Geometry & Topology Monographs
Volume X: Volume name goes here
Pages 1–XXX

The obstruction to finding a boundary for an open manifold of dimension greater than five

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Abstract For dimensions greater than five the main theorem gives necessary and sufficient conditions that a smooth open manifold W be the interior of a smooth compact manifold with boundary.

The basic necessary condition is that each end ϵ of W be *tame*. Tameness consists of two parts (a) and (b):

(a) The system of fundamental groups of connected open neighborhoods of ϵ is *stable*. This means that (with any base points and connecting paths) there exists a cofinal sequence

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$$

so that isomorphisms are induced

$$\mathrm{Image}(f_1) \xleftarrow{\cong} \mathrm{Image}(f_2) \xleftarrow{\cong} \cdots .$$

(b) There exist arbitrarily small open neighborhoods of ϵ that are dominated each by a finite complex.

Tameness for ϵ clearly depends only on the topology of W. It is shown that if W is connected and of dimension $\geqslant 5$, its ends are all tame if and only if $W \times S^1$ is the interior of a smooth compact manifold. However examples of smooth open manifolds W are constructed in each dimension $\geqslant 5$ so that W itself is not the interior of a smooth compact manifold although $W \times S^1$ is.

When (a) holds for ϵ, the projective class group

$$\widetilde{K_0}(\pi_1(\epsilon)) = \varprojlim_j G_j$$

is well defined up to canonical isomorphism. When ϵ is tame an invariant $\sigma(\epsilon) \in \widetilde{K_0}(\pi_1\epsilon)$ is defined using the smoothness structure as well as the topology of W. It is closely related to Wall's obstruction to finiteness for CW complexes (Annals of Math. 81 (1965) pp. 56-69).

Main Theorem. A smooth open manifold W^n, $n > 5$, is the interior of a smooth compact manifold if and only if W has finitely many connected components, and each end ϵ of W is tame with invariant $\sigma(\epsilon) = 0$. (This generalizes a theorem of Browder, Levine, and Livesay, A.M.S. Notices 12, Jan. 1965, 619-205).

For the study of $\sigma(\epsilon)$, a sum theorem and a product theorem are established for C.T.C. Wall's related obstruction.

Analysis of the different ways to fit a boundary onto W shows that there exist smooth contractible open subsets W of $\mathbb{R}^n$, n odd, $n > 5$, and diffeomorphisms of W onto itself that are smoothly pseudo-isotopic but not smoothly isotopic.

The main theorem can be relativized. A useful consequence is

Proposition. Suppose W is a smooth open manifold of dimension > 5 and N is a smoothly and properly imbedded submanifold of codimension $k \neq 2$. Suppose that W and N separately admit completions. If $k = 1$ suppose N is 1-connected at each end. Then there exists a compact manifold pair $(\overline{W}, \overline{N})$ such that $W = \operatorname{Int} \overline{W}$, $N = \operatorname{Int} \overline{N}$.

If W^n is a smooth open manifold homeomorphic to $M \times (0,1)$ where M is a closed connected topological $(n-1)$-manifold, then W has two ends ϵ_- and ϵ_+, both tame. With $\pi_1(\epsilon_-)$ and $\pi_1(\epsilon_+)$ identified with $\pi_1(W)$ there is a duality $\sigma(\epsilon_+) = (-1)^{n-1}\overline{\sigma(\epsilon_-)}$ where the bar denotes a certain involution of the projective class group $\widetilde{K_0}(\pi_1 W)$ analogous to one defined by J.W. Milnor for Whitehead groups. Here are two corollaries. If M^m is a stably smoothable closed topological manifold, the obstruction $\sigma(M)$ to M having the homotopy type of a finite complex has the symmetry $\sigma(M) = (-1)^m \overline{\sigma(M)}$. If ϵ is a tame end of an open topological manifold W^n and ϵ_1, ϵ_2 are the corresponding smooth ends for two smoothings of W, then the difference $\sigma(\epsilon_1) - \sigma(\epsilon_2) = \sigma_0$ satisfies $\sigma_0 = (-1)^n \overline{\sigma}_0$. Warning: In case every compact topological manifold has the homotopy type of a finite complex all three duality statements above are $0 = 0$.

It is widely believed that all the handlebody techniques used in this thesis have counterparts for piecewise-linear manifolds. Granting this, all the above results can be restated for piecewise-linear manifolds with one slight exception. For the proposition on pairs (W, N) one must insist that N be locally unknotted in W in case it has codimension one.

AMS Classification

Keywords

December 7, 2006

Introduction

The starting point for this thesis is a problem broached by W. Browder, J. Levine, and G. R. Livesay in [1]. They characterize those smooth open manifolds W^w, $w > 5$ that form the interior of some smooth compact manifold $\overline{W}$ with a simply connected boundary. Of course, manifolds are to be Hausdorff and paracompact. Beyond this, the conditions are

(A) There exists arbitrarily large compact sets in W with 1–connected complement.

