MATHEMATICAL INDUCTION

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Section 1. Historical Introduction

In philosophy and in the applied sciences the term *induction* is used to describe the process of drawing general conclusions from particular cases. For Mathematics, on the other hand, such conclusions should only be drawn with caution, because mathematics is a demonstrative science and any statement must be accompanied by a rigorous proof. For example John Wallis (1616-1703) was criticized strongly by his contemporaries because in his *Arithmetica Infinitorum* (1656), after inspecting the six relations,

$$\frac{0+1}{1+1}=\frac{1}{3}+\frac{1}{6}, \qquad \frac{0+1+4}{4+4+4}=\frac{1}{3}+\frac{1}{12},$$

$$\frac{0+1+4+9}{9+9+9+9}=\frac{1}{3}+\frac{1}{18}, \qquad \frac{0+1+4+9+16}{16+16+16+16+16}=\frac{1}{3}+\frac{1}{24},$$

$$\frac{0+1+4+9+16+25}{25+25+25+25+25+25}=\frac{1}{3}+\frac{1}{30}, \qquad \frac{0+1+4+9+16+25+36}{36+36+36+36+36+36+36}=\frac{1}{3}+\frac{1}{36}$$

stated without any further justification that the general rule, namely,

$$\frac{0^2+1^2+2^2+\ldots+n^2}{n^2+n^2+n^2\ldots+n^2}=\frac{1}{3}+\frac{1}{6n},$$

follows "*per modum inductionis*".

Although Wallis' claim is correct, amounting to the familiar statement (known to Archimedes) that

$$1^2+2^2+\ldots+n^2=\left(\frac{1}{3}+\frac{1}{6n}\right)n^2(n+1)=\frac{1}{6}n(n+1)(2n+1),$$

it nevertheless needed proof.

One way to deal with this problem is with the so-called method of *complete* or *mathematical induction.* This topic, sometimes called just *induction*, is the subject discussed below.

Induction is a simple yet versatile and powerful procedure for proving statements about integers. It has been used effectively as a demonstrative tool in almost the entire spectrum of mathematics: for example in as diverse fields as algebra, geometry, trigonometry, analysis, combinatorics, graph theory and many others.

The principle of induction has a long history in mathematics. For a start, although the principle itself is not explicitly stated in any ancient Greek text, there are several places that contain precursors of it. Indeed, some historians see the following passage from Plato's (427-347 BC) dialogue *Parmenides* (§147a7-c3) as the earliest use of an inductive argument:

Then they must be two, at least, if there is to be contact. - They must. - And if to the two terms a third be added in immediate succession, they will be three, while the contacts [will be] two. - Yes. - And thus, one [term] being continually added, one contact also is added, and it follows that the contacts are one less than the number of terms. For the whole successive number [of terms] exceeds the number of all the contacts as much as the first two exceed the contacts, for being greater in number than the contacts: for afterwards, when an additional term is added, also one contact to the contacts [is added]. - Right. - Then whatever the number of terms, the contacts are always one less. -True.

The previous passage is from a philosophical text. There are, however, several ancient mathematical texts that also contain quasi-inductive arguments. For instance Euclid (~330 - ~ 265 BC) in his *Elements* employs one to show that every integer is a product of primes.

An argument closer to the modern version of induction is in Pappus' (~290-~350 AD) *Collectio*. There the following geometric theorem is proved.

Let AB be a segment and C a point on it. Consider on the same side of AB three semi-circles with diameters AB, AC and CB, respectively. Now construct circles C_n as follows: C_1 touches the three semi-circles; C_{n+1} touches C_n and the semicircles on AB and AC. If d_n denotes the diameter of C_n and h_n the distance of its centre from AB, then $h_n = nd_n$.

The way Pappus proves the theorem is to show geometrically the recurrence relation $h_{n+1} /d_{n+1} = (h_n + d_n)/d_n$. Next, he invokes a result of Archimedes (287 - 212 BC) from his *Book of Lemma's* (Proposition 6) which states that the conclusion of the theorem above is true for the case n = 1. Coupling this with the recurrence relation, he is able to conclude the case for the general n.

After the decline of Greek mathematics, the Muses flew to the Islamic world. Although induction is not explicitly stated in the works of mathematicians in the Arab world, there are authors who reasoned using a preliminary form of it. For example al - Karaji (953-1029) in his *al-Fakhri* states, among others, the binomial theorem and describes the so called Pascal triangle after observing a pattern from a few initial cases (usually 5). He also knew the formula $1^3 + 2^3 + ... + n^3 = (1 + 2 + ... + n)^2$. About a century later we find similar traces of induction in al-Samawal's (~1130-~1180) book *al-Bahir*, where the identity $1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n+1)/6$ appears. Subsequently Levi Ben Gershon (1288-1344), who lived in France, uses quasi-inductive arguments in his book *Maasei Hoshev* written in Hebrew.

The first explicit inductive argument in a source written in a western language is in the book *Arithmeticorum Libri Duo* (1575) of Francesco Maurolyco (1495–1575), a mathematician of Greek origin who lived in Syracuse. For instance it is shown inductively in this text that the sum of the first n odd integers is equal to the n^{th} square number. In symbols, $1 + 3 + 5 + \dots + (2n - 1) = n^2$, a fact already known to the ancient Pythagoreans.

Another early reference to induction is in the *Traité du Triangle Arithmetique* of Blaise Pascal (1623–1662), where the pattern known to-day as 'Pascal's Triangle' is discussed. There the author shows that the binomial coefficients ${}^nC_k$ satisfy ${}^nC_k$: ${}^nC_{k+1} = (k + 1) : (n - k)$, for all n and k with $0 \le k < n$. Here the passage from n to n + 1 uses ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$.

All the above authors used an intuitive idea about the concept of natural number. This is sufficient for our purposes here, and below we shall follow suit. A characteristic of modern mathematics, however, especially from the late 19th century, was to develop the theory axiomatically. In particular, this was accomplished for the natural numbers by Giuseppe Peano (1858-1932) who published the so called

'Peano's axioms' in 1889, in a pamphlet entitled *Arithmetices principia, nova methodo exposita*. The exact procedure need not concern us here. We only mention that one of the axioms was so designed as to incorporate induction as a method of proof. In other words, the intuitive way to deal with induction below is actually a legitimate technique.

