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Mathematical methods of classical mechanics, by V. I. Arnold, translated from the 1974 Russian edition by K. Vogtmann and A. Weinstein, Graduate Texts in Math., Vol. 60, Springer-Verlag, New York and Berlin, 1978, x + 462 pp.

A course in mathematical physics, vol. 1: *Classical dynamical systems*, by Walter Thirring, translated from German by Evans M. Harrel, Springer-Verlag, New York and Berlin, 1978, xii + 258 pp., $19.80.

The science of mechanics is the oldest branch of applied mathematics. The principles underlying the kinematics of a particle were established early in the seventeenth century by Galileo, and their applications developed further by Christian Huygens after Galileo's death. The fundamental principles of particle dynamics were laid down by Newton and expounded by him in his great work *Philosophiae naturalis principia mathematica* (1687). The extension of these principles to cover the theory of the motion of rigid bodies was carried out by d'Alembert and the results published in his *Traité de mécanique* (1743). Although the foundations of analytical dynamics (or, as it is sometimes called, rational mechanics) were laid before that date as is evidenced by the publication in St. Petersburg in 1736 of Euler's *Mechanica, sive motus scientia analytice exposita*, the outstanding event in the early history of mechanics was the publication of Lagrange's *Mécanique analytique* in 1788.

The opening words of the *Avertisement* to Lagrange's treatise were: "On a déja plusieurs Traités de Mécanique". Of course, the number of treatises on analytical dynamics is vastly greater now than it was in 1788, for the reason that many distinguished mathematicians found the subject a rich source of research problems and the literature of the subject became correspondingly large. The theoretical work of the century and more after the death of Lagrange was crystallized by E. T. Whittaker in a treatise **[19]** which has not been superseded as the definitive account of classical mechanics. That work reveals that among those who have made significant contributions to the subject are Carathéodory, Cauchy, Darboux, Gauss, Jacobi, Lie, Liouville, Poincaré, and Weyl. In that era, mathematicians worked in both pure and in applied mathematics. That problems in analytical mechanics stimulated research in pure mathematics was shown by the work of G. D. Birkhoff whose Colloquium Lectures in 1920, **[5]**, led ultimately to the development of a new branch of abstract mathematics.

The position of classical mechanics within university curricula changed after World War I. No doubt because a knowledge of the principles of dynamics is basic to astronomers, physicists, and engineers–and because each group demands a different selection of topics from the many making up the subject–the responsibility for the teaching of classical mechanics was surrendered by mathematics departments in many universities and assumed by faculty members in other departments. (It is interesting to observe that, a few years ago, some mathematics departments, conscious of the gap left by the abandonment of classical mechanics, introduced courses on "modeling" –apparently ignorant of the fact that elementary mechanics provides the simplest examples of the modeling process!) Excellent accounts of the mathematical theory were given by Banach [**4**], Wintner [**20**], Lanczos [**8**], and Synge [**18**] but the majority of the many texts which appeared between 1920 and 1970 were directed to the needs of particular users. For that reason, Pars in the Preface to his comprehensive and elegant treatise [**14**] wrote: "In recent years there has been a marked decline among mathematicians of interest in the classical dynamics."

What Pars did not realize was that the impulse to reverse the swing of the pendulum had already been given by Kolmogorov in his invited address [**6**] to the International Congress of Mathematicians in Amsterdam in 1954 and in his paper [**7**]. The problem discussed in this latter paper arose from the difficulty that the methods devised by Poincaré [**15**] for calculating perturbations in celestial mechanics all led to *divergent* series and gave no information about the behaviour of a dynamical system as a whole over very long intervals of time. The divergence of the series arises from "small divisors" which vanish for exact resonance, so that close to resonance the terms of the series become very large. This difficulty is characteristic not only for problems of celestial mechanics but for all problems which are close to being integrable. Poincaré himself stated that the *fundamental problem of dynamics* was the study of the motions of a system with Hamiltonian.

$$H = H_0(I) + \epsilon H_1(I, \phi), \qquad \epsilon \ll 1,$$

in action-angle variables I and ϕ. Here H_0 is the Hamiltonian of the unperturbed system and ϵH_1 a perturbation which is a 2π-periodic function of the angle variables $\phi_1, \ldots, \phi_n$. Because Arnold [**2**] gave an alternative proof of Kolmogorov's result it has become known as the Kolmogorov-Arnold theorem; this theorem and the Moser twist theorem [**11**] together provided the solution of Poincaré's fundamental problem. A lucid exposition of the application of this method to celestial mechanics was given by Sternberg [**17**] and, much later, by Moser himself in [**16**] and [**12**].