(B) $H_*(W)$ is finitely generated as an abelian group.

I extend this characterization and give conditions that W be the interior of *any* smooth compact manifold. For the purposes of this introduction let W^w be a connected smooth open manifold, that has one end – i.e. such that the complement of any compact set has exactly one unbounded component. This end – call it ε – may be identified with the collection of neighborhoods of ∞ of W. ε is said to be *tame* if it satisfies two conditions analogous to (A) and (B):

(a) π_1 is *stable* at ε.

(b) There exists arbitrarily small neighborhoods of ε, each dominated by a finite complex.

When ε is tame an invariant $\sigma(\varepsilon)$ is defined, and for this definition, no restriction on the dimension w of W is required. The main theorem states that if $w > 5$, the necessary and sufficient conditions that W be the interior of a smooth compact manifold are that ε be tame and the invariant $\sigma(\varepsilon)$ be zero. Examples are constructed in each dimension $\geqslant 5$ where ε is tame but $\sigma(\varepsilon) \neq 0$. For dimensions $\geqslant 5$, ε is tame if and only if $W \times S^1$ is the interior of a smooth compact manifold.

The stability of π_1 at ε can be tested by examining the fundamental group system for any convenient sequence $Y_1 \supset Y_2 \supset \dots$ of open connected neighborhoods of ε with

$$\bigcap_i \text{closure}(Y_i) = \emptyset.$$

If π_1 is stable at ε, $\pi_1(\varepsilon) = \varprojlim \pi_1(Y_i)$ is well defined up to isomorphism in a preferred conjugacy class.

Condition (b) can be tested as follows. Let V be any closed connected neighborhood of ε which is a topological manifold (with boundary) and is small enough so that $\pi_1(\varepsilon)$ is a retract of $\pi_1(V)$ – i.e. so that the natural homomorphism

$\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ has a left inverse. (Stability of π_1 at ε guarantees that such a neighborhood exists.) It turns out that condition (b) holds if and only if V is dominated by a finite complex. No condition on the homotopy type of W can replace (b), for there exist contractible W such that (a) holds and $\pi_1(\varepsilon)$ is even finitely presented, but ε is, in spite of this, not tame. On the other hand, tameness clearly depends only on the topology of W.

The invariant $\sigma(\varepsilon)$ of a tame end ε is an element of the group $\widetilde{K_0}(\pi_1\varepsilon)$ of stable isomorphism classes of finitely generated projective modules over $\pi_1(\varepsilon)$. If, in testing (b) one chooses the neighborhood V of ε (above) to be a smooth submanifold, then

$$\sigma(\varepsilon) = r_*\sigma(V)$$

where $\sigma(V) \in \widetilde{K_0}(\pi_1 V)$ is up to sign C.T.C. Wall's obstruction [2] to V having the homotopy type of a finite complex, and r_* is induced by a retraction of $\pi_1(V)$ onto $\pi_1(\varepsilon)$. Note that $\sigma(\varepsilon)$ seems to depend on the smoothness structure of W. For example, every tame end of dimension at least 5 has arbitrarily small open neighborhoods each homotopy equivalent to a finite complex (use Theorem 8.6 and Theorem 6.5). The discussion of tameness and the definition for $\sigma(\varepsilon)$ is scattered in various sections. The main references are: Definitions 3.6, 4.2, Proposition 4.3, Definition 4.4, Lemma 6.1, Definition 7.7, §10, Corollary 11.6.

The proof of the main theorem applies the theory of non-simply connected handles bodies as expounded in Barden [31] and Wall [3] to find a collar for ε, viz. a closed neighborhood V which is a smooth submanifold diffeomorphic with $\mathrm{Bd}V \times [0,1)$. In dimension 5, the proof breaks down only because Whitney's famous device fails to untangle 2–spheres in 4–manifolds (c.f. §5). In dimension 2, tameness alone ensures that a collar exists (see Kerekjarto [26, p. 171]). It seems possible that the same is true in dimension 3 (modulo the Poincaré Conjecture) – c.f. Wall [30]. Dimension 4 is a complete mystery.

There is a striking parallelism between the theory of tame ends developed here and the well known theory of h–cobordisms. For example the main theorem corresponds to the s–cobordism theorem of B. Mazur [34][3]. The relationship can be explained thus. For a tame end ε of dimension $\geqslant 6$ the invariant $\sigma(\varepsilon) \in \widetilde{K_0}(\pi_1\varepsilon)$ is the obstruction to finding a collar. When a collar exists, parallel families of collars are classified relative to a fixed collar by torsions $\tau \in \mathrm{Wh}(\pi_1\varepsilon)$ of certain h–cobordisms (c.f. Theorem 9.5). Roughly stated, σ is the obstruction to capping ε with a boundary and τ then classifies the different ways of fitting a boundary on. Since $\mathrm{Wh}(\pi_1\varepsilon)$ is a quotient of $K_1(\pi_1\varepsilon)$ [17], the situation is very reminiscent of classical obstruction theory.

