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NEW APPLICATIONS OF METHOD OF COMPLEX NUMBERS IN THE GEOMETRY OF CYCLIC QUADRILATERALS

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Abstract. Any cyclic quadrilateral whose sides are not parallel can define a triangle with one vertex at the point of intersection of the quadrilateral's diagonals and the other vertices at the points of intersection of the continuations of the quadrilateral's pairs of opposite sides.
Using a cyclic quadrilateral and this triangle, the following four circles may be defined: the circumcircle of the quadrilateral, two circles with diameters that are the sides of the triangle that issue from the point of intersection of the quadrilaterals diagonals, and the Euler circle of the triangle.
In the current paper, we shall prove four properties that relate the quadrilateral, the triangle and these circles.
We shall show that the two circles whose diameters are the sides of the triangle are perpendicular to the circumcircle of the quadrilateral.
We shall prove equalities that relate the angle between the midlines in the quadrilateral with other angles.
We shall show that the point of intersection of the midlines of the quadrilateral belongs to the Euler's circle of the triangle defined using the quadrilateral.
In proving these properties we shall make use of the method of complex numbers in plane geometry, thereby illustrating different uses of this method of proof.

1. Introduction

The method of complex numbers in plane geometry is founded on the following principles:

(1) We choose a Cartesian system of coordinates in the plane. Any point, M, that belongs to the plane is given a pair of real coordinates (x, y) or a complex coordinate $m = x + yi$.
The number $\overline{m} = x - yi$ is the conjugate of m, and is the complex coordinate of point M', which is symmetrical to point M relative to the real axis of the system.

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(2) By using the coordinate m and the conjugate number $\overline{m}$ together, we can describe the properties of point M. For example: the product $m \cdot \overline{m}$ represents the square of the distance of point M from the origin, the equality $m = \overline{m}$ holds when m is a real number.
(3) There exist formulas that relate points that are located on geometrical shapes or which express properties of the shapes. These formulas include the coordinates of points (in the form of single letters without separation into real and imaginary parts) and the conjugate of the coordinates (also in the form of single letters).
(4) Most of the formulas can be simplified by using points that belong to a unit circle (a circle whose center lies at the origin and whose radius equals 1).

Some isolated formulas of the type described may be found in [7], [8], [10]; a system of formulas in can be found, for example, in [9, pp.154-181], [1].
Utilizing the method of complex numbers in plane geometry allows us to solve geometrical problems algebraically through defined formulas and technical calculations with polynomials.
The data and requirement of the problem determine which formulas of the method of complex numbers are suitable for solving this problem.
When a non-standard problem in geometry is given, its solution using the "ordinary" method (geometrical proof) often requires creativity and the ability to carry out a prolonged search for the gist of the solution; using the method of complex numbers in plane geometry allows transforming the problem into a standard algebraic.
The main shortcoming of the complex number method is that, during the process of solution, a stage may be reached where expressions are obtained that are so huge that it is difficult to simplify them manually. At this stage, we can make use of computer software such as *Mathematica* or *MATLAB*.
To prove the properties set forth in this paper, we shall make use of the method of complex numbers in the geometry of the plane. For proving properties 1 and 4, we shall make simplify manually; for proving properties 2 and 3, we shall make use of *Mathematica* software.

2. Properties that relate a cyclic quadrilateral to a triangle defined by it and to circles defined using the triangle

General data for all properties.
Let $ABCD$ be a quadrilateral inscribed in circle ε (O is the center of ε), in which:
E is the point of intersection of the diagonals;
F is the point of intersection of the continuations of sides BC and AD;
G is the point of intersection of the continuations of sides AB and CD;
T is the point of intersection of midlines PQ and VW (see Figure 1)*;*
O_1 is the middle of segment EF and O_2 is the middle of segment EG.

