A mini-course on crystalline cohomology

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Abstract

This is the lecture notes for the mini-course during June 11-15, 2018 at University of Michigan, about the crystalline cohomology. In this mini-course, we will give an overview about crystalline theory. We first give a mild introduction about the motivation and main results of crystalline cohomology, without anything technical. Then we start by looking at algebraic and geometric basics around the crystalline theory. We prove the comparison theorem between crystalline cohomology and de Rham cohomology, following Bhatt and de Jong [BdJ12]. After that, we turn to the study of the de Rham-Witt complex, a powerful tool in crystalline cohomology. At last, we apply our theory to several questions about rational points in arithmetic geometry.[1]

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1 Introduction

1.1 Motivations

Crystalline cohomology was invented by Grothendieck in 1966 , in order to find a "good" p-adic cohomology theory, to fill in the gap at p in the families of ℓ-adic étale cohomology, and to refine the notion of the de Rham cohomology in positive characteristic. It was started by Grothendieck's lectures at IHES in 1966 ([Gro68]), where he outlined the program, and was worked out by Berthelot in his thesis ([Ber74])

To motivate the story, we first look at some known cohomology theory at that time.

[1]The note is supposed to be in its final version right now, due to the lack of time of the author. And the author is responsible for whatever errors in this note. But comments are always welcome!

ℓ-adic cohomology Let $k = \mathbb{F}_q$ be a finite field of characteristic p, and $\ell \neq p$ be a prime number. For a smooth projective and geometrically connected scheme X over k, one of the most important numerical invariance of X is its zeta-function, defined by counting-points

$$\zeta_X(t) = \exp(\sum_{n=1}^{\infty} \frac{|X(\mathbb{F}_{q^n})|}{n} t^n) = \prod_{x \in |X(\overline{k})|} \frac{1}{1 - t^{\deg x}}.$$

It is clear from the second expression that $\zeta_X(t)$ is a power series of t. But it is the great result by Weil, Dwork, Grothendieck, Deligne and others that $\zeta_X(t)$ is actually a rational function, satisfying a functional equation, with zeros being of specific absolute values. This is the famous Weil Conjecture.

So how did people prove this? Motivated by singular cohomology for complex varieties, Weil suggested that the conjecture should follow "formally" from the existence of certain cohomology theory for varieties over finite fields, where the Lefschetz's fixed points theorem, Poincaré duality and other properties were supposed to hold. In this way, we can study the points-counting by looking at the Frobenius action. Then Grothendieck and Michael Artin managed to produce the ℓ-adic étale cohomology, which satisfies very good properties just like singular cohomology.

Precisely, the ℓ-adic cohomology is a Weil cohomology theory, in the sense that it is a contravariant functor H^* on the category of smooth projective varieties over k to the category of graded algebras over a field $\mathbb{Q}_\ell$ of characteristic 0, satisfying the following properties:

(i) Finiteness: $\dim \mathrm{H}^i(X)$ is finite.

(ii) Vanishing: $\mathrm{H}^i(X) = 0$ when $i < 0$ or $i > 2\dim(X)$.

(iii) Poincaré duality: isomorphism between $\mathrm{H}^i(X)$ and $\mathrm{H}^{2\dim(X)-i}(X)$.

(iv) Trace map: isomorphism $\mathrm{H}^{2\dim(X)}(X) \to \mathbb{Q}_\ell$.

(v) Künneth formula: $\mathrm{H}^*(X) \otimes \mathrm{H}^*(Y) \to \mathrm{H}^*(X \times Y)$.

(vi) Cycle map: $Z^i(X) \to \mathrm{H}^{2i}(X)$, compatible with intersections and cup product.

(vii) Weak Lefschetz Theorem.

(viii) Hard Lefschetz Theorem.

Moreover, when X comes from a smooth projective scheme over a finitely generated $\mathbb{Z}$-subalgebra R inside of $\mathbb{C}$, there exists a comparison theorem between ℓ-adic cohomology and singular cohomology, given by

$$\mathrm{H}^*(X_{\overline{k},\acute{e}t}, \mathbb{Z}_\ell) = \mathrm{H}^*_{sing}(X^{an}, \mathbb{Z}) \otimes \mathbb{Z}_\ell.$$

Here we note that the comparison does not kill the torsion; namely the ℓ-torsion information of the singular cohomology of X^{an} is covered by $\mathrm{H}^*(X, \mathbb{Z}_\ell)$.

However, things become quite different when $\ell = p$. In fact, almost non of the above properties hold for p-adic étale cohomology of varieties in characteristic p. So it is natural to ask if there is a "good" p-adic cohomology theory that can satisfies the above properties. Moreover, since ℓ-adic cohomology only encodes the Betti numbers and ℓ-torsion, there is no way to read any information about p-torsion from the ℓ-adic theory above. Thus, this motivates the need to find those "lost" p-torsion information.

De Rham cohomology, and infinitesimal theory When X is a smooth projective variety over $\mathbb{C}$, it is well-known that $\mathrm{H}^*(X^{an}, \mathbb{C})$ is a good cohomology theory. And there exists a comparison between singular cohomology and algebraic de Rham cohomology, given by

$$\mathrm{H}^*(X^{an}, \mathbb{C}) = \mathrm{H}^*(X^{an}, \Omega^\bullet_{X/\mathbb{C}}) = \mathrm{H}^*(X, \Omega_{X/\mathbb{C}}),$$

where the first equality follows from the Poincaré's Lemma, and the second the GAGA.

For a variety over k of positive characteristic, in order to find a p-adic cohomology theory, it is natural to consider its lifting over $\mathbb{Z}_p$ and look at the de Rham cohomology. Then there are several questions arise accordingly:

(a) Does every smooth proper variety over k admits a smooth proper lifting $\widetilde{X}$ over $\mathbb{Z}_p$?

(b) If so, is the de Rham cohomology $\mathrm{H}^*(\widetilde{X}, \Omega^\bullet_{\widetilde{X}/\mathbb{Z}_p})$ independent of liftings?

(c) Is there a canonical way to define those cohomology without referring to the liftings?

If those answers are yes, we will have a good p-adic cohomology theory. (though at that time it was already known that such a lifting may not exist, even when X is smooth projective over k ([Mum61]).) The way Grothendieck attacks those questions was to consider all possible infinitesimal local liftings, which then turns to a Grothendieck topology, called *infinitesimal site*. Precisely, for a variety X over $\mathbb{C}$, let $X/\mathbb{C}_{inf}$ be the category of all pairs (U, T), where $U \subset X$ is an open subset, and $U \to T$ is a closed nilpotent immersion, namely they have the same underlying topological space such that $\mathcal{O}_T \to \mathcal{O}_U$ is quotient by a nilpotent ideal. And the topology is given by Zariski-covering; i.e. (U_i, T_i) is a covering of (U, T), if $T_i \to T$ is a Zariski covering. A sheaf $\mathcal{F}$ on $X/\mathbb{C}_{inf}$ is given by a collection of Zariski sheaves $\mathcal{F}_T$ over T for each (U, T), such that for any map $f : (U, T) \to (U', T')$, it induces a morphism $f^*\mathcal{F}_{T'} \to \mathcal{F}_T$. If every induced map $f^*\mathcal{F}_{T'} \to \mathcal{F}_T$ is an isomorphism, we call $\mathcal{F}$ a *crystal*.

This topology somehow is suitable for all of those three questions above: every local nilpotent immersion can be regarded as a local infinitesimal lifting, and the value of sheaves on this topology are compatible with all possible transition maps between liftings, such that transitions induce isomorphisms when working at crystals. And the cohomology over this site is expected: when $X/\mathbb{C}$ is smooth proper, Grothendieck [Gro68] proved that

$$\mathrm{H}^*(X/\mathbb{C}_{inf}, \mathcal{O}_{X/\mathbb{C}}) \cong \mathrm{H}^*(X, \Omega^\bullet_{X/\mathbb{C}}).$$

More generally when $X \to Y$ is a closed immersion such that Y is smooth, we have

$$\mathrm{H}^*(X/\mathbb{C}_{inf}, \mathcal{O}_{X/\mathbb{C}}) \cong \mathrm{H}^*(\widehat{Y}, \widehat{\Omega}^\bullet_{Y/\mathbb{C}}),$$

where $\widehat{Y}$ is the formal completion of Y along X. We note here that the isomorphism tells us that we can compute the infinitesimal cohomology by formal liftings, but the cohomology group is independent of liftings we choose! This is partially motivated by the comparison between $\mathbb{C}$-singular cohomology and algebraic de Rham cohomology: the analytification X^{an} encodes the "differential" or "infinitesimal" geometry of the space, which locally can be regarded as a limit of all possible infinitesimal neighborhoods.

Now it is natural to apply this to when X is a variety over a field of positive characteristic. But there are still a minor issue about integration: in the complex coefficients, the comparison between infinitesimal cohomology and de Rham cohomology essentially follows from the Poincaré's Lemma, which fails in characteristic p:

Example 1.2. Let $X_0 = \mathrm{Spec}(k)$ be a point. Assume we $k = \mathbb{C}$, and consider its closed immersion $X \to \mathbb{A}^1_{\mathbb{C}} = Y$. We look at the de Rham cohomology of the formal completion $\widehat{Y}$ of Y along X. Then since it is affine, by the vanishing of higher coherent cohomology, the de Rham cohomology $\mathrm{H}^*_{dR}(X/k)$ is computed by the complex

$$0 \longrightarrow \mathbb{C}[[t]] \xrightarrow{d} \mathbb{C}[[t]]dt \longrightarrow 0.$$

Since each $t^n dt$ can be integrated to $\frac{t^{n-1}}{n}$, we know $\mathrm{H}^i_{dR}(\widehat{Y/X})$ is 0 except $i = 0$, which is k. This is exactly the same as $\mathrm{H}^i_{dR}(\widehat{Y/\mathbb{C}})$, for the $X \to Y = \mathrm{Spec}(\mathbb{C})$ being the trivial closed immersion.

