PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 106, Number 1, May 1989

SOME SPECIAL 2-GENERATOR KLEINIAN GROUPS

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(Communicated by Irwin Kra)

ABSTRACT. We explore the following question. Let G be the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by the elements A and B, where A has two fixed points, and B maps one fixed point of A onto the other; when is G discrete?

In conjunction with their work on the generalization of Jørgensen's inequality, Gehring and Martin [G-M] asked the following question (oral communication): Let $G_0 = \langle a, b\rangle$ be the subgroup of $\mathrm{PSL}(2, \mathbf{C})$ generated by an elliptic element a of order 6, and a parabolic element b, where b maps one fixed point of a onto the other; is G_0 discrete?

In this note, we give an affirmative answer to this question, describe some of the properties of this interesting group, and explore the following more general question:

Suppose a and b are elements of $\mathrm{PSL}(2, \mathbf{C})$, where a has exactly two fixed points, and b maps one of these fixed points to the other. When is $G_0 = \langle a, b\rangle$ discrete?

1. We start with a necessary condition.

Proposition 1. *If G_0 is discrete, then either a is elliptic of order* 2, 3, 4, *or* 6, *or b is elliptic of order* 2.

Proof. Normalize so that the fixed points of a are at 0 and ∞, where $b(0) = \infty$. If b is not a half-turn (i.e., an elliptic element of order 2), then $b(\infty) \neq 0$, so a and bab^{-1} have exactly the one fixed point at ∞ in common. Then the commutator $c = [a, bab^{-1}] = a(bab^{-1})a^{-1}(bab^{-1})^{-1}$ is parabolic with fixed point at ∞. In a discrete group, a loxodromic (including hyperbolic) element and a parabolic element cannot share a fixed point, hence a is elliptic. It follows that $\langle a, c\rangle$ is a discrete group of Euclidean motions; hence the order of a must be 2, 3, 4, or 6. □

2. If b is a half-turn, then G_0 preserves the fixed point set of a, so G_0 conjugates a into a^{-1}; hence G_0 is a $\mathbf{Z}_2$ extension of $\langle a\rangle$, the cyclic group

Received by the editors December 2, 1987 and, in revised form, April 27, 1988.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30F40.
Research supported in part by NSF Grant #DMS 8701774.

generated by a. It follows that, in this case, G_0 is discrete if and only if $\langle a \rangle$ is discrete. Of course $\langle a \rangle$ is discrete if and only if a is either loxodromic or elliptic of finite order.

3. From here on we assume that b is not a half-turn, and that a is a geometrically primitive elliptic transformation of order $\alpha = 2, 3, 4$, or 6 (an elliptic transformation is geometrically primitive if it is conjugate to a rotation of the form $z \to e^{2\pi i/q} z$, $q \in \mathbf{Z}$). We will also assume that b maps one fixed point of a onto the other. We will show below that $G_0 = \langle a, b \rangle$ is discrete if b is parabolic, hyperbolic, or a geometrically primitive elliptic transformation of finite order. When it is necessary to make it clear which of these groups we are referring to, we will label the group as $G_0(\alpha, \delta)$, where α is the order of a, and either $\delta = 0$, or $\delta = \infty$, or δ is an integer ≥ 3. This has the following meaning. If $\delta = 0$, then b is hyperbolic; if $\delta = \infty$, then b is parabolic; otherwise, b is a geometrically primitive elliptic transformation of order δ.

There are two other numbers that come up in conjunction with the number α. The first is β, defined by $1/\alpha + 1/\beta = 1/2$; if $\alpha = 2$, then $\beta = \infty$. The second is $\gamma = \beta/2$. Note that β is again an integer (including ∞); also γ is an integer for $\alpha = 2, 3$, or 4, but not for $\alpha = 6$.

We will show that these groups are discrete by explicitly constructing fundamental polyhedra, and using Poincaré's polyhedron theorem. A proof of this theorem, along with the requisite definitions, can be found in [**M**, p. 73 ff].

We will need the following computation.

