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GENERAL TOPOLOGY

On Kuratowski's Problem

by

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Note. This English translation of В. П. Солтан's paper, К Задаче Куратовского (in Russian), in Bull. Acad. Polon. Sci. Ser. Sci. Math., **28**(1980) 369–375, was prepared by Mark Bowron in July 2012. Some proofs have been modified.

Summary. The paper discusses some geometrical and combinatorial questions connected with the well-known problem of Kuratowski concerning the number of sets which can be constructed from a subset of a topological space by applications of the closure and complement operators in any order.

Kuratowski considered the following problem while investigating the topological closure axioms:

What is the largest number of sets that can be obtained from an arbitrary subset in a topological space by repeatedly applying the set operations of closure and complement in any order?

Kuratowski showed (e.g., see [**8,** 48–49]) that no more than 14 distinct sets can be obtained in this fashion. Later it was proved (see [**3,** 180] and [**5**]) that this result remains valid in more general spaces.

Recall that a mapping g is called a *closure operator* on a set X if for all subsets A, B of X we have

$$\begin{aligned} A \subset gA \quad & (g \text{ is } \textit{extensive}), \\ ggA \subset gA \quad & (g \text{ is } \textit{idempotent}), \\ A \subset B \Longrightarrow gA \subset gB \quad & (g \text{ is } \textit{increasing}). \end{aligned}$$

A set X and closure operator g on X are together called a *closure space*. Let the complement of A in X be denoted by cA. Kuratowski's result is a consequence of the following statement:

Any semigroup of operators generated by g and c contains at most 14 elements.

Sets obtained from $A \subset X$ by applying the operations g and c can be arranged into the following two sequences:

$$A,\ gA,\ cgA,\ gcgA,\ cgcgA,\ gcgcgA,\ cgcgcgA, \tag{1}$$

$$cA,\ gcA,\ cgcA,\ gcgcA,\ cgcgcA,\ gcgcgcA,\ cgcgcgcA. \tag{2}$$

As both $gcgcgcg = gcg$ and $gcgcgcgc = gcgc$ [**5**], each of the sequences (1) and (2) contains at most seven sets.

The following two diagrams display these 14 sets ordered by set inclusion ($\longrightarrow$ denotes $\subset$):

(3)
$$\begin{tikzcd}
 & & cgcgA \arrow[dr] & & \\
cgcA \arrow[r] \arrow[ddrr] & cgcgcgcA \arrow[ur] \arrow[dr] & & gcgcgA \arrow[r] & gA \\
 & & gcgcA \arrow[ur] & & \\
 & & A \arrow[uurr] & &
\end{tikzcd}$$

(4)
$$\begin{tikzcd}
 & & cgcgcA \arrow[dr] & & \\
cgA \arrow[r] \arrow[ddrr] & cgcgcgA \arrow[ur] \arrow[dr] & & gcgcgcA \arrow[r] & gcA \\
 & & gcgA \arrow[ur] & & \\
 & & cA \arrow[uurr] & &
\end{tikzcd}$$

It is easy to see that applying the operator c to each set in diagram (3) produces diagram (4). Introducing the interior operator $i = cgc$, the following two diagrams express (3) and (4) in terms of g, i, and c:

(5)
$$\begin{tikzcd}
 & & giA \arrow[dr] & & \\
iA \arrow[r] \arrow[ddrr] & igiA \arrow[ur] \arrow[dr] & & gigA \arrow[r] & gA \\
 & & igA \arrow[ur] & & \\
 & & A \arrow[uurr] & &
\end{tikzcd}$$

(6)
$$\begin{tikzcd}
 & & gi(cA) \arrow[dr] & & \\
i(cA) \arrow[r] \arrow[ddrr] & igi(cA) \arrow[ur] \arrow[dr] & & gig(cA) \arrow[r] & g(cA) \\
 & & ig(cA) \arrow[ur] & & \\
 & & cA \arrow[uurr] & &
\end{tikzcd}$$

We investigate the consequences when sets in (3) equal sets in (4). For brevity we set $B_1 = cgcA$, $B_2 = cgcgcgcA$, $B_3 = cgcgA$, $B_4 = A$, $B_5 = gcgcA$, $B_6 = gcgcgA$, $B_7 = gA$ and $C_j = cB_j$, $j = 1, \ldots, 7$.

