REPRESENTATIONS OF A SEMIGROUP

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The purpose of this paper is to develop for semigroups the notions of radical and semisimplicity similar to those which have been developed for rings. In 1961, E. J. Tully, Jr. published a paper dealing with transitive and 0-transitive operands of a semigroup. We shall be using his definitions and notation as a starting point in our considerations. The author wishes to express gratitude to Ancel Mewborn for many valuable suggestions.

In this paper we shall first define the radical of a semigroup S and investigate some of its properties. Just as in the case of rings, one finds that the radical of a semigroup is a quasi-regular two-sided ideal which contains each quasi-regular right ideal of the semigroup and that the radical of a semigroup contains each nil right ideal of the semigroup. If a semigroup with zero satisfies the minimum condition on left ideals and right ideals then its radical is also the left radical of the semigroup. One also finds that if S_n is the semigroup of all row-monomial $n \times n$ matrices over S then the radical of S_n is the semigroup of all row-monomial $n \times n$ matrices over the radical of S. If T is a two-sided ideal of S then the radical of T is the intersection of T and the radical of S.

In the latter part of the paper we shall study semisimple semigroups. If $s, t \in S$, let $s\rho t$ provided that, if m is contained in some 0-transitive operand of S, then $ms = mt$. Thus ρ is a two-sided congruence called the *radical congruence* of S. If the radical congruence of S is the identity relation then S is *semisimple*. Since M is a 0-transitive operand of S if and only if M is a 0-transitive operand of S/ρ, then it is easily seen that S/ρ is semisimple. If S is semisimple and T is a two-sided ideal of S, then T is semisimple. Finally, if S contains more than one element, then S is semisimple if and only if S is isomorphic to a subdirect sum of a set of semigroups each of which has a faithful 0-transitive operand.

1. **The radical of a semigroup.** If each of α and β is a right congruence of an operand M, then α *is contained in* β provided $m\beta n$ whenever $m\alpha n$. The right congruence α is *maximal* if α is not the universal relation and the only right congruence which properly contains α is the universal relation.

If σ is a right congruence on the operand M, x is a σ-class, and s is in S, let xs be the σ-class containing the set $\{ms\colon m \in x\}$. With this definition the set of

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σ-classes is itself an operand of S denoted by M/σ. If $m \in M$ then m_σ shall denote the element of M/σ which contains m.

THEOREM 1. *If σ is a right congruence on S and e is an element of S such that $e_\sigma S = S/\sigma$, then either S/σ is transitive or there is a maximal right congruence which contains σ.*

Proof. Suppose that S/σ is not transitive. If $s, t \in S$, let $s\gamma t$ if either $s\sigma t$ or $s_\sigma S \neq S/\sigma$ and $t_\sigma S \neq S/\sigma$. Thus γ is a right congruence on S. Let x be an element of S such that $x_\sigma S \neq S/\sigma$ and let $\mathscr{T} = \{\tau : \tau$ is a right congruence on S containing γ such that $e_\tau \neq x_\tau\}$. By Zorn's Lemma, $\mathscr{T}$ contains a maximal element α. Since $e_\alpha \neq x_\alpha$, then α is not the universal relation. Suppose that β is a right congruence which properly contains α. If $s \in S$, since $(xs)_\sigma S = x_\sigma sS \subseteq x_\sigma S \neq S/\sigma$, then $x\gamma xs$ and hence $x\beta xs$. Since $\beta \notin \mathscr{T}$, then $e_\beta = x_\beta$ and $s_\beta \in e_\beta S = x_\beta S = x_\beta$ for each $s \in S$. Thus β is the universal relation and α is a maximal right congruence which contains σ.

THEOREM 2. *If M is a primitive operand of S which contains at least three elements, then M is S-isomorphic to S/σ (denoted by $M \cong S/\sigma$) where σ is some maximal right congruence on S. Conversely, if σ is a maximal right congruence on S, then S/σ is primitive.*

Proof. From [5], we know that M is either transitive or 0-transitive. Let k be an element of M such that $kS = M$. If $s, t \in S$, let $s\sigma t$ if $ks = kt$; thus σ is a right congruence on S. If $m \in M$, let $m\theta = \{s : ks = m\}$. Thus θ is an S-isomorphism from M onto S/σ. Suppose that τ is a right congruence on S which properly contains σ. If $m, n \in M$, let $m\gamma n$ if each element of $m\theta$ is τ-congruent to each element of $n\theta$. Since γ is a right congruence on M, then γ is either the universal relation or the identity relation. Since τ properly contains σ, there exist m and n in M such that each element of $m\theta$ is τ-congruent to each element of $n\theta$ but $(m\theta) \neq (n\theta)$. Thus $m \neq n$, $m\gamma n$, and each of γ and τ is the universal relation. Therefore σ is maximal.