In what follows, the theory is presented in short sections, each with its own problems. These are rather easy especially at the beginning, but those in the last paragraph are more challenging. *Several questions can be solved by other means, but the idea is to use induction in all of them.*

Section 2. Basics

The principle of mathematical induction is a method of proving statements concerning integers. For example consider the statement "$1^2 + 2^2 + 3^2 + ... + n^2 = n(n+1)(2n + 1)/6$", which we denote by P(n). One can easily verify this for various n, for instance $1^2 = 1 = 1.(1+ 1)(2.1 + 1)/6$, $1^2 + 2^2 = 5 = 2.(2 + 1)(2.2 + 1)/6$, $1^2 + 2^2 + 3^2 = 14 = 3.(3 + 1)(2.3 + 1)/6$ and so forth. Here we verified the statement for the cases n = 1, n = 2 and n = 3 (in a while we shall see that the last two can be dispensed with) but assume that we have verified it up to the particular value n = k. The last statement means that we are certain that for this particular value k we have "$1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$". But is the formula true for the case of the next integer n = k + 1? We claim that it is. To see this, *making use of the fact that we have* $1^2 + 2^2 + 3^2 + ... + k^2 = k(k + 1)(2k + 1)/6$, we argue

$$\begin{aligned} 1^2 + 2^2 + 3^2 + ... + k^2 + (k + 1)^2 &= k(k + 1)(2k + 1)/6 + (k + 1)^2 \qquad \text{(by assumption)} \\ &= (k + 1)[k(2k + 1) + 6(k + 1)]/6 \\ &= (k + 1)(k + 2)(2k + 3)/6, \end{aligned}$$

and this last is precisely the original claim for n = k + 1.

Let us recapitulate: We wanted to prove that the statement P(n) is true for all integers $n \geq 1$. We first verified it for n = 1; then, *assuming* that it is true for n = k, we verified it for n = k + 1. In other words, reiterating our result, the validity of P(1) implies that of P(2); the validity of P(2) implies that of P(3); the validity of P(3) implies that of P(4), *and so on* for all integers $n \geq 1$.

The schema we use in the proof can be summarised symbolically as

$$\begin{array}{l} P(1) \\ P(k) \Rightarrow P(k+1) \\ \hline \Rightarrow\ P(n) \text{ true for all } n \in \mathbf{N} \end{array}$$

The step "$P(k) \Rightarrow P(k+1)$" in the proof is called the *inductive step*; the assumption that P(k) is true, is called the *inductive hypothesis*.

Here is another example.

Example 2.1 (Bernoulli's inequality). Show that if a is a real number with $a > -1$, then $(1 + a)^n \geq 1 + na$ for all $n \in \mathbf{N}$.

Solution. For n = 1 it is a triviality (in fact we get an equality). Assume now validity of the inequality for n = k; that is, assume $(1 + a)^k \geq 1 + ka$. This is our inductive hypothesis, and we are to show $(1 + a)^{k+1} \geq 1 + (k + 1)a$. We have

$$(1 + a)^{k+1} = (1 + a)(1 + a)^k$$

$\geq (1 + a)(1 + ka)$ (by the inductive hypothesis)

$= 1 + (k + 1)a + ka^2$

$\geq 1 + (k + 1)a$ (since $ka^2 \geq 0$).

This, by the principle of induction, completes the proof. □

As a final remark, the above examples start from n = 1. This need not be always the case and there are cases (see problems) that induction may start at any another integer. The situation is self explanatory and there is no need to qualify it any further.

The next problems require the verification of a variety of formulae. None of these should present the reader with any difficulty and the problems are there only to familiarise him/her with the idea of induction. In fact, the reader should try to do several of these problems mentally.

Problem 2.1.(Routine). Show inductively that each of the following formulae is valid for all positive integers n.

a) $1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n + 1)^2/4$,

b) $1^4 + 2^4 + 3^4 + \ldots + n^4 = n(n + 1)(2n + 1)(3n^2 + 3n - 1)/30$,

c) $1^5 + 2^5 + 3^5 + \ldots + n^5 = n^2(n + 1)^2(2n^2 + 2n - 1)/12$,

d) $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\ldots+\frac{1}{n(n+1)}=\frac{n}{n+1}$,

e) $\frac{1}{1\cdot 2\cdot 3}+\frac{1}{2\cdot 3\cdot 4}+\ldots+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$,

f) $\frac{3}{1^2 2^2}+\frac{5}{2^2 3^2}+\frac{7}{3^2 4^2}+\ldots+\frac{2n+1}{n^2(n+1)^2}=\frac{n(n+2)}{(n+1)^2}$,

g) $(n + 1)(n + 2)\ldots(2n - 1)(2n) = 2^n.1.3.5\ldots.(2n - 1)$,

h) $\sum_{k=1}^{n}\frac{(2k)!}{k!2^k}=\sum_{k=1}^{n}1\cdot 3\cdot 5\cdot\ldots\cdot(2k-1)$,

i) $1-\frac{x}{1!}+\frac{x(x-1)}{2!}-\ldots+(-1)^n\frac{x(x-1)\ldots(x-n+1)}{n!}=(-1)^n\frac{(x-1)(x-2)\ldots(x-n)}{n!}$,

j) $(\cos x)(\cos 2x)(\cos 4x)(\cos 8x)\ldots(\cos 2^{n-1}x) = \frac{\sin 2^n x}{2^n \sin x}$ (for $x\in\mathbf{R}$ with $\sin x \neq 0$),

k) $\sum_{k=1}^{n}\cos(2k-1)x=\frac{\sin 2nx}{2\sin x}$, (for $x\in\mathbf{R}$ with $\sin x \neq 0$),

l) $\underbrace{\sqrt{2+\sqrt{2+\ldots+\sqrt{2+\sqrt{2}}}}}_{n\ \text{radicals}}=2\cos\frac{\pi}{2^{n+1}}$,

m) $(1^5 + 2^5 + 3^5 + \ldots + n^5) + (1^7 + 2^7 + 3^7 + \ldots + n^7) = 2(1 + 2 + 3 + \ldots + n)^4$,

n) $\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\ldots+\frac{1}{2n-1}-\frac{1}{2n}$.

Problem 2.2. If a sequence (a_n) satisfies

a) $a_{n+1} = 2a_n + 1$ ($n\in\mathbf{N}$), show that $a_n + 1 = 2^{n-1}(a_1 + 1)$.

b) $a_1 = 0$ and $a_{n+1} = (1 - x)a_n + nx$ ($n \in \mathbf{N}$), where $x \neq 0$, show that

$$a_{n+1} = [nx - 1 + (1 - x)^n]/x.$$

Problem 2.3. Let (a_n) be a given sequence. Define new sequences (x_n), (y_n) by $x_1 = 1$, $x_2 = a_1$, $y_1 = 0$, $y_2 = 1$ and, for $n \geq 3$, $x_n = a_n x_{n-1} + x_{n-2}$, $y_n = a_n y_{n-1} + y_{n-2}$. Show that $x_{n+1}y_n - x_n y_{n+1} = (-1)^n$.

Problem 2.4. If each of $a_1, a_2, \ldots, a_n$, is a sum of two perfect squares, show that the same is true for their product.

