$=i_*(\pi_{n+1}(K^n))$ where $i:K^n \rightarrow K^{n+1}$ is the identity map. When, in addition, $\pi_{n+1}(K)=0$ and $n>3$, $\Gamma_{n+2}(K)$ is computed to be $H_n(K)/2H_n(K)$. This is equivalent to the statement that $\pi_{n+2}(K) \approx H_{n+2}(K)+H_n(K)/2H_n(K)$, and in this form it is proved again using the normal cell complexes of S. C. Chang.

There is a bibliography of 4 books and 42 papers and a rather complete combined index and glossary.

M. L. CURTIS

Lezioni sulle funzioni ipergeometriche confluenti. By F. G. Tricomi. Torino, Gheroni, 1952. 284 pp. 2000 Lire.

Die konfluente hypergeometrische Funktion mit besonderer Berücksichtigung ihrer Anwendungen. By Herbert Buchholz. (Ergebnisse der angewandten Mathematik, vol. 2.) Berlin-Göttingen-Heidelberg, Springer, 1953. 16+234 pp. 36.00 DM.

Confluent hypergeometric series arise when two of the three singularities of the hypergeometric differential equation coalesce in such a manner as to produce an irregular singularity at infinity. These series were introduced by Kummer in 1836. In 1904, E. T. Whittaker proposed new definitions and notations which clearly exhibit the symmetry and transformation properties of confluent hypergeometric functions, and facilitate the identification of many special functions, among them Bessel functions, Laguerre and Hermite polynomials, error functions, Fresnel integrals, sine and cosine integrals, exponential integrals, and the like, as particular instances of confluent hypergeometric functions. (As a matter of fact, the chapter on Bessel functions in all newer editions of Whittaker and Watson's *Modern analysis* follows upon, and leans heavily on, the chapter on confluent hypergeometric functions.)

In the first half of the present century a sizeable literature has grown up around these functions. Many of their properties were discovered (some of them several times), and they have been found useful in pure and applied mathematics alike. Fluid dynamics, nuclear physics, probability theory, elasticity, all offer problems which can be solved in terms of confluent hypergeometric functions, and these functions proved an excellent example for illustrating the technique of the Laplace transformation. In view of the hundreds of papers and dozens of applications it is somewhat strange that no monograph on these functions seemed to be available (although several well-known books, such as *Modern analysis*, Jeffreys and Jeffreys' *Methods of mathematical physics*, and Magnus and Oberhettinger's *Special functions* devote separate chapters to these functions). Now, within a short period, we have to record the publication of two books devoted to confluent hypergeometric functions, and a third book is in preparation. From the point of view of the mathematical public (and, one supposes, also from the point of view of the authors and publishers) it is fortunate that the two books differ to such an extent as to complement each other in a very useful manner. They differ considerably in the definition and notation of the functions they investigate, in the selection of the material, methods, presentation, in the applications they envisage, and most of all in their style.

Tricomi's book is based on a course of lectures given at the University of Torino, and it is written with that gift for exposition for which its author is so justly famous. The presentation is well planned, leisurely, and detailed, with very few computations "left to the reader" and no vague appeals to "it is easy to see." The book is self-contained, and those parts of mathematical analysis which are needed, and with which the student may not be familiar, form the topics of several digressions. There is no attempt at all-inclusiveness, but the topics which have been selected are discussed rather fully. Books and memoirs are quoted when they are needed in the text: otherwise there is no bibliography, and there is no index. The book naturally favors those parts of the theory of confluent hypergeometric functions, and these are many, to which its distinguished author contributed in recent years. Shortly, this book is an eminently readable introduction to confluent hypergeometric functions which takes the reader, in several directions, to topics of interest in current research.

Buchholz's book is a handbook, and it is the fruit of many years' diligent labor. Based on a bibliography of several hundred items, the book encompasses practically everything of importance that is known about confluent hypergeometric functions. Since the number of books and papers used in writing this volume exceeds the number of its pages, it is inevitable that the writing should be concise and highly condensed. Proofs are frequently given in outline only, some proofs are omitted, and some results are mentioned only (with a reference to the literature) without details and without discussion. The book abounds in formulas, and important formulas are repeated in several different forms for handy reference. An index of notations and a subject index help the reader to locate any desired information, and contribute to the usefulness of this volume as a work of reference. Except in the choice of the applications, the author's interests seem to have had little influence upon the selection of the material: everything is included with a fine impartiality.

There follows a brief description of each of the two books.

The basic differential equation in Tricomi's book is the confluent hypergeometric equation

$$(1)\qquad x\frac{d^2y}{dx^2} + (c - x)\frac{dy}{dx} - ay = 0$$

where a and c are constant parameters. There is a unique solution of (1) which is regular at the origin, and is equal to unity there: this is Kummer's series

$$(2)\qquad \Phi(a, c; x) = \sum \frac{\Gamma(c)\Gamma(a + n)x^n}{\Gamma(a)\Gamma(c + n)n!}$$

[also often denoted by ${}_1F_1(a; c; x)$].

Chapter I contains preliminary material on the Laplace transformation, on the analytic theory of ordinary linear differential equations, and on the gamma function (all of which is required later).

