NOTES ON THE OOGURI-VAFA METRIC

LORENZO FOSCOLO

The Ooguri-Vafa metric is a hyperkähler 4-dimensional metric. After introducing it via the Gibbons-Hawking ansatz, we will try to get more insight in its geometry by studying it in two different complex structures: in one case, we will talk about elliptic fibrations; in the second, the relevant structure is a fibration in special Lagrangian tori.

1. The Ooguri-Vafa metric via the Gibbons-Hawking ansatz

1.1. **The Gibbons-Hawking ansatz.** We start by giving the construction of the Ooguri-Vafa metric via the Gibbons-Hawking ansatz, as it was first introduced in [1]; we will follow the detailed description of [2].

Let X be a $U(1)$-bundle over an open set U of $\mathbb{R}^3$; then X admits a natural fibre-wise S^1-action and we want to find a hyperkähler metric on X for which the circle acts by tri-holomorphic isometries. It turns out that such a metric is completely described in terms of a harmonic function on U and a connection form on the bundle: this is known as the Gibbons-Hawking ansatz, [3].

Let $U \subset \mathbb{R}^3$ be an open subset with Euclidean metric and coordinates (u_1, u_2, u_3) and $\pi : X \to U$ a principal $U(1)$-bundle with S^1-action on the fibres generated by the vector field $\frac{\partial}{\partial t}$.

We choose a connection 1-form θ, i.e. an $\mathfrak{u}(1) = i\mathbb{R}$-valued 1-form on X which is S^1-invariant and such that $\theta\left(\frac{\partial}{\partial t}\right) = i$. Then the curvature $d\theta = \pi^*\alpha$ for a 2-form α on U such that $\frac{i}{2\pi}\alpha$ represents the first Chern class of the bundle $\pi : X \to U$.

Suppose that there exists a positive function V on U such that $*dV = \frac{\alpha}{2\pi i} = d\theta_0$, with $\theta_0 = \frac{\theta}{2\pi i}$. Note that, V is then harmonic.

Now define three 2-forms on X by

$$\begin{aligned} \omega_1 &= du_1 \wedge \theta_0 + V du_2 \wedge du_3 \\ \omega_2 &= du_2 \wedge \theta_0 + V du_3 \wedge du_1 \\ \omega_3 &= du_3 \wedge \theta_0 + V du_1 \wedge du_2 \end{aligned}$$

It is easily checked that $\omega_1^2 = \omega_2^2 = \omega_3^2 \neq 0$, $\omega_i \wedge \omega_j = 0$ if $i \neq j$ and $d\omega_j = 0$.

We want to show that the triple $(\omega_1, \omega_2, \omega_3)$ defines a hyperkähler structure on X. Since a 4-dimensional manifold is hyperkähler iff it is Calabi-Yau, let's consider the real and complex 2-forms

$$\omega_3 \text{ and } \Omega_3 = -\omega_1 - i\omega_2 = (\theta_0 - iV du_3) \wedge (du_1 + i du_2)$$

and show that they define a Calabi-Yau structure on X.

The forms above satisfy

(i) ω_3 is non-degenerate;
(ii) Ω_3 is locally decomposable and non-vanishing;
(iii) $\omega_1 \wedge Re(\Omega_3) = \omega_3 \wedge Im(\Omega_3) = 0$;
(iv) $\Omega_3 \wedge \overline{\Omega}_3 = \omega_3^2$;
(v) $d\omega_3 = 0 = d\Omega_3$.

We recall how such two forms define a complex structure and a metric on X such that ω_3 is the Kähler form and Ω_3 a holomorphic volume form (see for example [4]).

Consider the subspace W of the complexified cotangent bundle of X spanned by $(\theta_0 - iV du_3)$ and $du_1 + i du_2$; since $\Omega_3 \wedge \overline{\Omega}_3 \neq 0$, $T^* \otimes \mathbb{C} = W \oplus \overline{W}$ and this defines an almost complex structure J_3 such that λ is a 1-form of type $(1,0)$ iff $\lambda \wedge \Omega_3 = 0$. Moreover, (v) implies that $d\lambda \wedge \Omega_3 = 0$ if λ

is of type $(1,0)$ and by the Newlander-Niremberg theorem J_3 is integrable. Also (iii) and the fact that ω_3 is real imply that ω_3 is of type $(1,1)$.