Problem 2.5. Show that $2n^5/5 + n^4/2 - 2n^3/3 - 7n/30$ is an integer for all $n \in \mathbf{N}$.

Problem 2.6. Show that if $x \neq y$, then the polynomial $x - y$ divides $x^n - y^n$.

Problem 2.7. Show that a convex n-gon has ½ $n(n - 3)$ diagonals ($n \geq 3$).

Problem 2.8. Prove the binomial theorem inductively. Namely, show that

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^k b^{n-k}$$

where ${}^nC_k = \dfrac{n!}{k!(n-k)!}$. You may use ${}^{n+1}C_k = {}^nC_{k-1} + {}^nC_k$ ($1 \leq k \leq n$). (The binomial theorem was known to the Arabs. They did not have a complete proof, but after verifying it for few small n they stated the general form using in a quasi-inductive argument. Later the theorem was rediscovered by Isaac Newton (1654-1705), who included it in his celebrated *Philosophiae Naturalis Principia Mathematica* (1687). For the proof he used a combinatorial argument. The first inductive proof was by Jakob Bernoulli (1654-1705), published posthumously in his *Ars Conjectandi* (1713)).

Problem 2.9. It is easy to see that the number $(2+\sqrt{3})^n$ can be written in the form $a_n + b_n\sqrt{3}$. Show a) inductively and b) without induction, that the numbers a_n, b_n satisfy $a_n^2 - 3b_n^2 = 1$ ($n \in \mathbf{N}$).

Problem 2.10. Show that the number $2^{2^n} - 1$ is divisible by at least n distinct primes.

Problem 2.11. If $F_n = a^{2^n} + 1$ is the n^{th} Fermat number ($n = 0, 1, 2, \ldots$), show that $F_n - 2 = (a - 1)F_0F_1\ldots F_{n-1}$ ($n \in \mathbf{N}$).

Problem 2.12. Prove by induction that $n! > 3^n$ for $n \geq 7$.

Problem 2.13. If $a_0, a_1, a_2, \ldots$ is a sequence of positive real numbers satisfying $a_0 = 1$ and $a_{n+1}^2 > a_n a_{n+2}$ ($n = 0, 1, 2, \ldots$), show that $a_1 > a_2^{1/2} > a_3^{1/3} > a_4^{1/4} > \ldots > a_n^{1/n} > \ldots$.

Problem 2.14. A result of Ramanujan (whose proof is beyond the scope of this book) states that $\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\ldots}}}}}$ = 3. Use Ramanujan's result to show that for all $n \in \mathbf{N}$, $\sqrt{1+n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+(n+3)\sqrt{1+\ldots}}}}} = n+1.$

Section 3. Patterns

One of the disadvantages of the method of induction, as reflected by some of the examples portrayed above (especially in Problem 1), is that one needs to know *beforehand* the formula describing the situation considered. It is only then that one may embark on proving it. But this need for foreknowledge can often be remedied by detecting patterns after judicial evaluation of special cases. In practice it means that one needs to *conjecture* the underlying rule, and then verify whether it is, indeed, correct. In other words, we have to do some guessing. The following examples elucidate this point.

Example 3.1. For what values on n is $2^n + 1$ a multiple of 3?

Solution. By checking small values of the integer n one realizes that $2^n + 1$ is a multiple of 3 for n equals 1, 3, 5 and 7, but fails to be so when n equals 2, 4, 6 or 8. It seems reasonable to guess that $2^n + 1$ a multiple of 3 precisely when n is odd. This turns out to be correct, and the following inductive argument can be used (how?) to verify the claim: Write $a_n = 2^n + 1$. Then $a_{n+2} = 2^{n+2} + 1 = 4(2^n + 1) - 3 = 4a_n - 3$, which is a multiple of 3 precisely when a_n is. □

Example 3.2. If f(x) = 2x + 1, guess a formula for the nth term of the sequence $f_1 = f(x)$, $f_2 = f(f(x))$, $f_3 = f(f(f(x)))$, $f_4 = f(f(f(f(x))))$, … and then prove it by induction.

Solution. By direct calculation one verifies that $f_2 = 4x + 3$, $f_3 = 8x + 7$, $f_4 = 16x + 15$ and so on. If these examples are not adequate to guess the pattern, the reader should continue with further iterations of f. Sooner or later one suspects that $f_n = 2^n x + 2^n - 1$. It turns out that the guess is correct, as the reader should supply the missing portions of the following inductive argument that settles the matter: $f_{n+1} = f(f_n(x)) = f(2^n x + 2^n - 1) = 2(2^n x + 2^n - 1) + 1 = 2^{n+1}x + 2^{n+1} - 1$. □

Example 3.3. By considering the numerical sequence

$$2 - 1,\ 3 - (2 - 1),\ 4 - (3 - (2 - 1)),\ 5 - (4 - (3 - (2 - 1))),\ \ldots$$

guess and then prove inductively the numerical value of

$$n - (n - 1 - (n - 2 - (n - 3 - (\ldots - (3 - (2 - 1))\ldots)))).$$

Solution. The first few expressions simplify to 1, 2, 2, 3, 3 and 4 respectively. One may guess that the general pattern is

$$n - (n - 1 - (n - 2 - (n - 3 - (\ldots - (3 - (2 - 1))\ldots)))) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd} \end{cases}$$

This is easy to verify inductively and the details are left to the reader, who should consider separately the cases n even and n odd. □

A word of caution is necessary here: No matter how many initial cases we check in a particular situation, a pattern that seems to emerge is not sufficient to draw conclusions. A proof must *always* follow our guess and failure to devise such a proof may indicate that our conjecture is, perhaps, wrong. There are several examples showing that even first rate mathematicians were deceived by a few special cases. The great Fermat, for example, after observing that $2^{2^0} + 1 = 3$, $2^{2^1} + 1 = 5$, $2^{2^2} + 1 = 17$, $2^{2^3} + 1 = 257$ and $2^{2^4} + 1 = 65537$ are prime numbers, thought that $2^{2^n} + 1$ is a prime for each n. This turned out to be false, and the first counterexample was given by Euler who found that $2^{2^5} + 1 = 641 \times 6700417$.

Sometimes the first counterexample to what might appear to be a pattern is very far away. For instance, the numbers $n^{17}+9$ and $(n+1)^{17}+9$ are relatively prime for n =1, 2, 3, . . . successively, and for a *very* long time after that. But is this always the case? No, and the first counterexample is for

n = 8424432925592889329288197322308900672459420460792433.

There are two delightful articles by Richard Guy, entitled *The Strong Law of Small Numbers* (American Mathematical Monthly, (1988) 697-711) and *The Second Strong*

Law of Small Numbers (Mathematics Magazine, 63 (1990) 3 - 20) with numerous examples of sequences that *seem* to follow a pattern. But in some cases the reality is, against all intuition, very different. It is worth also looking at the web page

http://primes.utm.edu/glossary/page.php?sort=LawOfSmall

where the previous example appears.