(7)

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
C_1			${}^0X$	${}^*X$	${}^*X$	X	X
C_2			${}^0X$	${}^*X$	${}^*X$	X	X
C_3	${}^0\varnothing$	${}^0\varnothing$		${}^*X$	$=$	${}^*X$	${}^*X$
C_4	${}^*\varnothing$	${}^*\varnothing$	${}^*\varnothing$		${}^*X$	${}^*X$	${}^*X$
C_5	${}^*\varnothing$	${}^*\varnothing$	$=$	${}^*\varnothing$		${}^0X$	${}^0X$
C_6	$\varnothing$	$\varnothing$	${}^*\varnothing$	${}^*\varnothing$	${}^0\varnothing$		
C_7	$\varnothing$	$\varnothing$	${}^*\varnothing$	${}^*\varnothing$	${}^0\varnothing$		

Table (7) is interpreted as follows. If $C_i = B_j$, then $i < j \Longrightarrow C_i = B_j = X$ and $i > j \Longrightarrow C_i = B_j = \varnothing$. An equals sign means $C_i = B_j$ without any common value implied. An asterisk (zero) means that the relation $C_i = B_j$ is possible only when $g\varnothing = X$ ($g\varnothing = \varnothing$). Empty cells represent impossible equations. Symmetry with respect to the main diagonal holds since $C_i = B_j$ and $B_i = C_j$ are equivalent.

See the appendix for a proof of table (7).

We now examine what happens when sets within (3) or sets within (4) are equal. By duality it suffices to consider equalities within (5) alone. Note, our goal here is not to characterize the sets A for which given equalities hold in diagram (5) (this was done in **[10]** and **[4]** for topological spaces). We merely point out which equalities are equivalent to one another.

Theorem 1. *The following equivalencies hold for all $A \subset X$:*

1) $A = giA \Longleftrightarrow A = gigA$,
2) $A = igA \Longleftrightarrow A = igiA$,
3) $giA = igA \Longleftrightarrow igiA = gigA$,
4) $giA = gigA \Longleftrightarrow igA = igiA$,
5) $gA = X \Longleftrightarrow gigA = X$,
6) $iA = \varnothing \Longleftrightarrow igiA = \varnothing$.

Proof. 1) If $A = giA$, then $gA = ggiA = giA = A$ and $A = giA = gigA$. If $A = gigA$, then $gA = gigA = A$ and $A = giA$.

2) If $A = igA$, then $A = iA$ and $A = igiA$. If $A = igiA$, then $A = iA$ and $A = igA$.

3) $giA = igA \Longrightarrow gigA = giA = igA = igiA$; $igiA = gigA \Longrightarrow giA = gigiA = gigA = igiA = igigA = igA$.

4) If $giA = gigA$, then $igiA = igigA = igA$. If $A = igiA$, then $A = iA$ and $A = igA$.

Before proving parts 5) and 6), we note that $gcg\varnothing = X$. Indeed since $g\varnothing \subset gcg\varnothing$ and $g\varnothing \cup cg\varnothing = X$, it follows that $X \subset g(g\varnothing \cup cg\varnothing) \subset g(g\varnothing \cup gcg\varnothing) = gcg\varnothing$.

5) $gA = X \Longrightarrow gigA = giX = gcg\varnothing = X$. If $gigA = X$, then $gigA \subset gA = X$.

6) $iA = \varnothing \Longrightarrow igiA = ig\varnothing = cgcg\varnothing = cX = \varnothing$. If $igiA = \varnothing$, then $\varnothing = iA \subset igiA$.

Note that parts 1) – 4) of Theorem 1 are proved in **[4]** for topological spaces.

Definitions. For $A \subset X$ let $k(A)$ be the number of distinct sets generated by A by repeatedly applying the operations g and c. Let $l(A)$ be the number of distinct sets generated by A under g and i. Let $k'(A)$ be the number of sets in (1) that do not equal cA, or any sets to their left in (1). Let $k''(A)$ be the number of sets in (2) that do not equal any sets to their left in (2), or any of the first $k'(A)$ sets in (1).

From diagrams (3) – (6) it follows that for all $A \subset X$ we have:

$$2 \leq k(A) \leq 14, \quad 1 \leq l(A) \leq 7, \quad k(A) \leq l(A) + l(cA) = 2l(A).$$

Theorem 2. *For $A \subset X$ the following hold:*

1) *The number $k(A)$ is even.*

2) *The numbers $k'(A)$ and $k''(A)$ are both odd.*

Proof. Since $X \neq \varnothing$, no subset of X equals its own complement. Thus any finite family of subsets of X that is closed under c has even cardinality. This proves part (a). The initial complementary pair (A, cA) always has one set in (1) and the other in (2). All further complementary pairs reside within (1) or within (2). This proves part (b).

Examples showing that $k(A)$ can take any even value from 2 to 14 are presented in **[1]**. Necessary and sufficient conditions for A to satisfy $k(A) = 14$ are presented in **[9]** for both general and connected topological spaces.