Lemma 1. *Let a be an elliptic transformation of order α, with fixed points at x and y. Let b be a Möbius transformation where $b(x) = y$. If $\alpha = 2$, then the commutator $[a^{-1}, b^{-1}]$ is parabolic, and if $\alpha = 3$ or 4, then $[a^{-1}, b^{-1}]$ is elliptic of order γ.*

Proof. Normalize so that a has fixed points at 0 and ∞, and so that $b(0) = \infty$. Write

$$a = \begin{pmatrix} e^{\pi i/\alpha} & 0 \\ 0 & e^{-\pi i/\alpha} \end{pmatrix}, \qquad b = \begin{pmatrix} s & t \\ -t^{-1} & 0 \end{pmatrix},$$

and compute the trace of the commutator. □

One of the conclusions of Poincaré's polyhedron theorem is that one can read off a presentation, in the usual group theoretical sense, from the identifications of the sides of the polyhedron. One is also often interested in knowing which elements of a subgroup of $\mathrm{PSL}(2, \mathbf{C})$ are parabolic. Since every commutative subgroup of rank > 1 in a discrete group of hyperbolic motions is purely parabolic, it suffices to know the rank 1 parabolic subgroups. If one has a given (convex) fundamental polyhedron P for the discrete group G, then every fixed point of a parabolic element in G is equivalent to a point on the boundary of P [**M**, p. 123]. It follows that one can also read off generators for rank 1 parabolic subgroups from the identifications of the sides of P.

A *presentation* for a discrete subgroup H of $\mathrm{PSL}(2,\mathbf{C})$ is of the form $H = \langle a_1, a_2, \ldots : w_1^{\alpha_1} = w_2^{\alpha_2} \cdots = 1\rangle$, where the a_j are generators of H, the w_j are words in these generators, and the α_j are either non-negative integers or the symbol "∞". The presentation has the following meaning. If one ignores those words where $\alpha_j = 0$ or ∞, then one has an ordinary presentation of the algebraic structure of H. For each j where $\alpha_j = \infty$, w_j is parabolic, and the maximal commutative subgroup of H containing w_j has rank 1. Further, if g is a parabolic element of H, where the maximal commutative subgroup of H containing g has rank 1, then g is conjugate to a power of some w_j with $\alpha_j = \infty$. For the purpose of uniformity of presentation, one sometimes has a word of the form w^0 in the presentation; these have no meaning and should be ignored.

Our polyhedra will all be convex, and will be constructed in the upper half-space, $\mathbf{H}^3 = \{(z,t)|z \in \mathbf{C}, t > 0\}$, which we consider to be endowed with the hyperbolic metric:

$$ds^2 = t^{-2}(|dz|^2 + dt^2).$$

A (finite) convex polyhedron P in $\mathbf{H}^3$ is the intersection of a finite number of (hyperbolic) half-spaces D_j; the half-space D_j has boundary S_j in $\mathbf{H}^3$, and S_j has Euclidean boundary C_j in $\widehat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. We will describe D_j by B_j, the restriction of its Euclidean boundary to $\widehat{\mathbf{C}}$. We denote the face of D lying on the side S_j by the same letter S_j; this should cause no confusion.

4. In this section, we explicitly construct fundamental polyhedra with identifications for $\alpha = 2, 3$, and 4, and for $\delta = 0$, $\delta \geq 3$, and $\delta = \infty$.

We start with some definitions. A Kleinian group G is a discrete subgroup of $\mathrm{PSL}(2,\mathbf{C})$; it is of the first kind if every point of $\widehat{\mathbf{C}}$ is a limit point of G (that is, every point is the limit of a sequence of points of the form $\{g_m(x)\}$, where x is any point of $\mathbf{H}^3$); it is of the second kind otherwise.

If G is of the second kind, then the set of points on $\widehat{\mathbf{C}}$ at which G acts discontinuously is called the regular set, and is denoted by $\Omega = \Omega(G)$. If G is finitely generated with Ω/G connected (which is the only case that occurs here), then Ahlfors' finiteness theorem asserts that Ω/G can be represented as a closed Riemann surface of genus p, from which a finite number of points have been removed (these are identified as special points of order ∞), and the covering is branched over finitely many points (these are identified as special points of order s, where s is the order of branching). If in addition, every component of Ω is simply connected, then the signature of G is defined as $(p, n; s_1, \ldots, s_n)$, where n is the total number of special points, and $s_1, \ldots, s_n$ are their orders.