The time was now ripe for a presentation of the theory of classical mechanics in terms of the theory of differentiable manifolds which had been developed since the appearance of the fourth edition of Whittaker's treatise. Abraham's monograph [**1**] was the first to appear but it proved to be too difficult for the average student and it was due to the circulation of Saunders Mac Lane's lecture notes [**10**] and the publication of the textbook by Loomis and Sternberg [**9**] that the new formulation of classical mechanics reached a wider audience.

That formulation still had, of necessity, its roots in the work of the last century–in the realization that the equations of motion of a dynamical system, either in Newton's or in Lagrange's, could be derived from a simple variational principle (Hamilton's *principle of least action*). This stated that the motions of a conservative system coincided with the extremals of the functional

$$\int_{t_0}^{t} L(\mathbf{q}, \dot{\mathbf{q}}, t)\, dt$$

where $L = T - U$, the difference between the kinetic and potential energies, and called the *Lagrangian* of the system, is a function of the n generalized coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ used to describe the motion. Hamilton's principle is derived from Newton's second law of motion; then Lagrange's equations of motion are derived by means of the calculus of variations. By means of a Legendre transformation in which $p_j = \partial L/\partial \dot{q}_j$ the system of n second-order Lagrange equations may be converted to a remarkably symmetrical system of $2n$ first-order differential equations (called Hamilton's equations), involving the Hamiltonian function

$$H(\mathbf{p}, \mathbf{q}, t) = \sum_{j=1}^{n} p_j \dot{q}_j - L(\mathbf{q}, \dot{\mathbf{q}}, t).$$

The $\mathbf{q}$-space is called the *configuration space*, and the $(\mathbf{p}, \mathbf{q})$-space is called the *phase-space*. The *phase flow* is the one-parameter group of transformations of phase-space

$$g^t\colon \{\mathbf{p}(0), \mathbf{q}(0)\} \mapsto \{\mathbf{p}(t), \mathbf{q}(t)\}$$

where $\{\mathbf{p}(t), \mathbf{q}(t)\}$ is a solution of Hamilton's equations. Liouville's theorem, that the phase flow of Hamilton's equations preserves volume in the phase-space, is what allows the application of the methods of ergodic theory to classical mechanics [3].

These classical ideas can be cast in the language of the theory of differentiable manifolds. A Lagrangian system (M, L) is then defined in the following way: suppose that M is a differentiable manifold with tangent bundle $T(M)$ and that the map $L\colon T(M) \to \mathbf{R}$ is differentiable. We say that a map $\mathbf{x}\colon \mathbf{R} \to M$ is a *motion* in the Lagrangian system (M, L) if $\mathbf{x}$ is an extremal of the functional

$$\Phi(\mathbf{x}) = \int_{t_0}^{t} L(\dot{\mathbf{x}})\, dt$$

where $\dot{\mathbf{x}}$ denotes the velocity vector $\dot{x}(t) \in T_{\mathbf{x}(t)}(M)$. M is called the *configuraton manifold*, and L the *Lagrangian function* of the system.

The evolution of the local coordinates $\mathbf{q} = (q_1, \ldots, q_n)$ of a point $\mathbf{x}(t)$, under motion in a Lagrangian system (M, L) is governed by Lagrange's equations

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} = \frac{\partial L}{\partial \mathbf{q}},$$

where $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is the expression for the Lagrangian function in terms of the local coordinates $\mathbf{q}$, $\dot{\mathbf{q}}$ on $T(M)$.

If M is a Riemannian manifold, the quadratic form, defined by

$$T = \tfrac{1}{2}\langle \mathbf{v}, \mathbf{v}\rangle, \qquad \mathbf{v} \in T_{\mathbf{x}}(M),$$

on each tangent space, is called the *kinetic energy* of the system, and a differentiable map $U\colon M \to \mathbf{R}$ is called a *potential energy*. A Lagrangian system on a Riemann manifold is said to be a *natural system* if its Lagrangian function is equal to the difference $T - U$ of the kinetic energy T and a potential energy U. Natural systems, not unexpectedly, have more interesting properties than more general Lagrangian systems.