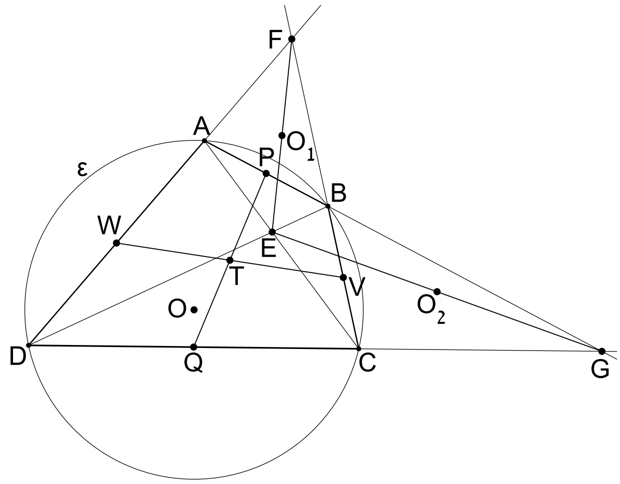

Figure 1

Property 1.
Additional data:
ω_{EF} is a circle whose diameter is segment EF,
ω_{EG} is a circle whose diameter is segment EG (see Figure 2),
H is the other point of intersection of circles ω_{EF} and ω_{EG} (in addition to point E). Then:

(a) circles ω_{EF} and ω_{EG} are each perpendicular to circumcircle ε of the quadrilateral;
(b) point H belongs to the straight line FG.

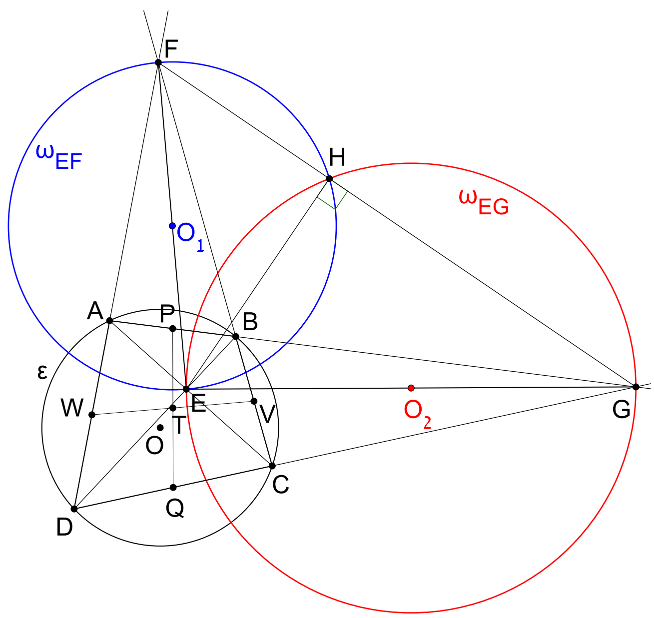

Figure 2

Proof.
(a) To prove that circles ε and ω_{EF} are perpendicular, it is sufficient to verify that the equality

$$|OO_1|^2 = r_\varepsilon^2 + r_{\omega_{EF}}^2 \tag{1}$$

is satisfied, where r_ε is the radius of circle ε and $r_{\omega_{EF}}$ is the radius of circle ω_{EF}.
We shall make use of the method of complex numbers in plane geometry.
We choose a system of coordinates such that circle ε is the unit circle: in other words, point O is the origin of the system and $r_\varepsilon = 1$. In this system, the equation of the circle ε is, $z \cdot \overline{z} = 1$, where z and $\overline{z}$ are a complex coordinate and its conjugate of an arbitrary point that is located on circle ε.
We denote the complex coordinates of points A, B, C and D by a, b, c and d, respectively. These points are located on the unit circle. Thus, the following relation holds between the coordinates of the points and their conjugates: $\bar{a} = \frac{1}{a}$, $\bar{b} = \frac{1}{b}$, $\bar{c} = \frac{1}{c}$ and $\bar{d} = \frac{1}{d}$.
Since segment O_1E is the radius of circle ω_{EF}, we can write down $r_{\omega_{EF}} = |O_1E|$.
We substitute the values of r_ε and $r_{\omega_{EF}}$ in equality (1) to obtain:

$$|OO_1|^2 = 1 + |O_1E|^2 \tag{2}$$

Using the complex coordinates of points E, O_1, and O (and their conjugates), equality (2) can be written as:

$$(o_1 - 0)\left(\overline{o}_1 - \overline{0}\right) = 1 + (e - o_1)\left(\overline{e} - \overline{o}_1\right).$$