Theorem 3. *For $A \subset X$ the following hold:*

1) *If $k(A) = 2$, then $1 \leq l(A) \leq 2$. In this case $l(A) = 2$ holds if and only if g is trivial ($g\varnothing = X$).*

2) *If $k(A) = 4$, then $2 \leq l(A) \leq 3$. In this case if $l(A) = 3$, then either $g\varnothing = X$, or $gcgA = cgA$ and $gcA = gA$.*

3) *If $k(A) = 6$, then $l(A) = 3$ or $l(A) = 5$. In this case $l(A) = 5$ holds if and only if both $gA = gcA = X$ and $g\varnothing \neq \varnothing$ hold.*

4) *If $k(A) \geq 8$, then $k(A) = 2l(A)$.*

Proof. Since $k(A) \leq 2l(A)$, the lower bounds for $l(A)$ have already been established. Since $l(cA) = l(A)$, we may assume without loss of generality that $k'(A) \geq k''(A)$.

1) Clearly $l(A) \leq k(A) = 2$. Suppose $l(A) = 2$. It follows from table (7) that the two sets in (5) are $\varnothing$ ($= cgcA$) and X ($= gA$). If $A = \varnothing$, then $g\varnothing = gA = X$. If $A = X$, then $cA = \varnothing$, hence $g\varnothing = gcA = X$. Conversely suppose $g\varnothing = X$. Since $l(A) \leq 2$, by (7) we have $A = \varnothing$ or $A = X$. If $A = \varnothing$ then $gA = X$, hence $l(A) = 2$. If $A = X$ then $cA = \varnothing$ and $gcA = X$. Hence $cgcA = \varnothing$. Conclude $l(A) = 2$.

2) Assume $k(A) = 4$. Theorem 2 implies $k'(A) = 3$ and $k''(A) = 1$. Note, when $k(A) \geq 4$, the set cA does not equal any sets in diagram (5), since otherwise by table (7) we would get $g\varnothing = X$ and either $A = \varnothing$ or $A = X$, implying $k(A) = 2$. Thus $l(A) \leq k(A) - 1 = 3$.

Claim $l(A) = 3 \iff$ either $cgA = iA = igA$ or $cgA = iA = giA$. Suppose the right side holds. We get $cgA = iA = \varnothing$ by (7). Thus $gA = X$. Since cA is not in (5), A cannot equal either X or $\varnothing$. Thus $l(A) = 3$. Conversely suppose $l(A) = 3$. Since $k'(A) = 3$ we have $cgA \neq cA$. Since $k(A) = 4$ and $l(A) = 3$, the set cgA must therefore be in (5). This implies $cgA = \varnothing$ by (7). The presence of $\varnothing$ in (5) implies that $iA = \varnothing$. It follows that neither A nor cA equals $\varnothing$ or X. Suppose $g\varnothing = B$ where $B \neq \varnothing$ and $B \neq X$. Since cA is not in (5) and $k(A) = 4$, the equation $g\varnothing = giA$ thus implies $g\varnothing = A$. But then $gA = gg\varnothing = g\varnothing = A$, contradicting $gA = X$. Hence either $g\varnothing = \varnothing$ (in which case $giA = \varnothing$) or $g\varnothing = X$ (in which case $igA = \varnothing$). This proves the claim.

Since $cgA = iA = igA \implies g\varnothing = X$, and $cgA = iA = giA \implies$ both $gcgA = cgA$ ($= \varnothing$) and $gcA = gA$ ($= X$), this completes the proof of part 2).

3) Suppose $k(A) = 6$. Then either $k'(A) = 5$ and $k''(A) = 1$, or $k'(A) = k''(A) = 3$.

(a) If $k'(A) = 5$ and $k''(A) = 1$ we claim the sets cgA and $gcgA$ either both belong to (5), or neither does. If cgA is in (5) then it is immediate that $gcgA$ is also in (5), since (5) is closed under g. On the other hand suppose $gcgA$ is in (5). By the same argument used below in part 4) (a) that deals with the case when $gcgA$ is in (5), not only do we get that cgA ($= \varnothing$) must also be in (5), we get that $gcgA = gcgcA = g\varnothing \neq \varnothing$. Thus the claim holds. Since cA is not in (5), we conclude either $l(A) = 3$ or $l(A) = 5$.

(b) If $k'(A) = k''(A) = 3$ we claim the sets cgA, gcA, and cA do not belong to (5). The set cA is not in (5) since $k(A) \geq 4$. Anytime cgA is in (5) we get $cgA = \varnothing$ by (7). The presence of $\varnothing$ in (5) implies $cgcA = \varnothing = cgA$, which by assumption cannot hold. Hence cgA is not in (5). The argument for gcA is similar. This proves the claim. Hence $l(A) = 3$.

(c) We now prove the stated equivalence in part 3).