Here are some problems along the above lines, where the reader is invited either (i) to discover a pattern and then prove his/her hypothesis correct, or (ii) to find a counterexample that contravenes the pattern that appears at first sight.

Problem 3.1. After guessing an appropriate formula by testing a few first values of n, use an inductive argument to find the following sums.

a) $1^2 - 2^2 + 3^2 - \ldots + (-1)^{n-1} n^2$,

b) $1\cdot(1!) + 2\cdot(2!) + 3\cdot(3!) + \ldots + n\cdot(n!)$,

c) $n^2 - [(n-1)^2 - [(n-2)^2 - [(n-3)^2 - [\ldots - [3^2 - (2^2 - 1^2)]\ldots]]]]$,

d) $\dfrac{1}{x(x+1)} + \dfrac{1}{(x+1)(x+2)} + \dfrac{1}{(x+2)(x+3)} + \ldots + \dfrac{1}{(x+n-1)(x+n)}$.

Problem 3.2. It is given that the sum $1^6 + 2^6 + 3^6 + \ldots + n^6$ can be simplified in the form $n(n+1)(2n+1)(An^4 + Bn^3 - 3n + 1)/42$, where A and B are constants independent of n. Guess appropriate values of A and B and then verify that they lead to a valid formula.

Problem 3.3. If (p_n) is the sequence of primes starting from $p_1 = 2$, show that the sequence of numbers $p_1 + 1$, $p_1 p_2 + 1$, $p_1 p_2 p_3 + 1$, ... , $p_1 p_2 p_3 \ldots p_n + 1$, used by Euclid in a proof in his *Elements*, consists of prime numbers for n = 1, 2, 3, 4, 5 but not for n = 6.

Problem 3.4. Given n points on the circumference of a circle, where n is successively 1, 2, 3, 4,... , draw (in separate figures) all chords joining them. For this make sure that the points are "in general position" in the sense that no three chords are concurrent. Now, count the regions into which each circle is partitioned by the chords. You will find that they are, successively 1, 2, 4, 8, 16, ... What pattern seems to emerge? Is the next answer 32? Show that it is not!

Problem 3.5. Guess the general term of the sequence (a_n) if $a_0 = 1$, $a_n = 2$ and for $n \geq 1$, $a_{n+1} = \sqrt{a_n + 6\sqrt{a_{n-1}}}$.

Problem 3.6. Guess the general term of the sequence (a_n) if $a_0 = 1$, and for $n \geq 1$ we have $\sqrt{a_1} + \sqrt{a_2} + \ldots + \sqrt{a_n} = \frac{1}{2}(n+1)\sqrt{a_n}$.

Section 4. Divisibility

The method of induction can be applied to an abundance of situations, not just proving formulae as, perhaps, most of the above examples suggest. In what follows we shall see some of these different circumstances. We start with a fairly easy situation, the case of divisibility of integers, of which we have already seen some problem s in Section 2. We shall use the notation a | b to signify that an integer a divides, or is a factor of, an integer b.

Example 4.1. Show that for each positive integer n we have $9 \mid 5^{2n} + 3n - 1$; that is, 9 divides the number $5^{2n} + 3n - 1$.

Solution. Let $a_n = 5^{2n} + 3n - 1$. It is clear that $a_1 = 27$ is divisible by 9. Assume now that for n = k, the number a_n is divisible by 9, that is, $5^{2k} + 3k - 1 = 9M$ for some integer M. We have to show that $a_{k+1} = 5^{2(k+1)} + 3(k+1) - 1 = 25.5^{2k} + 3k + 2$ is also divisible by 9. The idea is to somehow use our inductive hypothesis, and this can be done as follows:

$$\begin{aligned} a_{k+1} &= 25 \cdot 5^{2k} + 3k + 2 \\ &= 25 \cdot (5^{2k} + 3k - 1) - 72k + 27 \\ &= 25 \cdot 9M - 9(8n - 3) \quad \text{(by the inductive hypothesis)} \end{aligned}$$

i.e. a_{k+1} is a multiple of 9.

Therefore by the principle of induction $9 \mid a_n$ for all positive integers n.□

Problem 4.1. Redo the previous example more elegantly by considering $a_{k+1} - 25a_k$ in place of a_{k+1} alone.

Example 4.2. Show that all numbers in the sequence 1003, 10013, 100113, 1001113,… and so on, are divisible by 17.

Solution. We have 1003 = 17×59, moreover, the difference between two consecutive numbers of the sequence is of the form 9010…0, which is also a multiple of 17 (note 901 = 17×53). With this information the reader should be able to fill the details of a full inductive argument.□

Problem 4.2. Show that for each $n \in \mathbf{N}$, $7^{2n} - 48n - 1$ is a multiple of 2304.

Problem 4.3. Show that for each $n \in \mathbf{N}$, $3 \cdot 5^{2n+1} + 2^{3n+1}$ is a multiple of 17.

Problem 4.4. Show that the sum of cubes of any three consecutive integers is divisible by 9.

Section 5. Inequalities.

We have seen an inequality, Bernoulli's inequality (Example 2.1), that depends on a natural number n. This particular one was proved using induction and, sure enough, many inequalities that depend on n can be dealt with by induction. For instance the following generalisation of Bernoulli's inequality can be shown by a minor modification of the proof given above.

Example 5.1 (Weierstrass inequality). If a_n ($n \in N$) are real numbers that are either all positive or all in [-1, 0] then

$$\prod_{k=1}^{n} (1 + a_k) \geq 1 + \sum_{k=1}^{n} a_k$$

Proof. As mentioned, the proof follows closely that of Bernoulli's inequality given above, and the details are left to the reader: For the inductive step then one only needs to multiply both sides by the positive number $(1 + a_{n+1})$, but some care must be taken when all a_n are in [-1, 0], in which case the term involving the summation sign is negative but $a_{n+1}\left(\sum_{k=1}^{n} a_k\right)$ positive. □

There are several inequalities in the text and in the problems of what follows, but here is a preliminary set.

Problem 5.1. Prove by induction that a) $2^n > n^2$ for $n \geq 5$, b) $2^n > n^3$ for $n \geq 10$.

Problem 5.2. Prove by induction that $2!4!\ldots(2n)! > [(n + 1)!]^n$ ($n \in \mathbf{N}$).

Problem 5.3. Prove that $(2n)!(n + 1) > 4^n(n!)^2$ for all n > 1.