Now let us look at when $k = \mathbb{F}_p$, and we consider the closed immersion $X \to \mathbb{A}^1_{\mathbb{Z}_p} = Y$. The formal completion of Y along X is is $\mathrm{Spf}(\mathbb{Z}_p\{t\})$, where $\mathbb{Z}_p\{t\} = \mathbb{Z}_p[t]^\vee$ is the ring of convergent power series over $\mathbb{Z}_p$. And the de Rham cohomology is computed by

$$0 \longrightarrow \mathbb{Z}_p\{t\} \xrightarrow{d} \mathbb{Z}_p\{t\}dt \longrightarrow 0.$$

However, the H^1 is infinitely generated: there exists no element in $\mathbb{Z}_p\{t\}$ whose derivation is $t^{p-1}dt$. And obviously this is different from $\mathrm{H}^i_{dR}(\widehat{Y/X})$ for $X \to Y = \mathrm{Spec}(\mathbb{Z}_p)$ being the natural closed immersion.

If we look in more detail about the failure of the integration, then we find that we need elements of the form $\frac{t^n}{n!}$ (n-th integration of t). So what if we add those elements to the lifting? Consider the ring $\mathbb{Z}_p\langle t\rangle$, which is the p-adic completion of $\mathbb{Z}_p[t][\frac{t^n}{n!}, n]$. Then the derivation $d : \mathbb{Z}_p\langle t\rangle \to \mathbb{Z}_p\langle t\rangle dt$ is surjective, since each derivative $\sum_{n=0} a_n \frac{t^n}{n!}dt$ can be written as $d(\sum_{n=0} a_n \frac{t^{n+1}}{(n+1)!})$. So the de Rham cohomology $\mathrm{H}^i_{dR}(\mathbb{Z}_p\langle t\rangle/\mathbb{Z}_p)$ becomes what we expect.

We look in more detail about such an adjustment we made. In order to make the de Rham complex exact, instead of taking the completion along the closed immersion $X \to Y$ directly, we also add those elements which are of the form $\frac{a^n}{n!}$ for a belonging to the defining ideal of X in Y, which makes it possible to integrate the differential. This is exactly the adjustment we need in order to deal with the mix-characteristic situation, and the elements we add form a new multiplicative structure, called *divided power structure* (PD-structure in short). In this way, in order to allow the integration for the case we care, Grothendieck's idea was to add such a structure into the definition of infinitesimal sites: instead of looking at all possible nilpotent extensions, we look at all of those that are equipped with a PD structure on the defining ideal. And the new topology is called *crystalline site*, where the cohomology of this site is called *crystalline cohomology*. This marks the start of the whole story.

1.3 Foundational results

We then collect some results for crystalline cohomology of smooth proper varieties over k of characteristic p. Here we follows mostly the survey by Illusie [Ill94].

Let k be a perfect field of characteristic p. Denote by W_n to be n-th ring of Witt vectors of k, and $W = \varprojlim W_n$. Let K be the fraction field of W.

For a scheme X over k and for each $n \in \mathbb{N}$, we can associate a (truncated) crystalline site $X/W_{n_{\mathrm{crys}}}$ to it. The object of $X/W_{n_{\mathrm{crys}}}$ are pairs (U, T), where U is an open subset of X, and T is a W_n-scheme with a nilpotent closed immersion $i : U \to T$ such that the defining ideal of U in T has a PD-structure. The covering are given locally by Zariski covering, namely $\{(U_i, T_i)\}$ is a covering of (U, T) if $T_i \to T$ is a Zariski covering.

On the site $X/W_{n_{\mathrm{crys}}}$, we can define the structure sheaf $\mathcal{O}_{X/W_n}$, such that for each (U, T) the section on it is $\mathcal{O}(T)$. Then the *$n - th$ crystalline cohomology* is defined as

$$\mathrm{H}^*(X/W_{n_{\mathrm{crys}}}, \mathcal{O}_{X/W_n}).$$

And for different n, the above group form an inverse system, and the *crystalline cohomology of X* is defined as

$$\mathrm{H}^*(X/W_{\mathrm{crys}}) := \varprojlim_n \mathrm{H}^*(X/W_{n_{\mathrm{crys}}}, \mathcal{O}_{X/W_n}).$$

This is a graded W-module depending functorially on X/k.

Now let us look at its good properties.

Weil cohomology Let $\mathcal{C}$ be the category of proper smooth varieties over k. For each $X \in \mathcal{C}$, $\mathrm{H}^i(X/W_{\mathrm{crys}})$ is of finite type over W, which vanishes when $i > 2\dim(X)$. When X is projective (Katz-Messing [KM74]), or admits a proper smooth lifting over W, we have

$$\mathrm{rank}_W \mathrm{H}^i(X/W_{\mathrm{crys}}) = \dim_{\mathbb{Q}_\ell} \mathrm{H}^i(X_{\overline{k}, ét}, \mathbb{Q}_\ell),$$

for $\ell \neq p$. And when restricted to the subcategory of projective smooth varieties, the functor

$$X \longmapsto K \otimes_W \mathrm{H}^*(X/W_{\mathrm{crys}})$$

is a Weil cohomology theory. This started from Berthelot's thesis [Ber74], and was achieved after many people's work.

As a consequence of the formalism, when $k = \mathbb{F}_{p^a}$ is a finite field, we get a cohomological expression for the zata function of X/k:

$$\zeta_X(t) = \prod_{0 \leq i \leq 2d} \det(1 - F^{a*}t \mid \mathrm{H}^i(X/W_{\mathrm{crys}}) \otimes K)^{(-1)^{i+1}}.$$

And when X/k is projective, by Katz-Messing [KM74], from the Weil conjecture the crystalline characteristic polynomial coincide with ℓ-adic ones. In particular, they have the integer coefficients.

Comparison with de Rham cohomology There are two comparison of crystalline cohomology with de Rham cohomology, one is with that of X/k, another one is with the possible liftings to W. Let X be in $\mathcal{C}$.

When we look at coefficient W_n with $n = 1$, the crystalline cohomology coincides with the de Rham cohomology of X/k, namely

$$\mathrm{H}^i(X/k_{\mathrm{crys}}) = \mathrm{H}^i_{dR}(X/k).$$

And there exists a universal coefficient formula

$$0 \longrightarrow \mathrm{H}^i(X/W_{\mathrm{crys}}) \otimes k \longrightarrow \mathrm{H}^i(X/k_{\mathrm{crys}}) \longrightarrow \mathrm{Tor}_1^W(\mathrm{H}^{i+1}(X/W_{\mathrm{crys}}), k) \longrightarrow 0.$$

Suppose X has a proper smooth lifting Z over W. Then we have a canonical isomorphism

$$\mathrm{H}^i(X/W_{\mathrm{crys}}) \cong \mathrm{H}^i_{dR}(Z/W).$$

In particular, the de Rham cohomology of Z/W only depends on its special fiber over k. This answers exactly the second question given before, which asked if the de Rham cohomology of liftings would depend on the choice.

Frobenius and $F-$crystal One of the most important feature is the Frobenius action on crystalline cohomology. Let X be in $\mathcal{C}$. By the functoriality of X/k, the absolute Frobenius induces an action σ-linear map

$$\phi : \mathrm{H}^*(X/W_{\mathrm{crys}}) \to \mathrm{H}^*(X/W_{\mathrm{crys}}),$$

where σ is the F action on the Witt vector. This map is an isogeny, i.e. $\phi \otimes \mathbb{Q}$ is an isomorphism, so it induces a $F - crystal$ structure $(\mathrm{H}^i(X/W_{\mathrm{crys}})/torsion, \phi)$. It is a finitely generated W module together with a σ-linear action, such that its base change to $\mathbb{Q}$ is isomorphic. The F-crystal encodes many arithmetic structure. When $i = 1$ and the cohomology is torsion free, $(\mathrm{H}^1(X/W_{\mathrm{crys}}), \phi)$ is the Dieudonné module of the p-divisible group of the Albanese variety $Alb(X)$. Moreover, by classification result, it can be decomposed as a direct sum of $W[T]/(T^m - p^n)$, where $\frac{n}{m}$ is called its *slope*. And we can get a Newton polygon based on the eigenvalues and their multiplicities. One of the central theorem about Newton polygon was Katz conjecture, which asked the relative position about Newton polygon and Hodge polygon, where the latter is given by the hodge numbers of X/k. It was proved by Mazur and Ogus separately that "Newton is above Hodge". On the other hand, the Newton polygon plays an very important role in many modular problems of characteristic p, say that of curves, abelian varieties, K3 surfaces, and smooth complete intersections. In some sense, it generalizes the usual concept of "ordinary" and "supersingular" of an elliptic curve in positive characteristic, and gives a stratification of moduli.

2 Pd-structure and pd-algebra

In this section, we give a quick review about the pd-structure and pd-algebras.

Intuitively, pd-structure is introduced as an abstract version of elements with factorial denominators. It is defined as a collection of maps on a given ideal, satisfying whatever $\frac{x^n}{n!}$ should satisfy:

Definition 2.1. Let A be a ring, and I be a A ideal. We say a collection of maps $\{\gamma_n : I \to I, n \in \mathbb{N}\}$ is a $pd - structure$ of I if it satisfies:

- For $x \in I$, $\gamma_0(x) = 1$, $\gamma_1(x) = x$, $\gamma_i(x) \in I$ if $i \geq 1$.
- For $x, t \in I$, $\gamma_n(x + y) = \sum_{i=0}^{n} \gamma_i(x)\gamma_{n-i}(y)$.
- For $a \in A, x \in I$, $\gamma_n(ax) = a^n \gamma_n(x)$.
- For $x \in I$, $\gamma_n(x)\gamma_m(x) = \binom{n+m}{n}\gamma_{n+m}(x)$.
- For $x \in I$, $\gamma_p(\gamma_q(x)) = C_{p,q}\gamma_{pq}(x)$, where $C_{p,q} = \frac{(pq)!}{p!(q!)^p} \in \mathbb{N}$.