Point O_1 is the middle of segment EF, therefore for coordinate o_1 and number $\overline{o}_1$ there holds: $o_1 = \frac{1}{2}(f + e)$ and $\overline{o}_1 = \frac{1}{2}\left(\overline{f} + \overline{e}\right)$, as well as $\overline{0} = 0$.
Therefore, the last equality can be transformed into:

$$\frac{1}{4}(f + e)\left(\overline{f} + \overline{e}\right) = 1 + \frac{1}{4}(e - f)\left(\overline{e} - \overline{f}\right),$$

and finally we obtain:

$$f\overline{e} + e\overline{f} = 2 \tag{3}$$

Now, we express the complex coordinates of points E and F (and their conjugates) using the coordinates of points A, B, C, and D.
To this end, we shall make use of the following formulas:

Let $K(k)$, $L(l)$, $M(m)$, and $N(n)$ be four points that belong to the unit circle, and let $S(s)$ be the point of intersection of the straight lines that pass through the chords KL and MN in the unit circle.
For the complex coordinate of S and its conjugate, there holds:

$$\bar{s} = \frac{k + l - m - n}{kl - mn} \quad \text{and} \quad s = \frac{lmn + kmn - kln - klm}{mn - kl} \tag{4}$$

Using the formulas (4), the complex coordinates (and their conjugates) of points E and F can be expressed as:

$$\overline{e} = \frac{a + c - b - d}{ac - bd} \quad \text{and} \quad e = \frac{bcd + abd - acd - abc}{bd - ac} \tag{5}$$

$$\overline{f} = \frac{a + d - b - c}{ad - bc} \quad \text{and} \quad f = \frac{bcd + abc - acd - abd}{bc - ad} \tag{6}$$

Let us calculate the left-hand side of (3). We substitute in it expressions (5) and (6), and obtain:

$$f\overline{e} + e\overline{f} =$$
$$= \frac{bcd + abc - acd - abd}{bc - ad} \cdot \frac{a + c - b - d}{ac - bd} + \frac{bcd + abd - acd - abc}{bd - ac} \cdot \frac{a + d - b - c}{ad - bc}.$$

After adding and multiplying the algebraic fractions, and simplifying the numerator of the obtained fraction, we obtain the following:

$$\frac{-2b^2cd + 2abc^2 - 2a^2cd + 2abd^2}{(bc - ad)(ac - bd)} = 2 \cdot \frac{-b^2cd + abc^2 - a^2cd + abd^2}{abc^2 - b^2cd - a^2cd + abd^2} = 2 \cdot 1 = 2.$$

Since we have shown that equality (3) holds, it follows that equalities (2) and (1) also hold. In other words, circles ε and ω_{EF} are perpendicular.
The perpendicularity of circles ε and ω_{EG} is proven in a similar manner.

(b) The claim that H belongs to straight line FG follows from the fact that angles $\measuredangle EHF$ and $\measuredangle EHG$ are both inscribed angles resting on the diameter of the circle and therefore each one measures $90°$. Therefore $\measuredangle FHG$ is a straight angle.
Since $H \in FG$ and $EH \perp FG$, the following two properties hold:

(i) Segment EH is an altitude to side FG in triangle $\triangle EFG$.
(ii) Inversion relative to circle ε transforms each of the circles ω_{EF} and ω_{EG} into itself (since they are perpendicular to ε), and hence it transforms their points of intersection E and H one into the other.

□

Property 2.
The sum of two angles one of which is the angle between the midlines PQ and VW, and the other is the angle of triangle EFG whose vertex at point E equals $180°$ (in Figure 3, $\measuredangle PTV + \measuredangle FEG = 180°$).