($\Longrightarrow$) Suppose $l(A) = 5$. We showed in part (a) that this implies both $gcgA$ and cgA are in (5), with $gcgA = gcgcA = g\varnothing \neq \varnothing$. By (7) we get $cgA = \varnothing$; the presence of $\varnothing$ in (5) implies $iA = \varnothing$. Hence $gA = gcA = X$.

($\Longleftarrow$) If $gcA = gA = X$, then $cgA = \varnothing$ is in diagram (5). By parts (a) and (b), this implies $l(A) = 5$.

4) Suppose $k(A) = 8$. Then either $k'(A) = 7$ and $k''(A) = 1$, or $k'(A) = 5$ and $k''(A) = 3$.

(a) We claim if $k'(A) = 7$ and $k''(A) = 1$, then cA, cgA, $gcgA$, $cgcgcgA$ do not belong to (5); if $k'(A) = 5$ and $k''(A) = 3$, then cgA, $gcgA$, cA, gcA do not belong to (5). Clearly both cases imply $l(A) = 4$.

We know that cA cannot be in (5) in either case since $k(A) \geq 4$. If cgA is in (5) then $cgA = \varnothing$ by (7). But the presence of $\varnothing$ in (5) implies that $cgcA = \varnothing = cgA$. This gets cgA out of (5) in the second case. The first case is handled by noting that $cgA = \varnothing$ implies $gcgcgA = X = gA$. If gcA is in (5) then we have $gcA = X$ by (7). The presence of X in (5) implies that $gA = X = gcA$. This gets gcA out of (5) in the second case. If $cgcgcgA$ is in (5) then it equals $\varnothing$ by (7). This gets $cgcgcgA$ out of (5) in the first case since it implies $gcgcgA = X = gA$. If $gcgA$ is in (5) then by (7) it must equal $\varnothing$, X, or $gcgcA$. If $gcgA = \varnothing$, then $cgA = \varnothing$, hence $cgcgA = X = gA$. But this contradicts both case assumptions. Hence the relation $gcgA = \varnothing$ cannot hold in either case. If $gcgA = X$ is in (5), then $gA = X$. But then $gcgA = gA$, which cannot hold in either case. Finally if $gcgA = gcgcA$, then by Claim 3 (b) in the appendix, we have

$$\begin{gathered} B_1 = B_2 = \varnothing = C_6 = C_7, \\ C_1 = C_2 = X = B_6 = B_7, \\ B_3 = C_5,\ C_3 = B_5. \end{gathered}$$

Since this implies $k(A) \leq 6$, we conclude $gcgA$ is out of (5) in both cases. This completes the proof of (a).

Suppose $k(A) = 10$. Then either $k'(A) = 7$ and $k''(A) = 3$, or $k'(A) = k''(A) = 5$.

(b) We claim if $k'(A) = 7$ and $k''(A) = 3$, then cgA, $gcgA$, $cgcgcgA$, cA, gcA do not belong to (5); if $k'(A) = k''(A) = 5$, then cgA, $gcgA$, cA, gcA, $cgcgcA$ do not belong to (5). Clearly both cases imply $l(A) = 5$.

All of the arguments in the first and second cases in part 4) (a) respectively apply to the first and second cases here. Furthermore, by the same argument that got gcA out of (5) when $k'(A) = 5$ and $k''(A) = 3$, we get the same result for the first case here. Lastly, suppose the second case holds and $cgcgcA$ is in (5). By (7) it follows that $cgcgcA$ must equal $\varnothing$, X, or $cgcgA$. Since $k'(A) = k''(A) = 5$ we immediately have $cgcgcA \neq cgcgA$. If $cgcgcA = \varnothing$, then $gcgcA = X = gA$, contradicting $k'(A) = k''(A) = 5$. Hence $cgcgcA \neq \varnothing$. Similarly if $cgcgcA = X$, then $gcgcA = \varnothing = cgcA$, which by assumption does not hold. We conclude that in the second case, the set $cgcgcA$ cannot be in (5). This completes the proof of (b).

Suppose $k(A) = 12$. Then $k'(A) = 7$ and $k''(A) = 5$. By all of the arguments in part 4) (b), it follows that the sets cgA, $gcgA$, $cgcgcgA$, cA, gcA, $cgcgcA$ do not belong to (5). Therefore $l(A) = 6$.

When $k(A) = 14$ then, obviously, $l(A) = 7$. This completes the proof of Theorem 3.

Corollary 1. *If $g\varnothing = \varnothing$, then $k(A) = 2l(A)$ for all $A \subset X$, except as specified in Theorem 3 part 2).*

Corollary 2. ([**2**], [**9**]) *For any set A in a topological space, the relation $k(A) = 14$ holds if and only if $l(A) = 7$.*

Note that [**2**] gives necessary and sufficient conditions for a topological space X to satisfy

$$\max\{l(A) : A \subset X\} = p \quad (1 \leq p \leq 7).$$

We now consider what values $k(A)$ and $l(A)$ can take as $|X|$ varies.