Conversely, suppose that σ is a maximal right congruence on S and τ is a right congruence on S/σ. If $s, t \in S$, let $s\gamma t$ if $s_\sigma \tau t_\sigma$. Since γ contains σ, either $\gamma = \sigma$ or γ is the universal relation. Thus τ is either the identity relation or the universal relation and S/σ is primitive.

If M is a 0-transitive operand of S then the *zero element of M* is that element m in M such that $mS = m$.

THEOREM 3. *Suppose that σ is a right congruence on S and τ is a right congruence containing σ such that τ is not the universal relation. If S/σ is 0-transitive with zero element x_σ then S/τ is 0-transitive with zero element x_σ.*

Proof. If $s \in S$, since $x_\sigma s \subseteq x_\tau s$ and $x_\sigma s = x_\sigma \subseteq x_\tau$, then $x_\tau s = x_\tau$. Thus x_τ is a zero element of S/τ. Suppose that y is an element of S which is not contained in x_σ

and that z is an element of S. Since $y_\sigma S = S/\sigma$, then there is a t in S such that $y_\sigma t = z_\sigma$. Hence $z \in y_\tau t$ and $y_\tau t = z_\tau$. Therefore $y_\tau S = S/\tau$ and, since S/τ has more than one element, $y_\tau \neq x_\tau$. Thus $y \notin x_\tau$ and $x_\sigma = x_\tau$.

An element s of S is in the *radical* $N(S)$ of S if Ms is the zero element of M for each 0-transitive operand M of S. If S has no 0-transitive operand then $N(S) = S$.

Let $\{\sigma_\lambda : \lambda \in \Lambda\}$ be the family of maximal right congruences σ_λ on S with the following properties: (a) there is a left identity modulo σ_λ, and (b) S/σ_λ has a zero element z_λ.

THEOREM 4. *With the above notation,* $N(S) = \bigcap\{z_\lambda : \lambda \in \Lambda\}$.

Proof. Suppose that $r \in N(S)$, σ_λ is a maximal right congruence on S such that S/σ_λ has a zero element z_λ, and e is a left identity modulo σ_λ. Since S/σ_λ is 0-transitive, then $(S/\sigma_\lambda)r = z_\lambda$. Since $er\sigma_\lambda r$ and $er \in e_{\sigma_\lambda} r = z_\lambda$, then $r \in z_\lambda$.

Conversely, suppose that y is in $\bigcap\{z_\lambda : \lambda \in \Lambda\}$ and M is a 0-transitive operand of S. If z is the zero element of M, then $zy = z$. Suppose that m is an element of M such that $my \neq z$. Since $myS = M$, let $q \in S$ such that $myq = m$. If $s, t \in S$, let $s\tau t$ if $ms = mt$. Thus τ is a right congruence on S, τ is not the universal relation, yq is a left identity modulo τ, and $M \cong S/\tau$. If θ is an S-isomorphism from M onto S/τ, then S/τ is 0-transitive with zero element $z\theta$. There is a maximal right congruence γ which contains τ and S/γ is 0-transitive with zero element $z\theta$. The left identity modulo τ is also a left identity modulo γ. Hence $y \in z\theta$. Since yq is a left identity modulo τ, then $(yq)_\tau S = S/\tau$. But since $yq \in (z\theta)q = z\theta$, then $S/\tau = (yq)_\tau S = (z\theta)S = z\theta$; which contradicts the fact that τ is not the universal relation. Therefore $my = z$ for each m in M and y is in $N(S)$.

If $e \in S$, then e is a *right quasi-regular element of* S if there does not exist a right congruence σ on S with the following properties: (a) S/σ is a 0-transitive operand of S, and (b) e is a left identity modulo σ. A right ideal of S is *quasi-regular* if each element contained in it is a right quasi-regular element of S.

THEOREM 5. *$N(S)$ is a quasi-regular two-sided ideal which contains every quasi-regular right ideal of S.*

Proof. Suppose that $n \in N(S)$ and $s \in S$. If M is a 0-transitive operand of S, then $M(sn) = (Ms)n \subseteq Mn = 0$ and $M(ns) = (Mn)s = 0s = 0$. Thus $N(S)$ is a two-sided ideal of S.

