ABSTRACTS OF PAPERS

SUBMITTED FOR PRESENTATION TO THIS SOCIETY

The following papers have been submitted to the Secretary and the Associate Secretaries of the Society for presentation at meetings of the Society. They are numbered serially throughout this volume. Cross-references to them in the reports of the meetings will give the number of this volume, the number of this issue, and the serial number of the abstract.

67. Professor A. D. Campbell: *Pseudo-covariants of an n-ic in m variables in a Galois field that consists of terms of this n-ic.*

If we subject an n-ic in m variables in a Galois field of order p^s to linear homogeneous transformations with coefficients and variables in the same Galois field, we find sets of terms of this n-ic behaving in a strange fashion. We call these sets of terms pseudo-covariants. For example, we note that the only terms in the n-ic that can furnish terms in the transformed equation whose factors all have exponents less than p are those terms in the original equation of the n-ic whose factors all have exponents less than p. If the n-ic has no such terms as described above, we consider the terms in the n-ic that have each just one of the exponents u such that $p \leqq u < p^2$, while all the other exponents are less than p. In a similar manner we define the other pseudo-covariants of an n-ic. This paper makes a study of these pseudo-covariants and of the transformations of their coefficients caused by the linear transformations of the variables. (Received January 21, 1933.)

68. Mr. Morris Halperin: *The non-analytic character and the areas of Kasner's convex curves.*

A Kasner convex curve is defined as the limit of a sequence of convex polygons, each of which is obtained from the preceding one by measuring off from both ends of every one of its sides the rth part of its length and cutting off the corners; $r < 1/2$. Elsewhere in this Bulletin (abstract No. 39–1–63), Miss Lawrence points out that, except for $r = 1/4$, these curves are non-analytic. This property of these curves the present author has proved in two ways: (1) by means of differential equations, (2) by a consideration of the ratios between certain areas associated with these curves. The expression for the area of a Kasner convex curve is linear in the areas of two successive polygons of the defining sequence, and rational in r. (Received February 1, 1933.)

69. Dr. J. L. Doob (National Research Fellow): *The ranges of analytic functions.*

Let $f(z)$ be a function analytic in the interior of the unit circle of the z-plane Suppose that $f(0) = 0$, and that there is an arc A on $|z| = 1$ such that when

$\{z\}$ is a sequence of points in $|z|<1$ approaching a point of A, lim inf $|f(z_n)| \geqq 1$. Then the following is true. *If D_2 is an arbitrary closed point set in $|w|<1$ not containing $w=0$, and q an arbitrary integer, there is a neighborhood D_1 of $w=0$, depending only on D_2, mA, q, but not on $f(z)$, such that $w=f(z)$ assumes every value in D_2 at least q times if it does not assume every value in D_1. There is then a positive function $k(mA)$, such that $f(z)$ assumes every value in some circle of radius $k(mA)$.* With the help of these and more general results, we obtain theorems which contain as special cases well known theorems of Bloch, Bohr, Fekete, Montel, Valiron, and others. (Received February 3, 1933.)

70. Mr. R. A. Beaver: *Finite plane euclidean geometry.*

Veblen, Bussey, A. D. Campbell, and others have discussed finite geometries whose elements are defined in terms of marks of a Galois field. These papers, however, are almost exclusively devoted to finite *projective* geometry. In the present paper there is set up, by methods independent of projective geometry, a finite plane euclidean geometry, and its properties are extensively studied. (Received January 21, 1933.)

71. Dr. A. W. Tucker (National Research Fellow): *Cell-spaces.*

The cells of a complex K may be regarded as the elements of an abstract space; with stars of cells for neighborhoods such a cell-space satisfies the first three Hausdorff axioms. Certain subsets (of cells of K) are open, or closed; each subset X has a closure $Cl(X)$. Let $B(X)=Cl(X)-X$. Then equivalent to the Hausdorff axioms are the boundary axioms $X\cdot B(X)=0$; $B(0)=0$; $B(X+Y)=B(X)+B(Y)$, mod $X+Y$; $B(B(X))=0$, mod X, analogous to the chain boundary relations $F(\lambda C+\mu D)=\lambda F(C)+\mu F(D)$; $F(F(C))=0$. Let $K'=T(K)$ be a one-to-many correspondence of cells of K and K'. Then T^{-1} is "continuous" if T (open X) = open X'. Here T resembles a subdivision if $B(T(X))=T(B(X))$. All this has interesting interpretations in the dual complex K^*. (The terminology is that of Lefschetz's *Topology*; see also Tucker, Annals of Mathematics, April, 1933.) (Received February 2, 1933.)

72. Mr. Moses Richardson: *Invariants of symmetric product complexes.*

If a complex admits of a topological transformation of period p, it can be "collapsed" by regarding sets of p homologous points as single points. The Betti numbers (mod q) of the resulting complex are computed for every $q\neq p$. Applications yield all the Betti numbers of a p-fold symmetric product of a complex, except those mod p. (Received January 25, 1933.)

73. Professor P. A. Smith: *The torsion coefficients of symmetric products.*

The methods of M. Richardson (abstract No. 39–3–72) fail on account of certain exceptions to yield a complete knowledge of the homology characters of symmetric products; the special methods necessary to complete the results are considered in the present paper. (Received January 25, 1933.)