Theorem 4. *If $l(A) = 7$ for some $A \subset X$, then $|X| \geq 6$.*

Proof. The condition $l(A) = 7$ means that all seven sets in diagram (5) are distinct. Thus, clearly, $|igiA| > |iA|$, $|gigA| \geq |igiA| + 2$, $|gA| > |gigA|$, and therefore $|g(A)| \geq |iA| + 4$. By parts 5) and 6) of Theorem 1 we have $|iA| > 0$, $|X| > |gA|$. Therefore $|X| \geq 6$.

For $n = 1, \ldots, 7$ set $L(n) = \max\{l(A) : A \subset X,\ |X| = n\}$ and $K(n) = \max\{k(A) : A \subset X,\ |X| = n\}$. It is easy to verify the following values of $L(n)$ and $K(n)$ for $1 \leq n \leq 6$:

(8)

n	1	2	3	4	5	6
$L(n)$	2	3	5	5	6	7
$K(n)$	2	4	8	10	12	14

Assuming $g\varnothing = \varnothing$, table (8) takes the form:

(9)

n	1	2	3	4	5	6
$L(n)$	1	3	3	4	6	7
$K(n)$	2	4	6	8	12	14

The values for $K(n)$ in (8) are given in **[1]** for algebraic closure. Given these values, the corresponding values of $L(n)$ for $4 \leq n \leq 6$ (and upper bounds for $L(n)$ for $1 \leq n \leq 3$) are supplied by Theorem 3. An example with $n = 3$ and $l(A) = 5$ appears in Claim 3 part (a) in the appendix. Hence $L(3) = 5$. It is easy to find examples that prove $L(1) = 2$ and $L(2) = 3$.

See the appendix for a treatment of the case $g\varnothing = \varnothing$.

For the case of a topological space it is known (see **[1]**, **[6]**) that $K(n) = 2L(n) = 2n$ $(n = 1, \ldots, 7)$.

A nontrivial example of a closure operator is the convex hull operator in a linear space. In **[7]** it is shown that for the convex hull g in n-dimensional Euclidean space E^n $(n \geq 2)$ we have $\max\{k(A) : A \subset E^n\} = 10$. From this result and Theorem 3 we get the following corollary.

Corollary 3. *The convex hull g in E^n satisfies the relation* $\max\{l(A) : A \subset E^n\} = 5$ $(n \geq 2)$.

It is easy to verify that the closed convex hull operator in a linear space is also a closure operator. A proof similar to the above-mentioned one in **[7]** yields the following result:

Theorem 5. *The closed convex hull g in E^n satisfies*

$$\max\{k(A) : A \subset E^n\} = 2 \max\{l(A) : A \subset E^n\} = 10.$$

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V. P. Soltan, **On Kuratowski's Problem**

For any set A in a closure space let $k(A)$ be the number of different sets obtained from A via the closure operator g and the complement operator applied in any order and let $l(A)$ be the number of different sets obtained from A by applying the operators g and $i = cgc$. Kuratowski, and later Birkhoff and Hammer, have shown that $2 \leq k(A) \leq 14$, $1 \leq l(A) \leq 7$, $k(A) \leq l(cA) + l(A) = 2l(A)$.

Result 1. *The number $k(A)$ is always even.*

Result 2. *When $k(A) = 2$, then $1 \leq l(A) \leq 2$; $k(A) = 4$, then $2 \leq l(A) \leq 3$; $k(A) = 6$, then $l(A) = 3$ or $l(A) = 5$; $k(A) \geq 8$, then $k(A) = 2l(A)$.*

We also consider some geometric issues and study the numbers $k(A)$, $l(A)$ as $|X|$ varies.

Appendix

The following proofs did not appear in the original paper.

Section 1. Proof of table (7).

Consider the following table of values of i and j satisfying $gB_n = B_i,\ gC_n = C_j$ $(1 \le n \le 7)$:

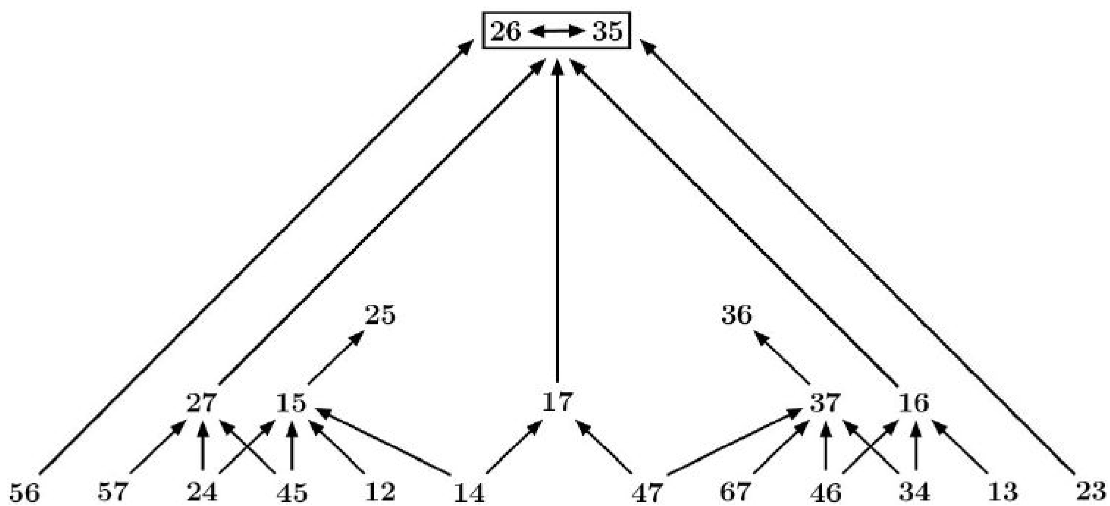
(10)

n	1	2	3	4	5	6	7
i	5	5	6	7	5	6	7
j	1	2	3	1	2	3	3

$gB_n = B_i,\ gC_n = C_j$

Let "ij" denote the (equivalent) equations $B_i = C_j$ and $B_j = C_i$ $(1 \le i < j \le 7)$. The following logical implications ($\longrightarrow$ denotes $\Longrightarrow$) are easy to verify using (10):

(11)



Claim 1. For all $i < j$ such that $B_i \subset B_j$ in diagram (3), equations ij imply that $B_i = C_j = \varnothing$ and $B_j = C_i = X$.

Proof. This follows immediately from the inclusions $B_i = C_j = cB_j \subset cB_i$ and $B_j = C_i = cB_i \supset cB_j$.

Claim 2. The property stated in Claim 1 holds for all equations ij except 35.

Proof. Besides equations 35, the only equations ij not handled by Claim 1 are those involving $B_4 = A$ and neither B_1 nor B_7. These are: 24, 34, 45, 46.

Suppose equations 24 hold. Then so do equations 27 by diagram (11). Thus by Claim 1, $C_2 = B_7 = X$ and $B_2 = C_7 = \varnothing$. Conclude $B_2 = C_4 = \varnothing$ and $B_4 = C_2 = X$.

The proofs for the other three sets of equations are similar to the one above, since $34 \Longrightarrow 37$, $45 \Longrightarrow 15$, and $46 \Longrightarrow 16$. This completes the proof of Claim 2.

Claim 3.
(a) Equations 35 $\not\Longrightarrow$ either $B_3 = C_5 = \varnothing$ or $B_3 = C_5 = X$.
(b) Equations 35 $\Longrightarrow B_1 = B_2 = \varnothing = C_6 = C_7$.

Proof. Let $X = \{1,2,3\}$. Define g by setting $g\varnothing = \{2\} = g(\{2\})$ and $gE = X$ for all other E. It is easy to verify g is a closure operator. For $A = \{1\}$ we have $B_3 = cgcg(\{1\}) = \{1,3\} = cgcgc(\{1\}) = C_5$. This proves part (a). Equations 35 imply $B_3 = C_5 = cB_5$. Thus $B_1 \subset B_2 \subset B_3 \cap B_5 = \varnothing$. Similarly $C_7 \subset C_6 \subset C_3 \cap C_5 = \varnothing$. This proves part (b).

Claim 4. Equations 25 and 36 each imply $g\varnothing = X$.

Proof. By Claim 2, equations 25 imply $B_2 = \varnothing$ and $gB_2 = B_5 = X$. Similarly, equations 36 imply $B_3 = \varnothing$ and $gB_3 = B_6 = X$. This proves Claim 4.

Note, by Claim 4, any equations ij in diagram (11) that imply either 25 or 36 also imply $g\varnothing = X$.

Claim 5. Equations 56, 57, and 67 each imply $g\varnothing = \varnothing$.

Proof. By Claim 2, equations 56 and 57 each imply $gB_5 = B_5 = \varnothing$. Similarly, equations 67 imply $gB_6 = B_6 = \varnothing$. This proves Claim 5.

Note, by substituting cA for A in Claim 5, we get that equations 12, 13, and 23 also imply $g\varnothing = \varnothing$.

Note that equations 12 and 67 each imply both $g\varnothing = \varnothing$ and $g\varnothing = X$. Hence, since $X \neq \varnothing$, neither equations 12 nor equations 67 can hold in any closure space.