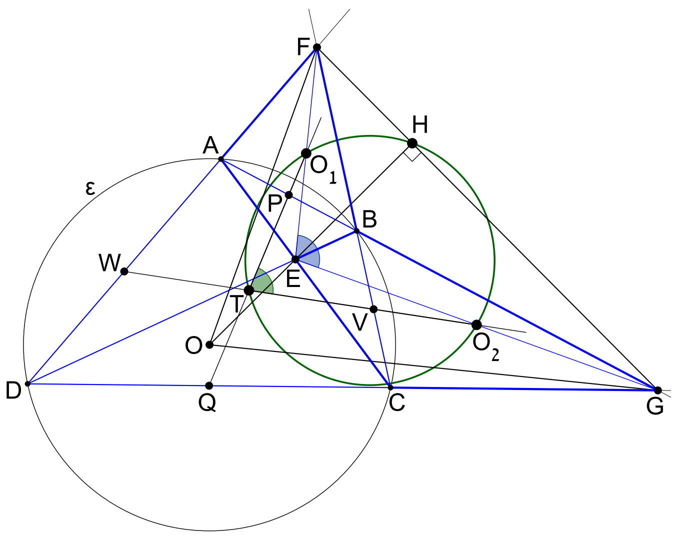

Figure 3

Proof. In the first phase, we prove that straight lines PQ and VW pass through points O_1 and O_2, respectively.
We use the following property of the complete quadrilateral (see [6, Section 194]: *"in the complete quadrilateral, the middles of the diagonals are on the same straight line"*.
In our case, in the complete quadrilateral $AFBECD$ (see Fig. 3), points P, Q and O_1 are the middles of diagonals AB, CD and EF, respectively. Therefore, they are on the same straight line.
Similarly, in the complete quadrilateral $BGCEDA$ the points V, W and O_2 are the middles of the diagonals BC, AD, and EG, respectively, and therefore they lie on the same straight line.
We now prove that the four points, O_1, T, O_2, and H, lie on the same circle.
Again we employ the method of complex numbers. We again choose a system where circle ε is the unit circle, and express the complex coordinates of points G, O_1, T, O_2, and H (and their conjugates) using the coordinates of points A, B, C, and D.
Using the formulas (4) which were presented in the proof of Property 1, the complex coordinates (and their conjugates) of point G can be expressed as:

$$\bar{g} = \frac{c+d-a-b}{cd-ab} \quad \text{and} \quad g = \frac{abd+abc-bcd-acd}{ab-cd} \tag{7}$$

The points P and Q are the middles of sides AB and CD.
Therefore, $p = \frac{1}{2}(a+b)$ and $q = \frac{1}{2}(c+d)$.
The point T is the middle of segment PQ. Therefore, the coordinate of T (and its conjugate) can be expressed as:

$$\begin{aligned} t = \frac{1}{2}(p+q) = \frac{1}{4}(a+b+c+d) \quad \text{and} \\ \bar{t} = \frac{1}{4}\overline{(a+b+c+d)} = \frac{bcd+acd+abd+abc}{4abcd} \end{aligned} \tag{8}$$

The complex coordinates of points O_1 and O_2 (which are the middles of segments EF and EG), and their conjugates shall be:

$$o_1 = \frac{1}{2}(f+e) \quad \text{and} \quad \bar{o}_1 = \frac{1}{2}(\bar{f}+\bar{e}) \tag{9}$$

$$o_2 = \frac{1}{2}(g+e) \quad \text{and} \quad \bar{o}_2 = \frac{1}{2}(\bar{g}+\bar{e}) \tag{10}$$

Points E and H are the points of intersection of circles ω_{EF} and ω_{EG}, which are perpendicular to circle ε. Inversion relative to circle ε transforms each of the circles ω_{EF} and ω_{EG} into itself. Therefore, it transforms their points of intersection E and H one into the other (see for example [2, chapter 5, paragraph 5]), and therefore, the complex coordinates of points E and H and their conjugates satisfy the following relation (see [10, paragraph 13]):

$$h = \frac{1}{\bar{e}} \quad \text{and} \quad \bar{h} = \frac{1}{e} \tag{11}$$

We use the following property of four points which belong to the same circle (see [10, paragraph 7]): *Points $K(k)$, $L(l)$, $M(m)$ and $N(n)$ belong to the same circle if and only if the complex coordinates of these points and their* *conjugates satisfy the following relation:*

$$\frac{k-m}{l-m}:\frac{k-n}{l-n}=\frac{\overline{k}-\overline{m}}{\overline{l}-\overline{m}}:\frac{\overline{k}-\overline{n}}{\overline{l}-\overline{n}}.$$