A closer analysis of the ways of fitting a boundary onto an open manifold gives the first counterexample of any kind to the conjecture that pseudo-isotopy of diffeomorphisms implies (smooth) isotopy. Unfortunately open (rather than closed) manifolds are involved.

§6 and §7 give sum and product theorems for Wall's obstruction to finiteness for CW complexes. Here are two simple consequences for a smooth open manifold W with one end ε. If ε is tame, then Wall's obstruction $\sigma(W)$ is defined and $\sigma(W) = i_*\sigma(\varepsilon)$ where $i\colon \pi_1(\varepsilon) \longrightarrow \pi_1(W)$ is the natural map. If N is any closed smooth manifold then the end $\varepsilon \times N$ of $W \times N$ is tame if and only if ε is tame. When they are tame

$$\sigma(\varepsilon \times N) = \chi(N) j_* \sigma(\varepsilon)$$

where j is the natural inclusion $\pi_1(\varepsilon) \longrightarrow \pi_1(\varepsilon \times N) = \pi_1(\varepsilon) \times \pi_1(N)$ and $\chi(N)$ is the Euler characteristic of N. Then, if $\chi(N) = 0$ and $W \times N$ has dimension > 5, the main theorem says that $W \times N$ is the interior of a smooth compact manifold.

The sum and product theorems for Wall's obstruction mentioned above have counterparts for Whitehead torsion (§§6,7; [19]). Likewise the relativized theorem in §10 and the duality theorem in §11 have counterparts in the theory of h–cobordisms. Professor Milnor has pointed out that examples exist where the standard duality involution on $\widetilde{K_0}(\pi)$ is not the identity. In contrast no such example has been given for $\mathrm{Wh}(\pi)$. The examples are for $\pi = \mathbb{Z}_{229}$ and $\mathbb{Z}_{257}$; they stem from the remarkable research of E. E. Kummer. (See appendix.)

It is my impression that the P.L. (= piecewise-linear) version of the main result is valid. This opinion is based on the general consensus that handlebody theory works for P.L. manifolds. J. Stallings seems to have worked out the details for the s–cobordism theorem in 1962–93. B. Mazur's paper [35] (to appear) may be helpful. The theory should be formally the same as Wall's exposition [3] with P.L justifications for the individual steps.

For the same reason it should be possible to translate for the P.L. category virtually all other theorems on manifolds given in this thesis. However the theorems for pairs Theorem 10.3–Theorem 10.10 must be re-examined since tubular neighborhoods are used in the proofs and M. Hirsch has recently shown that tubular neighborhoods do not generally exist in the P.L. category. For Theorem 10.3 in codimension $\geqslant 3$ it seems that a more complicated argument employing only regular neighborhoods does succeed. It makes use of Hudson and Zeeman [36, Cor. 1.4, p. 73]. It also succeeds in codimension 1 if one assumes that the given P.L. imbedding $N^{n-1} \longrightarrow W^n$ is locally unknotted

[36, p. 72]. I do not know if Theorem 10.6 holds in the P.L. category. Thus Theorem 10.8 and Corollary 10.9 are undecided. But it seems Theorem 10.7 and Theorem 10.10 can be salvaged.

Professor J. W. Milnor mentioned to me, in November 1964, certain grounds for believing that an obstruction to finding a boundary should lie in $\widetilde{K}_0(\pi_1\varepsilon)$. The suggestion was fruitful. He has contributed materially to miscellaneous algebraic questions. The appendix, for example, is his own idea. I wish to express my deep gratitude for all this and for the numerous interesting and helpful questions he has raised while supervising this thesis.

I have had several helpful conversations with Professor William Browder, who was perhaps the first to attack the problem of finding a boundary [51]. I thank him and also Jon Sondow who suggested that the main theorem (relativized) could be applied to manifold pairs. I am grateful to Dr. Charles Giffen for his assistance in preparing the manuscript.

Contents

Some Notation

$\cong$	diffeomorphism.
$\simeq$	homotopy equivalence.
$X \hookrightarrow Y$	inclusion map of X into Y.
$\mathrm{b}S$	frontier of the subspace S.
$\mathrm{Bd}\, M$	boundary of a manifold M.
$M(f)$	mapping cylinder of f.
$\widetilde{X}$	universal covering of X.
χ	Euler characteristic.
$\mathcal{D}$	the class of Hausdorff spaces with homotopy type of a CW complex and dominated by a finite CW complex.
$f.g.$	finitely generated.
$K_0(\pi)$	Grothendieck group of finitely generated projective modules over the group π.
$\widetilde{K_0}(\pi)$	the group of stable isomorphism classes of finitely generated projective modules over the group π.
$[P]$	the stable isomorphism class of a module $[P]$.

