: when q=1, q-Poisson curve (14) degenerates to the classical degree n Poisson curve.

Theorem 5. The degree n q-Poisson curves are variation diminishing, which means that the number of intersection points between any straight line and q-Poisson curve is no more than the number of intersection points between the straight line and its control polygon.

Proof. Let C denote a planar q-Poisson curve p(t; q) defined by (14), and L be any straight line, I(C, L) be the number of times C crosses L. Let L be the abscissa axis and establish the coordinate system. Because q-Poisson curves are geometric invariant, we can denote the new coordinates of the control points by (x_i, y_i). Thus, the number of intersection points between L and p(t; q) is equal to the number of the roots of new q-Poisson curve p*(t; q) = $\sum_{k=0}^{\infty} y_k b_k^n(t;q)$ within the domain D := {$t \mid 0 \le t < 1/(1-q^n)$}. Let P denote the control polygon and let I(P, L) be the number of times P crosses L. For any polynomial $f(t)$, we denote Zt∈ I⊆(0,∞)[$f(t)$] as the number of positive roots of $f(t)$ on the interval I. For any vector V$=(v_0, v_1, \cdots, v_k, \cdots)$, S$(v_0, v_1, \cdots, v_k, \cdots)$ denotes the strict sign changes in V. We have

$$
\begin{aligned}
I(C,L) = Z_D[\bar{p}] &= Z_D\left\lfloor \sum_{k=0}^{\infty} y_k b_k^n(t;q) \right\rfloor \\
&\le S\left(\tfrac{1}{[0]!} y_0, \tfrac{1}{[1]!} y_1, \cdots, \tfrac{1}{[k]!} y_k, \cdots\right) \\
&\le S(y_0, y_1, \cdots, y_k, \cdots) = I(P,L)
\end{aligned}
$$

Thus, the q-Poisson curves hold the variation diminishing property.

From Theorem 5, we can easily deduce the following properties:

Property 1. The degree n q-Poisson curves are convexity-preserving, which means that when the control polygon of planar q-Poisson curve is convex, the planar q-Poisson curve will also be convex.

Property 2. The degree n q-Poisson curves are Monotonicity-preserving, which means that when the control polygon of q-Poisson curve is monotonically increasing (or decreasing) in a direction, the q-Poisson curve will also be monotonically increasing (or decreasing) in that direction.

Theorem 6. (Degree elevation for q-Poisson curve) A degree n q-Poisson curve can be expressed as a combination of degree n+1 q-Poisson curves

$$
\boldsymbol{p}(t;q) = \sum_{k=0}^{\infty} \boldsymbol{P}_k b_k^n(t;q) = \sum_{\boldsymbol{k}=\boldsymbol{0}}^{\infty} \boldsymbol{P}_{\boldsymbol{k}}^{*} b_k^{n+1}(t;q), \tag{10}
$$

with $\boldsymbol{P}_{\boldsymbol{k}}^{*} = \sum_{m=0}^{k} \boldsymbol{P}_m \left([n]/[n+1]\right)^m \left(1-[n]/[n+1]\right)_q^{k-m} \begin{bmatrix} k \\ m \end{bmatrix}$.

Proof. The result is obvious by using Theorem 2, we omit the details of the proof.

Theorem 7. (De Casteljau algorithm)

$$\boldsymbol{p}(t;q)=\sum_{k=0}^{\infty}\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{1}}b_k^{n-1}(t;q)=\cdots=\sum_{\boldsymbol{k=0}}^{\infty}\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{j}}b_k^{n-j}(t;q)=\cdots=\boldsymbol{P}_{\boldsymbol{0}}^{\boldsymbol{n}}, \tag{11}$$

with

$$\boldsymbol{P}_{\boldsymbol{k}}^{\boldsymbol{j}}=\begin{cases}\boldsymbol{P}_{\boldsymbol{k}}, & j=0,\, k=0,1,2,\cdots\\ \sum_{m=0}^{k}\boldsymbol{P}_{\boldsymbol{m}}^{\boldsymbol{j-1}}\left([n+1-j]/[n-j]\right)^m\left(1-[n+1-j]/[n-j]\right)_q^{k-m}\begin{bmatrix}k\\m\end{bmatrix}, & j=1,2,\cdots,\, k=0,1,2,\cdots\end{cases}$$

are the control points of j-th iteration.

Proof. According to the degree reduction algorithm of the q-Poisson base, we obtain Eq. (11).

SHAPE CONTROL OF *q* POISSON CURVE

In this section, we discuss the effect of parameter q. The parameter q can be regard as the shape parameter. If we want to adjust the shape of the curve, we can just change the value of parameter q. For $0 < q < 1$, as q increases, the curve moves closer to the control polygon. The effect of the shape parameter of the q Poisson curve is clear. Figure 1 shows the effect on the shape of the curve by altering the value q for $0 < q < 1$.

CONCLUSIONS

In this paper we present a new class of q-Poisson bases by applying the q-calculus to Poisson distribution function. The q-Poisson bases have some good properties same as classical Poisson bases. With q-Poisson bases, the corresponding q-Poisson curves can be constructed. Compared to the classical Poisson curves, the q-Poisson curves have one more degree of freedom, which makes the adjustment of the curve easier. Experimental results show that the curve moves closer to the control polygon as q becomes greater.

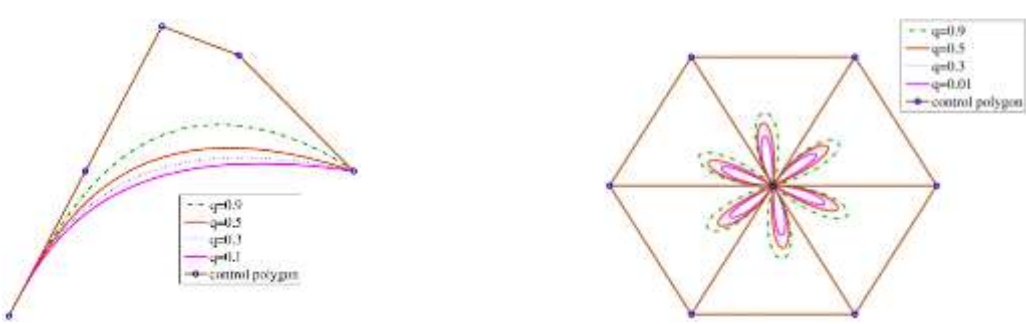

Figure 1. The effects of parameter q on the shape of q-Poisson curves.

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