Therefore, in order to prove that points O_1, T, O_2, and H lie on the same circle, it is enough to show that the following relation is satisfied:

$$\frac{o_1-t}{o_2-t}:\frac{o_1-h}{o_2-h}=\frac{\overline{o}_1-\overline{t}}{\overline{o}_2-\overline{t}}:\frac{\overline{o}_1-\overline{h}}{\overline{o}_2-\overline{h}} \tag{12}$$

By substituting the expressions for the letters t, $\overline{t}$, o_1, $\overline{o}_1$, o_2, $\overline{o}_2$, h, and $\overline{h}$ obtained above into (12), we get:

$$\left(\frac{\dfrac{\left(\dfrac{bcd+abc-acd-abd}{bc-ad}\right)+\left(\dfrac{bcd+abd-acd-abc}{bd-ac}\right)}{2}-\left(\dfrac{a+b+c+d}{4}\right)}{\dfrac{\left(\dfrac{abd+abc-bcd-acd}{ab-cd}\right)+\left(\dfrac{bcd+abd-acd-abc}{bd-ac}\right)}{2}-\left(\dfrac{a+b+c+d}{4}\right)}\right)\times$$

$$\times\left(\frac{\dfrac{\left(\dfrac{abd+abc-bcd-acd}{ab-cd}\right)+\left(\dfrac{bcd+abd-acd-abc}{bd-ac}\right)}{2}-\left(\dfrac{ac-bd}{a+c-b-d}\right)}{\dfrac{\left(\dfrac{bcd+abc-acd-abd}{bc-ad}\right)+\left(\dfrac{bcd+abd-acd-abc}{bd-ac}\right)}{2}-\left(\dfrac{ac-bd}{a+c-b-d}\right)}\right)=$$

$$=\left(\frac{\dfrac{\left(\dfrac{a+d-b-c}{ad-bc}\right)+\left(\dfrac{a+c-b-d}{ac-bd}\right)}{2}-\left(\dfrac{bcd+acd+abd+abc}{4abcd}\right)}{\dfrac{\left(\dfrac{c+d-a-b}{cd-ab}\right)+\left(\dfrac{a+c-b-d}{ac-bd}\right)}{2}-\left(\dfrac{bcd+acd+abd+abc}{4abcd}\right)}\right)\times$$

$$\times\left(\frac{\dfrac{\left(\dfrac{c+d-a-b}{cd-ab}\right)+\left(\dfrac{a+c-b-d}{ac-bd}\right)}{2}-\left(\dfrac{bd-ac}{bcd+abd-acd-abc}\right)}{\dfrac{\left(\dfrac{a+d-b-c}{ad-bc}\right)+\left(\dfrac{a+c-b-d}{ac-bd}\right)}{2}-\left(\dfrac{bd-ac}{bcd+abd-acd-abc}\right)}\right)$$

After simplifying the two sides of the equality (for this, we used *Mathematica* ® software), we obtain the same expression,
$-\dfrac{(b-c)(a-d)\left(a^2cd+b^2cd+ab\left(c^2-4cd+d^2\right)\right)}{(a-b)(c-d)\left(a^2bc+a\left(b^2-4bc+c^2\right)d+bcd^2\right)}$, on both sides.

Therefore, equality (12) holds, and points O_1, T, O_2 and H lie on the same circle.

Hence, it follows that quadrilateral O_1TO_2H is inscribable (see Fig. 4), and therefore the opposite angles in this quadrilateral satisfy:
$\measuredangle O_1TO_2+\measuredangle O_1HO_2=180^\circ$ or $\measuredangle PTV+\measuredangle O_1HO_2=180^\circ$ (because $\measuredangle O_1TO_2=\measuredangle PTV$).