A function group is a finitely generated Kleinian group where some component of Ω is kept invariant by the entire group. Function groups have more complicated signatures (see [**M**, p. 271 ff]).

Theorem 1. *For* $\alpha = 2, 3$, *or* 4, *the group* $G_0(\alpha, \delta)$ *is discrete. It has the (Kleinian group) presentation*:

$$G_0 = \langle a, b \colon a^\alpha = b^\delta = [a^{-1}, b^{-1}]^\gamma = 1 \rangle .$$

Further, $G_0(\alpha, \delta)$ *is of the first kind if* $(\alpha, \delta) = (4, 4)$, $(4, 3)$, *or* $(3, 3)$; *it is of the second kind otherwise. For the groups that are of the second kind, if* $\delta \neq 0$, *then every component of* Ω *is a circular disc,* $\Omega(G_0)/G_0$ *is connected and has signature* $(0, 3; \gamma, \delta, \delta)$. *If* $\delta = 0$, *then* $\Omega(G_0)$ *is connected but not simply connected; it can be topologically described by a function group signature as follows: the signature has one part of basic signature* $(0, 3; \alpha, \alpha, \gamma)$, *and there is an* α*-connector between the two special points of order* α.

Proof. We first remark that the statement about function group signatures can be restated as follows: $\Omega(G_0)/G_0$ is a single Riemann surface of signature $(1, 1; \gamma)$, and there is a nondividing simple loop on this surface so that α is the smallest positive power for which this loop lifts to a loop in Ω.

Construction of $G_0(\alpha, 0), \alpha = 2, 3, 4$

Normalize so that $b(z) = \lambda^2 z$, $\lambda > 1$, and so that a has its fixed points at 1 and λ^2. If $\alpha \neq 2$, choose a, as opposed to a^{-1}, so that the center of the isometric circle of a lies in the upper half-plane. Define P by the sides $B_1 = \{z \mid |z| > 1\} \cup \{\infty\}$; $B_2 = \{z \mid |z| < \lambda\}$; B_3 is the outside of the isometric circle of a; and B_4 is the outside of the isometric circle of a^{-1}; if $\alpha = 2$, then $S_3 = S_4$ (see Figure 1, for $\alpha = 3$). Observe that $b(S_1) = S_2$, and $a(S_3) = S_4$.

For $\alpha \neq 2$, the faces S_3 and S_4 meet at an angle of $2\pi/\alpha$; this edge is not equivalent to any other edge. The faces S_1 and S_2 meet the faces S_3 and S_4 at an angle of π/β; these four edges are equivalent, and form a cycle of

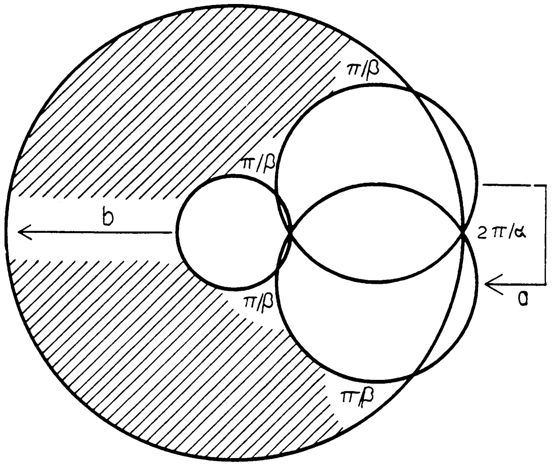

FIGURE 1

edges, the sum of the angles in this cycle is $2\pi/\gamma$. Lemma 1 shows that the corresponding cycle transformation, $[a^{-1}, b^{-1}]$, has order γ.

For $\alpha = 2$, the polyhedron has no edges; there is one reflection relation, namely $a^2 = 1$; and there is one infinite cycle transformation, $[a^{-1}, b^{-1}]$, which is parabolic.

The information above shows that our polyhedron with identifications satisfies the hypotheses of Poincaré's polyhedron theorem. We conclude that $G_0(\alpha, 0)$ is discrete, and has the given presentation.