It is a striking fact that various "conservation laws" in mechanics are particular cases of a general theorem, due to Emmy Noether [13], which states that, to every one-parameter group of diffeomorphisms of the configuration manifold of a Lagrangian system which preserves the Lagrangian, there corresponds a first integral of the equations of motion. To put Noether's theorem into the language of the theory of differentiable manifolds, we need another definition: we say that a Lagrangian system (M, L) *admits the mapping* h: $M \to M$ if $L(h_* \mathbf{v}) = L(\mathbf{v})$, for any tangent vector $\mathbf{v} \in T(M)$; h_* denotes the derivative of h. Noether's theorem then states that if a Lagrangian system (M, L) admits the one-parameter group of diffeomorphisms h^s: $M \to M$, $s \in \mathbf{R}$, then the system of equations, corresponding to the Lagrangian function L, has a first integral I: $T(M) \to \mathbf{R}$. In local coordinates $\mathbf{q}$ on M, the first integral I can be calculated by means of the formula

$$I(\mathbf{q}, \dot{\mathbf{q}}) = \frac{\partial L}{\partial \dot{\mathbf{q}}} \cdot \frac{dh^s(\mathbf{q})}{ds}\Big|_{s=0}.$$

Hamiltonian mechanics may be thought of as the geometry of the phase-space, which has the structure of a symplectic manifold. For that reason, the study of Hamiltonian mechanics depends basically on the theory of differential forms. The phase-space can be seen as an even-dimensional differentiable manifold, M^{2n}, on which a symplectic structure ω^2 has been defined.

We can define an isomorphism I, mapping the cotangent bundle $T_x^*(M^{2n})$ to $T_x(M^{2n})$ as follows: with each vector $\mathbf{u}$, tangent to a symplectic manifold (M^{2n}, ω^2) at the point $\mathbf{x}$, we associate a 1-form $\omega_{\mathbf{u}}^1$ on $T_x(M^{2n})$ by the rule

$$\omega_u^1(\mathbf{v}) = \omega^2(\mathbf{v}, \mathbf{u}), \qquad \forall \mathbf{v} \in T_x(M^{2n}).$$

If H is a function on the symplectic manifold, then dH is a differential 1-form on M^{2n}, and at every point there is a tangent vector to M^{2n} associated with it. In this way there is established on M^{2n}, a *Hamiltonian vector field* IdH; H is called the *Hamiltonian function* of this vector field. The one-parameter group of diffeomorphisms g_H^t: $M^{2n} \to M^{2n}$, defined by

$$\frac{d}{dt} g_H^t \mathbf{x}\Big|_{t=0} = IdH(\mathbf{x})$$

is called the *Hamiltonian phase flow* corresponding to H. It can be proved that a Hamiltonian phase flow preserves the symplectic structure, that the form ω^2, giving the symplectic structure, is an integral invariant of a Hamiltonian phase flow, and that the function H is a first integral of the Hamiltonian phase flow g_H^t.

Every pair of vector fields F, H on a manifold determines a new vector field, called the *Poisson bracket* (F, H), defined by the equation

$$(F, H)(\mathbf{x}) = \frac{d}{dt} F\{g_H^t(\mathbf{x})\}\Big|_{t=0}.$$

It is easily shown that a function F is a first integral of the Hamiltonian phase flow associated with H, if and only if $(F, H) = 0$, and that (F, H) is equal to the value of the 1-form dF on the vector field IdH of the phase flow associated with H:

$$(F, H) = dF(IdH).$$

The Poisson bracket operation makes the Hamiltonian vector fields on a symplectic manifold into a subalgebra of the Lie algebra of all fields.

Lagrangian mechanics is, or course, contained in Hamiltonian mechanics as a special case–in which the phase-space is the cotangent bundle of the configuration space, and the Hamiltonian function is the Legendre transform of the Lagrangian function.

The Hamiltonian approach enables us to solve completely a series of dynamical problems which do not yield solutions by other means, and has great value in establishing the approximate methods of perturbation theory in celestial mechanics, for analyzing the general character of motion in the complicated systems encountered in statistical mechanics, and in connection with optics and quantum mechanics.

The two books under review–one written by a distinguished mathematician, the other by a distinguished theoretical physicist–are the first textbooks successfully to present to students of mathematics and physics, classical mechanics in a modern setting.