Claim 6.
(a) Neither $g\varnothing = \varnothing$ nor $g\varnothing = X$ is implied by any of the following: 16, 17, 26, 27.
(b) Each of the following holds in at least one (nonempty) closure space: 13, 14, 23, 24, 34.

Proof. Let $A = \{1\}$. Suppose $X = \{1\}$ and $g\varnothing = X$. Then:

$$B_1 = cgcA = \varnothing = cgcgcgcA = B_2, \tag{12}$$

$$C_6 = cgcgcgA = \varnothing = cgA = C_7, \text{ and} \tag{13}$$

$$C_4 = cA = \varnothing = cgcA = B_1 = cgcgcgcA = B_2 = cgcgA = B_3. \tag{14}$$

Now suppose $X = \{1, 2\}$, $g\varnothing = \varnothing$, and $gE = X$ for $E \neq \varnothing$. In this case equations (12) and (13) again hold. The following equations also hold:

$$C_3 = gcgA = \varnothing = cgcA = B_1 = cgcgcgcA = B_2. \tag{15}$$

Equations (12) and (13) show 16, 17, 26, and 27 each holding within nonempty closure spaces, one of which satisfies $g\varnothing = \varnothing$ and the other $g\varnothing = X$. This proves part (a). Equations (14) and (15) imply part (b). This completes the proof of Claim 6.

Note that by diagram (11) and duality, Claim 6 implies that the following equations also hold within at least one (nonempty) closure space: 15, 25, 36, 37, 45, 46, 47.

Thus, where "=" denotes equality with no specific value implied, table (7) is as follows:

(16)

	B_1	B_2	B_3	B_4	B_5	B_6	B_7
C_1			${}^0X$	${}^*X$	${}^*X$	X	X
C_2			${}^0X$	${}^*X$	${}^*X$	X	X
C_3	${}^0\varnothing$	${}^0\varnothing$		${}^*X$	$=$	${}^*X$	${}^*X$
C_4	${}^*\varnothing$	${}^*\varnothing$	${}^*\varnothing$		${}^*X$	${}^*X$	${}^*X$
C_5	${}^*\varnothing$	${}^*\varnothing$	$=$	${}^*\varnothing$		${}^0X$	${}^0X$
C_6	$\varnothing$	$\varnothing$	${}^*\varnothing$	${}^*\varnothing$	${}^0\varnothing$		
C_7	$\varnothing$	$\varnothing$	${}^*\varnothing$	${}^*\varnothing$	${}^0\varnothing$		

Translator's note. Table (7) in the original contains six incorrect entries.

Section 2. Proof of table (9).

In this section it will be assumed throughout that $g\varnothing = \varnothing$. Closure spaces satisfying this additional axiom are called *Fréchet V-spaces*.

Case 1. ($n = 1$) It is trivial that $K(1) = 2$. Theorem 3 implies $L(1) = 1$.

Case 2. ($n = 2$) When $X = \{1, 2\}$ and $gA = X$ for $A \neq \varnothing$, we have $k(\{1\}) = 4$ and $l(\{1\}) = 3$. Hence $K(2) = 4$ and $L(2) = 3$.

Case 3. ($n = 3$) Suppose $|X| = 3$. Without loss of generality we can assume $X = \{1, 2, 3\}$. Claim $k(A) \leq 6$ for all $A \subset X$. It is immediate that $k(X) = k(\varnothing) = 2$. Suppose $A = \{1\}$. It is easy to verify that this implies $k''(A) \leq 3$. Thus, since $gA = X$ and $gA = A$ each imply $k'(A) \leq 3$, we can assume without loss of generality that $gA = \{1, 2\}$. Hence $cgA = \{3\}$. Since $cA = \{2, 3\}$, the only possible values of $gcgA$ that might satisfy $k'(A) = 5$, are $\{1, 3\}$ and X. If $gcgA = X$ then $k''(A) = 1$, hence $k(A) = 6$. If $gcgA = \{1, 3\}$, then $g\{1\} \not\subset g\{1, 3\}$ $(= \{1, 3\})$, contradicting the definition of a closure operator. Hence $gcgA \neq \{1, 3\}$. This proves the claim for $A = \{1\}$. Since $k(A) = k(cA)$, the claim holds for all subsets of cardinality 2 as well. Conclude $k(A) \leq 6$.

Hence $K(3) \leq 6$. It is easy to check that when $X = \{1, 2, 3\}$, $g\varnothing = \varnothing$, $gA = \{1, 2\}$ for all nonempty $A \subset \{1, 2\}$, and $gA = X$ otherwise, then $k(\{1\}) = 6$. Hence $K(3) = 6$. By Theorem 3 we get $L(3) = 3$.