Since the boundary of this fundamental polyhedron is a fundamental domain for the action of G on $\widehat{\mathbf{C}}$ [M, p. 116], we see at once that $\Omega(G_0)/G_0$ is connected; hence G_0 is a function group. Identifying the sides of this fundamental domain, we see that $\Omega(G_0)/G_0$ is a torus with one special point of order γ. To obtain the Kleinian group signature, note that there is a simple loop in $\Omega(G_0)$ that is precisely invariant under $\langle a \rangle$; the projection of this loop is a simple loop, which, when raised to the αth power, lifts to a loop. It follows from the planarity theorem (see [M, p. 251]) that since we are on a torus with one boundary component, this planar regular covering is completely determined by the ramification number at the special point, and this one loop, together with the number α.

Construction of $G_0(\alpha, \delta)$, $\alpha = 2, 3, 4$, $2 < \delta < \infty$

We next take up the case that b is a geometrically primitive elliptic transformation of order $\delta > 2$. We normalize so that $b(z) = e^{2\pi i/\delta} z$, and so that the fixed points of a are at $e^{\pm \pi i/\delta}$. If $\alpha \neq 2$, then we further normalize, by perhaps replacing a with a^{-1}, so that $a^{-1}(\infty)$ lies inside the unit disc.

We define the sides of P by $B_1 = \{z\colon -\pi/\delta < \arg(z) < -\pi/\delta + \pi\}$; $B_2 = \{z\colon \pi/\delta - \pi < \arg(z) < \pi/\delta\}$; C_3 is the circle passing through the fixed points of a and making an angle of π/α with the unit circle (this angle is measured outside C_3 and inside the unit circle); then a is the composition of reflection in C_3 followed by reflection in the unit circle. Let B_3 be the outside of C_3. For $\alpha = 2$, set $C_4 = C_3$; for $\alpha > 2$, let $C_4 = a(C_3)$; in either case, define B_4 so that a maps B_3 onto the complement of B_4. Note that C_4 also makes an angle of π/α with the unit circle. These define a polyhedron P; observe that $b(S_1) = S_2$, and $a(S_3) = S_4$ (see Figure 2 for the case $\alpha = 3$, $\delta = 6$).

It is an easy computation to see that the origin is contained inside C_3 if $\alpha > \delta$; C_3 passes through the origin if $\alpha = \delta$; and the origin lies outside C_3 if $\alpha < \delta$. If $\alpha \geq \delta$, which can occur only when $(\alpha, \delta) = (4, 4)$, $(4, 3)$, or $(3, 3)$, the boundary of P intersects the boundary of $\mathbf{H}^3$ in a finite number of points, so the group is of the first kind; if $\alpha < \delta$, then the boundary of P intersects the boundary of $\mathbf{H}^3$ in a fundamental domain for the action of G_0 on $\widehat{\mathbf{C}}$.

If $\alpha \neq 2$, then, exactly as above, the sides S_1 and S_2 meet the sides S_3 and S_4 at the angle π/β, and these four edges form a cycle of order γ. Also, the sides S_3 and S_4 meet at an angle of π/α; this one edge is its own cycle of order α.

If $\alpha = 2$, then P has no edges, and has one infinite cycle; as above, Lemma 1 shows that the corresponding infinite cycle transformation is parabolic.

In either case, we can conclude from Poincaré's polyhedron theorem that $G_0(\alpha, \delta)$ is discrete, and has the stated presentation. If $(\alpha, \delta) \neq (4, 4)$, $(4, 3)$, or $(3, 3)$, then the fundamental domain for G_0 at the boundary of P has two components (these are shaded in Figure 2). These two components are identified by a, so one easily sees, by folding together the sides of this fundamental domain, that $\Omega(G_0)/G_0$ has signature $(0, 3; \gamma, \delta, \delta)$. Since every uniformization of a surface with this signature is given by a Fuchsian group, every component of $\Omega(G_0)$ is a circular disc.