Suppose that $r \in N(S)$, σ is a right congruence on S such that S/σ is a 0-transitive operand of S, and r is a left identity modulo σ. Let τ be a maximal right congruence which contains σ. Thus S/τ is a 0-transitive operand of S and r is a left identity modulo τ. But by the previous theorem r is contained in the zero of S/τ, which contradicts the fact that τ is not the universal relation. Therefore $N(S)$ is quasi-regular.

Suppose that T is a quasi-regular right ideal of S, $t \in T$, and M is a 0-transitive operand of S such that $Mt \neq 0$. Let $m \in M$ such that $mt \neq 0$. Since $mtS = M$, let $s \in S$ such that $mts = m$. If $u, v \in S$, let $u\gamma v$ if $mu = mv$. Then γ is a right congruence on S such that $M \cong S/\gamma$, S/γ is a 0-transitive operand, and ts is a left identity modulo γ. Thus ts is not a right quasi-regular element of S, which contradicts the fact that ts is an element of the quasi-regular right ideal T. Therefore if M is a 0-transitive operand of S, then $Mt = 0$. Thus t is in $N(S)$.

THEOREM 6. *If S is a semigroup with zero and n is a positive integer, then* $N(S_n) = [N(S)]_n$.

Proof. Let $T = \{t : t \text{ is an entry in some element of } N(S_n)\}$. Since $N(S_n)$ is a two-sided ideal of S_n, then T is a two-sided ideal of S. Suppose that T is not quasi-regular. Let s_{km} be the (k, m) entry of some element $((s_{ij}))$ of $N(S_n))$ such that s_{km} is not right quasi-regular. Since s_{km} is not right quasi-regular, let σ be a right congruence on S such that S/σ is a 0-transitive operand of S and s_{km} is a left identity modulo σ. If $((t_{ij}))$, $((p_{ij})) \in S_n$, let $((t_{ij}))\tau_k((p_{ij}))$ if $t_{kj}\sigma p_{kj}$ for each j. We shall now show that τ_k is a right congruence on S_n.

Suppose that $((t_{ij}))\tau_k((p_{ij}))$ and $((v_{ij})) \in S_n$. If the kth row of $((t_{ij}))$ does not contain a nonzero entry, then $p_{kj}\sigma 0$ for each j; thus $p_{kh}v_{hj}\sigma 0$ for each h and j and $((t_{ij}))((v_{ij}))\tau_k((p_{ij}))((v_{ij}))$. If $t_{kh} \neq 0$ and $p_{kh} \neq 0$, then, since $t_{kh}v_{hj}\sigma p_{kh}v_{hj}$ for each j, $((t_{ij}))((v_{ij}))\tau_k((p_{ij}))((v_{ij}))$. If $t_{kh} \neq 0$ and $p_{kh} = 0$, then $t_{kh}v_{hj}\sigma p_{kh}v_{hj} = 0$ for each j. If $q \neq h$, since $((t_{ij}))$ is row-monomial, then $p_{kq}v_{qj}\sigma t_{kq}v_{qj} = 0$ $v_{qj} = 0$ for each j. Thus $((t_{ij}))((v_{ij}))\tau_k((p_{ij}))((v_{ij}))$ and τ_k is a right congruence.

To see that S_n/τ_k is a 0-transitive operand of S_n, let $((x_{ij}))_{\tau_k}$ be a nonzero element of S_n/τ_k and let $((y_{ij})) \in S_n$. If each element in the kth row of $((y_{ij}))$ is zero, then $((x_{ij}))((0_{ij}))\tau_k((y_{ij}))$. Suppose that $y_{kh} \neq 0$. Since $((x_{ij}))_{\tau k} \neq 0$, then x_{kq} is not σ-congruent to 0 for some q. Since S/σ is 0-transitive, let p be an element of S such that $x_{kq}p\sigma y_{kh}$. Let $((p_{ij})) \in S_n$ such that $p_{qh} = p$ and $p_{ij} = 0$ if $(i,j) \neq (q, h)$. Since $x_{kq}p_{qh}\sigma y_{kh}$, then $((x_{ij}))((p_{ij}))\tau_k((y_{ij}))$. Therefore S_n/τ_k is a 0-transitive operand of S_n.

Now let $((r_{ij})) \in S_n$ such that $r_{kk} = s_{km}$ and $r_{ij} = 0$ if $(i,j) \neq (k,k)$; and let $((w_{ij})) \in S_n$ such that $w_{mk} = s_{km}$ and $w_{ij} = 0$ if $(i,j) \neq (m,k)$. Since $N(S_n)$ is a two-sided ideal then $((z_{ij})) = ((r_{ij}))((s_{ij}))((w_{ij}))$ is in $N(S_n)$. One sees that $((z_{ij}))$ is a left identity modulo τ_k. But S_n/τ_k is a 0-transitive operand of S_n, which contradicts the fact that since $((z_{ij})) \in N(S_n)$, then $((z_{ij}))$ is right quasi-regular. Therefore T is quasi-regular, $T \subseteq N(S)$, and $N(S_n) \subseteq [N(S)]_n$.