We now prove that angles $\measuredangle O_1HO_2$ and $\measuredangle O_1EO_2$ are equal.
By symmetry transformation relative to straight line O_1O_2, each of the points O_1 and O_2 is transformed into itself.
In triangle EFG, segment EH is an altitude to side FG and segment O_1O_2 is the midline. Therefore, segment EH is bisected by O_1O_2 and is perpendicular to it. It follows from here that points E and H are two symmetric points relative to the line O_1O_2. We obtained that symmetry relative to line O_1O_2 transforms angles $\measuredangle O_1HO_2$ and $\measuredangle O_1EO_2$ one into the other.
Therefore, $\measuredangle O_1HO_2 = \measuredangle O_1EO_2$, and hence: $\measuredangle PTV + \measuredangle O_1EO_2 = 180°$, and finally, we obtain: $\measuredangle PTV + \measuredangle FEG = 180°$.

□

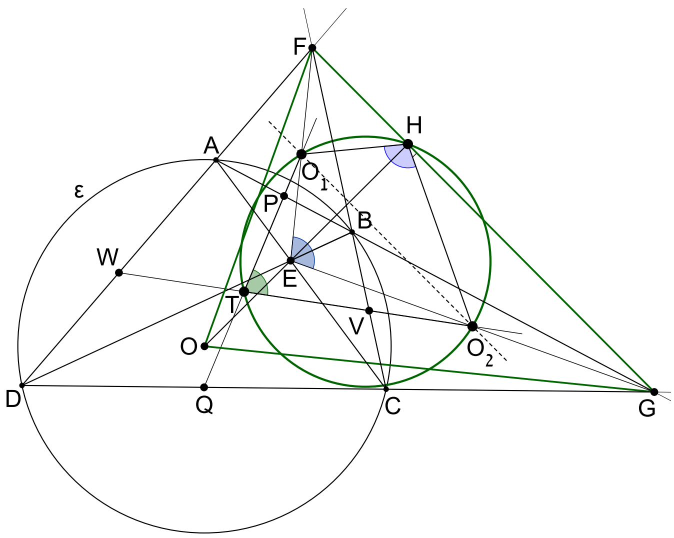

Figure 4

Property 3.
The point of intersection of the midlines in a cyclic quadrilateral $ABCD$ (point T in Figure 4) *belongs to the nine-point circle (the Euler circle) of triangle EFG.*

Proof. As is well known the nine-point circle of a triangle is a circle that passes through the following nine points: the middles of the three sides, the three feet of the altitudes (the points of intersection of the altitudes and the sides), and the middles of the three segments between the triangle's vertices and the orthocenter. Any three points of these nine may be used to define the Euler's circle.
Points O_1 and O_2 are the middles of sides EF and EG, respectively, of triangle EFG. Point H is the foot of altitude OH to side FG. Therefore points O_1, H, and O_2 are three of the nine points that belong to Euler's circle of triangle EFG, and therefore they define the Euler circle of triangle EFG.
In Property 2 we proved that the points O_1, T, O_2, and H are on the same circle. Therefore point T belongs to Euler's circle of triangle EFG.

□

Property 4.
The following equality holds: $\measuredangle PTV = \measuredangle FOG$.

Proof. We make use of the following formula which expresses the cosine of an angle through the complex coordinates of the points that define the angle:
Let $\measuredangle BAC = \alpha$ *be an angle where point* $A(a)$ *is the vertex and where points* $B(b)$ *and* $C(c)$ *are located on the sides of the angle in such a manner that rotation around point* A *at angle* α *counterclockwise transfers ray* AB *to ray* AC.
Then there holds:

$$\cos\alpha = \cos\left(\widehat{\overrightarrow{AB},\overrightarrow{AC}}\right) = \frac{(c-a)\left(\overline{b}-\overline{a}\right)+(\overline{c}-\overline{a})(b-a)}{2\sqrt{(c-a)(\overline{c}-\overline{a})}\cdot\sqrt{(b-a)\left(\overline{b}-\overline{a}\right)}}$$

From this formula, for angle FOG there holds:

$$\cos\left(\widehat{\overrightarrow{OG},\overrightarrow{OF}}\right) = \frac{(f-0)\left(\overline{g}-\overline{0}\right)+\left(\overline{f}-\overline{0}\right)(g-0)}{2\sqrt{(f-0)\left(\overline{f}-\overline{0}\right)}\cdot\sqrt{(g-0)\left(\overline{g}-\overline{0}\right)}} = \frac{f\cdot\overline{g}+\overline{f}\cdot g}{2\sqrt{f\cdot\overline{f}\cdot g\cdot\overline{g}}} \tag{13}$$