Problem 5.4. Prove for all integers n > 1 the inequality

$$\frac{1}{\sqrt{1}}+\frac{1}{\sqrt{2}}+...+\frac{1}{\sqrt{n}}>2\sqrt{n+1}-2.$$

Problem 5.5. Prove that if a_k satisfies $0 < a_k < 1$ for $1 \le k \le n$, then

$$(1 - a_1)(1 - a_2)\ldots(1 - a_n) > 1 - (a_1 + a_2 + \ldots + a_n).$$

Problem 5.6. Prove that if a_k satisfies $0 \le a_k \le 1$ for $1 \le k \le n$, then

$$2^{n-1}(1 + a_1 a_2 \ldots a_n) \ge (1 + a_1)(1 + a_2)\ldots(1 + a_n).$$

Section 6. Variations of induction

Up to now the proof of a statement P(n) for all positive integers n, proceeded by verifying P(1) and then P(k +1) from the assumption that P(k) is true. There are variants of the inductive argument as will be shown in the following paragraphs.

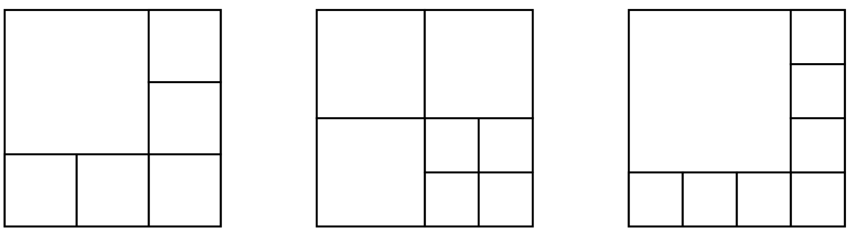
a) Jumps: In this method we prove the validity of a statement P(n) by proceeding, say, 2 steps at a time. In other words, the inductive step establishes the validity of P(k + 2) from the assumption that P(k) is true. If, in addition, we verify that P(1) and P(2) are true, then we reach our goal as we clearly have the implications P(1) $\Rightarrow$ P(3) $\Rightarrow$ P(5) $\Rightarrow$ P(7) $\Rightarrow$ … and P(2) $\Rightarrow$ P(4) $\Rightarrow$ P(6) $\Rightarrow$ P(8) $\Rightarrow$ … which, collectively, cover all cases of P(n). Similarly we may proceed in jumps of any fixed $t\in$**N.** This requires showing the validity of P(k) $\Rightarrow$ P(k + t) and of P(1), P(2), …, P(t).

Example 6.1. Show that each $n\in$**N**, the equation $a^2 + b^2 = c^n$ has a solution in positive integers.

Solution. We work using jumps of 2: The cases for n = 1 and 2 are clear. Now, if $a_1^2 + b_1^2 = c_1^k$ is a particular positive integer solution of the given equation for n = k, then a solution for the case n = k + 2 is obtained from $(c_1a_1)^2 + (c_1b_1)^2 = c_1^{k+2}$. □

Example 6.2. It is clear that a square can be divided into subsquares by drawing segments parallel to the sides. Show that it can be divided into n squares (of not necessarily equal size) whenever $n \ge 6$.

Solution. The figures below show how to divide the square into 6, 7 or 8 subsquares. Since a square can be further divided into four smaller ones, application of this operation increases the total number of squares in a subdivision by three (4 new ones and one lost). So we can use (how?) an inductive argument, jumping in 3's, to complete the proof. □



Note that the leaps need not be constant. Here is an example.

Example 6.3. Show that there exists an infinite number of triangular numbers that are perfect squares. (Recall, triangular numbers are the integers of the form $T_n = 1 + 2 + \ldots + n = \frac{1}{2} n(n + 1)$).

Solution. $T_1 = 1 = 1^2$. Suppose now that the triangular number T_k is a perfect square. Our problem is to utilize this information and find a bigger one that is also a perfect square. Somehow T_{k+1}, T_{k+2} etc. do not seem to work and we need to do better than that. A moment's reflection gives us a better choice: $T_{4k(k+1)} = 4k(k + 1)[4k(k + 1) + 1]/2 = 4(4k^2 + 4k + 1)\, T_k = 4(2k + 1)^2 T_k$ is clearly a perfect square along with T_k. □

Problem 6.1. Use an inductive argument in jumps of 2 to show that for all $n \in \mathbf{N}$,

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1}(1 + 2 + \ldots + n).$$

Problem 6.2. (Eötvös Competition 1901). Use an inductive argument in jumps of 4 to show that $1^n + 2^n + 3^n + 4^n$ is divisible by 5, if and only if n is not divisible by 4.

Problem 6.3. Use an inductive argument in jumps of 3 to show that no number of the form $2^n + 1$ is a multiple of 7.

Problem 6.4. After verifying the simple equations $\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{6^2}=1$, $\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}+\frac{1}{4^2}=1$ and $\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{7^2}+\frac{1}{14^2}+\frac{1}{21^2}=1$, show using an inductive argument with jumps of 3 that for every $n \geq 6$ there exist integers a_1, a_2, … , a_n such that $\dfrac{1}{a_1^2}+\dfrac{1}{a_2^2}+\ldots+\dfrac{1}{a_n^2}=1.$

Problem 6.5. (Erdös-Suranyi theorem). After verifying the simple equations $1 = 1^2$, $2 = -1^2 - 2^2 - 3^2 + 4^2$, $3 = - 1^2 + 2^2$ and $4 = - 1^2 - 2^2 + 3^2$ show that for each natural number N there is an n and an appropriate choice of + and - signs (which we write as $\pm$ in short) such that $N = \pm 1^2 \pm 2^2 \pm 3^2 \pm \ldots \pm n^2$.

b) Strong induction: In Euclid's *Elements* it is shown that every integer k > 1 is a product of (one or more) primes. His proof is essentially the following. The statement is clearly true for k = 2. Suppose now that we have proved that all integers *up to and including* k are products of primes. Then for k+1 we can argue that it is either a prime number, in which case we are done, or a product of two smaller numbers. But each of these two smaller numbers are, by assumption, products of primes and hence so is k+1. By iterating the argument we conclude the corresponding property for k+2, k+3, etc, and eventually for all integers.

In other words, Euclid's argument is a stronger version of induction where a) we verify P(1) and b) prove statement P(k+1) by assuming that *all* of P(1), P(2), … , P(k) are true (not just the last one P(k)). The inductive schema we invoke is then the validity of the implications P(1) $\Rightarrow$ [P(1) and P(2)] $\Rightarrow$ [P(1) and P(2) and P(3)] $\Rightarrow$ [P(1) and P(2) and P(3) and P(4)], and so on, finally covering all P(n).

A simpler version of strong induction is to prove P(k+1) from the validity of P(k-1) and P(k) (and not utilizing still smaller integers). In other words we first verify the validity of P(1), P(2) and then of the implication [P(k-1) and P(k)] $\Rightarrow$ [P(k+1). Note that here we proceed, essentially, by the steps [P(1) and P(2)] $\Rightarrow$ [P(2) and P(3)] $\Rightarrow$ [P(3) and P(4)], and so on.