We call the pair (A, I, γ) a pd-structure, and A is a $pd - algebra$.

Here are some basic examples:

Example 2.2. For any $\mathbb{Q}$-algebra A and any ideal I, there exists unique pd-structure of I, given by

$$\gamma_n(x) = \frac{x^n}{n!}.$$

Example 2.3. For a finite extension $K/\mathbb{Q}_p$, the pair $(\mathcal{O}_K, (\pi_K))$ has a pd-structure (unique) if and only if the ramification index satisfies $e_{K/\mathbb{Q}_p} \leq p - 1$. Under the condition, the element $\frac{\pi_K^n}{n!}$ belongs to $\mathcal{O}_K$ for any $n \in \mathbb{N}$.

We will not discuss the pd-structure itself in detail, just mention one of the property we need here.

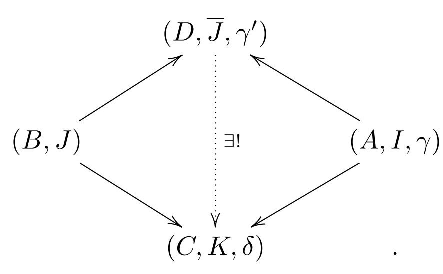
Proposition 2.4 ([Sta], 07H1)**.** *Let (A, I, γ) be a pd algebra, and $f : A \to B$ be a ring homomorphism. Then the pd-structure extends to (B, IB) (unique if so) when we have any of the following conditions:*

- *$IB = 0$;*
- *I is principal;*
- *f is flat.*

Corollary 2.5. *The pd-structure can be globalized to the general schemes and its closed subschemes.*

One of the most important constructions is the pd-envelope. The idea of the pd-envelope is to add the pd-structure on the given ideal that preserves the original one. It is the pd-algebra satisfying a universal property among pd-algebras:

Theorem 2.6. *Let (A, I, γ) be an pd-algebra, B be an A algebra, and J be an ideal of B. Then there exists a unique pd-algebra $(D, \overline{J}, \gamma')$, with a morphism $(B, J) \to (D, \overline{J})$, compatible with the pd-structure on A, such that for any morphism of pairs $(B, J) \to (C, K, \delta)$ compatible with (A, I, δ), there exists a unique pd-homomorphism $(D, \overline{J}, \gamma') \to (C, K, \delta)$.*



It is called the pd − envelope *of* Spec(B/J) *in* Spec(B).

Sketch of the construction. (1) Let $M = J+IB$. Consider the graded B-algebra Λ, such that $\Lambda_0 = B$, and $\Lambda_1 = M$, and Λ_n is the free B module generated by the symbols $[x_1]_{r_1} \cdots [x_s]_{r_s}, x \in M$, such that $\sum r_j = n$. To endow it with a pd-structure, we quotient the ring by the ideal generated by all of the axioms of pd-structure above, assuming $[\,]_n$ is the supposed pd-structure on the ideal M. So we get the quotient ring $\Gamma_B(M)$, such that $(\Gamma_B(M), \Gamma_B^+(M), [\,])$ is a pd-algebra.

(2) We quotient the ring $\Gamma_B(J + IB)$ by the ideal generated by:

$$\begin{cases} \gamma_n(a) - [a]_n, a \in I, n \in \mathbb{N}; \\ x - [x]_1, x \in J + IB. \end{cases}$$

Let D be the quotient ring, and $\overline{J}$ be the image of $\Gamma_B^+(J + IB)$.

□

We denote the ring constructed in the theorem by $\mathcal{D}_{B,\gamma}(J)$. Here are two examples of the pd-envelope

Example 2.7. 1. Let A be a ring with trivial pd-structure on 0, M be an A-module, $B = \mathrm{Sym}_A M$, $J = \mathrm{Sym}_A^+ M$. Then $\mathcal{D}_B(J) = (\Gamma_B(M), \Gamma_B^+(M))$, as the quotient ring of B in the construction above.

2. Let $A = \mathbb{Z}_p/p^n$ and $I = (p)$, equipped with the canonical p-adic pd-structure δ. Then the pd-envelope $\mathcal{D}_{B,\delta}(J)$ for $B = A[t]$, $J = [t]$ is the *$pd-polynomial$ ring*

$$A\langle t\rangle := B[\gamma_n(t), t \in \mathbb{N}]/\sim = A[\gamma_n(t), t \in \mathbb{N}]/\sim.$$

It should be read as adding all of the divided powers of J to the B. We note that since every t^n can be generated by $\gamma_n(t)$, we could instead start with the base A, and add the divided powers of the t to it.

We still do not discuss so much about it, just to mention several "extension" properties.

Proposition 2.8. *Let (A, I, γ) be a pd-algebra, and let B be an A algebra with J an ideal of B.*

(a) Assume we have a surjective pd morphism $(A, I, \gamma) \to (A', I', \gamma')$. Let $B' = B \otimes_A A'$, $J' = JB'$. Then we have an isomorphism

$$\mathcal{D}_{B',\gamma'}(J') \cong \mathcal{D}_{B,\gamma}(J) \otimes_A A'.$$

(b) Let B' be a flat B-algebra, then we have the isomorphism

$$\mathcal{D}_{B',\gamma}(JB') = \mathcal{D}_{B,\gamma}(J) \otimes_B B'.$$

As a corollary, the pd-envelope can also be globalized. Here we fix a notation. Let $X \to Y$ be a closed immersion of schemes, and $Y \to S$ be a morphism such that S is equipped with a pd-structure γ We denote by $D_{X,\gamma}(Y)$ (or $D_X(Y)$ in short if without confusion) to be the scheme corresponding to the pd-envelope of X in Y, compatible with γ on S. In other words, $D_{X,\gamma}(Y)$ locally is given by the spectrum of the pd-envelope of $\mathcal{O}_Y$ at the ideal defining X, compatible with that on S.

3 Crystalline geometry

In this section, we define the crystalline site and the crystalline cohomology.

Grothendieck topology First recall that given a fibered category $\mathcal{C}$, a *Grothendieck (pre)topology* on $\mathcal{C}$ is an assignment for each $X \in \mathcal{C}$ a collection of families of morphisms $\{X_i \to X\}$, called the *covering families*, satisfying:

- Identity: $id : X \to X$ is a covering;
- Base change: If $Y \to X$ is a morphism, and $\{U_i \to X\}$ is a covering of X, then $\{Y \times_X U_i \to Y$ is a covering of Y.
- Refinement: If $\{U_i \to X, i\}$ is a covering, $\{V_{ij} \to U_i, j\}$ are coverings for each i, then the composition of morphisms $\{V_{ij} \to X, i, j\}$ is a covering of X.

We call the category together with a Grothendieck topology a *site*. It was Grothendieck who observed that in order to define a cohomology theory, what we really need is a category with a covering as above. The objects in this category serve as open subsets of a topological space, and those collection of coverings are analogous to an assignment of open base in a topological space (i.e. how those subsets become open). As an example, when $\mathcal{C} = \mathrm{Sch}$ is the category of schemes, the families of all Zariski open coverings on Sch consists of a Grothendieck topology on $\mathcal{C}$. And if we consider the small category consists of all open subsets of a given scheme X, we get the (small) Zariski site X_{Zar}.

Crystalline site and crystalline topos Now we back to the crystalline theory. Let $W = \mathrm{Spec}(\mathbb{Z}_p)$, and X be a finite type W-scheme where p is locally nilpotent. We say the closed immersion $X \to T$ is a $pd-thickening$ of X if the defining ideal of X in T has a pd-structure compatible with the canonical pd-structure $(\mathbb{Z}_p, (p))$. Since X is assumed to be locally p-nilpotent, any pd-thickening is a nil-immersion. This is because locally $T = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(A/I)$, such that (A, I, γ) is a pd-structure, such that $p^N = 0$ in A. So for any $x \in I$, $x^n = n! \cdot \gamma_n(x) \in n! \cdot A = 0$ for $n \geq p^N$, and the ideal I is nilpotent in A.

Definition 3.1. The *crystalline site* of X over W, denoted by X/W_{crys}, is defined on the category of all pd-thickenings (U, T) for $U \subset X$ open, such that $\{(U_i, T_i) \to (U, T)\}$ is a covering if $\{T_i \to T\}$ is a Zariski covering.

And we call $X/W_{n,\text{crys}}$ the *truncated crystalline site*, whose objects are pd-thickenings (U, T) with T being a W_n-scheme, with the same coverings as above.

One of the most important types of ph-thickening are pd-envelopes between locally p-nilpotent schemes: assume $X \to Y$ is a closed immersion of finite type schemes over $S = W_N$. Then the pd-envelope $D_X(Y)$ of X in Y is an object in X/W_{crys}. We will see that when Y is smooth over S, such a pd-thickening plays a very important role in calculation, for it covers the final object in the crystalline site.

It can also be showed that the crystalline site admits finite inverse limits, in particular it admits finite products. Here we note that for (U_i, T_i, δ_i) in X/W_{crys}, their product is given by

$$D_{U_1 \cap U_2, \delta_1, \delta_2}(T_1 \times T_2).$$

A sheaf $\mathcal{F}$ on X/W_{crys} is given by associating an abelian group (or a set) $\mathcal{F}((U, T))$ to each $(U, T) \in X/W_{\text{crys}}$, such that it is exact with respect to any covering. Equivalently, it can be described as follows (check it!): For each $(U, T) \in X/W_{\text{crys}}$, we assign a Zariski sheaf $\mathcal{F}_T$ on T, such that for each morphism $f : (U, T) \to (U', T')$, we have a map of sheaves on T

$$f^{-1}\mathcal{F}_{T'} \to \mathcal{F}_T.$$

The category of sheaves on the crystalline site is called the *crystalline topos*. It is easy to check that the crystalline topos is subcanonical, namely the representable presheaf is actually a sheaf.