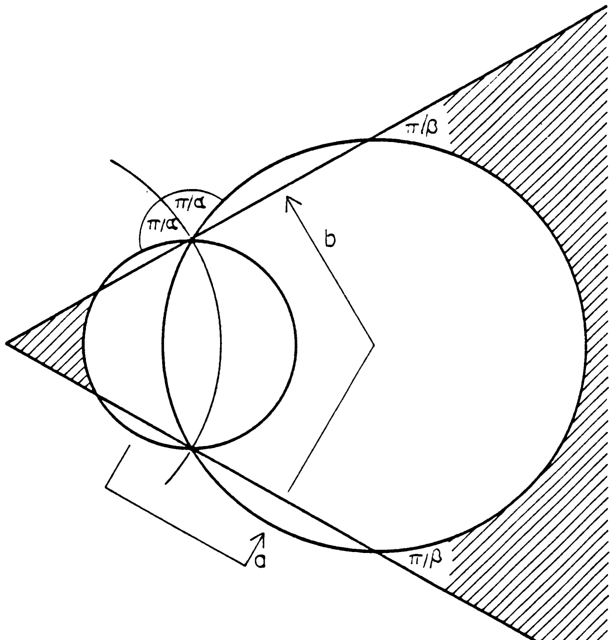

FIGURE 2

CONSTRUCTION OF $G_0(\alpha, \infty)$, $\alpha = 2, 3, 4$

We normalize so that $b(z) = z + 2$, so that a has its fixed points at ± 1, and so that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. We construct the polyhedron P as follows. $B_1 = \{z \mid \mathrm{Re}(z) > -1\}$; $B_2 = \{z \mid \mathrm{Re}(z) < 1\}$; B_3 is the outside of the isometric circle of a; and B_4 is the outside of the isometric circle of a^{-1}. These form a polyhedron P, where the sides S_1 and S_2 are identified by b, and the sides S_3 and S_4 are identified by a (See Figure 3 for the case $\alpha = 3$).

As in the two cases above, if $\alpha \neq 2$, the sides S_1 and S_2 meet the sides S_3 and S_4 in four edges, each with an angle of π/β; these four edges form a cycle

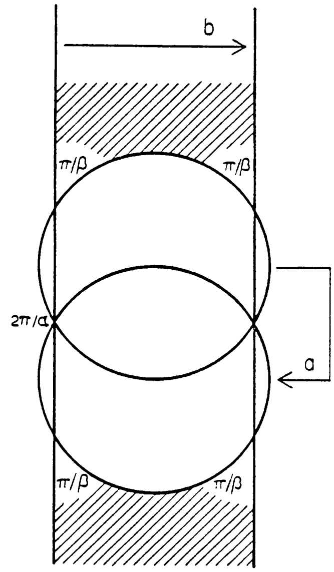

FIGURE 3

of order γ. There is also an edge between S_3 and S_4, where the cycle consists of only this one edge, and has order α. If $\alpha = 2$, then a^2 is a reflection relation; P has no edges, and has one infinite cycle; the infinite cycle relation is: $[a^{-1}, b^{-1}]^\infty = 1$.

The fundamental domain on the boundary of P on the sphere at infinity has two components that are connected via a. Hence, $\Omega(G_0)/G_0$ is connected. Folding together the sides of this fundamental domain, we see that $\Omega(G_0)/G_0$ has signature $(0, 3; \gamma, \infty, \infty)$. As above, this implies that every component of $\Omega(G_0)$ is a circular disc. □

5. The above construction does not work for $\alpha = 6$, for in this case the sides S_1 and S_2 meet the sides S_3 and S_4 at an angle of $\pi/3$, so that the sum of the angles in the cycle of these edges is $4\pi/3$. We get around this difficulty by constructing a larger group generated by reflections. We start by constructing a hyperbolic polyhedron with four sides, where the sides meet at certain specified angles. If S_m and S_n are sides of the polyhedron P, then we denote the interior angle between S_m and S_n by θ_{mn}.

Theorem 2. *Let a be elliptic of order $\alpha = 2, 3, 4$, or 6, and let b be hyperbolic, parabolic, or a geometrically primitive elliptic transformation of finite order $\delta > 2$, where b maps one fixed point of a onto the other. Then there is a polyhedron P, bounded by four sides, $S_1, \dots, S_4$, where $\theta_{12} = \pi/\delta$ (if $\delta = 0$, then S_1 and S_2 do not intersect, even on the sphere at infinity; if $\delta = \infty$, then S_1*

and S_2 *are tangent at the sphere at infinity*); $\theta_{13} = \pi/2$*;* $\theta_{14} = \pi/\beta$ (*these sides are tangent at the sphere at infinity if* $\alpha = 2$); $\theta_{23} = \pi/2$; $\theta_{24} = \pi/2$; $\theta_{34} = \pi/\alpha$. *Denote reflection in* S_j *by* r_j. *Then the following hold.*

(i) $a = r_3 r_4$, *and* $b = r_1 r_2$.