Conversely, we shall show by induction that $[N(S)]_n \subseteq N(S_n)$. Clearly $[N(S)]_1 \subseteq N(S_1)$. Suppose that $[N(S)]_k \subseteq N(S_k)$ for some k, $((e_{ij})) \in [N(S)]_{k+1}$, and σ is a right congruence on S_{k+1} such that S_{k+1}/σ is a 0-transitive operand of S_{k+1} with left identity $((e_{ij}))$. Now suppose that the hth column of $((e_{ij}))$ contains no nonzero entry. Let

$$((f_{ij})) = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1,h-1} & 0 & e_{1,h+1} & \cdots & e_{1,k+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e_{h-1,1} & e_{h-1,2} & \cdots & e_{h-1,h-1} & 0 & e_{h-1\ h+1} & \cdots & e_{h-1,k+1} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ e_{h+1,1} & e_{h+1,2} & \cdots & e_{h+1,h-1} & 0 & e_{h+1,h+1} & \cdots & e_{h+1,k+1} \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ e_{k+1,1} & e_{k+1,2} & \cdots & e_{k+1,h-1} & 0 & e_{k+1,h+1} & \cdots & e_{k+1,k+1} \end{bmatrix}.$$

Since $((e_{ij}))$ is a left identity modulo σ and since its hth column contains no nonzero entry then $((e_{ij}))\sigma((e_{ij}))((e_{ij})) = ((e_{ij}))((f_{ij}))\sigma((f_{ij}))$. If $((t_{ij}))$, $((s_{ij})) \in S_k$, let $((t_{ij}))\tau((s_{ij}))$ if

$$\begin{bmatrix} t_{11} & \cdots & t_{1,h-1} & 0 & t_{1,h} & \cdots & t_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_{h-1,1} & \cdots & t_{h-1,h-1} & 0 & t_{h-1,h} & \cdots & t_{h-1,k} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ t_{h,1} & \cdots & t_{h,h-1} & 0 & t_{h,h} & \cdots & t_{h,k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ t_{k1} & \cdots & t_{k,h-1} & 0 & t_{kh} & \cdots & t_{kk} \end{bmatrix} \sigma$$

$$\begin{bmatrix} s_{11} & \cdots & s_{1,h-1} & 0 & s_{1,h} & \cdots & s_{1k} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ s_{h-1,1} & \cdots & s_{h-1,h-1} & 0 & s_{h-1,h} & \cdots & s_{h-1,k} \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ s_{h1} & \cdots & s_{h,h-1} & 0 & s_{hh} & \cdots & s_{hk} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ s_{k1} & \cdots & s_{k,h-1} & 0 & s_{kh} & \cdots & s_{kk} \end{bmatrix}.$$

Show that τ is a right congruence on S_k with left identity

$$((a_{ij})) = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1,h-1} & e_{1,h+1} & \cdots & e_{1,k+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{h-1,1} & e_{h-1,2} & \cdots & e_{h-1,h-1} & e_{h-1,h+1} & \cdots & e_{h-1,k+1} \\ e_{h+1,1} & e_{h+1,2} & \cdots & e_{h+1,h-1} & e_{h+1,h+1} & \cdots & e_{h+1,k+1} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ e_{k+1,1} & e_{k+1,2} & \cdots & e_{k+1,h-1} & e_{k+1,h+1} & \cdots & e_{k+1,k+1} \end{bmatrix}.$$

Since $((f_{ij}))\sigma((e_{ij}))$ is not σ-congruent to 0, then $((a_{ij}))$ is not τ-congruent to 0 and τ is not the universal relation. Let α be a maximal right congruence on S_k which contains τ. If S_k/α contains only two elements, 0 and $((a_{ij}))_\alpha$, since $((a_{ij}))_\alpha S_k = S_k/\alpha$, then S_k/α is 0-transitive. If S_k/α contains more than two elements, then, since S_k/α is primitive, S_k/α is 0-transitive by (5). But $((a_{ij}))$ is a left identity modulo α; thus $((a_{ij}))$ is not right quasi-regular. However, $((a_{ij})) \in [N(S)]_k = N(S_k)$, which contradicts Theorem 5. Therefore each column of $((e_{ij}))$ contains a nonzero entry. Since $((e_{ij}))$ is row-monomial, then each row and each column of $((e_{ij}))$ contains exactly one nonzero entry. Thus some power of $((e_{ij}))$ is diagonal. Let q be a positive integer such that $((e_{ij}))^q = ((c_{ij}))$ where $c_{ij} = 0$ if $i \neq j$. If $((s_{ij}))$, $((t_{ij})) \in S_k$, let $((s_{ij}))\beta((t_{ij}))$ if there exist positive integers m and n such that if p is a positive integer, then