74. Professor W. A. Wilson: *A separation theorem.*

It is shown that, if A and B are connected sets lying in the locally connected compact metric space Z and $\overline{A} \cdot \overline{B}$ is totally disconnected or void, then Z is the union of two locally connected continua M and N, such that $M \cdot (\overline{B} - \overline{A} \cdot \overline{B})$ and $M \cdot (\overline{A} - \overline{A} \cdot \overline{B})$ are void, $\overline{A} \cdot \overline{B}$ is a subset of $M \cdot N$, and each component of $M \cdot N$ is locally connected. (Received January 16, 1933.)

75. Dr. Selby Robinson (National Research Fellow): *A metrization theorem of Chittenden and Pitcher.*

In 1919 (Transactions of this Society, vol. 20, pp. 229–230) Chittenden and Pitcher proved that for a Hausdorff space to be compact and metric it is necessary and sufficient that it have three properties (A_ω), (G), and (H). In the present paper it is shown that property (G) is a necessary condition for metrizibility and that in a compact Hausdorff space it is also sufficient. By the well known Urysohn theorem (1924), perfect separability is also a necessary and sufficient condition for metrizibility of compact Hausdorff spaces. Perfect separability is independent of property (G) in general Hausdorff spaces, but is a consequence of property (G) and the Lindelöf property. Property (G) is implied by Axiom 1 of R. L. Moore's book (this is still true if part (4) of Moore's axiom is omitted) and equivalent to it in locally compact Hausdorff spaces. A V-space satisfying Moore's axiom is a Hausdorff space which is regular, has a sequence G_n of families of regions which has a weaker form of the uniformity property of part (3) of Moore's axiom, and also a sequence G_n' satisfying part (4) of Moore's axiom. This uniformity property is implied by property (G), but together with regularity it implies property (G). (Received February 4, 1933.)

76. Professor Oystein Ore: *On a special class of polynomials.*

In this paper is given a discussion of a special class of polynomials, which the author has called p-polynomials. They are of interest because of their applications in number theory and also because they form an algebraic analogue to the linear differential polynomials. It is found that their roots form a modulus, and that adjoint polynomials, multipliers, and the Picard-Vessiot group of rationality can be introduced. A new proof of a theorem of Dickson's on the complete set of invariants of the full linear group (mod p) is given. (Received January 25, 1933.)

77. Mr. E. H. Hanson: *A note on compactness.*

This paper deduces the conditions for compactness of a set of measurable functions from the general criterion for compactness in complete metric spaces, which says that a set S of elements of a complete metric space is compact if and only if, for every positive ϵ, S is inclosable in a finite number of spheres of radius ϵ. Such procedure suggests itself immediately, and it succeeds without difficulty. Fréchet has obtained the result for measurable functions (Fundamenta Mathematicae, vol. 9, p. 25) but without making the deduction in the direct manner made possible by the use of the general criterion; the deduction given in this note, it would seem, is the simplest possible. (Received January 21, 1933.)

78. Dr. N. E. Rutt: *Concurrence and uncountability.*

Suppose that Z is a plane bounded continuum expressible as the sum of a continuum X and an orderable class G of continua no two of which have in common any point not belonging to X and no one of which has no point in X. Series of elements of G are defined in a natural way, and it is explained under what circumstances two such series may be called the same or opposite in sense, and when concurrent. It is then proved that no collection H of series no two of which are concurrent or opposite in sense can be both well ordered and uncountable. It is proved also that if H, as above, consists as well of series each of which has a limit point in some definite element E of G, then H is well ordered and consequently countable. These results are then applied to the problem of expressing the set consisting of X and certain subcollections of G as the sum of a countable collection of continua. (Received January 21, 1933.)

79. Dr. N. E. Rutt: *Prime ends and order.*

Suppose that Z is a plane bounded continuum expressible as the sum of a continuum X and an orderable class G of continua no two of which have any common point in the complement of X and all of which have points in common with X. Natural correspondences between the order of the elements of G and the order of the prime ends of the connected domain ξ complementary to X containing them are derived. These are applied to a study of the limiting sets of points of X and of prime ends of ξ which certain subcollections of G may have. The paper concludes with an application consisting of an improvement upon that notion of order among the points upon the boundary of ξ which is derived merely from the prime ends of ξ. (Received January 21, 1933.)

80. Dr. G. T. Whyburn: *Sets of local separating points of a continuum.*

Let L be the set of local separating points of a locally compact metric continuum M. In this paper it is shown that if H is any connected subset of M such that $\overline{H} \subset L+C$, where C is some countable set, then H is a locally connected G_δ. With this result it is proved that a necessary and sufficient condition for every connected subset of M to be a G_δ is that $M-L$ be countable. Thus if M is locally connected and the non-local-separating points of each cyclic element of M are countable, then every connected subset of M is arcwise connected. An example is given of a regular curve C having no continuum of condensation and containing nevertheless a connected subset which is not arcwise connected. (Received January 7, 1933.)

81. Professor A. A. Albert: *A note on the equivalence of algebras of degree two.*

In this Bulletin, vol. 36 (1930), the author gave necessary and sufficient conditions for the equivalence of any two rational generalized quaternion division algebras. These conditions were unnecessarily complicated, and were, moreover, not applicable to the case of algebras over a general field. The purpose of the present short note is to prove that the results of the paper cited

above may be simplified by a change of point of view to give very simple necessary and sufficient conditions for the equivalence of any two normal simple algebras of degree two. (Received January 19, 1933.)