When talking about the functoriality, the crystalline site is too strict to work sometimes. Assume we have a morphism of W-schemes $f : X \to Y$

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & W & \end{array} .$$

Then it is natural to ask if this induces a morphism between two crystalline sites $X/W_{\text{crys}} \to Y/W_{\text{crys}}$. This actually fails in most of times: there is no canonical way to pullback a nilpotent extension $Y \to T$ along $X \to Y$. However, the map does induces a morphism f_{crys} between crystalline topos $\text{Sh}(X/W_{\text{crys}}) \to \text{Sh}(Y/W_{\text{crys}})$. Namely we have a natural way to define the adjoint pair f_{crys}^{-1}, and $f_{\text{crys}*}$, such that f_{crys}^{-1} commutes with finite limits. It is given as follows: For each $(V, T) \in Y/W_{\text{crys}}$, define the presheaf $f^{-1}(V, T)$ on X/W_{crys} by

$$(U, S) \longmapsto pd - \text{Hom}_f((U, S), (V, T)).$$

Then it is actually a sheaf on X/W_{crys}. We then define the pushout to be

$$f_{\text{crys}*}\mathcal{F}(V, T) := \text{Hom}_{\text{Sh}(X/W_{\text{crys}})}(f^{-1}(V, T), \mathcal{F}).$$

This is actually a sheaf on Y/W_{crys}. On the other hand, we define $f_{\text{crys}}^{-1}\mathcal{G}$ to be the sheaf associated to

$$(U, S) \longmapsto \varinjlim_{\substack{U \xrightarrow{f} V \\ \downarrow \quad\ \ \downarrow \\ S \longrightarrow T}} \mathcal{G}((V, T)).$$

Morphisms with Zariski sites There are two canonical morphisms connecting X/W_{crys} and X_{Zar}. The first one is the projection morphism

$$u_{X/W} : X/W_{\text{crys}} \longrightarrow X_{\text{Zar}};\ u_{X/W}^{-1}(U) = (U, U).$$

The induced morphisms on topoi are given by

$$u_{X/W,*}\mathcal{F}(U) = \Gamma(U/W_{\text{crys}}, \mathcal{F});\ u_{X/W}^{*}\mathcal{G}((U, T)) = \mathcal{G}(U).$$

Another is its section, given by

$$i_{X/W} : X_{\text{Zar}} \longrightarrow X/W_{\text{crys}};\ i_{U/W}^{-1}((U, T)) = U.$$

And induced morphisms on topoi are given by

$$i_{X/W,*} = u_{X/W}^{*};\ i_{X/W}^{*} = u_{X/W!} : \mathcal{F} \mapsto \mathcal{F}_X.$$

Here we note that since $i_{X/W*} = u_{X/W}^{*}$ has both adjoints, it is exact. Thus the Leray special sequence of it degenerates.

Crystals Like the scheme theory, we can define a structure sheaf on the crystalline site X/W_{crys}. Denote by $\mathcal{O}_{X/W}$ to be the presheaf given by

$$\mathcal{O}_{X/W}(U, T) = \mathcal{O}_T(T).$$

Then it is actually a sheaf on X/W_{crys}, called the *structure sheaf*. This makes X/W_{crys} become a ringed site, and its topos a ringed topos.

We could then define a sheaf $\mathcal{F}$ on X/W_{crys} to be *quasi−coherent* if for each $(U,T) \in X/W_{\text{crys}}$, $\mathcal{F}_T$ is a quasi-coherent sheaf of $\mathcal{O}_T$-module. For example, the sheaf of differential $\Omega^i_{X/W_{\text{crys}}}$ on X/W_{crys} is a quasi-coherent sheaf on X/W_{crys}. What we care mostly are a special types of quasi-coherent sheaves:

Definition 3.2. We call a quasi-coherent sheaf of $\mathcal{O}_{X/W}$-module $\mathcal{F}$ a *crystal* if for each $f : (U,T) \to (U',T')$ in X/W_{crys}, we have an isomorphisms

$$f^{-1}\mathcal{O}_T \otimes_{\mathcal{O}'_T} \mathcal{F}'_T \longrightarrow \mathcal{F}_T.$$

Exercise 3.3. Check that $\mathcal{O}_{X/W}$ is a crystal.

Example 3.4. A non-trivial example is given by pushout along closed immersions: Let $i : Z \to X$ be a closed immersion of W-schemes. Then we can define a morphism of topos as before.

$$i_{\text{crys}} : \text{Sh}(Z/W_{\text{crys}}) \longrightarrow \text{Sh}(X/W_{\text{crys}})$$

Then it can be showed that $i_{\text{crys}*}(\mathcal{O}_{Z/W})$ is actually a crystal of $\mathcal{O}_{X/W}$-algebra.

Crystalline cohomology It is by a general nonsense that the category of abelian sheaves on a site admits enough injectives. In particular, for any abelian sheaf $\mathcal{F}$ on X/W_{crys}, we could define its cohomology

$$\text{H}^*(X/W_{\text{crys}}, \mathcal{F}).$$

As a convention, if we does not mention the specific sheaf, we most assume the crystalline cohomology of X/W_{crys} is given by

$$\text{H}^*(X/W_{\text{crys}}, \mathcal{O}_{X/W}).$$

Our next goal is to study this cohomology group, or more generally the cohomology of crystals on X/W_{crys}. We will show that they can be computed by using the de Rham complex of pd-envelopes.

4 Crystalline cohomology and de Rham cohomology

In this section, we will introduce the divided power de Rham complex to compute the cohomology of crystals. In particular, we will compare the de Rham cohomology and crystalline cohomology when X/k admits a smooth proper lifting to W. Throughout this section, we follow the method by Bhatt and de Jong [BdJ12].

4.1 Affine case

Let P be a polynomial over $\mathbb{Z}_p$, and $P \to A$ a surjection of $\mathbb{Z}_p$-algebras, such that $p^N = 0$ in A. Denote by X to be the scheme $\text{Spec}(A)$, and $Y = \text{Spec}(P)$. For each $n \in \mathbb{N}$, let $\mathcal{J}(n)$ be the kernel of $P \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} P \to A$ ($n+1$-copies), and let $\mathcal{D}(n)$ be the p-adic completion of the pd-envelope $\mathcal{D}_{P\otimes\cdots\otimes P}(J(n))$. Here we use the Roman letters $D(n)$ to denote the formal $\text{Spf}(\mathcal{D}(n))$. When $n = 0$, $\mathcal{D}$ is the completed pd-envelope of A in P. We define $\Omega^i_{X/W_{\text{crys}}}$ to be the sheaf on X/W_{crys}, whose restriction on $(U,T) \in X/W_{\text{crys}}$ is the quotient of $\Omega^i_{T/W}$, by $d\gamma_n(x) = \gamma_{n-1}(x) \otimes dx$ for x being in the defining ideal of U in $\mathcal{O}_T$. Note that $\Omega^i_{X/W_{\text{crys}}}$ is simply a coherent $\mathcal{O}_{X/W}$-module, not a crystal.

Here is our main theorem in the affine case:

Theorem 4.2. *Assume the notation as above. Let $\mathcal{F}$ be a crystal on X/W_{crys}, and let $M = \varprojlim_e \mathcal{F}_{D_e}$ be its restriction on D. Then we have*

$$R\Gamma(X/W_{\text{crys}}, \mathcal{F}) = M \to M \widehat{\otimes}_{\mathcal{D}} \Omega^1_{D/W} \to M \widehat{\otimes}_{\mathcal{D}} \Omega^2_{D/W} \to \cdots.$$

We now prove it.

Čech-Alexander complex So how do we compute the cohomology of a sheaf on the Grothendieck topology X/W_{crys}? In the scheme case, what we often do is to choose some Čech covers and compute the corresponding complex. Here we play the similar game, except that we will need a generalized formalism: hypercovers and Čech-Alexander complex.

Lemma 4.3. *Under the condition of 4.2, we have*

$$R\Gamma(X/W_{\text{crys}}, \mathcal{F}) = M \to M\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(1) \to M\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(2) \to \cdots .$$

Proof. We first note that the crystalline site X/W_{crys} can be regarded as the colimit of sites $X/W_{e_{\text{crys}}}$: since for each pd-thickening (U, T), p is locally nilpotent in T, such a pd-thickening will be an object in $X/W_{e_{\text{crys}}}$ for a large $n \geq N$. Thus by the general derived functor theory, we have

$$R\Gamma(X/W_{\text{crys}}, \mathcal{F}) = R\lim_{e} R\Gamma(X/W_{e_{\text{crys}}}, \mathcal{F}).$$

Then we will need some general result about hypercovers: since D_e cover the final object, the augmented simplicial object $D_e(\bullet) \to *$ is a hypercover of the final object in the topos $\text{Sh}(X/W_{e_{\text{crys}}})$, where $D_e(n) = D_e \times_W \cdots \times_W D_e$ is the $n+1$-th product of D_e in $X/W_{e_{\text{crys}}}$. So the cohomology truncated by p^e is given by the Čech-Alexander complex, i.e. applying $R\Gamma(-, \mathcal{F})$ to this hypercover

$$\begin{aligned} R\Gamma(X/W_{e_{\text{crys}}}, \mathcal{F}) &= R\Gamma(D_e, \mathcal{F}) \to R\Gamma(D_e(1), \mathcal{F}) \to R\Gamma(D_e(2), \mathcal{F}) \to \cdots \\ &= \mathcal{F}(D_e) \to \mathcal{F}(D_e(1)) \to \mathcal{F}(D_e(2)) \to \cdots , \end{aligned}$$

where the last equality follows from the vanishing of quasi-coherent cohomology on affines. And we can take this back to our previous formula, and get

$$\begin{aligned} R\Gamma(X/W_{\text{crys}}, \mathcal{F}) &= R\lim_{e} R\Gamma(X/W_{e_{\text{crys}}}, \mathcal{F}) \\ &= R\lim_{e} (\mathcal{F}(D_e) \to \mathcal{F}(D_e(1)) \to \mathcal{F}(D_e(2)) \to \cdots). \end{aligned}$$

At last, we will hope the higher $R\lim_e$ vanishes, so we can reduce the computation to the usual complex. And for a quasi-coherent $\mathcal{O}_{X/W}$-module $\mathcal{F}$, it is true when satisfies one of the following:

- It is coherent;
- It is a crystal.