$$\begin{bmatrix} s_{11} \cdots s_{1k} & 0 \\ \vdots \quad\ \ \vdots & \vdots \\ s_{k1} \cdots s_{kk} & 0 \\ \\ 0 \ \ \cdots \ \ 0 & c^{m+p}_{k+1,k+1} \end{bmatrix} \sigma \begin{bmatrix} t_{11} \cdots t_{1k} & 0 \\ \vdots \quad\ \ \vdots & \vdots \\ t_{k1} \cdots t_{kk} & 0 \\ \\ 0 \ \ \cdots \ \ 0 & c^{n+p}_{k+1,k+1} \end{bmatrix}.$$

Show that β is a right congruence on S_k with left identity

$$((b_{ij})) = \begin{bmatrix} c_{11} & 0 & \cdots 0 \\ \\ 0 & c_{22} & \cdots 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & \cdots c_{kk} \end{bmatrix}.$$

We shall now show that $((b_{ij}))$ is not β-congruent to 0. Suppose that each of m and n is a positive integer such that

$$((c_{ij}))^m \sigma \begin{bmatrix} 0 \cdots & 0 & 0 \\ \vdots & \cdots & \vdots \\ 0 \cdots & 0 & c^n_{k+1,k+1} \end{bmatrix}.$$

If s, $t \in S$, let $s\psi t$ if

$$\begin{bmatrix} 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots & 0 & s \end{bmatrix} \sigma \begin{bmatrix} 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots & 0 & t \end{bmatrix}.$$

Thus ψ is a right congruence on S with left identity $c^n_{k+1,k+1}$. Since

$$\begin{bmatrix} 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 \cdots & 0 & c^n_{k+1,k+1} \end{bmatrix} \sigma\, ((c_{ij}))^m \sigma((c_{ij})) \text{ is not } \sigma\text{-congruent to 0,}$$

then $c^n_{k+1,k+1}$ is not ψ-congruent to 0. Just as we saw above for $((a_{ij}))$, $c^n_{k+1\ k+1}$ is not right quasi-regular. But $c^n_{k+1,k+1} \in N(S)$, which contradicts Theorem 5. Therefore $((c_{ij}))^m$ is not σ-congruent to

$$\begin{bmatrix} 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & c^n_{k+1,k+1} \end{bmatrix}$$

for any m and n.

If there exist positive integers m and n such that

$$\begin{bmatrix} c_{11} & 0 & \cdots & 0 & 0 \\ 0 & c_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & c_{kk} & 0 \\ 0 & 0 & \cdots & 0 & c^m_{k+1,k+1} \end{bmatrix} \sigma \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & c^n_{k+1,k+1} \end{bmatrix},$$

then

$$((c_{ij}))^{m+1} = \begin{bmatrix} c_{11} & 0 & \cdots & 0 & 0 \\ 0 & c_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & c_{kk} & 0 \\ 0 & 0 & \cdots & 0 & c^m_{k+1,k+1} \end{bmatrix} \begin{bmatrix} c^m_{11} & 0 & \cdots & 0 & 0 \\ 0 & c^m_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & c^m_{kk} & 0 \\ 0 & 0 & \cdots & 0 & c_{k+1,k+1} \end{bmatrix}$$

$$\sigma \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & c^n_{k+1,k+1} \end{bmatrix} \begin{bmatrix} c^m_{11} & 0 & \cdots & 0 & 0 \\ 0 & c^m_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & c^m_{kk} & 0 \\ 0 & 0 & \cdots & 0 & c_{k+1,k+1} \end{bmatrix} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \\ 0 & \cdots & c^{n+1}_{k+1,k+1} \end{bmatrix}$$

which contradicts our previous work. Thus $((b_{ij}))$ is not B-congruent to 0.

Just as we saw above for $((a_{ij}))$, $((b_{ij}))$ is not right quasi-regular but $((b_{ij})) \in N(S_k)$, which contradicts Theorem 5. Therefore $((e_{ij}))$ is right quasi-regular, $[N(S)]_{k+1}$ is quasi-regular, and $[N(S)]_{k+1} \subseteq N(S_{k+1})$.