82. Dr. L. E. Bush: *On Young's definition of an algebra.*

J. W. Young has given a definition of a *general algebra* (Annals of Mathematics, (2), vol. 29, pp. 54–60). Every *linear algebra* is a general algebra, but not conversely, and a large part of the theory of linear algebras is valid for the general algebra. The operations made use of in Young's definition, namely addition and multiplication, are operations *within* the algebra, i.e., they are applied to the elements of the algebra and yield an element of it. The usual definition of linear algebra presupposes a number field over which the algebra is taken, and makes use of an operation, namely scalar multiplication, not within the algebra. In the present paper certain additional restrictions are placed on Young's general algebra which make it equivalent to the usual definition of linear algebra. It is noted that these restrictions are equivalent to the postulation of a field over which the algebra is taken, and to the usual postulates on scalar multiplication. However, no use is made of an operation external to the algebra, although with these restrictions it is evidently possible to define a field and a scalar multiplication for the algebra. (Received December 22, 1932.)

83. Dr. L. E. Bush: *On a generalization of certain theorems on algebras with a modulus.*

Certain of the theorems which lead up to the arithmetic of a linear associative algebra with a modulus are here generalized to algebras without a modulus. In particular, it is shown that every non-nilpotent algebra over an algebraic field contains a maximal domain of integrity which includes a specified principal idempotent and all of whose elements satisfy integral equations. (Received December 22, 1932.)

84. Dr. R. D. James (National Research Fellow): *The representation of integers as sums of pyramidal numbers.*

It has been shown by K. C. Yang (abstract No. 34–4–27, this Bulletin) that every integer can be represented as the sum of at most nine pyramidal numbers $(x^3-x)/6$, $x=1, 2, \cdots$. In this paper it is proved that all large integers require at most eight pyramidal numbers. The proof is similar to Landau's proof (see, for example, his *Handbuch der Lehre von der Verteilung der Primzahlen*) of the theorem that all large integers are sums of eight cubes. (Received January 30, 1933.)

85. Mr. Solomon Kullback: *An application of characteristic functions to statistics.* Preliminary report.

P. Lévy's definition (*Calcul des Probabilités*, p. 161) of a characteristic function when extended to n variables gives the characteristic function of a function U of the variables $x_i (i=1, 2, \cdots, n)$ whose law of distribution is given by

$F(x_1, x_2, \cdots, x_n)dx_1dx_2 \cdots dx_n$, where F is, on a certain n-dimensional manifold, R, a single-valued, non-negative, continuous function such that we have $\int_R F(x_1, x_2, \cdots, x_n)\, dx_1dx_2 \cdots dx_n = 1$. A further extension has been made to the case of several functions $U_j(j=1, 2, \cdots, m)$ of the variables $x_i(i=1, 2, \cdots, n)$ whose law of distribution is the same as that above given. We show that the same results may be obtained by going back to fundamentals with the use of the discontinuity factor $(1/2\pi))\int_{-\infty}^{\infty}\int_{\omega_1}^{\omega_2} e^{i\vartheta(t-x)}dt\, d\theta$. We verify and extend the results of Charlier (Arkiv för Matematik, Astronomi och Fysik, vol. 8, No. 4, pp. 1–39) and T. Kameda (Proceedings of the Tokyo Mathematico-Physical Society, vol. 9, pp. 155–159). The general theory is applied for the derivation of the distribution function of means, standard deviations, simultaneous distribution of means and standard deviations, correlations, Student's ratio, χ^2, etc. for samples of $n(n \geqq 1)$ drawn from various parent distributions. (Received January 10, 1933.)

86. Professor M. H. Stone: *On the structure of Boolean algebras.*

This paper presents a theory of ideals and homeomorphisms for Boolean algebras and related systems which may be described as unitless Boolean algebras. The analogy with ideal theory in abstract rings (see van der Waerden's *Moderne Algebra*) is brought out by means of new sets of postulates for these algebras. Under the assumption that such an algebra A can be well-ordered, the existence and properties of prime ideals in A are established. Every ideal is shown to be the product of all its prime ideal divisors (it may be the product of fewer). The algebra A is shown to be isomorphic with an algebra A^* whose elements are subsets of the set E of all prime ideals in A. Each set in A^* can be regarded as a neighborhood (in the sense of Hausdorff) of every prime ideal belonging to it. When E is thus topologized, algebraic properties of A are interpretable as topological properties of E and conversely. In particular, the algebras homeomorphic with A are determined (up to isomorphisms) by means of the topological space E. It is of interest to note that all analogues of the maximal and minimal conditions so frequently postulated by algebraists lead in the present instance to finite algebras. (Received January 24, 1933.)

87. Dr. E. F. Beckenbach (National Research Fellow): *Bloch's theorem for minimal surfaces.*

There exists a positive absolute constant B with the following property. Let the unit circle be mapped conformally on a minimal surface with unit area deformation ratio at the origin; then on the minimal surface there is an open geodesic circle of radius at least B, containing no singular points, which is the one-to-one map of a portion of the unit circle. (Received January 9, 1933.)