For angle PTV there holds:

$$\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right) = \frac{(p-t)\left(\overline{v}-\overline{t}\right)+\left(\overline{p}-\overline{t}\right)(v-t)}{2\sqrt{(p-t)\left(\overline{p}-\overline{t}\right)}\cdot\sqrt{(v-t)\left(\overline{v}-\overline{t}\right)}}.$$

Let us calculate each of the parentheses that appear in the previous formula:

$$v-t = \frac{b+c}{2}-\frac{a+b+c+d}{4} = \frac{b+c-a-d}{4} = \frac{b+c-a-d}{bc-ad}\cdot\frac{bc-ad}{4} = \overline{f}\cdot\frac{bc-ad}{4}.$$

$$\overline{v}-\overline{t} = \frac{b+c}{2bc}-\frac{bcd+acd+abd+abc}{4abcd} = \frac{abd+acd-bcd-abc}{4abcd} = \frac{abd+acd-bcd-abc}{ad-bc}\cdot\frac{ad-bc}{4abcd} = f\cdot\frac{ad-bc}{4abcd}$$

$$p-t = \frac{a+b}{2}-\frac{a+b+c+d}{4} = \frac{a+b-c-d}{4} = \frac{a+b-c-d}{ab-cd}\cdot\frac{ab-cd}{4} = \overline{g}\cdot\frac{ab-cd}{4}.$$

$$\overline{p}-\overline{t} = \frac{a+b}{2bc}-\frac{bcd+acd+abd+abc}{4abcd} = \frac{acd+bcd-abd-abc}{4abcd} = \frac{acd+bcd-abd-abc}{cd-ab}\cdot\frac{cd-ab}{4abcd} = g\cdot\frac{cd-ab}{4abcd}.$$

Now, we substitute the expressions obtained in the formula for $\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)$, and obtain:

$$\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)=\frac{\overline{g}\cdot\dfrac{ab-cd}{4}\cdot f\cdot\dfrac{ad-bc}{4abcd}+g\cdot\dfrac{cd-ab}{4abcd}\cdot\overline{f}\cdot\dfrac{bc-ad}{4}}{2\sqrt{\overline{g}\cdot\dfrac{ab-cd}{4}\cdot g\cdot\dfrac{cd-ab}{4abcd}}\cdot\sqrt{\overline{f}\cdot\dfrac{bc-ad}{4}\cdot f\cdot\dfrac{ad-bc}{4abcd}}}=$$

$$=\frac{\dfrac{bc-ad}{4}\cdot\dfrac{cd-ab}{4abcd}\left(f\cdot\overline{g}+\overline{f}\cdot g\right)}{2\sqrt{\left(\dfrac{bc-ad}{4}\right)^2\cdot\left(\dfrac{cd-ab}{4abcd}\right)^2\cdot f\cdot\overline{f}\cdot g\cdot\overline{g}}}=\frac{\dfrac{bc-ad}{4}\cdot\dfrac{cd-ab}{4abcd}\left(f\cdot\overline{g}+\overline{f}\cdot g\right)}{2\left|\dfrac{bc-ad}{4}\cdot\dfrac{cd-ab}{4abcd}\right|\sqrt{f\cdot\overline{f}\cdot g\cdot\overline{g}}}.$$

An absolute value sign appears in the denominator of the last expression, therefore one of the following options holds for $\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)$:

$$\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)=\frac{f\overline{g}+\overline{f}\cdot g}{2\sqrt{f\cdot\overline{f}\cdot g\cdot\overline{g}}},\quad\text{or:}\tag{14}$$

$$\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)=-\frac{f\cdot\overline{g}+\overline{f}\cdot g}{2\sqrt{f\cdot\overline{f}\cdot g\cdot\overline{g}}}\tag{15}$$