Considering that $J_3(du_1) = -du_2$ and $J_3(du_3) = -V^{-1}\theta_0$, we see that the metric g defined by the formula $g(u,v) = \omega_3(u, J_3 v)$ is

$$g = V d\mathbf{u} \cdot d\mathbf{u} + V^{-1}\theta_0^2 \, .$$

Example: gravitational instantons of type ALF.. Start with the standard projection $\mathbb{C}^2 \setminus \{0\} \to \mathbb{CP}^1 = S^2$ and restrict it to the 3-sphere:

$$(z_1, z_2) \mapsto \big(2Re(z_1\overline{z}_2), 2Im(z_1\overline{z}_2), |z_1|^2 - |z_2|^2\big) \, ;$$

extend it radially to $\mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ and compose with $z_2 \mapsto \overline{z}_2$. We obtain the S^1-bundle $p : \mathbb{C}^2 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$

$$p(z_1, z_2) = \big(2Re(z_1 z_2), 2Im(z_1 z_2), |z_1|^2 - |z_2|^2\big)$$

with S^1-action $(e^{it}z_1, e^{-it}z_2)$, which is just the Hopf map $S^3 \to S^2$ extended radially.

We want to look for a harmonic function on $\mathbb{R}^3 \setminus \{0\}$ such that $-\int_{S^2} *dV = \pm 1$, the Chern number of the bundle (the sign depend on the chosen orientation on S^2), and a connection 1-form θ such that $\frac{d\theta}{2\pi i} = *dV$. Such a couple is for example given by, for $e \geq 0$,

$$V = e + \frac{1}{4\pi|\mathbf{u}|} \text{ and } \theta = \frac{iIm(\overline{z}_1 dz_1 - \overline{z}_2 dz_2)}{|z_1|^2 + |z_2|^2} \, .$$

By a change of coordinates one can check that the corresponding metric g extends to $\mathbb{R}^4$ and all these examples are ALF metrics, the Euclidean one when $e = 0$ and the Taub-NUT when $e = 1$.

1.2. **S^1-invariant Ricci-flat metrics.** Consider now a principal $U(1)$-bundle over $B \times S^1$, where B is an open set of $\mathbb{C}$; a solution V, θ of the Gibbons-Hawking ansatz periodic in $u = u_3$ defines a metric that descends on the $U(1)$-bundle iff $\omega_1, \omega_2, \omega_3$ are invariant under changing u by a period, i.e. the periodicity of u is independent of $y = u_1 + iu_2 \in B$.

Take B to be a disc in $\mathbb{C}$ and define $U := \big[(B \times \mathbb{R}) \setminus (\{0\} \times 2\pi\mathbb{Z})\big]/2\pi\mathbb{Z}$. We want to construct a harmonic function V on U with singular behaviour at the point 0 analogous to the Taub-NUT case: in this way, by applying the Gibbons-Hawking ansatz, we will obtain a hyperkähler metric on the one-point compactification of the $U(1)$-bundle over $\overline{U} = D \times S^1$.

Since in the Taub-NUT case we take V to be $\frac{1}{4\pi|\mathbf{u}|}$, we simply make periodic this function in the third variable $u = u_3$:

$$V(y,u) := \frac{1}{4\pi} \sum_{m \in \mathbb{Z}} \frac{1}{\sqrt{|y|^2 + (u + 2\pi m)^2}} - a_{|m|}$$

with $a_m = 1/(2\pi m)$ if $m > 0$ and $a_0 = (-\gamma + \log 2)/\pi$ (γ is the Euler constant).

An alternative and useful expression for V_0 can be obtained by Fourier analysis in the circle variable u. Recall that we are looking for a harmonic function on U with fixed singular behaviour at the puncture; we can reformulate this, asking for a solution of the Poisson's equation $\triangle V = -\delta(y,u)$ with the Dirac delta on the right-hand side. Decomposing in Fourier modes we have

$$V(y,u) = \sum_{m \in \mathbb{Z}} V_m(y) e^{imu} \quad \text{and} \quad \delta(y,u) = \frac{1}{2\pi\varepsilon} \sum_{m \in \mathbb{Z}} \delta(y) e^{imu}$$

and therefore the equations we have to solve are

$$\triangle V_m - m^2 V_m = -\delta$$

in $\mathbb{R}^2$. It follows that $V_0(y) = -\frac{1}{2\pi}\log|y| + f(y)$ for a harmonic function f on the whole disc B and $V_m(y) = \frac{1}{2\pi}K_0(|my|)$, with K_0 the second modified Bessel function of the second kind; the particular choice of constant a_0 in the previous formula implies that in this case $f \equiv 0$.