88. Professor L. E. Ward: *A third-order irregular boundary value problem and the associated series.*

This paper is concerned with the characteristic functions defined by the differential system $u'''+[\rho^3+r(x)]u=0$, $\alpha_{12}u''(0)+\alpha_{11}u'(0)+\alpha_{10}u(0)=0$, $\alpha_{22}u''(0)+\alpha_{21}u'(0)+\alpha_{20}u(0)+\beta_{22}u''(\pi)+\beta_{21}u'(\pi)+\beta_{20}u(\pi)=0$, $\alpha_{31}u'(0)+\alpha_{30}u(0)=0$, where the α's and β's are real costants, the determinant

of the α's is not zero, not all the β's are zero, and $r(x)$ is analytic at $x=0$. There are derived certain properties of the function to which an infinite series of these characteristic functions converges if the convergence is uniform on any segment of the interval $(0, \pi)$. In brief, the function is analytic at $x=0$, and it and its derivatives satisfy an infinite set of equations at $x=0$, whose form depends on $r(x)$, its derivatives, and the α's of the first and third boundary conditions. Sufficient conditions are exhibited for the formal series corresponding to a given function to converge to that function, and a proof of such sufficiency is given. (Received January 6, 1933.)

89. Dr. R. E. A. C. Paley and Professor Norbert Wiener: *Notes on the theory and application of Fourier transforms.* I–II.

(I) The authors give a new proof of Carleman's theorem, that a necessary and sufficient condition that the class of functions defined by the inequalities $\int_{-\infty}^{\infty} |f^{(\nu)}(x)|^2 dx \leqq BA_\nu^2$ should be quasi-analytic is that $\int_0^\infty \log \left(\sum_{\nu=0}^{\infty} r^{2\nu}/A_\nu^2\right) \cdot [1/(1+r^2)]dr = \infty$. (II) The following theorem is proved: let $f(\theta)$ be an odd function non-decreasing in the range $(-\pi, \pi)$, and absolutely integrable in that range. Then the conjugate function $\tilde{f}(A)$ of $f(\theta)$ is also absolutely integrable. The condition that $f(\theta)$ should be odd cannot be omitted. (Received February 11, 1933.)

90. Dr. R. M. Gut: *On the decomposition of prime ideals in relative icosahedron fields.*

Let k be an algebraic field and K an extension field of k, the relative group G of which is the icosahedron group. Because G is simple, all prime ideals of k decompose in K, and according to the theorems of Hilbert on relative Galois fields, there are 15 possible types of decomposition. A. Speiser has shown that if we suppose that k contains the 5th roots of unity and eventually a further square root, there exists an equation of a certain normal type, defining K with respect to k. By means of certain normal resolvents and using theorems of Dedekind and Ore and certain auxiliary theorems, the present author has now determined the type of decomposition of any prime ideal of k which is prime to the relative degree 60. As a corollary there follows a necessary and sufficient condition that the relative different of K with respect to k is prime to the relative degree. This paper will appear in the Commentarii Mathematici Helvetici. (Received February 1, 1933.)

91. Professor Jesse Douglas: *A Jordan space-curve which bounds no finite simply-connected area.*

The author, in collaboration with Philip Franklin, recently gave an example of a skew polygon of a denumerable infinity of sides, every surface bounded by which has infinite area (Proceedings of the National Academy of Sciences, January, 1933). The present paper gives a simpler and more striking example of a Jordan space-curve such that every simply-connected surface bounded by the curve has infinite area. The brief description and proof will appear in the February, 1933, issue of the Proceedings of the National Academy of Sciences. (Received February 1, 1933.)

92. Mr. M. M. Flood: *On the division transformation for matric polynomials.*

The matric polynomial $h(\lambda)$ is said to be *associated* with the matric polynomial $b(\lambda)$ if their product $h(\lambda)b(\lambda)$ is of the same degree in λ as $b(\lambda)$ and if the coefficient of the highest power of λ in the product is a non-singular matrix. The conditions that the polynomial $h(\lambda)$ shall be linear, quadratic, or cubic are determined in this paper, and a solution for $h(\lambda)$ is exhibited in each case. It is shown, further, that if one such associated polynomial is linear, quadratic, or cubic, then all associated polynomials are linear, quadratic, or cubic respectively. These results are used in defining the division transformation for matric polynomials in case the leading coefficient of the divisor is singular but the determinant of the divisor not identically zero. (Received January 18, 1933.)

93. Professor C. A. Garabedian: *The uniformly loaded thick rectangular plate with at least two opposite edges supported.*

The problem suggested by the title comprises precisely the six cases that were solved by Estanave (Thèse, Annales de l'Ecole Normale, 1900) for the *thin* plate. Passing to the *thick* plate, the case of four supported edges (problem of Navier) was solved by Mesnager for central load (Comptes Rendus, vol. 164 (1917), p. 721) and by the author for uniform load (Comptes Rendus, vol. 178 (1924), p. 619). The case of two free edges (problem of the beam) was also solved by the author (Comptes Rendus, vol. 179 (1924), p. 381). A recent paper by Galerkin (Comptes Rendus, vol. 193 (1931), p. 568) revives interest in our sixfold problem by describing a new possible method of attack, but does not give explicit solutions. The aim of the present paper is to exhibit in concise and usable form the complete solutions of all six cases. (Received January 6, 1933.)

94. Dr. G. B. Price: *A study of certain dynamical systems with applications to the generalized double pendulum.*

The first part of this paper investigates the analytic continuation of periodic orbits in the neighborhood of a position of equilibrium of a reversible dynamical system with n degrees of freedom. Two existence theorems establish the existence of periodic orbits in certain new cases, and show that these orbits join two points of the oval of zero velocity. The second part of the paper continues the study in the case of two degrees of freedom by means of surfaces of section and surface transformations. Results on periodic orbits are obtained. The results are more specific when the system is symmetric in the origin. In the third part of the paper the theory developed in the first two parts is applied to the generalized double and n pendulums, systems of which the ordinary double pendulum is a special case. Most of the results of the paper hold only in the small, but it is shown in the third part that for the symmetric generalized double pendulum there exists in the large at least one periodic orbit which joins two points of the oval of zero velocity. (Received February 2, 1933.)