THEOREM 7. *The radical of $S/N(S)$ is zero, where $S/N(S)$ is defined as in* (3).

Proof. This follows from the fact that the set M is a 0-transitive operand of S if and only if M is a 0-transitive operand of $S/N(S)$.

Suppose that S has a zero element 0. The element s in S is *nilpotent* if there is a positive integer n such that $s^n = 0$. A right ideal of S is *nil* if each element contained in it is nilpotent. The right ideal T is *nilpotent* if there is a positive integer n such that $T^n = 0$.

THEOREM 8. *If S has a zero element and s is a nilpotent element of S, then s is a right quasi-regular element of S.*

Proof. Suppose that s is nilpotent but not right quasi-regular. Let σ be a right congruence on S which is not the universal relation such that s is a left identity modulo σ. For each positive integer n, $s^n \sigma s$. Thus $s\sigma 0$, which contradicts the fact that σ is not the universal relation. Therefore s is right quasi-regular.

COROLLARY 8. *If S has a zero element, then $N(S)$ contains each nil right ideal of S.*

Proof. This follows from Theorems 5 and 8.

THEOREM 9. *If T is a two-sided ideal of S and $t \in T$, then t is a right quasi-regular element of S if and only if t is a right quasi-regular element of T.*

Proof. Suppose that σ is a right congruence on T such that T/σ is a 0-transitive operand of T and t is a left identity modulo σ. If $r, s \in S$, let $r\gamma s$ if $tsp\sigma trp$ for each p in S. Thus γ is a right congruence on S such that t is a left identity modulo γ. Let z be an element in the zero of T/σ. To show that $z_\gamma S = z_\gamma$, suppose that each of s and p is an element of S such that $tzsp$ is not σ-congruent to tzp. Therefore either $(zsp)T = T/\sigma$ or $(zp)T = T/\sigma$. But if $n \in T$, then $zspn \in z_\sigma T = z_\sigma$ and $zpn \in z_\sigma T = z_\sigma$, which is a contradiction. Thus $tzsp\sigma tzp$ for each s and p in S and $z_\gamma S = z_\gamma$. Let τ be a maximal right congruence on S containing γ. Since S/τ is primitive, t is a left identity modulo τ, and $z_\tau S = z_\tau$, then S/τ is a 0-transitive operand of S. Therefore each element of T which is a right quasi-regular element of S is also a right quasi-regular element of T.

Conversely, suppose that σ is a right congruence on S such that S/σ is a 0-transitive operand of S and t is a left identity modulo σ. If $x, y \in T$, let $x\gamma y$ if $x\sigma y$. Thus γ is a right congruence on T with left identity t. Let τ be a maximal right congruence on T which contains γ. Thus T/τ is 0-transitive with left identity t. Therefore each element of T which is a right quasi-regular element of T is also a right quasi-regular element of S.

COROLLARY 9. $N[N(S)] = N(S)$.

Proof. Since $N(S)$ is a quasi-regular ideal of S then $N(S)$ is a quasi-regular ideal of $N(S)$. Thus by Theorem 5, $N(S) \subseteq N[N(S)]$.

LEMMA 10. *If each of S and T is a semigroup and θ is a homomorphism from S onto T then $[N(S)]\theta \subseteq N(T)$.*

Proof. Suppose that $r \in N(S)$ and $(r)\theta$ is not a right quasi-regular element of T. Let σ be a right congruence on T such that T/σ is a 0-transitive operand of T and $(r)\theta$ is a left identity modulo σ. If $s, p \in S$, let $s\tau p$ if $(s)\theta\sigma(p)\theta$. One sees that τ is a right congruence on S. If $z \in S$ such that $(z)\theta$ is contained in the zero element of T/σ, then z_τ is the zero element of S/τ. Suppose that $n \in S$ and $n \notin z_\tau$. Since $(n_\tau)\theta$ is an element of T/σ which is not $[(z)\theta]_\sigma$, then $[(n_\tau)\theta]T = T/\sigma$. Suppose that $m_\tau \in S/\tau$. Let $t \in T$ such that $[(n_\tau)\theta]t = (m_\tau)\theta$. Let $s \in S$ such that $(s)\theta = t$. Then $n_\tau s = m_\tau$ and S/τ is a 0-transitive operand of S. If $s \in S$, since $(rs)\theta = (r)\theta(s)\theta\sigma(s)\theta$, then $rs\tau s$. But this contradicts the fact that $N(S)$ is a quasi-regular ideal. Thus if $r \in N(S)$, then $(r)\theta$ is a right quasi-regular element of T. Thus $\{(r)\theta\} \cup \{(r)\theta T\}$ is a quasi-regular right ideal of T and $\{(r)\theta\} \cup \{(r)\theta T\} \subseteq N(T)$. Therefore $[N(S)]\theta \subseteq N(T)$.