Using well-known facts about Bessel functions, one can prove that there exists a constant C such that for any $0 < r_0 < 1$, if $|y| > r_0$ then

$$\left| V + \frac{1}{2\pi}\log|y| \right| \le \frac{C}{\varepsilon} e^{-|y|} \, .$$

As we saw, we have the freedom to consider $V + f$ in the Gibbons-Hawking ansatz, for some harmonic function f defined on the whole disc B; the only condition one has to check is that, taking the diameter of B small enough, $V + f$ stays positive on $B \times \mathbb{R}$.

2. From the Gibbons-Hawking ansatz to holomorphic coordinates and elliptic fibration

Taking the harmonic function V defined in the previous section, by the Gibbons-Hawking we construct a hyperkähler metric on a S^1-bundle over U that compactifies to a manifold X with a projection over $\overline{U}$.

Let's look at this manifold with complex structure J_3, Kähler form ω_3 and holomorphic volume form $\Omega_3 = (\theta_0 - iVdu) \wedge dy$. As we have already explained, this means that $\theta_0 - iVdu$ and dy span the space of $(1,0)$ form and therefore y is a holomorphic coordinate on X and the projection $\pi : X \to B$ is an elliptic fibration in complex structure J_3.

Moreover, locally there exists a holomorphic coordinate x such that $dx = \theta_0 - iVdu$ $(mod\ dy)$. Pass to the universal cover $B \times \mathbb{R}^2$ of X and fix a holomorphic section σ_0; then integrating $\theta_0 - iVdu$ along the fibres from the base-point σ_0, we can define a global holomorphic coordinate x on $B \times \mathbb{R}^2$ and see X as a quotient of the holomorphic cotangent bundle $\mathcal{T}^*_B \to B$ by a degenerating family of lattices $\Lambda(y)$ (because the elliptic fibration π is singular at the origin); notice also that, since σ_0 was chosen to be holomorphic, the holomorphic volume form Ω_3 on X is induced by the standard complex symplectic form $dx \wedge dy$ on $\mathcal{T}^*_B$.

We want to compute the periods of the elliptic fibration $\pi : X \to B$. Recall that in the Gibbons-Hawking ansatz we started with an S^1-bundle over U with a natural action of the circle on the fibres. Choose a basis of $H_1(X_y, \mathbb{Z})$ taking γ_1 to be the S^1-orbit and γ_2 the circle parametrised by u in the base; then

$$\int_{\gamma_1} dx = \int_{\gamma_1} \theta_0 - iVdu = \int_{\gamma_1} \theta_0 = 1$$

while

$$\tau(y) := \int_{\gamma_2} \theta_0 - iVdu \, .$$

Consider the imaginary part of $\tau(y)$: $\int_{\gamma_2} Im(dx) = -\int_{\gamma_2} Vdu = -\frac{1}{2\pi}\log|y| + f(y)$ if we started with $V + f/\varepsilon$ in the Gibbons-Hawking construction of the metric. It follows that $\int_{\gamma_2} Re(dx)$ is a harmonic conjugate of $-\frac{1}{2\pi}\log|y| + f(y)$ and therefore

$$\tau(y) = \frac{1}{2\pi i}\log y + ih(y) + C$$

where $h = f + ig$ is a holomorphic function on B; moreover, shifting θ_0 by a factor $a\,du$ (which does not change the equation $*dV = d\theta_0$), we can assume that $C = 0$.

Following Kodaira's classification of singular fibres of elliptic fibrations, to understand what is the type of singular fibre $\pi^{-1}(0)$ we need to compute the monodromy of the fibration, i.e. how the periods change by topological parallel transport along a path winding 1 around $0 \in B$: since $\tau(e^{i\theta}y) = \tau(y) + \frac{\theta}{2\pi}$, we see that the monodromy is given by

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} .$$

This implies that $\pi^{-1}(0)$ is a type I_1 singular fibre, i.e. a pinching torus.

3. Hyperkähler rotation: from elliptic to SYZ fibration

One of the most interesting features of hyperkähler geometry is the presence of a whole 2-sphere of complex structures compatible with the same Kähler metric. We saw that in the complex structure J_3 the Ooguri-Vafa metric is defined on a neighborhood of a I_1 singular fibre of an elliptic fibration over a disc $B \subset \mathbb{C}$.