95. Dr. C. F. Luther: *Concerning primitive groups of class u. Paper II.*

Limits to the degree of multiply transitive groups of class $u(>3)$ containing a substitution of order 2 and degree $u+\epsilon$ were given by the author in the American Journal of Mathematics, vol. 55 (1933), p. 77. The present paper is concerned with the limits of the degree n of the same groups that happen to be more than $2^\alpha+p_1+p_2+\cdots+p_r$ times transitive, where $\alpha\geqq 2$, and p_1, p_2, p_3, $\cdots$, p_r are distinct odd primes. It is found that $n<2^\alpha p_1p_2\cdots p_r u/(2^\alpha p_1p_2\cdots p_r-2)+2^\alpha\epsilon/(2^\alpha-2)+1$. If the condition upon the transitivity is merely that it exceed $2^\alpha(\alpha\geqq 2)$, the hypothesis that it be of class u is redundant. It can then be said that if the group contains a substitution of order 2 and degree v, it will follow that $n\leqq 2^\alpha v/(2^\alpha-2)$, provided the group is not alternating or symmetric. (Received February 4, 1933.)

96. Professor Pauline Sperry: *Ernest Julius Wilczynski.*

Professor Sperry's paper is devoted to a short description of the life and work of Ernest Julius Wilczynski, whose long illness and recent death at a comparatively early age have inflicted so severe a loss upon mathematicians the world over, and deep sorrow upon a host of his friends and pupils. His work as the founder of projective differential geometry is compared briefly with later developments in Europe, especially in Italy, and his method is outlined and an illustration of it given. (Received February 8, 1933.)

97. Professor B. A. Bernstein: *Simplification of the set of four postulates for Boolean algebras in terms of rejection.*

The author simplifies the set of four postulates previously obtained by him as a reduction of Sheffer's set of five postulates for Boolean algebras expressed in terms of the operation "rejection." The simplification consists in replacing one of the postulates, P4, by a proposition in which the negative elements are fewer, and more symmetrically distributed, than in P4. In connection with the proof of the sufficiency of the new postulates, the author derives from them not only his old set, but also Sheffer's set and the Whitehead-Huntington set. The consistency and the independence of the new postulates are established in connection with the complete existential theory furnished for the postulates. (Received February 9, 1933.)

98. Dr. A. H. Diamond: *The complete existential theory of the Whitehead-Huntington set of postulates for the algebra of logic.*

In this paper the author establishes the *complete existential theory* of the Whitehead-Huntington set of ten postulates for the algebra of logic, expressed in terms of logical addition and logical multiplication. The theory consists of propositions stating the existence or non-existence of concrete systems corresponding to all the affirmations and denials of the individual postulates. The propositions and concrete systems required to establish the theory far outnumber those of complete existential theories hitherto published. The systems are all algebras of not more than three elements which, by means of the *modular* theory of Professor B. A. Bernstein, are expressed with a conciseness

and a simplicity not existing in proof-systems employed before the development of this theory. Further, the systems relate to postulates expressing laws found in many other mathematical theories and so may prove to be valuable as a source of proof-systems to be used in other sets of postulates. (Received February 14, 1933.)

99. Professor E. T. Bell: *The Latin square functions.*

The generalizations of the partial differential equations of mathematical physics recently discussed by G. Humbert and others make use of the functions introduced by Appell in 1877 which generalize circular and hyperbolic functions by being based on an imaginary nth root of unity. Passing at once to the general case we construct functions based on exponential functions involving an arbitrary algebraic number of degree n. When the algebraic field concerned is cyclotomic, the functions degenerate to Appell's. The degenerate functions are related by means of the norm in a cyclotomic field. The norm is replaced in the general case by a set of relations connected with the general Latin square of degree n. The generalization is not immediately obvious. In Appell's case the very degenerate Latin square is invariant under the substitutions of the cyclic group of degree n on the elements of its first row. In the general case there is no group connected with the Latin squares. (Received February 16, 1933.)

100. Dr. A. E. Ross: *A note on three equivalent theorems.*

Legendre's classical theorem on the indeterminate equation $ax^2+by^2+cz^2=0$ can be derived by means of quite elementary considerations (see, for example, Dedekind's proof in Dirichlet's *Zahlentheorie* or in Dickson's *Introduction to the Theory of Numbers*). Next, it can be shown that conditions sufficient for the existence of a non-trivial solution insure the existence of a *proper* solution even with the omission of the requirement that a, b, c should contain no square factors. Legendre's theorem thus augmented is used (ibid.) to prove, quite simply, Gauss's celebrated *theorem on duplication* for binary quadratic forms. H. J. S. Smith (Collected Mathematical Papers, vol. 1, pp. 480–83) employs the theorem on duplication to prove that any two ternary quadratic forms of the same genus can be transformed into one another by means of a transformation of determinant unity and with rational coefficients whose greatest common denominator is prime to $2\Omega\Delta$. It may seem at first sight that in passing from the first to the second theorem and then finally to the third one we each time pass from one statement to another of different depth. Therefore it may be of interest to note that assuming the last mentioned theorem of Smith we can deduce very simply the first mentioned criteria for solvability of $ax^2+by^2+cz^2=0$, and thus to show that the three theorems are actually equivalent. (Received February 16, 1933.)