Of course there are further variations, such as proving P(k+1) from the validity of P(k-2), P(k-1) and P(k), after verifying the statement for small n.

The following paradigms exemplify these ideas.

Example 6.4. A sequence (a_n) satisfies $a_1 = a_2 = 4$ and $a_{n+1}a_{n-1} = (a_n - 6)(a_n - 12)$ for n = 2, 3, ... Show that it is constant.

Solution. Of course we expect the constant to be 4, the common value of a_1 and a_2. We assume then that $a_{k-1} = a_k = 4$. Using now *both* these assumptions, we conclude from the recursion that $4a_{k+1} = (4 - 6)(4 - 12) = 16$, so that $a_{k+1} = 4$. Since by assumption $a_1 = a_2 = 4$, it is easily seen that, for all n, we have $a_n = 4$. □

Example 6.5. Recall that the natural numbers satisfy $\sum_{k=1}^{n} k^3 = (\sum_{k=1}^{n} k)^2$ for all $n\in\mathbf{N}$. Show that, conversely, if $a_n > 0$ is a sequence of real numbers such that $\sum_{k=1}^{n} a_k^3 = (\sum_{k=1}^{n} a_k)^2$ for all $n\in\mathbf{N}$, then $a_n = n$ ($n\in\mathbf{N}$).

Solution. The case n = 1 gives $a_1^3 = a_1^2$, so that $a_1 = 1$ (as $a_n > 0$). Assume now that *for all values of k up to m* we have $a_k = k$, in other words $a_1 = 1, \dots, a_m = m$. This is our strong inductive hypothesis and we are going to use every bit of it. For n = m + 1 we have, by assumption,

$$1^3 + 2^3 + \dots + m^3 + a_{m+1}^3 = (1 + 2 + \dots + m + a_{m+1})^2$$

$$= (1 + 2 + \dots + m)^2 + 2(1 + 2 + \dots + m)a_{m+1} + a_{m+1}^2$$

so clearly $a_{m+1}^3 = m(m + 1)a_{m+1} + a_{m+1}^2$ and so $a_{m+1}(a_{m+1} + m)(a_{m+1} - m - 1) = 0$, from which the claim follows. □

Problem 6.6. For the Fibonacci sequence, defined by $F_1 = F_2 = 1$, $F_{n+2} = F_{n+1} + F_n$, show that a) $F_nF_{n+1} - F_{n-2}F_{n-1} = F_{2n-1}$, b) $F_{n+1}F_{n+2} - F_nF_{n+3} = (-1)^n$.

Problem 6.7. Let (a_n) be the sequence of Example 6.4 with the only difference that $a_1 = 2$ and $a_2 = 20$. Show that $a_n = 9n^2 - 9n + 2$ ($n\in\mathbf{N}$). If, instead, we had $a_1 = 2$ and $a_2 = 5$, show $a_n = 4 + (-\frac{1}{2})^{n-2}$.

Problem 6.8. Given an angle a, define x by x + 1/x = 2cosa. Show that $x^n + 1/x^n = 2\cos na$ ($n\in\mathbf{N}$).

Problem 6.9. Show that if a, b satisfy a + b = 6 and ab = 1, then the number $a^n + b^n$ a) is always an integer and b) is never divisible by 5.

Problem 6.10. Let $a_n = (3+\sqrt{5})^n + (3-\sqrt{5})^n$. Show that a_n is an integer and that $2^n | a_n$.

Problem 6.11. Prove the statement of Pascal in his *Traité du Triangle Arithmetique* referred to in Section 1 above.

Problem 6.12. Let $a_1, a_2, a_3, \dots$ be positive integers chosen such that $a_1 = 1$ and $a_n < a_{n+1} \le 2a_n$ ($n\in\mathbf{N}$). Show that every positive integer can be written as a sum of distinct a_n's.

Problem 6.13. Let a sequence (b_n) satisfy $b_1 = 1$, $b_{2n} = b_n$ and $b_{2n+1} = b_{2n} + 1$. Show that b_n equals the number of ones in the binary representation of n.

c) Double induction: There are cases where the proof of the inductive step requires, in its own right, an inductive argument. The following examples illustrate this point.

Example 6.6. Show that for each $n\in\mathbf{N}$, $2\cdot 7^n + 3\cdot 5^n - 5$ is a multiple of 24.

Solution. Writing $a_n = 2\cdot 7^n + 3\cdot 5^n - 5$, the claim is clear for n = 1. Assuming it true for n = k then as $a_{k+1} = 7\cdot a_k - 6\cdot 5^k + 30$, the inductive argument would be complete if we could prove that $6\cdot 5^k - 30$ is always a multiple of 24. We can now start a new inductive argument to prove the last statement, an easy task left to the reader. □

d) Two dimensional induction: So far we have considered statements P(n) depending on a single integer n. But sometimes we meet statements depending on two (or more) integers. A useful inductive way to deal with such a statement, for simplicity call it P(m,n), is to proceed in stages, intermingling the m's and n's. For instance, we can prove the validity of a) P(1,1), then b) of P(2,1) and P(1,2), then c) of P(3,1), P(2,2) and P(1,3), and so on. This particular description moves, so to speak, 'diagonally'. Any other way which covers all (m,n) in stages, is just as acceptable.

Example 6.7. (IMO 1972). Prove that (2m)!(2n)! is a multiple of m!n!(m+n)! for any non-negative integers m and n.

Solution. We are to show that C(m, n) = (2m)!(2n)!/(m!n!(m+n)!), for m, n $\geq$ 0, is integral. This is certainly the case for C(m, 0) = (2m)!/(m!m!) (we leave this to the reader: one way to see it is to recognize it as a binomial coefficient). Finally it is easy to verify that C(m, n) = 4C(m, n – 1) - C(m + 1, n – 1), from which, using the previous, we can in turn verify that C(m,1) is integral for all m, then C(m, 2) for all m, C(m, 3) for all m, and so on. □

Problem 6.14. Prove inductively that the product of r consecutive integers is divisible by r!

Problem 6.15. If (F_n) denotes the Fibonacci sequence, prove that $F_n^2 + F_{n+1}^2 = F_{2n+1}$ and $2F_nF_{n+1} + F_{n+1}^2 = F_{2n+2}$. (Hint: Let P(n) be the first identity and Q(n) the second. Induction proceeds via P(1) $\Rightarrow$ Q(1) $\Rightarrow$ P(2) $\Rightarrow$ Q(2) $\Rightarrow$ P(3) $\Rightarrow$ ...).

e) Back and forth: This variant of the usual inductive procedure is in two steps. First one shows $P(1) \Rightarrow P(n_1) \Rightarrow P(n_2) \Rightarrow P(n_3) \Rightarrow \dots$ for some chosen but fixed sequence $1 < n_1 < n_2 < n_3 < \dots$. Then shows the *backward* step P(k) $\Rightarrow$ P(k-1). A moment's reflection shows that the backward step fills the gaps between the numbers 1, n_1 , n_2 , n_3 , … left unattended in the first step, completing the proof. Here is an example of such a proof of the AM-GM inequality. The first step uses the sequence $1 < 2 < 2^2 < 2^3 < \dots$.