Thus under our assumption ,$R\lim_e = \lim_e$, and we get

$$\begin{aligned} R\Gamma(X/W_{\text{crys}}, \mathcal{F}) &= \lim_{e} (\mathcal{F}(D_e) \to \mathcal{F}(D_e(1)) \to \mathcal{F}(D_e(2)) \to \cdots) \\ &= \lim_{e} \mathcal{F}(D_e) \to \lim_{e} \mathcal{F}(D_e(1)) \to \lim_{e} \mathcal{F}(D_e(2)) \to \cdots \\ &= \mathcal{F}(D) \to \mathcal{F}(D(1)) \to \mathcal{F}(D(2)) \to \cdots \\ &= \mathcal{F}(D) \to \mathcal{F}(D)\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(1) \to \mathcal{F}(D)\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(2) \to \cdots , \end{aligned}$$

where the last equality follows from the fact that $\mathcal{F}$ is a crystal, and $\mathcal{F}(D_e(n)) = \mathcal{F}(\mathcal{D}_e) \otimes_{\mathcal{D}_e} \mathcal{D}_e(n)$. □

Connecting the Čech-Alexander with de Rham Since our goal is to compute the cohomology by the de Rham complex, it is natural to relate the Čech-Alexander complex with the de Rham complex together. A natural framework is the double complex consists of those two

$$M^{n,m} = M\widehat{\otimes}_{\mathcal{D}}\Omega^m_{D(n)/W}$$

This is a double complex in the first quadrangle such that the first row ($m = 0$) is the Čech-Alexander complex

$$M \to M\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(1) \to M\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(2) \to \cdots ,$$

while the first column ($n=0$) is the de Rham complex we want

$$M \to M\widehat{\otimes}_{\mathcal{D}}\Omega^1_{D/W} \to M\widehat{\otimes}_{\mathcal{D}}\Omega^2_{D/W} \to \cdots .$$

By the general formalism about the spectral sequence and the double complex, there are two E_1-spectral sequences converging to the cohomology of the total complex

$${}^{I}E_1^{n,m} = \mathrm{H}^m(M\widehat{\otimes}_{\mathcal{D}}\Omega^\bullet_{D(n)/W}),\ {}^{II}E_1^{n,m} = \mathrm{H}^n(M\widehat{\otimes}_{\mathcal{D}}\Omega^m_{D(\bullet)/W}).$$

So we will need to know more about those two spectral sequences. Here are two results about the degeneracy:

Lemma 4.4 (A)**.** *For any $m>0$, the cosimplicial module $M\widehat{\otimes}_{\mathcal{D}}\Omega^m_{D(\bullet)/W}$ is homotopic to* 0*.*

Lemma 4.5 (B)**.** *For any map $[0]\to[n]$, the induced morphism between complexes $M\widehat{\otimes}_{\mathcal{D}}\Omega^\bullet_{D/W} \to M\widehat{\otimes}_{\mathcal{D}}\Omega^\bullet_{D(n)/W}$ is a quasi-isomorphism.*

By the Lemma A4.4, the spectral sequence ${}^{II}E_1^{n,m} = \mathrm{H}^n(M\widehat{\otimes}_{\mathcal{D}}\Omega^m_{D(\bullet)/W})$ degenerates into

$$\mathrm{H}^n(M\widehat{\otimes}_{\mathcal{D}}\mathcal{D}(\bullet)).$$

On the other hand, by the Lemma B4.5, since the cosimplicial boundary map is given by $\delta^n = \sum_{0\le i\le n}(-1)^i d^i$, each row in the ${}^{I}E_1^{n,m}$ becomes

$$\mathrm{H}^m(M^{0,\bullet}) \xrightarrow{0} \mathrm{H}^m(M^{1,\bullet}) \xrightarrow{1} \mathrm{H}^m(M^{2,\bullet}) \xrightarrow{0} \cdots .$$

Hence its E_2 page has 0 for each column except for the degree 0, where it becomes the $\mathrm{H}^m(M\widehat{\otimes}_{\mathcal{D}}\Omega^\bullet_{D/W})$. Thus we are done, granting the above two lemmas.

Proof of the degeneracy lemmas At last, we complete the proof of the above two lemmas.

Proof of the Lemma A. First note that it suffices to show $\Omega^1_{D(\bullet)/W}$ that is homotopic to 0. This is because the complex $M\widehat{\otimes}_{\mathcal{D}}\Omega^m_{D(\bullet)/W}$ is obtained from $\Omega^1_{D(\bullet)/W}$ by

(i) m-th wedge product;

(ii) tensor product with M;

(iii) p-adic completion.

And note that all of those operations preserve the homotopic to 0.

Furthermore, note that the simplicial module $\Omega^1_{D(\bullet)/W}$ is produced by

(i) tensor product of the cosimplicial module $\Omega^1_{P^{\otimes\bullet+1}/W}$ along a morphism of cosimplicial rings $P^{n+1}\to\mathcal{D}(n)$;

(ii) quotient by the pd-differential relation: $d\gamma_n(x) = \gamma_{n-1}(x)\otimes dx$;

(iii) p-adic completion.

Thus it reduces to show that the cosimplicial module $\Omega^1_{P^{\otimes\bullet+1}/W}$ is homotopic to 0.

We also observe that since P is a polynomial ring, the differential is a free of finite rank module over P. So the complex is a direct sum of

$$A_\bullet,\ A_n = \oplus_{i=0}^n Pe_i,$$

where the transition map is given by for $f:[n-1]\to[n]$, $e_i\mapsto e_{f(i)}$.

At last, we make the following claim

Claim 4.6. The cosimplicial module $A_\bullet$ is homotopic to 0.

Since the boundary map of the corresponding Moore complex is $\delta^n = \sum_{0\le i\le n}(-1)^i d^i$, we could check the exactness by hand, or use construct a homotopy between identity and 0-map of the cosimplicial modules $A_\bullet$ by

$$h : A_\bullet \longrightarrow \mathrm{Hom}(\Delta[1], A_\bullet),$$

given by for $\alpha_j^n : [n] \to [1]$ with $\alpha_j^n(i) = 0 \Leftrightarrow i < j$, $h_n(e_i)(\alpha_j^n) = \begin{cases} e_i, \ \alpha_j^n(i) \neq 0; \\ 0, \ otherwise. \end{cases}$

Or we could do it directly: define $h : A_n \to A_{n-1}$ to be the identity except 0 at e_n. And we are done.

□

Proof of the Lemma B. Again, it reduces to prove that for each $f : [0] \to [n]$, the induced map

$$\Omega_{D/W} \to \Omega^\bullet_{D(n)/W}$$

is a quasi-isomorphism.

Then we note that since $D \to D(n)$ has a section by diagonal, the morphism $\Omega^1_{D/W} \to \Omega^1_{D(n)/W}$ is injective. And note that the Hodge filtration on $\Omega^\bullet_{D/W}$ induces a filtration on $\Omega^\bullet_{D(n)/W}$, such that the graded factor is given by

$$\Omega^i_{D/W} \longrightarrow \Omega^i_{D/W} \widehat{\otimes}_{\mathcal{D}} \Omega^\bullet_{D(n)/D},$$

So we reduce to show that the map

$$\mathcal{D} \longrightarrow \Omega^\bullet_{D(n)/D}$$

is a quasi-isomorphism. This is in fact the pd-Poincaré Lemma for pd-polynomial, which can be checked explicitly. We refer to [BO78], Theorem 6.13 for detail. □

Remark 4.7. The above result is true for general smooth algebra P over $\mathbb{Z}_p$, not necessarily polynomial. The reason follows from the existence of local coordinates: locally any smooth scheme is étale over affine space. Such a "framing" will preserve differential along étale base change, and all the above proof can be applied to the general situation.

4.8 Global case

Now we work toward the global result.

We first give some notation for the global settings. Let X be scheme of finite type over W_N, and let $i : X \to Y$ be a closed immersion such that Y is smooth and finite type over W. Denote by $\mathcal{D}_e$ to be the pd-envelope of $\mathcal{O}_Y/p^e \to \mathcal{O}_X$, and D_e be its corresponding scheme. We let D be the colimit of D_e, which is the formal scheme over $\mathrm{Spf}(\mathbb{Z}_p)$. Denote by X/W_{crys} to be the crystalline site of X over W.