We now want to understand the geometry of the Ooguri-Vafa metric in complex structure J_1; we will follow here mainly Chan's paper [7]. Consider the complex symplectic form $\Omega_1 = \omega_3 - i\omega_2 = (i\theta_0 + V dy_1) \wedge (dy_2 + idu) = (i\theta_0 + V dy_1) \wedge \frac{d\eta}{\eta}$, where $\eta := e^{iy_2}e^{iu}$ defines a holomorphic projection $\kappa : X \to \mathbb{C}^*$.

Once again recall that we started from a $U(1)$-bundle over an open set U, thus there is a natural S^1-action on X. Let's compute the moment map of this action with respect to the symplectic form ω_1. First, if t is a local coordinate on the S^1-fibre over U, the S^1-action is given by $e^{is} \cdot (y_1, y_2, u, t) = (y_1, y_2, u, t + s)$. It follows that the moment map $\mu : X \to i\mathbb{R}$ satisfies $d\mu = -i\frac{\partial}{\partial t} \lrcorner\, \omega_1 = \frac{i}{2\pi} dy_1$ and therefore $\mu(y_1, y_2, u, t) = y_1$ (after dropping the factor $i/2\pi$).

Consider now the Kähler quotient $X_r = \mu^{-1}(r)/S^1$. It is a complex curve endowed with a holomorphic $(1, 0)$-form defined by

$$\Omega_r := -i\frac{\partial}{\partial t} \lrcorner\, \Omega_1 = \frac{i}{2\pi} d(\log \eta) .$$

Consider the integral real curves C of the 1-form $Im(\Omega_r)$: these are the curves $Re(\log \eta) = y_2 = const$. The inverse image of such a C in X is a surface $T_{r,s} = \{y_1 = r, y_2 = s\}$ such that $\omega_1|_{T_{r,s}} \equiv 0$ (because $\mu|_{T_{r,s}} \equiv r$ and ω_1 is non-degenerate) and $Im(\Omega_1)|_{T_{r,s}} \equiv 0$. In other words, each $T_{r,s}$ is a special Lagrangian submanifold of $(X, J_1, \omega_1, \Omega_1)$ and $\pi : X \to B$ is a SYZ fibration, i.e. a fibration in special Lagrangian tori. When $(r, s) \neq (0, 0)$ $T_{r,s}$ is a torus embedded in X and $T_{0,0}$ is nodal.

3.1. **The Ooguri-Vafa metric via a toric construction, the complement of a conic in $\mathbb{C}^2$ and holomorphic discs.** Now we want to identify the SYZ fibration we have just described with a standard special Lagrangian fibration of the complement of a conic in $\mathbb{C}^2$. As an intermediate step, we will give a toric description of the Ooguri-Vafa space following [5] that will be useful also in the following.

Consider the standard 2-simplex with vertexes $\tau = \{e_1, e_2\}$ in a 3-dimensional lattice $N \simeq \mathbb{Z}^3$; fix also a 2-simplex σ in the dual lattice N^* with the property that $\langle \sigma, \tau \rangle = 1$: for example we can choose $\sigma = \{e_1^* + e_2^*, e_1^* + e_2^* + e_3^*\}$. Let X_τ be the toric 3-dimensional variety associated to τ which with our choice is simply $\mathbb{C}^2 \times \mathbb{C}^*$; we define a hypersurface $Z_{\tau,\sigma}$ as the closure in X_τ of the hypersurface $XY(1 - Z) = 1$ in $(\mathbb{C}^*)^3$. The circle $\sigma^\perp \otimes \mathbb{R}/\sigma^\perp = \mathbb{R}(e_1 - e_2)/\mathbb{Z}(e_1 - e_2)$ acts on $Z_{\tau,\sigma}$ and taking a moment map μ for this action and the natural projection $\kappa : X_\tau \to \mathbb{C}^*$ gives a map

$$(\mu, \kappa) : Z_{\tau,\sigma} \longrightarrow \mathbb{R} \times \mathbb{C}^*$$

which is a principal S^1-bundle over $(\mathbb{R} \times \mathbb{C}^*) \setminus \{(0, 1)\}$.