Case 4. ($n = 4$) Suppose $A \subset X = \{1, 2, 3, 4\}$. Claim $k(A) \leq 8$. If $cgA = cgcA = \varnothing$, it follows that every set in diagrams (5) and (6) except possibly A and cA must equal X or $\varnothing$. Hence we may assume without loss of generality that $cgA \neq \varnothing$.

Suppose $iA = \varnothing$. It follows that $giA = igiA = iA$ $(= \varnothing)$. Suppose $k(A) \geq 10$. Since this implies $l(A) \leq 5$, no further equalities can hold within (5). Thus $1 \leq |igA| < |gigA| < |gA| < 4$. Hence also $1 \leq |cgA| < |cgigA| < |cigA| < 4$. These inequalities fix the cardinalities of all six sets, including $|gcgA| = |cigA| = 3$. Since $|X| = 4$, we get that either $A \subset gcgA$ or $cA \subset gcgA$. Hence either $gA \subset gcgA$ or $gcA \subset gcgA$. Since $gcA = X$, the only possibility is $gA \subset gcgA$. But since $|gA| = |gcgA| = 3$, we get $gA = gcgA$. By (7), this implies that $gcgA$ equals X, $\varnothing$ or giA $(= \varnothing)$. This contradicts $|gcgA| = 3$. Conclude $k(A) \leq 8$.

Suppose $iA \neq \varnothing$. Suppose $k(A) \geq 10$. Then $l(A) \geq 5$ by Theorem 3. By diagram (5) we have the following inclusions, where the notation "$\xrightarrow{\alpha_i}$" represents a variable whose possible values are "$\subsetneq$" or "$=$":

$$\begin{array}{rcccccccc}
iA & \xrightarrow{\alpha_1} & igiA & \xrightarrow{\alpha_2} & giA & \xrightarrow{\alpha_3} & gigA & \xrightarrow{\alpha_4} & gA \\
iA & \xrightarrow{\alpha_5} & A & \xrightarrow{\alpha_6} & gA & & & & \\
igiA & \xrightarrow{\alpha_7} & igA & & & & & & \\
igA & \xrightarrow{\alpha_8} & gigA & & & & & &
\end{array}$$

For $i = 1, \ldots, 8$ let $\alpha_i = 0$ if inclusion α_i is an equality and $\alpha_i = 1$ otherwise. Since $iA \neq \varnothing$ and $gA \neq X$, each of the following sums is bounded above by $|gA| - |iA| \leq 2$:

$$\begin{array}{rl}
\text{I.} & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\
\text{II.} & \alpha_1 + \alpha_7 + \alpha_8 + \alpha_4 \\
\text{III.} & \alpha_5 + \alpha_6
\end{array}$$

On the other hand, $l(A) \geq 5$ implies $\sum_{i=1}^{8} \alpha_i \geq 6$. The only possible values of α_i are therefore $\alpha_1 = \alpha_4 = 0$ and $\alpha_i = 1$ for $i = 2, 3, 5, 6, 7, 8$. Hence, since $l(A) \geq 5$, the sets A, giA, and igA are distinct subsets of gA that satisfy $|A| = |giA| = |igA| = 2$. Furthermore each contains iA, but this is impossible since $|gA| = 3$ and $|iA| = 1$. Conclude $k(A) \leq 8$. This proves the claim.

The claim implies $K(4) \leq 8$. It is easy to check that the space $X = \{1, 2, 3, 4\}$ with closure g defined in table (17) satisfies $k(\{2, 3\}) = 8$. This proves $K(4) = 8$. It follows by Theorem 3 that $L(4) = 4$.

(17)

A	gA
$\varnothing$	$\varnothing$
$\{1\}$	$\{1,2\}$
$\{2\}$	$\{1,2\}$
$\{3\}$	$\{3\}$
$\{4\}$	$\{3,4\}$
$\{1,2\}$	$\{1,2\}$
$\{1,3\}$	$\{1,2,3\}$
$\{1,4\}$	$\{1,2,3,4\}$
$\{2,3\}$	$\{1,2,3\}$
$\{2,4\}$	$\{1,2,3,4\}$
$\{3,4\}$	$\{3,4\}$
$\{1,2,3\}$	$\{1,2,3\}$
$\{1,2,4\}$	$\{1,2,3,4\}$
$\{1,3,4\}$	$\{1,2,3,4\}$
$\{2,3,4\}$	$\{1,2,3,4\}$
$\{1,2,3,4\}$	$\{1,2,3,4\}$

Case 5. ($n = 5$, $n = 6$) The examples given in **[1]** for $n = 5$ and $n = 6$ both satisfy $g\varnothing = \varnothing$, hence the last two columns in (8) are preserved when $g\varnothing = \varnothing$.

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