Here is our main result

Theorem 4.9. *Let $\mathcal{F}$ be a crystal over X/W_{crys}, and $\mathcal{M}$ be the inverse limit of $\mathcal{F}_{D_e}$. Then $R\Gamma(X/W_{\mathrm{crys}}, \mathcal{F})$ can be computed by the $R\Gamma(D, -)$ of the complex*

$$\mathcal{M} \longrightarrow \mathcal{M}\widehat{\otimes}_{\mathcal{D}}\Omega^1_{D/W} \longrightarrow \mathcal{M}\widehat{\otimes}_{\mathcal{D}}\Omega^2_{D/W} \longrightarrow \cdots .$$

Proof. Recall that the projection morphism $u_{X/W_{\mathrm{crys}}} : X/W_{\mathrm{crys}} \to X_{\mathrm{Zar}}$ induces a morphism on topos, such that

$$u_{X/W_{\mathrm{crys}}*}\mathcal{G}(U) = \Gamma(U/W_{\mathrm{crys}}, \mathcal{G}).$$

We note that the derived functor $R\Gamma(X/W_{\mathrm{crys}}, -)$ is the composition $R\Gamma(X_{\mathrm{Zar}}, -) \circ Ru_{X/W_{\mathrm{crys}}*}$, so we get

$$R\Gamma(X/W_{\mathrm{crys}}, \mathcal{F}) = R\Gamma(X_{\mathrm{Zar}}, Ru_{X/W_{\mathrm{crys}}*}(\mathcal{F})).$$

Then we make the following claim:

Claim 4.10. We have the isomorphism in the derived category:

$$Ru_{X/W_{\mathrm{crys}}*}(\mathcal{F}) = Ru_{X/W_{\mathrm{crys}}*}(\mathcal{F} \to \mathcal{F}\otimes\Omega^1_{X/W_{\mathrm{crys}}} \to \mathcal{F}\otimes\Omega^2_{X/W_{\mathrm{crys}}} \to \cdots),$$

where the morphism is induced by the truncation at 0.

The claim follows from the vanishing of the cohomology of $\mathcal{F}\otimes\Omega^m_{X/W_{\mathrm{crys}}}$ for $m > 0$, which is true locally by the Lemma 4.3 and the Lemma 4.4 before.

Then we reduce to compute

$$R\Gamma(X_{\mathrm{Zar}}, Ru_{X/W_{\mathrm{crys}}*}(\mathcal{F} \to \mathcal{F}\otimes\Omega^1_{X/W_{\mathrm{crys}}} \to \mathcal{F}\otimes\Omega^2_{X/W_{\mathrm{crys}}} \to \cdots)).$$

By the Theorem 4.2 and the Remark 4.7, the above becomes $\mathcal{M}\widehat{\otimes}_{\mathcal{D}}\Omega^{\bullet}_{D/W}$ when X is affine. Assume X is separated. Then we can cover X be a finite many of affine open subset $U_i \to X$, and computed by the Čech-complex. Since the intersection between finite many of affine is still affine, we apply the Theorem 4.2 and get

$$R\Gamma(X_{\mathrm{Zar}}, \mathcal{M}\widehat{\otimes}_{\mathcal{D}}\Omega^{\bullet}_{D/W}).$$

For the general case, since all smooth finite type scheme X over W is quasi-compact and quasi-separated, we apply the Čech complex again by a refinement to get the required equality. ☐

Corollary 4.11 (Comparison theorem)**.** *Assume X admits a smooth proper lifting of X' over W. Then we have*

$$\mathrm{H}^i(X/W_{\mathrm{crys}}, \mathcal{O}_{X/W}) = \mathrm{H}^i_{dR}(X'/W).$$

In particular, the de Rham cohomology of this lifting depends only on its special fiber X.

Proof. When X admits a lifting X', the completed pd-envelope can be the formal completion of X' itself. We only note that the complex in the Theorem 4.9, the complex is p-adic completed de Rham complex, not the usual one. Their equality then follows from the formal function theorem, since X' is proper. ☐

Remark 4.12. Pointed in [BdJ12], the proof of the Theorem can be applied to the proof for the Grothendieck's infinitesimal cohomology for schemes over $\mathbb{C}$. The only difference is instead of using the pd-Poincaré Lemma in Lemma 4.5, we will need the Poincaré Lemma for formal completion in characteristic 0.

5 De Rham-Witt complex

In this section, we will introduce the basic of de Rham-Witt complex. Many of the materials below, including related discussion, can be found in [Ill11].

Witt vectors We first recall the Witt vectors. Let A be a ring. Denote by $W(A)$ to be the infinite product set $\prod_{n=0}^{\infty} A$, indexed by $\mathbb{N}$. It is a priori a set without any structure. For each $r = (r_0, r_1, r_2, \cdots) \in W(A)$, we define the element $w_n(r)$ for $n \in \mathbb{N}$ as

$$w_n(r) = r_0^{p^n} + pr_1^{p^{n-1}} + \cdots p^{n-1}r_{n-1^p} + p^n r_n.$$

Then we have

Theorem/Definition 5.1. *There exists a unique commutative ring structure on $W(A)$, which is functorial on A, such that the* ghost map *from $W(A)$ to the infinite product ring $A^{\mathbb{N}}$*

$$W(A) \longrightarrow A^{\mathbb{N}};$$
$$r = (r_0, r_1, \cdots) \longmapsto (w_0(r), w_1(r), w_2(r), \cdots)$$

is a homomorphism of rings.

We call the ring $W(A)$ with this structure the Witt vectors of A.

In the ring $W(A)$, $1 = (1, 0, \ldots)$, $0 = (0, 0, \ldots)$. We also have a *truncated Witt vectors* $W_n(A)$, given by the first $n+1$ entries of $W(A)$ with the induced ring structure. The addition and multiplication is given by

$$\begin{aligned} r + r' &= (s_0(r, r'), s_1(r, r'), \ldots); \\ r \cdot r' &= (p_0(r, r'), r_1(r, r'), \ldots). \end{aligned}$$

Here for $r = (x_0, x_1, \ldots)$ and $r'(y_0, y_1, \ldots)$, the first several ghost maps are $s_0(r, r') = x_0 + y_0$, $s_1(r, r') = x_1 + y_1 + \sum_{0<i<p} \frac{\binom{p}{i}}{p} x_0^i y_0^{p-i}$; $p_0(r, r') = x_0 y_0$, $p_1(r, r') = x_0^p y_1 + x_1 y_0^p + p x_1 y_1$.

For the following discussion throughout the note, we consider those $W(A)$ for A being a $\mathbb{F}_p$-algebra.

There are several important operators on $W(A)$. The very first is the Teichmüller lifting $A \to W(A)$, which is multiplicative (but not additive), given by $x \mapsto [x] = (x, 0, \ldots)$. We also have *Frobenious* and *Verschiebung* maps:

$$\begin{aligned} F &: W_n(A) \longrightarrow W_{n-1}(A),\ (a_0, \ldots, a_n) \mapsto (a_0^p, \ldots, a_{n-1}^p); \\ V &: W_{n-1}(A) \longrightarrow W_n(A),\ (a_0, \ldots, a_{n-1}) \mapsto (0, a_0, \ldots, a_n). \end{aligned}$$

Note that since the Frobenius map on $\mathbb{F}_p$-algebras are homomorphism, by the functoriality of $W(-)$, the Frobenius map F is also a homomorphism of rings. The V is only additive.

Those operators satisfy some basic relations:

$$\begin{aligned} FV = VF &= p; \\ V((Fx)y) &= xV(y); \\ F[x] &= [x^p]. \end{aligned}$$

Here are two important examples.

Example 5.2. 1. When $A = k$ is a perfect field of characteristic p, $W(A)$ is the unique complete absolute unramified lifting of k, In the sense that it is a p-adically complete, p-torsion free discrete valuation ring whose maximal ideal is generated by p. This can be extended to all perfect ring A over $\mathbb{F}_p$, where $W(A)$ is called the (unique) strict p-ring over A. The functor produce an equivalence of categories

$$\{perfect\ ring\ over\ \mathbb{F}_p\} \Longleftrightarrow \{strict\ p-rings\}.$$

In this condition, we could write $V = pF^{-1}$, which makes sense since $W(A)$ is p-torsion free.

We note here that the Witt vector can be used to construct the untilt of perfectoid algebras.

2. Assume $A = \mathbb{F}_p[t]$. Then there exists an injection of ring $\mathbb{Z}_p[T] \to W(A)$, mapping T onto the Teichmüller lifting $[t]$. The action of F and V induces that on T, such that $F(T) = T^p$, and $V(T) = pT^{\frac{1}{p}}$. Denote by E^0 to be the subset $\sum_{n=0}^{\infty} V^n \mathbb{Z}_p[T]$. Then we have

$$W_N(\mathbb{F}_p[t]) = E^0 / V^N(E^0) = \sum_{n=0} V^n \mathbb{Z}_p[T] / \sum_{n=N} V^n \mathbb{Z}_p[T].$$

It admits the following decomposition depending on the exponent of T being integral or rational:

$$W_N(\mathbb{F}_p[t]) = \bigoplus_{n \in \mathbb{N}} \mathbb{Z}_p / p^N \cdot T^n \oplus \bigoplus_{\substack{n = \frac{m}{p^a}, \\ p \nmid m}} \mathbb{Z}_p / p^{N-a} \cdot V^a(T^m).$$

The similar holds for $\mathbb{F}_p[t_1, \ldots, t_r]$.

One of the important property of the construction of the Witt vector is that it can be globalized:

Proposition 5.3. *Let X be a scheme. Then the presheaf of ring $U \mapsto W_n(\mathcal{O}_X(U))$ is a sheaf on X.*

The above is true for a general ring on a topos. Here we denote by $W_n(X)$ to be the scheme with the structure sheaf $W_n(\mathcal{O}_X)$. We note here that when p is nilpotent in X, $VW_n(\mathcal{O}_X)$ is also nilpotent in $W_n(\mathcal{O}_X)$, and the pair

$$(W_n(\mathcal{O}_X), VW_n(\mathcal{O}_X))$$

has a canonical pd- structure compatible with $(\mathbb{Z}_p, (p))$, given by

$$\gamma_m(Vx) = \frac{p^{m-1}}{m!}V(x^m).$$

Construction and basic properties Now we can introduce the de Rham-Witt complex.

We first give the construction.