Comparing (14) and (15) with (13) shows that one of the following two equations holds true:

$$\cos\left(\widehat{\overrightarrow{OG},\overrightarrow{OF}}\right)=\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right)\quad\text{or}\quad\cos\left(\widehat{\overrightarrow{OG},\overrightarrow{OF}}\right)=-\cos\left(\widehat{\overrightarrow{TV},\overrightarrow{TP}}\right),$$

therefore one of the following two equalities holds true for the angles:

$$\measuredangle FOG=\measuredangle PTV \text{ or } \measuredangle FOG=180^\circ-\measuredangle PTV.$$

Let us prove that the second equality cannot hold.
From Property 2, it follows that $\measuredangle FEG=180^\circ-\measuredangle PTV$. Therefore, if the second equality holds, there also holds $\measuredangle FOG=\measuredangle FEG$. But, since these two angles rest on segment FG and their vertexes (points E and O) lie on the same perpendicular OH to FG, and there holds: $OH>EH$, and therefore $\measuredangle FOG<\measuredangle FEG$.
Since we have a contradiction, and therefore the assumption that $\measuredangle FOG=180^\circ-\measuredangle PTV$ is satisfied is not true.
Therefore, the first equality, $\measuredangle FOG=\measuredangle PTV$, is the true one.

□

Note:
In the "Theory of a convex quadrilateral and a circle that forms Pascal points on the sides of the quadrilateral" (see [3], [4] , [5]) it is proven that in the case of a cyclic quadrilateral, the middles of a pair of opposite sides are Pascal points formed by the circle whose diameter is the segment that connects the point of intersection of the diagonals of the quadrilateral and the point of intersection of the other pair of opposite sides (see [4, Theorem 3]).
For example, in Figure 5, points P and Q are Pascal points that are formed on sides AB and CD by the circle whose diameter is segment EF, and points V and W are Pascal points formed on sides BC and AD by the circle whose diameter is segment EG.

Therefore, properties (2) and (3) we proved above can be formulated as follows:

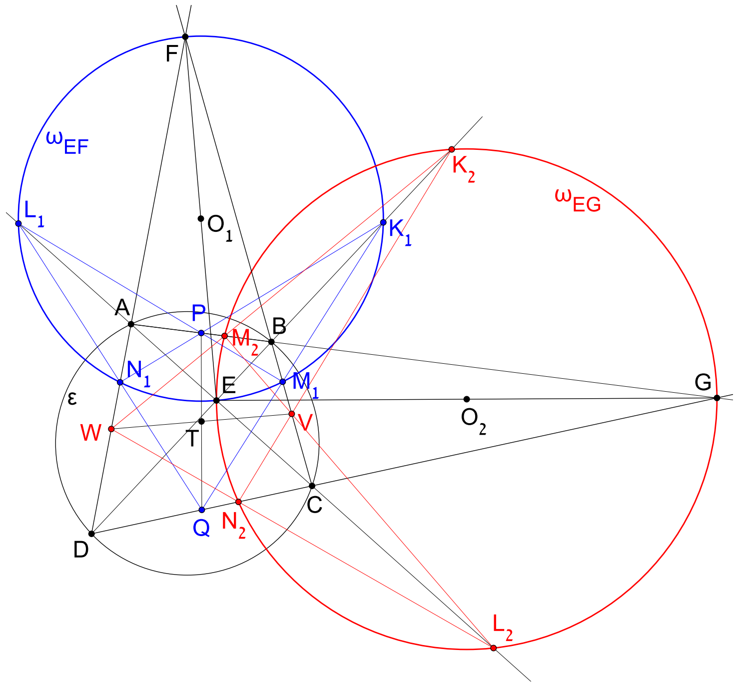

Figure 5

Let $ABCD$ be a quadrilateral inscribed in circle ε (O is the center of ε), where:
E is the point of intersection of the diagonals;
F is the point of intersection of the continuations of sides BC and AD;
G is the point of intersection of the continuations of sides AB and CD;
P and Q are Pascal points on sides AB and CD formed using the circle whose diameter is segment EF;
V and W are Pascal points on sides BC and AD formed using the circle whose diameter is segment EG;
Then:

(i) The sum of two angles one of which is the angle between the lined PQ and VW, and the other is the angle of triangle EFG whose vertex at point E equals $180°$.
(ii) Point of intersection of lines PQ and VW belongs to the nine-point circle of triangle EFG.

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