Example 6.8. Show that for any sequence (a_n) of positive numbers we have

$$\left(\frac{a_1 + a_2 + \ldots + a_n}{n}\right)^n \geq a_1 a_2 \cdot \ldots \cdot a_n \qquad (n \in \mathbf{N})$$

Solution. The case for n = 1 is trivial. Assuming validity of P(k) for all sequences (a_n) of positive numbers, verification of P(2k) is as follows: Apply P(k) to the sequence $((a_{2n-1} + a_{2n})/2)$. We get

$$\left(\frac{\frac{a_1+a_2}{2}+\frac{a_3+a_4}{2}+\ldots+\frac{a_{2k-1}+a_{2k}}{2}}{k}\right)^k \geq \frac{a_1+a_2}{2}\cdot\frac{a_3+a_4}{2}\cdot\ldots\cdot\frac{a_{2k-1}+a_{2k}}{2}$$

$$\geq \sqrt{a_1a_2}\cdot\sqrt{a_3a_4}\cdot\ldots\cdot\sqrt{a_{2k-1}a_{2k}}$$

which is easily rewritten as P(2k), namely

$$\left(\frac{a_1+a_2+\ldots+a_{2k}}{2k}\right)^{2k} \geq a_1a_2\cdot\ldots\cdot a_{2k}.$$

So we now know the inequality for the cases P(1), P(2), P(2^2), P(2^3), … .

For the backward step, we assume P(k) and derive P(k-1). For this purpose, by P(k) applied to the k numbers a_1, a_2, …, a_{k-1} and (a_1+ a_2 + …+ a_{k-1})/(k-1) , we have

$$\left(\frac{a_1+a_2+\ldots+a_{k-1}+\frac{a_1+a_2+\ldots+a_{k-1}}{k-1}}{k}\right)^k \geq a_1a_2\cdot\ldots\cdot a_{k-1}\cdot\frac{a_1+a_2+\ldots+a_{k-1}}{k-1}$$

which is easily seen to reduce to statement P(k-1) after collecting terms. □

Problem 6.16. Redo the step P(k) ⇒ P(2k) in the proof of Example 6.7 by making use of the equality $\sqrt[2k]{a_1a_2\cdot\ldots\cdot a_{2k}} = \sqrt{\sqrt[k]{a_1a_2\cdot\ldots\cdot a_k}\cdot\sqrt[k]{a_{k+1}a_{k+2}\cdot\ldots\cdot a_{2k}}}$, thereby giving a slightly different proof of the AM-GM inequality.

Problem 6.17. (Jensen's inequality). If f : I → **R**, where I ⊆ **R** is an interval, is a concave function, then $f\left(\frac{a_1+a_2+\ldots+a_n}{n}\right) \geq \frac{f(a_1)+f(a_2)+\ldots+f(a_n)}{n}$ for all a_1, a_2, …, a_n in I. The reverse inequality is true for convex functions. (Recall, concave functions satisfy $f\left(\frac{x+y}{2}\right) \geq \frac{f(x)+f(y)}{2}$ and convex ones the reverse inequality).

f) Strengthening: The following example illustrates this curious technique, which will be explained immediately after.

Example 6.9. Show that for n ≥ 2, $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq \frac{3}{4}$.

Solution. Here the inductive step does not work, so will modify our approach. We show instead the *stronger* inequality P(n): $\frac{1}{2^2}+\frac{1}{3^2}+\ldots+\frac{1}{n^2} \leq \frac{3}{4}-\frac{1}{n}$.

The case n = 2 is immediate, and for the inductive step we can clearly argue along the lines $\frac{1}{2^2}+\frac{1}{3^2}+...+\frac{1}{k^2}+\frac{1}{(k+1)^2}\le\frac{3}{4}-\frac{1}{k}+\frac{1}{(k+1)^2}\le\frac{3}{4}-\frac{1}{k+1}$, which completes the proof. □

The curiosity is that although we failed to prove a statement, we managed to prove a stronger one! The mystery clarifies if we realize that proof of the inductive step the second time was based on a *stronger hypothesis*. So it is not surprising that the conclusion was also stronger. In the failed attempt, the inductive hypothesis was too weak to prove the full statement.

Problem 6.18. Prove the inequality $\frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot...\cdot\frac{(2n-1)^2}{(2n)^2}<\frac{1}{3n}$.

Problem 6.19. Prove the inequality $\frac{1}{2\sqrt{1}}+\frac{1}{3\sqrt{2}}+...+\frac{1}{(n+1)\sqrt{n}}<2$.

Problem 6.20. Show that $(1+\frac{1}{2^3})(1+\frac{1}{3^3})\cdot...\cdot(1+\frac{1}{n^3})\le 3$.

The technique of strengthening has so far been used only to prove inequalities. One should not draw the conclusion that this is the only place that it can be used. Here is an example.

Example 6.10. Show that for every n there exist n distinct divisors of n! whose sum is n!

Solution. Before enunciating which exactly is the strengthened statement, let us attempt an inductive proof: The case n = 1 is clear. Assuming that there are k distinct divisors d_1, d_2, …, d_k of k! whose sum is k!, we seek k + 1 distinct divisors of (k + 1)! whose sum is (k + 1)!. Consider $(k + 1)d_1$, $(k + 1)d_2$, … , $(k + 1)d_k$. They are divisors of (k + 1)!, they are distinct, they sum up to (k + 1)! but the problem is that they are only k of them. If we replace $(k + 1)d_1$ by kd_1 and d_1, we have k + 1 numbers but now one of them, namely kd_1, may not be a divisor of (k + 1)!. There is a way out of this difficulty, and this is by taking $d_1 = 1$, but are we allowed to do this? The answer is yes if we start all over again, but this time we strengthen our original statement to showing that "for every n there exist n distinct divisors of n! whose sum is n! *and such that one of the divisors is 1*". The procedure is now clear and the details are left to the reader. □

Section 7. Subtleties

At the beginning of this chapter we talked about the versatility of induction as a proving device. With the examples below we will see more clearly the diversity and adaptability of the versatile tool we are discussing. Here the application of the inductive hypothesis will be slightly more intricate.