Construction 5.4 (Langer-Zink)**.** For a homomorphism $A \to B$ of $\mathbb{F}_p$-algebra, the de Rham Witt complex of B/A is a projective system of differential graded algebra (dg-algebra):

$$W_{n+1}\Omega^\bullet_{B/A} \to W_n\Omega^\bullet_{B/A} \to \cdots \to W_1\Omega^\bullet_{B/A},$$

equipped with maps $F : W_{n+1}\Omega^\bullet_{B/A} \to W_n\Omega^\bullet_{B/A}$ and $V : W_n\Omega^\bullet_{B/A} \to W_{n+1}\Omega^\bullet_{B/A}$. It is constructed as follows:

1. First we quotient the projective system of dg-algebras $\{\Omega^\bullet_{W_n(B)/W_n(A)}\}_n$ by the ideal generated by relations corresponding to the canonical pd-structure $(W_n(B), VW_n(B), \gamma)$:

   $$d\gamma_n(V(x)) - \gamma_{n-1}(V(x)) \otimes dV(x),\ x \in W_{n-1}(B).$$

   We get a projective system of differential graded algebra, denoted by $\{\widetilde{\Omega}^\bullet_{W_n(B)/W_n(A)}, n\}$.

2. Then by the universal property of pd-differential, the operator F on $W_n(B)$ can be extended to

   $$F : \widetilde{\Omega}^\bullet_{W_n(B)/W_n(A)} \to \widetilde{\Omega}^\bullet_{W_{n-1}(B)/W_{n-1}(A)};\ Fd([x] + Vy) = [x^{p-1}]d[x] + dy.$$

3. We quotient the $\{\widetilde{\Omega}^\bullet_{W_n(B)/W_n(A)}, n\}$ by extending the operator V to the differential

   $$V(xdy_1 \cdots dy_r) = Vx \cdot dVy_1 \cdot \cdots \cdot dVy_r.$$

   Then the quotient system of dg-algebras is the de Rham-Witt complex $\{W_n\Omega^\bullet_{B/A}, n\}$. And we denote by $W\Omega^\bullet_{B/A}$ to be the inverse limit $\varprojlim_n W_n\Omega^\bullet_{B/A}$.

From the construction, we have

Proposition 5.5. *1. The de Rham-Witt complex $W_\bullet\Omega^\bullet_{B/A}$ is the initial object in both the category of Witt complex and the category of pro $-V-$ complex, in the sense of [Ill11], 4.1.*

2. $W_n\Omega^0_{B/A} = W_n(B), \forall n$.

3. $W_1\Omega^\bullet_{B/A} = \Omega^\bullet_{B/A}$.

Here are some basic formulas of the $W_\bullet\Omega^\bullet_{B/A}$.

Proposition 5.6. *For operators F, V, d on $W_\bullet\Omega^\bullet_{B/A}$, we have*

- $dF = pFd$;
- $Fd[x] = [x^{p-1}]d[x]$;

- $FdV = d$;
- $FxFy = F(xy)$;
- $Vd = pdV$;
- $xVy = V((Fx) \cdot y)$;
- $FV = p$.

Here we note that since the Witt vector can be globalized to schemes, we could also globalize the de Rham-Witt complex to the general relative setting. Also, by the construction above, the $W_\bullet \Omega^\bullet_{X/S}$ is functorial with respect to X/S. We only mention the quasi-coherence and étale base change:

Proposition 5.7. *Assume S is a $\mathbb{F}_p$-scheme.*

1. *For any S-scheme X, $W_n\Omega^i_{X/S}$ is quasi-coherent over $W_n(X)$.*
2. *For an étale morphism $X \to Y$ of S-schemes, the induced morphism of Witt schemes $W_n(X) \to W_n(Y)$ is also étale. Moreover, we have the following isomorphism*

$$W_n\mathcal{O}_X \otimes_{W_n\mathcal{O}_Y} W_n\Omega^i_{Y/S} = W_n\Omega^i_{X/S}.$$

Key computation We then look at the de Rham-Witt complex for affine spaces.

Let $A = \mathbb{F}_p$, and $B = \mathbb{F}_p[t_1, \dots, t_r]$. Define the *complex of integral forms* $E^\bullet$ to be the subsets of ω in $\Omega^\bullet_{\mathbb{Q}_p[T_1^{\frac{1}{p^\infty}}, \dots, T_r^{\frac{1}{p^\infty}}]/\mathbb{Q}_p}$ such that

$$\omega \in E^\bullet \Longleftrightarrow \omega \textit{ and } d\omega \textit{ are integral (have coefficients in } \mathbb{Z}_p).$$

We then define the action of F and V on $E^\bullet$, by

$$FT_i = T_i^p, V = pF^{-1}, V(xdy_1 \dots dy_s) = Vx \cdot Vdy_1 \cdots Vdy_s.$$

Then based on the observation by Deligne, we have

$$W_N\Omega^\bullet_{B/A} = E^\bullet/(V^N E^\bullet + dV^N E^{\bullet-1}).$$

This can be proved by looking at the grading structure indexed by $\mathbb{Z}[\frac{1}{p}]$.

As an example, we look at the case for affine line, i.e. when $r = 1$. For E^0, it is straightforward to see that for $\mathbb{Z}_p[T] \in E^0$. Moreover, for $p \nmid m$, $bT^{\frac{m}{p^a}} \in E^0$ if and only if $p^a | b$. So

$$E^0 = \sum V^i \mathbb{Z}_p[T] = \sum_{a>0, p\nmid m} V^a T^m,$$

as in the Example 5.2. For E^1, since there is no degree 2 differential forms, it is easy to check that

$$\begin{aligned} E^1 &= \mathbb{Z}_p[T^{\frac{1}{p^\infty}}]dlog[T] \\ &= \mathbb{Z}_p[T]dT + \sum_{a>0, p\nmid m} \mathbb{Z}_p dV^a T^m. \end{aligned}$$

From this, due to the observation above, we get

$$\begin{aligned} W_N\Omega^0_{B/A} &= \bigoplus_{n\in\mathbb{N}} \mathbb{Z}_p/p^N \cdot T^n \oplus \bigoplus_{a>0, m\nmid p} \mathbb{Z}_p/p^{N-a} \cdot V^a T^m; \\ W_N\Omega^1_{B/A} &= \bigoplus_{n\in\mathbb{N}_+} \mathbb{Z}_p/p^N \cdot T^n d\log T \oplus \bigoplus_{a>0, m\nmid p} \mathbb{Z}_p/p^{N-a} \cdot dV^a T^m; \\ W_N\Omega^i_{B/A} &= 0, \ i \geq 2. \end{aligned}$$

Thus we reach the key observation:

Observation 5.8 (Deligne)**.** *The de Rham Witt complex $W_N\Omega^\bullet_{B/A}$ of the affine space contains the de Rham complex $\Omega^\bullet_{(\mathbb{Z}_p/p^n)[T]}$ as a direct summand, while the complement is acyclic.*

Comparison theorem Thanks to the Deligne's observation, we get the comparison between the de Rham Witt complex and the crystalline cohomology:

Theorem 5.9 (Crystalline and De Rham-Witt comparison). *Let k be a perfect field of characteristic p, X/k smooth. Then there exists a canonical isomorphism in the derived category of projective W_n-modules over X:*

$$Ru_{X/W_{n_{\mathrm{crys}}}*}\mathcal{O}_{X/W_n} \cong W_n\Omega^\bullet_{X/k}.$$

Proof. The case when X is an affine scheme that admits a smooth lifting X' follows from the truncated version of the affine version of the Comparison Theorem 4.2: The left side is computed by

$$U \mapsto R\Gamma(U/W_n, \mathcal{O}_{X/W_n}) = \Omega^\bullet_{(\mathcal{D}/p^n)/\mathbb{Z}_p}$$

But note that since X admits a smooth lifting X', the truncated pd-envelope $\mathcal{D}/p^n$ can be exactly lifting. So the left side is given by the de Rham complex of X'/W_n.

On the other hand, by the étale base change 5.7 and the Observation of Deligne, the right side is (locally) quasi-isomorphic to

$$\Omega^\bullet_{(\mathbb{Z}_p/p^n)[T_i]} \otimes_{\mathbb{Z}/p^n[T_i]} \mathcal{O}_{X'} = \Omega^\bullet_{X'/W_n}.$$

So we get the comparison.

For the general case, we only need to note that any smooth scheme over k locally always admits such a lifting, so we could use a hypercover by affines and the cohomological descent. □

Corollary 5.10. *For X/k smooth and proper, we have the isomorphism*

$$R\Gamma(X/W_{\mathrm{crys}}, \mathcal{O}_{X/W}) \cong R\Gamma(X, W\Omega^\bullet_{X/k}).$$

And $R\Gamma(X/W_{\mathrm{crys}}, \mathcal{O}_{X/W})$ is a perfect complex, with

$$R\Gamma(X/W_{\mathrm{crys}}, \mathcal{O}_{X/W}) \otimes^L_W k = R\Gamma(X, \Omega^\bullet_{X/k}).$$

Here we note that the above comparison theorem does not require the existence of the lifting (compare with 4.11).

6 Some results of the De Rham-Witt complex

In this section, we give some known and useful results about the de Rham-Witt complex.

The slope spectral sequence The initial goal for constructing the de Rham-Witt complex was to study the Frobenius action on the crystalline cohomology. And this leads to the study of the slope spectral sequence of the de Rham-Witt complex.

Let X/k be a proper smooth scheme. Then the absolute Frobenius morphism of X/k induces an σ-linear action Φ on $W\Omega^\bullet_{X/k}$, such that it is p^iF on the degree i $W\Omega^i_{X/k}$. Here σ denotes the restriction of F on $W(k)$, which is the ring endomorphism of $W(k)$ induced its Frobenius.