101. Dr. D. C. Duncan: *A self-dual septimic possessing seven of each kind of the simple singularities, autopolar by seven rectangular hyperbolas and a circle.*

The equation of the locus is developed in homogeneous rectangular coordinates x, y, z under the assumptions: (1) the seven cusps are distributed at

equal intervals about the unit circle; (2) the curve is invariant under rotations through angles $2k\pi/7 (k=1, 2, \cdots, 6)$; (3) each cuspidal tangent is an axis of symmetry. It is then shown that the locus is invariant under 14 collineations and 14 correlations of which 8 are polarities by real conics. The locus is unipartite, elliptic, and approximately realized by drawing secant lines through alternate points of the seven points equally spaced about the unit circle and then removing the chords from the interior of the circle. (Received February 16, 1933.)

102. Dr. D. C. Duncan: *Three autopolar rational quintic curves.*

The self-dual rational quintic is in general not invariant under collineations and correlations (L. H. Swinford, Thesis, University of California Publications, 1929). The quintic admitting the maximum number of correlations is invariant under a G_{12} consisting of six collineations, and six correlations of which four are polarities (L. E. Wear, this Bulletin, June, 1919, p. 405). In that exposition only one polarity is effected by a real conic and of the twelve collineations and correlations eight are imaginary. In the present paper three forms of the self-dual rational quintic are discussed: (1) completely symmetric, singularities all distinct, invariant under the G_{12} including three of the four polarities by real conics; (2) quintic with three rhamphoid cusps, invariant under G_6, 3 collineations and three polarities by three real conics; (3) quintic with two rhamphoid cusps, other singularities distinct, invariant under two collineations and two polarities by real conics. (Received February 16, 1933.)

103. Dr. R. D. James: *The value of the number $g(k)$ in Waring's problem.*

Let $g(k)$ be the least value of s such that every integer n is the sum of s non-negative kth powers; let $G(k)$ be the least value of s such that every integer $n > C$ is the sum of s non-negative kth powers. In the present paper an upper bound for $g(k)$ in terms of k is deduced from the known results of the Hardy Littlewood analytic method for $G(k)$ when $k \geqq 6$. (See E. Landau, *Vorlesungen über Zahlentheorie*, vol. 1, Part 6.) The method consists in first determining the number C as a function of k, s, and ϵ, where ϵ is a small positive quantity; thus every integer $n > C(k, s, \epsilon)$ is a sum of s kth powers when $s \geqq s_1(k, \epsilon)$. Then, using a theorem proved by L. E. Dickson (this Bulletin, 37 (1931), page 551), it is shown that every integer $n \leqq C(k, s, \epsilon)$ is a sum of s kth powers when $s \geqq s_2(k, \epsilon)$. The quantity ϵ is chosen as a function of k so that $s_1(k) = s_2(k)$ and hence $g(k) \leqq s_1(k) = s_2(k)$. In particular it is proved that *all* numbers are sums of at most 183 sixth powers, 353 seventh powers, 665 eighth powers, 1252 ninth powers, 2504 tenth powers, $\cdots$. The best results previously known were 478 sixth, 3806 seventh, 31353 eighth, (ninth not known), 140004 tenth powers, $\cdots$. (Received February 17, 1933.)

104. Dr. Anne D. B. Andrews: *The space quartic of the second kind by synthetic methods.*

One set of rulings of a quadric surface κ and the tangent planes to a quadric cone K are put into projective one to one correspondence by means of a pencil

of planes Σ_A of the first order. The locus Q of points of intersection of corresponding elements is a space quartic of the second kind. To each point A of Q corresponds a Σ_A, the locus of whose axes is a quadric cone Σ. The various relations between K, Q, and Σ are developed. (Received February 18, 1933.)

105. Professor W. M. Whyburn: *An existence theorem for differential systems.*

Perron and others (see Müller, Mathematische Zeitschrift, vol. 26, pp. 637–645) have given existence theorems for differential systems that are of a type different from the ones that ordinarily occur in mathematical literature. These theorems outline a procedure which yields the existence of solutions on certain intervals and, at the same time, gives many properties of the solution functions. The present paper establishes a further theorem of the general type introduced by Perron. The paper also strives to remove some of the notation difficulties of former theorems and to show that this group of existence theorems constitutes a powerful tool in applied mathematics. (Received February 20, 1933.)

106. Mr. E. D. Rainville: *On the representation of numbers modulo m.*

The classical Kronecker-Dirichlet extension of the notion of primitive root permits a unique representation modulo m for any n such that $(n, m)=1$. In another investigation the removal of the condition $(n, m)=1$ was necessary. That is the contribution of this note. If $m=2^{\alpha_0}\rho_1^{\alpha_1}\cdots\rho_\nu^{\alpha_\nu}$, the representation in question is $n\equiv 2^{\sigma_0}\lambda^{i}\lambda_0^{i_0}\prod_{k=1}^{k=\nu}\rho_\kappa^{\sigma_\kappa}\lambda_\kappa^{i\kappa}(\text{mod } m)$ where the exponents in the congruence are subject to a system of simple restrictions. (Received February 13, 1933.)