Theorem 10. *If T is a two-sided ideal of S, then $N(T) = N(S) \cap T$.*

Proof. By Theorem 7, $N[S/N(S)]$ is zero. Let $Q = [T \cup N(S)]/N(S)$. Since $[N(Q)]Q \subseteq N(Q)$, then by Theorems 5 and 9, each element of $[N(Q)]Q$ is a right quasi-regular element of Q and of $S/N(S)$. Thus $[N(Q)]Q \subseteq N[S/N(S)] =$ zero. Let $P = \{p : p$ is an element of $T \cup N(S)$ such that $[p/N(S)]Q =$ zero$\}$. Since $P/N(S)$ is an ideal of Q such that $[P/N(S)]^2 =$zero, by Corollary 8, $P/N(S) \subseteq N(Q)$. Thus $N(Q) = P/N(S)$. If $R = \{r : r \in S$ such that $[r/N(S)]Q =$ zero$\}$, then $R/N(S)$ is a two-sided ideal of $S/N(S)$ and $N(Q) = [R/N(S)] \cap Q$. Thus $N(Q)$ is a two-sided ideal of $S/N(S)$. By the previous theorem, $N(Q)$ is a quasi-regular two-sided ideal of $S/N(S)$. Therefore, by Theorems 5 and 7, $N(Q)$ is zero. Let θ be the natural homomorphism from $T \cup N(S)$ onto Q. By the previous lemma, $\{N[T \cup N(S)]\}\theta \subseteq N(Q) = 0 = N(S)/N(S)$ and thus $N[T \cup N(S)] \subseteq N(S)$. By Theorems 5 and 9, $N(T)$ is a quasi-regular ideal of $T \cup N(S)$. Hence $N(T) \subseteq N[T \cup N(S)$ and $N(T) \subseteq N(S) \cap T$.

Since $N(S) \cap T$ is an ideal of S, then, by the previous theorem, each element of $N(S) \cap T$ is a right quasi-regular element of $N(S) \cap T$. Thus $N(S) \cap T \subseteq N[N(S) \cap T]$. Since $N(S) \cap T$ is an ideal of the semigroup T then, by the above work, $N[N(S) \cap T] \subseteq N(T) \cap [N(S) \cap T] = N(T)$. Thus $N(S) \cap T \subseteq N(T)$.

In a similar manner one can define a left operand of a semigroup and one can prove the preceding theorems for this case. We shall denote the left radical of S by $L(S)$.

Theorem 11. *If S has a zero element* 0, *S satisfies the minimum condition on right ideals and T is a quasi-regular right ideal of S then T is nilpotent.*

Proof. Suppose that T is not nilpotent. Let $\mathscr{R} = \{R : R$ is a right ideal of S which is not nilpotent and $R \subseteq T\}$. $\mathscr{R}$ is nonempty since it contains T. Let P be a minimal element of $\mathscr{R}$. Since $P^2 \in \mathscr{R}$ and $P^2 \subseteq P$, then $P^2 = P$. Let $\mathscr{M} = \{M : M$ is a right ideal of S such that $MP \neq 0$ and $M \subseteq P\}$. Thus $P \in \mathscr{M}$. Let N be a minimal element of $\mathscr{M}$ and $n \in N$ such that $nP \neq 0$. Since $(nP)P \neq 0$, then $nP \in \mathscr{M}$, $nP \subseteq N$, and therefore $nP = N$. Let $p \in P$ such that $np = n$. If s, $t \in S$, let $s\sigma t$ if $ns = nt$; σ is a right congruence on S with left identity p. If σ were the universal relation on S, then $0 = n0 = np = n$, which contradicts the fact that $nP \neq 0$. Thus there is a maximal right congruence τ which contains σ. Since p is a left identity modulo τ, if S/τ contains only the two elements p_τ and o_τ, then S/τ is 0-transitive. If S/τ contains more than two elements then, since S/τ is primitive, S/τ is 0-transitive. Thus p is an element of T which is not a right quasi-regular element of S.

Corollary 11.1. *If S is as in Theorem 11 and T is a nil right ideal of S, then T is nilpotent.*

Proof. Since $N(S)$ is quasi-regular, then $N(S)$ is nilpotent. By Corollary 8, $T \subseteq N(S)$; hence T is nilpotent.