If we want to define a hyperkähler metric on this bundle by the Gibbons-Hawking ansatz, by direct inspection of the topological behaviour at the singular point $(0, 1)$, then we have to solve $\triangle V = -\delta(y_1, y_2, u)$, where we are taking y_1 to be the moment map coordinate and $e^{y_2}e^{iu}$ a coordinate on $\mathbb{C}^*$. It follows that $Z_{\tau,\sigma}$, or better the subset where the chosen harmonic function $V + f$ is positive, is the Ooguri-Vafa space.

On the other hand, from the description we have given, it is clear that we can identify $Z_{\tau,\sigma}$ with $\mathbb{C}^2 \setminus \{XY = 1\}$. We will then use work of Auroux, for example in [6], to describe a natural SYZ fibration on the complement of a conic in $\mathbb{C}^2$; this coincides with the previous description (of course, restricting to the subset where $V + f$ is positive).

We equip $\mathbb{C}^2 \setminus \{XY = 1\}$ with the symplectic form induced by the standard one on $\mathbb{C}^2$ and with the holomorphic volume form $\Omega = \frac{dX \wedge dY}{XY - 1}$. Let the circle act on $\mathbb{C}^2$ by $(e^{it}x, e^{-it}y)$ with moment map $\mu(X,Y) = |X|^2 - |Y|^2$; then $\mathbb{C}^2 \setminus \{XY = 1\}$ is fibred by special Lagrangian tori

$$T_{r,s} = \{(X,Y) \in \mathbb{C}^2 \mid XY \neq 1\,,\ \mu(X,Y) = r\,,\ \log|XY - 1| = s\}\ .$$

A useful way to see these tori is to consider the projection $\kappa : (X,Y) \mapsto xy \in \mathbb{C} \setminus \{1\}$ (its fibres are conics, $\kappa^{-1}(0)$ being the nodal $XY = 0$); then $T_{r,s}$ is contained in the inverse image by κ of a circle of radius e^s centred in $1 \in \mathbb{C}$ and consists of one single S^1-orbit in each fibre of κ. To see that $T_{r,s}$ is special Lagrangian, as before, notice that it is the lift of an integral curve of the form $Im(\Omega_{red})$, where $id\log(XY-1)$, in the Kähler reduction $\mu^{-1}(r)/S^1$.

We conclude with a discussion of holomorphic discs with boundary on $T_{r,s}$. Suppose to have a holomorphic disc $\varphi : (D^2, \partial D^2) \to (X, T_{r,s})$; then the composition with κ gives a holomorphic map $(D^2, \partial D^2) \to (\mathbb{C} \setminus \{1\}, \{|z| = e^s\})$ and by the maximum principle $\kappa \circ \varphi$ has to be constant, i.e. a holomorphic disc is contained in a single fibre of κ. Since $\kappa^{-1}(z)$ is a smooth conic if $z \neq 0$, there cannot be any holomorphic disc in such a fibre. On the opposite, the torus $T_{r,0}$ intersects the X-axis (Y-axis) in a circle $|Y|^2 = -r$ if $r < 0$ ($|X|^2 = r$ if $r > 0$) and therefore it bounds a holomorphic disc.

3.2. **Affine coordinates.** From the previous discussion we can see the base B of the SYZ fibration as an open set in the moduli space (or better at its boundary, because of the presence of the nodal torus) of such deformations.

Hitchin showed in [4] that the moduli space of smooth special Lagrangian deformations of a given compact special Lagrangian submanifold L of a Calabi-Yau n-fold X carries two natural sets of affine coordiantes, one related to the symplectic geometry and the other to the complex structure. We briefly recall this construction in the second case and then compute the affine coordinates in the Ooguri-Vafa case.

Let $(t_1, \dots, t_m)$ be coordinates on the moduli space of smooth special Lagrangian deformations of a compact special Lagrangian submanifold $L \subset X$; it is then known that $m = b_1(L)$. Consider the interior product

$$\frac{\partial}{\partial t_j} \lrcorner\, Im\,\Omega\ ,$$

where, by a little abuse of notation, we are identifying the vector field $\frac{\partial}{\partial t_j}$ on the moduli space with the normal vector field on L corresponding to the infinitesimal variation in the direction t_j.