107. Dr. T. S. Peterson: *An integral equation with symmetric kernels.*

A solution, $X(x, y)$, of the integral equation $\int_a^b A(x, s)\,X(s, y)ds+\int_a^b X(x, s)\cdot B(s, y)ds=C(x, y)$, with symmetric kernels $A(x, y)$ and $B(x, y)$, may not in general exist. In this paper necessary restrictions on the function $C(x, y)$ are determined in order to insure the existence of solutions; and with these restrictions satisfied the nature of solutions is given. A discussion of "characteristic values" with respect to a pair of kernels plays an important part in the restrictive conditions on the function $C(x, y)$ and the solutions of the above equation. (Received February 14, 1933.)

108. Dr. T. S. Peterson: *The analogue of Weyl's conformal curvature tensor in a Michal functional geometry.*

In a series of papers A. D. Michal has investigated the properties and invariants of a certain type of functional geometry. It is the purpose of this paper to obtain a set of conformally invariant functionals which by analogy with current n-dimensional geometries may be called the functional conformal curvature tensor of the space. (Received February 14, 1933.)

109. Professor W. A. Manning: *The order of primitive groups (V).*

If n is the degree and $u(>3)$ the class of an 8-ply transitive group, $n<3u/2$. By means of this new theorem it is possible to show that if a primitive group of degree n (not alternative or symmetric) contains a substitution of prime order p on qp letters ($q>7$; $p>2q-3$), then $n<qp+3q-6$. (Received February 24, 1933.)

110. Professor Raymond Garver: *The transformation $y=f'(x)$.*

The transformation $y=f'(x)$ on the roots of an nth degree equation $f(x)=0$ leads to a transformed equation whose constant term, except for a constant factor, is the discriminant of $f(x)=0$. If the transformation can be applied easily we have a convenient method of finding the discriminant. The method is applied to the cubic, quartic, Bring-Jerrard quintic, Brioschi quintic, principal quintic and general trinomial equation. (Received February 25, 1933.)

111. Professor B. A. Bernstein: *On Section A of Principia Mathematica.*

Section A of Whitehead and Russell's *Principia Mathematica*, which is concerned with the theory of deduction, contains "formal" propositions and "informal" statements. The writer discusses the logic of Section A in the light of Huntington's new sets of postulates for the algebra of logic. The main results are that the "formal" Section A, whether in Huntington's T-form or in the writer's 1-form, is derivable from the logic of classes; that this "formal" Section A is not the *whole* of the logic of classes; and that neither of Huntington's two systems for the combined "formal" and "informal" Section A is sufficient for the logic of classes or for the logic of propositions. (Received March 9, 1933.)

112. Professor A. R. Williams: *Analogues of the Steiner surface and their double curves.*

The rational surface $x_1:x_2:x_3:x_4=x^n:y^n:z^n:w^n$, where x, y, z, w, are linear functions of 3 homogeneous parameters, and $x+y+z+w\equiv 0$, is of order n^2. There are 3 multiple right lines of order n, concurrent if n is even, coplanar if n is odd. The order of the double curve is $(n^2-1)(n^2-2)/2-(n-1)(n-2)/2$. In addition to the multiple lines, it consists of $6(n-1)(n-2)/2$ plane curves, each of order n, lying in planes through the 6 edges of the tetrahedron, and $(n-1)(n-2)(n-3)/2$ skew curves of order n. Each of these curves is isolated, except at one point, from the continuous portion of the surface, and its image in the plane is a pair of imaginary lines. (Received March 15, 1933.)

113. Professor W. A. Manning: *Degree and class of multiply transitive groups.*

At the 1932 summer meeting of the Society the author announced that if all the substitutions of degree $\leqq u+e$ of a t-ply transitive group ($t>6$), of degree n and class u (>3) are of odd order, then $n<2u-4e-2t+13$. This

limit can be replaced by $2u-4e-5t+37$, without change in the hypothesis of the theorem. By means of this new limit and certain results due to Dr. C. F. Luther, it can now be proved that if a group of degree n and class $u(>3)$ is t-ply transitive ($t>23$), then $n<6u/5+u/t-t$. In the former communication it was shown that for such a group, $n<3u/2$ if $t>11$. But this is now shown to be true if $t>7$. Other results of the present paper are that $n<4u/3$ if $t>11$, and $n<5u/4-t$, if $t>21$; it being understood that the t-ply transitive groups in question are of class u and not alternating or symmetric. (Received February 9, 1933.)

114. Dr. E. F. Beckenbach and Professor Tibor Radó: *On the isoperimetric inequality.*

If a is the area and l the length of the boundary curve of a simply-connected plane region S, then $a\leqq(1/4\pi)l^2$. This classical theorem has been generalized by Carleman who showed that the inequality holds also if S denotes a minimal surface. It is proved in the present paper that the inequality remains valid for every surface S whose Gaussian curvature K is $\leqq 0$. Conversely, if the inequality holds for every portion of a surface S, then its K is $\leqq 0$. The proofs depend upon inequalities involving simple and double integral means of subharmonic functions. Some generalizations of these latter inequalities, which seem to be interesting in themselves, are also developed in the paper. (Received February 10, 1933.)

115. Professor J. V. Collins: *An unused system of coordinates in analytic geometry.*

This paper points out some of the advantages in analytic geometry of the (r, x) coordinate system which defines a point by its distance from the origin and its abscissa. The (r, x) system is especially well adapted to the discussion of conics and of certain higher plane curves, as Cassini's oval and the conchoid of Nicomedes. It is also useful in the discussion of tangents, radii vectores, and radii of curvature. (Received February 19, 1933.)