Before coming to the first example, recall that so far the variable n to which we applied an inductive argument was pretty clear from the premises of the problem. There are, however, some interesting cases where the choice of the variable with which we chose to work, is rather subtle.

Example 7.1. Show that for any set $\{a_1, a_2,\ldots, a_n\}$ of nonnegative integers, the expression $X = \frac{(a_1 + a_2 + \ldots + a_n)!}{a_1!a_2!\ldots a_n!}$ is an integer.

Solution. Induction will not be on n but rather on the number $N = a_1 + a_2 +\ldots+ a_n$. If N = 1, in which case (without loss of generality) $a_1 = 1$, $a_2 = \ldots = a_n = 0$, the result is trivial. Suppose now that for some $k \geq 1$, X is an integer whenever the sum $a_1 + a_2 +\ldots+ a_n = k$. We show the same thing for n nonnegative integers whose sum is k + 1. Note that we may assume that $a_j \geq 1$ for all $1 \leq j \leq n$ (if some $a_j = 0$, it gives no contribution in X, so we may delete it).

Let then $a_1 + a_2 +\ldots+ a_n = k + 1$. By the inductive hypothesis applied to the numbers $a_1 - 1, a_2, \ldots , a_n$, we have that

$$\frac{a_1X}{a_1 + a_2 + \ldots + a_n} = \frac{(a_1 - 1 + a_2 + \ldots + a_n)!}{(a_1 - 1)!a_2!\ldots a_n!}$$

is an integer. Similarly $a_2X/(a_1 + a_2 +\ldots+ a_n)$, … , $a_nX/(a_1 + a_2 +\ldots+ a_n)$ are also integers, and hence so are their sum

$$X = \sum_{j=1}^{n} \frac{a_jX}{a_1 + a_2 + \ldots + a_n}. \square$$

Problem 7.1. Give another proof of Example 7.1 using the identity $\frac{(a + b + \ldots + c)!}{a!b!\ldots c!} = \frac{((a + b) + \ldots + c)!}{(a + b)!\ldots \cdot c!} \cdot \frac{(a + b)!}{a!b!}$.

In the next two examples we apply our inductive hypothesis in a more dexterous way.

Example 7.2. Let A be any subset of {1, 2, 3, ... , 2n - 1} with n elements, where $n\in\mathbf{N}$. Show that there are elements x and y of A (not necessarily distinct) with x + y = 2n.

Solution. For n = 1 the result is clear. Assume the conclusion true for n = k and consider now a subset A of {1, 2, 3, ... , 2k + 1} with k + 1 elements. We are to show that there exist x and y in A with x + y = 2(k + 1). If 1 and 2k + 1 are both in A, we are done, so we may assume that at least one of the two is missing. Delete from A the other. What remains is a set A' of at least k elements such that $A' \subseteq \{2, 3, \ldots , 2k\}$. Subtract 1 from each element of A', to get a subset of {1, 2, 3, ... , 2k -1} with (at least) k elements. *We apply our inductive hypothesis to this last set*: Thus there are x and y in A such that (x - 1) + (y - 1) = 2k, and so x + y = 2(k + 1). □

Problem 7.2. (Hermite's identity) If n is a positive integer and x a real number, prove that

$$[x] + [x + \frac{1}{n}] + [x + \frac{2}{n}] + \ldots + [x + \frac{n-1}{n}] = [nx],$$

where [.] denotes 'integer part'. (Hint: induction is not on n but rather on the unique $k\in N$ with $k/n \leq x < (k+1)/n$).

Problem 7.3. There are n fuelling stations on a circular track and the total gas among them is just enough for a car to complete the circuit. Show that there is a fuelling station from which the car can start and manage to complete the circuit. The car is allowed to use only the gas provided at the fuelling stations, which it can collect only as it goes along.

Section 8. Harder Questions

Problem 8.1. If x is a real number not of the form n + ½ for an integer n, let {x} denote the nearest integer to x (so that for example {e} = {π} = 3). Show $\sum_{k=1}^{n^2+n}\{\sqrt{k}\} = 2\sum_{k=1}^{n}k^2$.

Problem 8.2. Let n be an integer. Consider all points (a,b) of the plane with integer co-ordinates such that 0 ≤ a, 0 ≤ b, a + b ≤ n. Show that if these points are covered by straight lines then there are at least n + 1 such lines.

Problem 8.3. Given a positive integer N perform the following operation to obtain a new integer s(N): First write N in its decimal form as N = $\overline{a_n a_{n-1} \ldots a_0}$ and then set s(N) = $\sum a_k^2$. Show that repeated application of this operation will eventually lead to the number 1 or to the cycle 4, 16, 37, 58, 89, 145, 42, 20. (Remark: One can check by hand the validity of the claim for all three figure numbers, a fact which you may take for granted. Induction on N starts thereafter.)

Problem 8.4. Show that every member of the sequence defined by $a_1 = a_2 = a_3 = 1$ and

$a_{n+3} = (1 + a_{n+1}a_{n+2})/a_n$ (n ≥ 1) is an integer.

Problem 8.5. If m and n are positive integers, show that so is $\frac{(mn)!}{m!(n!)^m}$.

Problem 8.6 (Chebychev inequality). Let $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$ and $b_1 \geq b_2 \geq \ldots \geq b_n \geq 0$. Show that $\frac{1}{n}(\sum_{k=1}^{n} a_k).\frac{1}{n}(\sum_{k=1}^{n} b_k) \leq \frac{1}{n}(\sum_{k=1}^{n} a_k b_k)$. What is the corresponding inequality if $0 \leq a_1 \leq a_2 \leq \ldots \leq a_n$ and $0 \leq b_1 \leq b_2 \leq \ldots \leq b_n$?

Problem 8.7. (Putnam 1968, slightly differently worded). Let S be a set of n elements and let P be the set of all subsets of S. Show that we can label the elements of P as $A_1, A_2, \ldots, A_{2^n}$ so that $A_1 = \emptyset$ and such that any two consecutive sets in this labeling differ by exactly one element of S.

Problem 8.8. (Putnam 1956, slightly differently worded). Given any 2n points (n ≥ 2) that are joined by $n^2 + 1$ segments, show that at least one triangle is formed from these segments.

Problem 8.9. Let A be a subset of {1, 2, … , 2n} with n + 1 elements. Show inductively that there exist x, y in A such that x divides y.

Problem 8.10. Show that for any n > 1 there exists a finite set A_n of points on the plane such that for any $x \in A_n$ there are points $x_1, x_2, \ldots, x_n$ in A_n each of which is at a distance 1 from x.

Problem 8.11. (Adapted from IMO 1997). Show that there exist infinitely many values of n for which we can find an n×n matrix whose entries come from the set S = {1, 2, ... , 2n-1} and, for each k = 1, 2, ... , n, its k^{th} row and k^{th} column together contain all elements of S.