(ii) *The group* $\widetilde{G}(\alpha, \delta)$, *generated by* $r_1, \ldots, r_4$ *is discrete and has the presentation* $\langle r_1, \ldots, r_4 \colon r_1^2 = \cdots = r_4^2 = (r_1 r_2)^\delta = (r_1 r_3)^2 = (r_1 r_4)^\beta = (r_2 r_3)^2 = (r_2 r_4)^2 = (r_3 r_4)^\alpha = 1 \rangle$.

(iii) *If* $\alpha \geq \delta$, *then* $\widetilde{G}(\alpha, \delta)$ *is of the first kind; it is of the second kind otherwise. For* $\delta \neq 0$, *the action of* $\widetilde{G}(\alpha, \delta)$ *on* Ω *is that of the* $(2, \beta, \delta)$*-triangle group, generated by* r_1, r_2, *and* r_4. *For* $\delta = 0$, *the fundamental domain for the action of* $\widetilde{G}(\alpha, 0)$ *on its regular set is a quadrilateral with angles* $\pi/2$, $\pi/2$, $\pi/2$, *and* π/β.

Proof. We start with the observation, again using Poincaré's polyhedron theorem, that for conclusions (i) and (ii), it suffices to construct the polyhedron P, with the sides meeting at the correct angles, where a and b are as described in conclusion (i).

Construction of $\widetilde{G}(\alpha, 0)$.

Normalize a and b as above so that $b(z) = \lambda^2 z$, $\lambda > 1$, and so that a has its fixed points at 1 and λ^2; we can also assume that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. Let $B_1 = \{z \mid |z| < \lambda^2\}$; $B_2 = \{z \mid |z| > \lambda\} \cup \{\infty\}$; B_3 is the upper half-plane; and B_4 is the outside of the isometric circle of a (the fundamental domain for conclusion (iii) is shaded in Figure 4).

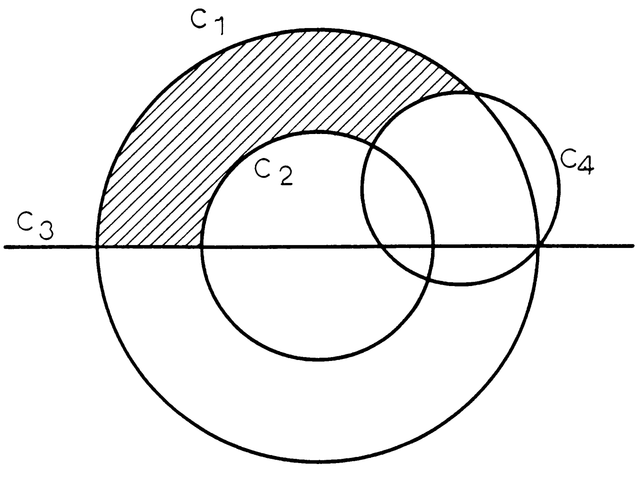

FIGURE 4

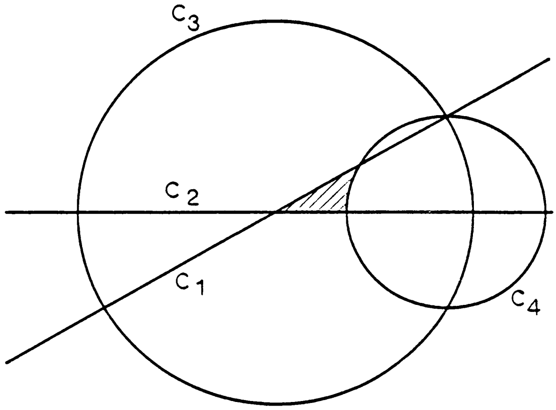

FIGURE 5

CONSTRUCTION OF $\widetilde{G}(\alpha, \delta)$, $2 < \delta < \infty$

Normalize so that $b(z) = e^{2\pi i/\delta} z$; a has its fixed points at $e^{\pm \pi i/\delta}$; and if $\alpha > 2$, the center of the isometric circle of a lies inside the unit circle. Let $B_1 = \{z | \pi/\delta - \pi < \arg(z) < \pi/\delta\}$; B_2 is the upper half-plane; B_3 is the unit disc; and B_4 is the outside of the circle passing through the fixed points of a, and making an angle of π/α with the (inside of the) unit circle (see Figure 5 for the case that $\alpha = 3$, and $\delta = 6$; the fundamental domain for conclusion (iii) is shaded).