116. Professor M. H. Ingraham: *On the rational solutions of the matric equations* $P(X)=A$.

Consider a polynomial P with coefficients in a field F and an $n\times n$ square matrix A with elements in F. The $n\times n$ square matric solutions X of $P(X)=A$, where X is a polynomial in A, were first found by Roth, who with others studied the general solution. This paper gives a new development of both these solutions and in addition gives a finite process for finding all the solutions, X, whose elements are rational in the coefficients of P and the elements of A. (Received Februry 21, 1933.)

117. Mr. H. H. Alden: *Solution of* $f(x, y)\partial z/\partial x+g(x, y)\partial z/\partial y=0$ *in a neighborhood of a singular point.*

The real functions f and g are assumed to have continuous first partial derivatives in an open region R containing just one singular point. By a solution

of the given equation is understood a function $z(x, y)$ which is constant in no circle however small, and for which first partial derivatives exist and reduce the equation to an identity in x and y. If the singular point is a spiral point or a knot point of the system $dx/dt=f(x, y)$, $dy/dt=g(x, y)$, it is shown that no solution of the partial differential equation exists over the singular point. If the orbits of the system consist of a sequence of closed curves converging to the singular point, and spirals between these closed curves, no solution exists over the singular point. If the orbits are all closed, a solution over the singular point is exhibited, and is defined as follows: $z(x, y)$ equals the square of the least distance from the singular point to the orbit through (x, y). If the orbits form a saddle, the same construction yields a solution over the singular point and all orbits not having the singular point as a limit point. The partial derivatives of $z(x, y)$ are continuous except possibly at the singular point. (Received March 1, 1933.)

118. Dr. Francis Regan: *The application of the theory of admissible numbers to time series with constant probability.*

Admissible numbers furnish a method for testing the consistency of the assumptions of the theory of probability. In this paper, the author extends the concept of admissibility to time series having a constant probability. An ideal set of points representing the time series is set up on the time axis in such a manner that there is a definite probability of getting a point in any interval. This set possesses the property necessary to use the statistical point of view of probability without any modifications, and if it is to satisfy the fundamental assumptions of the theory of probability, it is necessary that sequences of successes and failures be represented by the digits of admissible numbers. Hence, the number of conditions imposed upon these points has the power of the continuum since, for every interval, a different set of conditions is obtained. This paper shows the manner in which this set of points is obtained in order that these conditions be consistent and be satisfied. (Received March 3, 1933.)

119. Professor L. M. Graves: *The existence of an extremum in problems of Mayer.*

This paper gives sufficient conditions for the existence of an absolute minimum in certain classes of Mayer problems with variable or fixed end-points. These classes are much more inclusive than those treated by the author in a recent note (this Bulletin, February, 1933, p. 101), although they do not include the difficult types of isoperimetric problems. The Mayer problem is treated both in parametric and in non-parametric form. As an essential preliminary to the existence proof, theorems are given on the semi-continuity of solutions of systems of differential equations which depend on arbitrary curves. (Received March 3, 1933.)

120. Mr. D. S. Nathan: *Regular infinitesimal transformations in a composite function space.*

The results obtained by Kowalewski (Wiener Sitzungsberichte, vol. 120 (1911), p. 1435) for the space C of real continuous functions of a real variable defined on (0, 1) are here extended to the composite function-space (C, E_n), in which there serve as additional coordinates n independent real variables. Through the introduction of composite integral power series, regular infinitesimal transformations in (C, E_n) are defined. The commutator of two such transformations is defined, and the group property is expressed in terms of it. The infinitesimal projective and conformal groups in (C, E_n) are obtained. The following generalization of a theorem by Barnett (Proceedings of the National Academy of Sciences, vol. 15 (1929), p. 96) is proved: The subgroup of the infinitesimal projective group in (C, E_n) leaving invariant the unit sphere in (C, E_n) is transformed by a stereographic projection into the infinitesimal conformal group in (C, E_{n-1}). (Received March 6, 1933.)

121. Mr. D. S. Nathan: *Linear transformations in a composite function space.*

The non-singular finite linear transformations, and the infinitesimal linear transformations, in (C, E_n) (see abstract, No. 39–3–120) are shown to have the group properties. A given infinitesimal linear transformation in (C, E_n) generates a one-parameter group of non-singular linear transformations in (C, E_n). The coefficients of the generated transformations are determined. The application of these results to the obtaining of the finite projective and conformal groups in (C, E_n) is made possible by the introduction of homogeneous coordinates in the former case and of homogeneous sphere coordinates in the latter. A further application is in the proof that a given infinitesimal projective transformation in (C, E_n) generates a one-parameter group of finite projective transformations in (C, E_n) and in the determination of the coefficients of the generated transformations, with an analogous application to the conformal group. (Received March 6, 1933.)

122. Mr. N. E. Steenrod: *Finite arc-sums.*

It is shown that for a bounded and closed subset of M of E_n to be a subset of the sum of *two* arcs of E_n, it is necessary and sufficient that M have no continuum of condensation. The problem of obtaining conditions under which a continuous curve is the sum of a finite number of arcs is reduced to the problem for the true cyclic elements as follows: In order that a continuous curve M be the sum of a finite number of arcs, it is necessary and sufficient that (1) the end-points of M be finite in number, and (2) each true cyclic element of M be the sum of a finite number of arcs, in every case numbering not more than a fixed integer k, and, in all but a finite number of the cyclic elements, beginning and ending at cut-points of M. In the special case of the boundary of a plane domain, a necessary and sufficient condition is that the nodes be finite in number. (Received March 6, 1933.)