We then have the following deep result

Theorem 6.1. *[Ill79] Assume the assumption above.*

1. *For any (i,j), the following inverse limit gives an isomorphism:*

$$\mathrm{H}^j(X, W\Omega^i_{X/k}) = \varprojlim_n \mathrm{H}^j(X, W_n\Omega^i_{X/k}).$$

2. *The V action on $\mathrm{H}^j(X, W\Omega^i_{X/k})$ is complete and separated.*

3. *The torsion part $T^{i,j}$ of $\mathrm{H}^j(X, W\Omega^i_{X/k})$ can be killed by a finite p-power, such that the torsion free quotient is finite rank over W.*

Sketch of the proof.

Step 1 We consider the truncated complex:

$$0 \longrightarrow W\Omega^i_{X/k}[-i] \longrightarrow W\Omega^{\le i}_{X/k} \longrightarrow W\Omega^{\le i-1}_{X/k} \longrightarrow 0.$$

By the LES associated to it, suffices to show the properties of V_i on $\mathrm{H}^i(W\Omega^{\le i}_{X/k})$, where V_i is a endomorphism of complex such that the map on degree j is $p^{i-j}V$.

Step 2 Then we consider the filtration defined by $Fil^n = \ker(W\Omega^\bullet_{X/k} \to W_n\Omega^\bullet_{X/k})$ and the graded factor to show that

Claim 6.2. (i) $\mathrm{H}^j(W\Omega^{\le i}_{X/k}/V_i W\Omega^{\le i}_{X/k})$ is finite length W-module;

(ii) $\mathrm{H}^j(W\Omega^\bullet_{X/k})$ is V-separated and complete.

Step 3 Granting the Claim, the extended action of $W_\sigma[[V_i]][\Phi]$ with $V_i\Phi = \Phi V_i = p^{i+1}$ leads to the conclusion.

□

Corollary 6.3. *The slope spectral sequence*

$$E_1^{ij} = \mathrm{H}^j(W\Omega^i_{X/k}) \Longrightarrow \mathrm{H}^{i+j}(W\Omega^\bullet_{X/k})$$

degenerates after quotient p^∞-torsion, with

$$\mathrm{H}^{*-i}(W\Omega^i_{X/k}) \otimes K = (\mathrm{H}^*(W\Omega^\bullet_{X/k}) \otimes K)_{[i,i+1)}$$

being the component such that the slope of Φ are within $[i, i+1)$.

Proof. The degeneracy follows from the decomposition, since those slopes are disjoint for different i. Then we note that we already has $p^i|\Phi$ on the degree i, such that the action of V is topologically nilpotent. Granting this, since $\Phi V = p^{i+1}$, the slope of Φ on $\mathrm{H}^j(W\Omega^i_{X/k})/T^{i,j}$ must be strictly smaller than $i+1$. □

Corollary 6.4. *By taking $i = 0$, for any j, we have*

$$\mathrm{H}^j(W\mathcal{O}_{X/k}) \otimes K = (\mathrm{H}^*(X/W_{\mathrm{crys}}) \otimes K)_{[0,1)}.$$

Remark 6.5. Here we remark that in many situations the torsion can be infinitely generated over W. So it would be surprising to see them being able to be killed by a uniform p-power.

The higher Cartier isomorphism We first recall the Cartier isomorphism in characteristic p:

Theorem/Definition 6.6. *Let X/k be a smooth scheme over a perfect field of characteristic p Then there exists a functorial isomorphism*

$$C^{-1} : \Omega^i_{X^{(p)}/k} \longrightarrow \mathcal{H}^i(F_{X/k*}\Omega^\bullet_{X/k}),$$

where $F_{X/k}$ is the relative Frobenius, and $X^{(p)}$ is the Frobenius twist. On the local coordinates, the map for degree 0 is $C^{-1}(x \otimes 1) = x^p$, and for degree 1 it is $C^{-1}(dy \otimes 1) = y^{p-1}dy$.

The isomorphism C^{-1} is called the Cartier isomorphism.

Here we note that the Cartier map can be lift to the $F : W_2\Omega^i_{X/k} \to \Omega^i_{X/k}$ of the truncated de Rham Witt complex. Moreover, we have the following generalized result

Theorem 6.7 (Higher Cartier isomorphism)**.** *For $n \ge 1$, the map $F^n : W_{2n}\Omega^i_{X/k} \to W_n\Omega^i_{X/k}$ induces an isomorphism*

$$W_n\Omega^i_{X/k} \longrightarrow \mathcal{H}^i(W_n\Omega^\bullet_{X/k}),$$

that is compatible with products, and equals to C^{-1} when $n = 1$.

Remark 6.8. The Cartier isomorphism, or more generally the F-map are essential for the de Rham Witt complex. Recently Bhatt, Lurie and Mathew gave another construction of the de Rham-Witt complex, with the input being only the Cartier isomorphism and some homological algebra. Due to the lack of time, we will not discuss anything here.

Applications: rational points in characteristic p Now we apply the de Rham-Witt complex to the study of rational points for varieties in characteristic p.

Let $k = \mathbb{F}_q$ be a finite field for $q = p^a$. Then for any X/k separated of finite type, recall from the beginning that the zeta function of X is given by

$$\zeta_X(t) = \exp(\sum_{n \in \mathbb{N}_+} |X(\mathbb{F}_{q^n})| \frac{t^n}{n}),$$

which is a rational function by

$$\zeta_X(t) = \prod_{i=0}^{2\dim(X)} \det(1 - F_*^a | \mathrm{H}^i(X/W_{\mathrm{crys}}) \otimes K)^{(-1)^{i+1}}.$$

Then there is an easy consequence of the slope spectral sequence:

Proposition 6.9. *Assume X/k satisfy:*

- *geometrically connected;*
- *proper and smooth;*
- $\mathrm{H}^i(X, W\mathcal{O}_{X/k}) \otimes K = 0$*, any* $i > 0$.

Then for any finite extension $\mathbb{F}_{q^n}$ of k, we have

$$|X(\mathbb{F}_{q^n})| \equiv 1, \ \mathit{mod}\ q^n.$$

Proof. By the Corollary 6.4 to the slope spectral sequence, we have

$$(\mathrm{H}^i(X/W_{\mathrm{crys}}) \otimes K)_{[0,1)} = \mathrm{H}^i(X, W\mathcal{O}_{X/k}) \otimes K = 0.$$

Moreover, since X is geometrically connected, the action of F^a on $\mathrm{H}^0(X/W_{\mathrm{crys}}) \otimes K$ is trivial. Thus except for degree 0 cohomology, Frobenius eigenvalues of all the other cohomology groups are divided by q. Hence by expending the power series, we get the result from the expression of the zeta function. □

Corollary 6.10. *Let X/k be a smooth Fano hypersurface in a projective space $\mathbb{P}^n_k$. Then $|X(\mathbb{F}_{q^n})| \equiv 1$, mod q^n, for any $n \geq 1$.*

Proof. It suffices to check the vanishing of $\mathrm{H}^i(X, W\mathcal{O}_{X/k}) = 0$ for $i \geq 1$. Then recall the first item in the Theorem 6.1 that

$$\mathrm{H}^i(X, W\mathcal{O}_{X/k}) = \varprojlim_n \mathrm{H}^i(X, W_n\mathcal{O}_{X/k}).$$

Moreover, we have a short exact sequence of sheaves over X

$$0 \longrightarrow F_*^n\mathcal{O}_X \longrightarrow W_n\mathcal{O}_X \longrightarrow W_{n-1}\mathcal{O}_X \longrightarrow 0.$$

Thus by induction, it suffices to show that

$$\mathrm{H}^i(X, F_*^n\mathcal{O}_X) = 0, i > 0.$$

Now note that by assumption, the Frobenius map is finite. In particular, the higher pushout of F^n vanishes. So from the degenerated Leray spectral sequence we get

$$\mathrm{H}^i(X, F_*^n\mathcal{O}_X) = \mathrm{H}^i(X, \mathcal{O}_X).$$

Moreover, by assumption X is a smooth Fano hypersurface in $\mathbb{P}^n$. Thus $\mathrm{H}^i(X, \mathcal{O}_X) = 0$ for $1 \leq i \leq n-2$, and when $i = n-1$ we get

$$\begin{aligned} \mathrm{H}^{n-1}(X, \mathcal{O}_X) &= \mathrm{H}^0(X, \omega_X) \\ &= \mathrm{H}^0(X, \mathcal{O}_X(d-n-1)) = 0, \end{aligned}$$

since $d - n - 1 < 0$. So we are done. □

References

[BdJ12] B. Bhatt, A.J. de Jong. Crystalline cohomology and de Rham cohomology. Preprint.

[Ber74] P. Berthelot. Cohomologie cristalline des schmas de caractristique p>0. (French) Lecture Notes in Mathematics, Vol. 407. Springer-Verlag, Berlin-New York, 1974. 604 pp.

[BO78] P. Berthelot, A. Ogus. Notes on crystalline cohomology, Math. Notes, vol. 21, Princeton Univ. Press, Princeton, NJ, 1978.

[Gro68] A. Grothendieck. Crystals and the de Rham cohomology of schemes. Notes by I. Coates and O. Jussila. Adv. Stud. Pure Math., 3, Dix exposs sur la cohomologie des schmas, 306358, North-Holland, Amsterdam, 1968.

[Ill79] L. Illusie. Complexe de deRham-Witt et cohomologie cristalline. Ann. Sci. École Norm. Sup. (4), 12(4):501661, 1979.

[Ill94] L. Illusie. Crystalline cohomology. Motives (Seattle, WA, 1991), 4370, Proc. Sympos. Pure Math., 55, Part 1, Amer. Math. Soc., Providence, RI, 1994.

[Ill11] L. Illusie. De Rham-Witt complexes and p-adic Hodge theory, Pre-notes for Sapporo seminar, March 2011.

[KM74] N. Katz. W. Messing. Some Consequences of the Riemann Hypothesis for Varieties over Finite Fields. Inventiones mathematicae 23 (1974): 73-78.

[Mum61] D. Mumford. Pathologies of modular algebraic surfaces. Amer. J. Math. 83 (1961), pp. 339342.

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