CONSTRUCTION OF $\widetilde{G}(\alpha, \infty)$

Normalize so that $b(z) = z + 2$, so that a has its fixed points at ± 1, and so that if $\alpha > 2$, then the center of the isometric circle of a lies in the upper half-plane. Let $B_1 = \{z | \operatorname{Re}(z) < 1\}$; B_2 is the right half-plane; B_3 is the upper half-plane; and B_4 is the outside of the isometric circle of a (the fundamental domain for this group is shaded in Figure 6). □

6. Let $G(\alpha, \delta)$ be the orientation preserving half of $\widetilde{G}(\alpha, \delta)$. Then $G(\alpha, \delta)$ is generated by $a = r_3 r_4$, $b = r_1 r_2$, and $c = r_1 r_4$, and has the presentation $G(\alpha, \delta) = \langle a, b, c \colon a^\alpha = b^\delta = c^\beta = (ac^{-1})^2 = (bc^{-1})^2 = (a^{-1}b^{-1}c)^2 = 1 \rangle$.

Using the explicit constructions of the fundamental polyhedra for G_0 and $\widetilde{G}$, it is easy to see that for $\alpha = 2, 3$, or 4, $G_0(\alpha, \delta)$ is a subgroup of index 2 in $G(\alpha, \delta)$. We can make this explicit by observing that, up to the boundaries, we obtain a fundamental polyhedron for $G(\alpha, \delta)$ from that of $\widetilde{G}(\alpha, \delta)$ by reflection in S_2, and then we obtain the given fundamental polyhedron for $G_0(\alpha, \delta)$ by reflecting that fundamental polyhedron in S_3. We have shown the following.

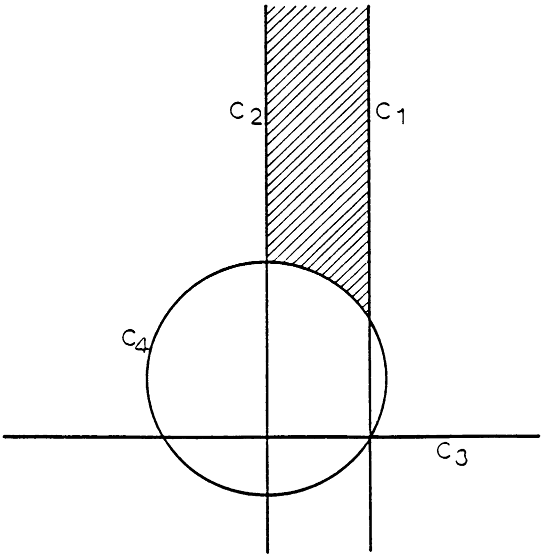

FIGURE 6

Proposition 2. *For* $\alpha = 2, 3$, *and* 4, *and for all possible* δ,

$$[G(\alpha, \delta) \colon G_0(\alpha, \delta)] = 2.$$

Proposition 3. *For every possible* δ, $G(6, \delta) = G_0(6, \delta)$.

Proof. We first prove this result for $\delta = \infty$. We renormalize so that $a(z) = e^{-\pi i/3} z$, so that $b(z)$ has its fixed point at 1, and so that $b(0) = \infty$. Then $b(z) = (2z - 1)/z$. With this new normalization, we have $C_1 = \{z \mid \mathrm{Re}(z) = 1\}$, $C_2 = \{z \mid |z| = 1\}$, $C_3 = \{z \mid \mathrm{Im}(z) = 0\}$, and $C_4 = \{z \mid \arg(z) = \pi/6\}$ (see Figure 7). It is easy to compute reflections in these lines. Writing these reflections as matrices in $\mathrm{PGL}(2, \mathbf{C})$, we obtain the following.