Choose a basis $B_1, \dots, B_m$ of $H_{n-1}(L, \mathbb{Z})$ and define the period matrix to be

$$\mu_{ij} = \int_{B_i} \frac{\partial}{\partial t_j} \lrcorner\, Im\,\Omega\ .$$

Since the 1-forms $\xi_i := \sum_{j=1}^m \mu_{ij} dt_j$ are closed there exists local coordinates $(v_1, \dots, v_m)$ on the moduli space such that $dv_i = \xi_i$.

Let's compute them for the Ooguri-Vafa metric and the SYZ-fibration we described. We have $Im\,\Omega_1 = \theta_0 \wedge dy_2 + V dy_1 \wedge du$. We can also decompose $V = V^{sf} + V^{inst}$, where $V^{sf} = -\frac{1}{2\pi}\log|y|$ and V^{inst} depends on u (this is the infinite sum of Bessel functions), and accordingly decompose the connection 1-form θ_0 as $\frac{dt}{2\pi} + A^{sf}du + \alpha\,dy$, where α is a complex valued function depending on u, and A^{sf} is the multi-valued function $\frac{1}{2\pi}\arg y$.

With respect to the basis $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ of the tangent space of B and γ_1, γ_2 (S^1-orbit, circle parametrised by u) of $H_1(T_{r,s}, \mathbb{Z})$, we easily compute

$$\mu_{11} = 0, \mu_{12} = -1, \mu_{21} = -\frac{1}{2\pi}\log|y|, \mu_{22} = \frac{1}{2\pi}\arg y\ ;$$

thus the affine coordinates are given by

$$v_1 = -y_2\,,\ v_2 = -Re\,(y\log y - y)\ .$$

It is clear that the affine coordinates, defined on $B \setminus 0$, depend only on the semi-flat (sf) part of V and θ_0 and therefore they do not extend over the origin; in other words they do not encode the correct complex structure of X. To have the correct information one has to take in account the instanton correction V^{inst} and $\alpha\, dy$, as the following theorem of Gross well explains (we won't be extremely precise in the statement of the theorem).

Theorem 3.1 (M. Gross, [8]). *Suppose to have a symplectic manifold (X, ω) and a fibration $\pi : X \to B$. Then giving a non-vanishing n-form Ω such that*

- *Ω is locally decomposable,*
- *$\frac{i^n}{n!}\Omega \wedge \overline{\Omega} = \omega^2$,*
- *ω is a positive $(1,1)$-form in the almost complex structure defined by Ω,*
- *$\pi : X \to B$ is special Lagrangian on the smooth part $X^\sharp$,*

is equivalent to giving a metric h on the fibres of π and a splitting $T_{X^\sharp} = T_{X^\sharp/B} \oplus \mathcal{F}$.

Here $\mathcal{F}$ corresponds to the choice of the subspace $J(T_{X^\sharp/B})$. The proof is an explicit calculation in Darboux coordinates.

It is then clear that the information we are missing is encoded precisely in the instanton part of V and θ_0, because in the Ooguri-Vafa case

$$h = V du^2 + V^{-1}(dt + A^{sf} du)^2$$

and $\mathcal{F}$ is defined using the full connection θ_0.

References

[1] H. Ooguri and C. Vafa, *Summing up Dirichlet instantons*, Phys. Rev. Lett. 77 (1996), 3296-3298.
[2] M. Gross and P. M. H. Wilson, *Large complex structure limits of K3 surfaces*, J. Differential Geometry 55 (2000), 475-546.
[3] G. W. Gibbons and S. W. Hawking, *Gravitational multi-instantons*, Phys. Letters B, 78 (4), 1978, 430-432.
[4] N. J. Hitchin, *The moduli space of special Lagrangian submanifolds*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 25 (1997), no. 3-4, 503-515.
[5] I. Zharkov, *Limiting behaviour of local Calabi-Yau metrics*, Adv. Theor. Math. Phys. 8 (2004), 395-420.
[6] D. Auroux, *Special Lagrangian fibrations, wall-crossing and mirror symmetry*, Surveys in Differential Geometry, Vol. 13, H.D. Cao and S.T. Yau Eds., Intl. Press, 2009, 1-47.
[7] K. Chan, *The Ooguri-Vafa metric, holomorphic discs and wall-crossing*, preprint 2009, arXiv:0909.3608 .
[8] M. Gross, *Special Lagrangian fibrations, II: geometry*, Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds (Cambridge, MA, 1999), AMS/IP Stud. Adv. Math. 23, Amer. Math. Soc., Providence, 2001, pp. 95-150 (math.AG 9809072).