Corollary 11.2. *If S has a zero and satisfies the minimum condition on left ideals and on right ideals, then $N(S) = L(S)$.*

Proof. Since $N(S)$ is quasi-regular, then $N(S)$ is nilpotent. By Corollary 8, $N(S) \subseteq L(S)$. Similarly $L(S) \subseteq N(S)$.

2. Semisimple semigroups.

Theorem 1. *If S is semisimple and T is a two-sided ideal of S then T is semisimple.*

Proof. Let ρ be the radical congruence of T, and t, s be distinct elements of T. Since S is semisimple, then S has a 0-transitive operand M which contains an element m such that $mt \neq ms$. If $n, j \in M$, let $n\sigma j$ if either $n = j$ or $nT = jT = 0$; thus σ is a right congruence on M. Since $mt \neq ms$, then $m_\sigma \neq 0_\sigma$. Suppose that that $k_\sigma \in M/\sigma$ and $k_\sigma \neq 0_\sigma$. Then there is a $u \in T$ such that $ku \neq 0$. Thus $M/\sigma = (kuS)/\sigma = k_\sigma uS \subseteq k_\sigma T$. Therefore M/σ is a 0-transitive operand of T. At least one of mt or ms is nonzero. Suppose that $mt \neq 0$. Thus $mtS = M$. Let $r \in S$ such that $mtr = m$. Then $0 \neq mt = mtrt \in mtT$. Since $m_\sigma t = (mt)_\sigma \neq (ms)_\sigma = m_\sigma s$ then t is not ρ-congruent to s. Therefore ρ is the identity relation.

Theorem 2. *The semigroup S which contains more than one element is semisimple if and only if S is isomorphic to a subdirect sum of a set of semigroups each of which has a faithful* 0*-transitive operand.* (*Subdirect sum is defined as in* (2) *and a faithful operand is defined in* (5).)

Proof. Suppose that S is semisimple. If M is a 0-transitive operand of S, let ρ_M be the two-sided congruence on S such that if s, $t \in S$ then $s\rho_M t$ if $ms = mt$ for each m in M. Thus S/ρ_M is a semigroup and M is a faithful 0-transitive operand of S/ρ_M.

Let τ be a function from S into $\sum_M \oplus S/\rho_M$ such that if $s \in S$ and M is a 0-transitive operand of S then $M[(s\tau)] = s_{\rho M}$. The image of τ is a subdirect sum of the set $\{S/\rho_M$: M is a 0-transitive operand of $S\}$. Since $M[(st)\tau] = (st)\rho_M = s\rho_M t\rho_M = M[(s)\tau]M[(t)\tau]$, then $(st)\tau = (s)\tau(t)\tau$. Suppose that $(s)\tau = (t\tau_\tau)$. Then $s_{\rho M} = t_{\rho M}$ for each 0-transitive operand M. Thus $s\rho t$ where ρ is the radical congruence on S. Since S is semisimple, then $s = t$ and τ is reversible. Thus S is isomorphic to a subdirect sum of a set of semigroups each of which has a faithful 0-transitive operand.

Conversely, suppose that τ is an isomorphism from S onto a subdirect sum of a set $\{S_M : S_M$ is a semigroup which has a faithful 0-transitive operand $M\}$. Let ρ be the radical congruence on S and s, $t \in S$ such that $s\rho t$. If M is one of the above 0-transitive operands, $m \in M$, and $r \in S$, let $mr = m[M(r\tau)]$; thus M is a 0-transitive operand of S. Since $s\rho t$, then $m[M(s\tau)] = ms = mt = m[M(t\tau)]$ for each m in M. Since M is a faithful operand of S_M, then $M(s\tau) = M(t\tau)$. Thus $(s)\tau = (t)\tau$. Since τ is reversible, then $s = t$ and S is semisimple.

Bibliography

1. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Vol. I, Math. Surveys No. 7, Amer. Math. Soc., Providence, R. I., 1961.

2. Nathan Jacobson, *Structure of rings*, Amer. Math. Soc. Colloq. Publ. Vol. 37, Amer. Math. Soc., Providence, R. I., 1956.

3. D. Rees, *On semigroups*, Proc. Cambridge Philos. Soc. **36** (1940), 387–400.

4. R. R. Stoll, *Representation of finite simple semigroups*, Duke Math. J. **11** (1944), 251–265.

5. E. J. Tully, *Representation of a semigroup by transformations acting transitively on a set*, Amer. J. Math. **83** (1961), 533–451.

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