$$r_1 = \begin{pmatrix} -1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad r_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$r_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad r_4 = \begin{pmatrix} e^{i\pi/6} & 0 \\ 0 & e^{-i\pi/6} \end{pmatrix},$$

from which we compute

$$a = \begin{pmatrix} e^{-i\pi/6} & 0 \\ 0 & e^{i\pi/6} \end{pmatrix}, \qquad b = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix},$$
$$c = \begin{pmatrix} -e^{-i\pi/6} & 2e^{i\pi/6} \\ 0 & e^{i\pi/6} \end{pmatrix}.$$

Now set $e = [a, bab^{-1}]$, so that $e(z) = z - 2$. Then a straightforward computation shows that $e^{-1}aea = c^{-1}$. Hence $G(6, \delta) = G_0(6, \delta)$.

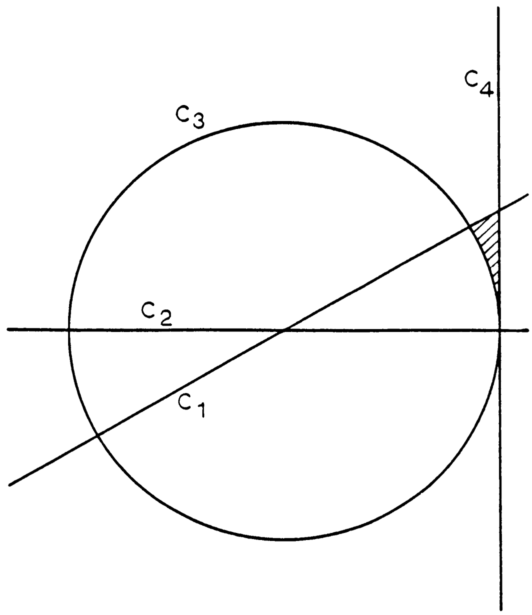

FIGURE 7

There is an obvious homomorphism, given by conclusion (ii) of Theorem 2, from $G(6,\infty)$ onto $G(6,\delta)$ (this is an isomorphism for $\delta = 0$), from which we conclude that $G(6,\delta)$ is generated by a and b for every δ. $\square$

7. We conclude with some remarks.

(i) We first note that $G(6,0)$ is a function group representing a surface of signature $(0,4;2,2,2,3)$, so the fact that this group has only two generators is related to the observation that the Fuchsian group of signature $(0,4;2,2,2,3)$ can be generated by two hyperbolic elements [P-R].

(ii) We note that for $\delta \neq 0$, the group $\widetilde{G}(\alpha,\delta)$ contains the following four triangle groups, obtained by taking the defining reflections three at a time: the $(2,2,\delta)$-triangle group, which is finite for δ finite, and Euclidean for $\delta = \infty$; the $(2,\beta,\delta)$-triangle group, which is finite if $1/\beta + 1/\delta > 1/2$, Euclidean if $1/\beta + 1/\delta = 1/2$, and Fuchsian otherwise; the Euclidean $(2,\alpha,\beta)$-triangle group; and the finite $(2,2,\alpha)$-triangle group. Of course, $G(\alpha,\delta)$ contains the orientation preserving half of these groups.

(iii) Since every $\widetilde{G}(\alpha,\delta)$ with $\delta \neq 0$ contains the Euclidean $(2,\alpha,\beta)$-triangle group, none of the groups we consider here has a compact fundamental polyhedron in $\mathbf{H}^3$.

(iv) The above analysis omits two cases. One expects that, in general, if a and b are as above, where b is elliptic of finite order, but not geometrically primitive, then $\langle a,b\rangle$ is not discrete. It is not clear if there are any exceptions.

(v) If a and b are as above, and b is hyperbolic, then any quasiconformal deformation of $\langle a, b\rangle$ is again discrete, and in general b is loxodromic. There are also limits of such groups that are discrete. It is not known if there are any other such discrete groups, where b is loxodromic.

REFERENCES

[G-M] F. W. Gehring and G. Martin, (to appear).

[M] B. Maskit, *Kleinian groups.* Springer-Verlag, 1988.

[P-R] N. Purzitsky and G. Rosenberger, *Correction to "Two generator Fuchsian groups of genus one"*, Math. Z. **132** (1973), 261–262.

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