In Chapter II it is first shown that any ordinary linear differential equation of the second order whose coefficients are linear functions of the independent variable may be reduced to (1). If c is not an integer, the general solution is a linear combination of $\Phi(a, c; x)$ and $x^{1-c}\Phi(a-c+1, 2-c; x)$. The properties of (2), or rather of

$$(3)\qquad \Phi^*(a, c; x) = \frac{1}{\Gamma(c)}\Phi(a, c; x),$$

which is an entire function of all three of its arguments, are discussed, the discussion including Kummer's transformation, recurrence and differentiation formulas, Laplace transforms. In the case $c = 2a$ there is a connection with Bessel functions, and in the case of a negative integer a with Laguerre and Hermite polynomials. This chapter concludes with the author's well known expansion of Φ in series of Bessel functions.

Chapter III introduces the "second solution"

$$(4)\qquad \begin{aligned}\Psi(a, c; x) = {} & \frac{\Gamma(1 - c)}{\Gamma(a - c + 1)}\Phi(a, c; x) \\ & + \frac{\Gamma(c - 1)}{\Gamma(a)}x^{1-c}\Phi(a - c + 1, 2 - c; x),\end{aligned}$$

gives integral representations for Φ and Ψ, recurrence relations, etc., for Ψ, and the asymptotic properties of Φ and Ψ, showing that $\Psi \to 0$ as $x \to +\infty$. For real a and c, the real zeros of Φ and the positive zeros of Ψ are counted, and a description of $\Phi(a, c; x)$ for real a, c, x is given.

Chapter IV is devoted to the incomplete gamma functions and related functions. The author puts

$$(5) \qquad \int_0^x e^{-t}t^{\alpha-1}dt = \gamma(\alpha, x) = x^\alpha\Gamma(\alpha)\gamma^*(\alpha, x) = \Gamma(\alpha) - \Gamma(\alpha, x),$$

and points out that γ^*, being an entire function of both α and x, is the basic function. The connection with confluent hypergeometric functions with $a=1$ or $c=a+1$ is used to obtain most properties of incomplete gamma functions and of such functions as the error functions, Fresnel integrals, exponential integral, sine and cosine integrals which are related to incomplete gamma functions. This chapter contains a table of most of the important particular cases of confluent hypergeometric functions.

Chapter V contains applications. The two body problem of wave mechanics, bending of elastic plates, resultant of a large number of random vectors, and water waves are discussed and show the variety of problems in which confluent hypergeometric functions may be used with advantage.

The book is reproduced from a typescript.

Turning now to Buchholz's book, we find that Chapter I opens with the derivation of (1) and (2), but the discussion turns very soon to Whittaker's equation

$$(6) \qquad \frac{d^2y}{dx^2} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{1-\mu^2}{4z^2}\right)y = 0$$

and to its solutions

$$(7) \qquad M_{\kappa,\mu/2}(x) = x^{(\mu+1)/2}e^{-x/2}\Phi\left(\frac{1+\mu}{2} - \kappa,\ 1+\mu;\ x\right),$$

$$(8) \qquad W_{\kappa,\mu/2}(x) = x^{(\mu+1)/2}e^{-x/2}\Psi\left(\frac{1+\mu}{2} - \kappa,\ 1+\mu;\ x\right).$$

The function corresponding to Φ^* is

$$(9) \qquad \mathcal{M}_{\kappa,\mu/2}(x) = M_{\kappa,\mu/2}(x)/\Gamma(1+\mu)$$

and is frequently used in Buchholz's book. Chapter I further contains an investigation of the basic properties of "parabolic functions" (thus Buchholz refers to M, W, $\mathcal{M}$), differential equations which can be reduced to (6) (including Weber's equation of parabolic cylinder functions), an inhomogeneous Whittaker equation, and the separation of the wave equation in coordinates of the parabolic cylinder and of the paraboloid of revolution.

Chapter II brings a thorough discussion of integral representations, with application to recurrence and differentiation formulas. A few integrals representing products of parabolic functions are also given.

Chapter III is devoted to a study of the asymptotic behavior of parabolic functions, most of the results being obtained by the method of steepest descents applied to suitable integral representations.

In Chapter IV a large number of integrals involving parabolic functions are given: both indefinite and definite integrals occur, and the Laplace, Mellin, and Hankel transforms are included, as are also some infinite series involving parabolic functions.

Chapter V contains a discussion of polynomials related to parabolic functions. Laguerre and Hermite polynomials are given in detail (with many series and integrals involving these polynomials), others more briefly.

The material in Chapter VI is of great importance for the application of parabolic functions in wave propagation problems: it is the expression of several types of waves (planes waves, spherical waves, etc.) as a superposition of solutions of the wave equation in the coordinates introduced in Chapter I.

In Chapter VII the zeros of M and W are investigated both as k, m are fixed and z varies, and as m, z are fixed and k varies (in the latter case only M being considered). Vibrations of an inhomogeneous elastic string, and Green's function of the wave equation for a region bounded by two confocal paraboloids of revolution are used to illustrate the application of parabolic functions in eigenvalue problems.

Appendix I is a very useful summary of particular cases of parabolic functions, with information regarding numerical tables of these functions. Appendix II is an extensive bibliography. An index of symbols and notations is at the beginning of the book, and there is a subject index at the end.

In their very different ways both books may be expected to be of considerable assistance to those having occasion to employ these functions. Between them they provide all the information needed in the application of these functions, with the possible exception of nuclear physics where a different notation is used (under the name of Coulomb wave functions), and the selection of a second solution is different.

A. Erdélyi