The larger book is based on a year-and-a-half long required course on classical mechanics, taught by Arnold to third and fourth year students of mathematics in Moscow State University in 1966–1968. In it, the author develops–as he needs them–the many different mathematical concepts and techniques which lie at the foundations of classical mechanics. The reader is not assumed to have any previous knowledge beyond that contained in standard courses in analysis (calculus and differential equations), geometry (vectors and vector spaces) and linear algebra. The main text of the book (300 pages) examines all the basic problems of dynamics, including the theory of small oscillations, the theory of the motion of a rigid body, and the Hamiltonian formalism, while 13 Appendices (155 pages) explore the connections between classical mechanics and other branches of mathematics. The material contained in these appendices did not form part of the required course and, whereas in the main body of the book Arnold has developed all the proofs without reference to other sources, the appendices consist on the whole of summaries of results, whose proofs are to be found in the cited literature. The final product is an attractive well-written book.

The book by Thirring is, by contrast, much shorter though it aims to cover relativistic as well as nonrelativistic mechanics. It is the first volume of a textbook which presents mathematical physics in its chronological order, originating in a four-semester course given by the author in the Institute for Theoretical Physics in the University of Vienna to both mathematicians and physicists who in Thirring's words "were only required to have taken the conventional introductory courses". Thirring covers much the same mathematical material as Arnold, but he restricts applications to nonrelativistic mechanics to particle dynamics. Of the six chapters in Thirring's book, two have no parallel in Arnold's. One deals with the relativistic motion of a point mass. The other, entitled "The Structure of Space and Time", seems to be out of place in a book entitled "Classical Dynamical Systems" although it undoubtedly contains much of value to a student of theoretical physics going on to the study of the theory of relativity.

It is fascinating to compare the approaches of the two authors. The mathematician has written a connected account of classical dynamics following the pattern of previous treatises and discussing all the classic problems of the subject, stopping only to develop mathematical techniques as he needs them and therefore showing all the more clearly the historical links between mechanics and mathematical analysis. The physicist, on the other hand, has written a concise account of analysis on manifolds to which he has appended some few examples from particle mechanics. It is almost as though they had changed roles!

Both books are well written (and the translation so idiomatic that it is easy to forget that they were originally written in Russian and German respectively) but a student will undoubtedly find that Arnold's book presents the fewer difficulties of the two. Arnold's treatment of Hamiltonian systems is the superior and this is of great importance at a time when the Hamiltonian aproach is favoured in so many branches of applied mathematics. Anyone offering a course in advanced dynamics (at graduate level) would find both of these texts invaluable though students would find them difficult unless they had already attended a course based, for instance, on Chapters 9, 10, 11, and 13 of [9]. Indeed the ideal approach might be to supplement this latter material with specific examples–and further general theory–taken from Arnold's book. It would amply demonstrate to young pure mathematicians that much of what they study in global analysis has its roots in physical problems of the past, and to their applied counterparts that modern pure mathematics can provide them with invaluable tools with which to develop general theories and tackle special problems.

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Elliptic curves: Diophantine analysis, by Serge Lang, Grundlehren der mathematischen Wissenschaften, vol. 231, Springer-Verlag, Berlin-Heidelberg-New York, 1978, xi + 261 pp., $37.40.

The study of the arithmetical properties of elliptic curves has been one of the most exciting areas of mathematical research for at least the past fifty years. It is customary to divide the modern theories according as one is dealing with rational points or with integer points; and both aspects of the subject can be regarded as having been initiated in 1922 by some remarkable discoveries of Mordell.

It had long been known that the rational points on an elliptic curve, defined over the rationals, form a group Γ under a chord and tangent construction; Mordell proved that Γ has a finite basis. The proof was most ingenious. It began with a demonstration that the group $\Gamma/2\Gamma$ is finite and then proceeded by a method of infinite descent (for references, see [5]). A far-reaching generalization of the finite basis theorem concerning abelian varieties was established by Weil in 1928; much important work arose therefrom, and an excellent survey of the subject as it existed in 1966 was given by Cassels [3]. Here there are discussions of the celebrated conjectures of Birch and Swinnerton-Dyer, of the theorems of Lutz and Nagell, of the Tate-Shafarevich and Selmer groups, and of a great deal besides. In another direction, Mordell showed that the Diophantine equation

$$y^2 = ax^3 + bx^2 + cx + d \qquad (*)$$

where a, b, c, d denote integers and the cubic on the right has distinct zeros, has only finitely many solutions in integers x, y. The proof involved the theory of the reduction of binary quartic forms followed by an application of a famous theorem of Thue (again, see [5]). Another proof, and indeed one that was applicable more generally to the hyperelliptic equation, was given by Siegel in 1926. Furthermore, in 1929, in a most profound work, Siegel succeeded in combining the Mordell-Weil theorem with a refinement of Thue's theorem that he had proved earlier, to show that any curve, defined over the rationals, with genus at least 1, has only finitely many integer points.