- *topological manifold*: Hausdorff and paracompact topological manifold.
- *proper map*: continuous map such that the preimage of every compact set is compact.
- *nice Morse function*: Morse function such that the value of critical points is an increasing function of the index.

1 Ends in general

The interval $(0,1)$ has two (open) ends while $[0,1)$ has one. We must make this idea precise. Following Freudenthal [5] we define the ends of an arbitrary Hausdorff space X in terms of open sets having compact frontier. Consider collections ε of subsets of X so that:

(i) Each $G \in \varepsilon$ is a connected open non-empty set with compact frontier $\mathrm{b}G = \overline{G} - G$;

(ii) If $G, G' \in \varepsilon$, there exists $G'' \in \varepsilon$ with $G'' \subset G \cap G'$;

(iii)
$$\bigcap \left\{ \overline{G} \ : \ G \in \varepsilon \right\} = \emptyset.$$

Adding to ε every open connected non-empty set $H \subset X$ with $\mathrm{b}H$ compact such that $G \subset H$ for some $G \in \varepsilon$, we produce a collection ε' satisfying (i), (ii), (iii), which we call *the end of X determined by ε*.

Lemma 1.1 *With ε as above, let H be any set with compact frontier. Then there exists $G \in \varepsilon$ so small that either $\overline{G} \subset H$ or $\overline{G} \cap H = \emptyset$.*

Proof Since $\mathrm{b}H$ is compact, there exists $G \in \varepsilon$ so small that $\overline{G} \cap \mathrm{b}H = \emptyset$ (by (ii) and (iii)). Since G is connected, $\overline{G} \subset H$ or else $\overline{G} \cap H = \emptyset$ as asserted. □

It follows that if $\varepsilon_1 \supset \varepsilon$ also satisfies (i), (ii), (iii), then every member H of ε_1 contains a member G of ε, i.e. $H \in \varepsilon'$. For, the alternative $\overline{G} \cap H = \emptyset$ in Lemma 1.1 is here ruled out. Thus $\varepsilon' \supset \varepsilon_1 \supset \varepsilon$, and so we can make the more direct.

Definition 1.2 An *end* of a Hausdorff space X is a collection ε of subsets of X which is maximal with respect to the properties (i), (ii), (iii) above.

From this point ε will always denote an end.

Definition 1.3 A *neighborhood* of an end ε is any set $N \subset X$ that contains some member of ε.

As the neighborhoods of ε are closed under intersection and infinite union, the definition is justified. Suppose in fact we add to X an ideal point $\omega(\varepsilon)$ for each end ε and let $\left\{ \widehat{G} \ : \ G \in \varepsilon \right\}$ be a basis of neighborhoods of $\omega(\varepsilon)$, where $\widehat{G} = G \cup \{\omega(\varepsilon') \ : \ G \in \varepsilon'\}$. Then a topological space $\widehat{X}$ results. It is Hausdorff because

Lemma 1.4 *Distinct ends ε_1, ε_2 of X have distinct neighborhoods.*

Proof If $G_1 \in \varepsilon_1$ by Lemma 1.1, for all sufficiently small $G_2 \in \varepsilon_2$, either $G_1 \supset \overline{G_2}$ or $G_1 \cap \overline{G_2} = \emptyset$. The first alternative does not always hold since that would imply $\varepsilon_1 \supset \varepsilon_2$, hence $\varepsilon_1 = \varepsilon_2$. □

Observation 1 If N is a neighborhood of an end ε of X, $\overline{G} \subset N$ for sufficiently small $G \in \varepsilon$ (by Lemma 1.1). Thus ε determines a unique end of N.

Observation 2 If $Y \subset X$ is closed with compact frontier $\mathrm{b}Y$ and ε' is an end of Y, then ε' determines an end of X. Further Y is a neighborhood of ε and ε determines an end ε' of Y as in Observation (1).

(Explicitly, if $G \in \varepsilon'$ is sufficiently small, the closure of G (in Y) does not meet the compact set $\mathrm{b}Y$. Then as a subset of X, G is non-empty, open and connected with $\mathrm{b}G$ compact. Such $G \in \varepsilon'$ determine the end ε of X.)

Definition 1.5 An end of X is *isolated* if it has a member H that belongs to no other end.

From the above observation it follows that $\overline{H}$ has one and only one end.

Example 1.1 The universal cover of the figure 8 has $2^{\aleph_0}$ ends, none isolated.

Observe that a compact Hausdorff space X has no ends. For, as $\bigcap \{\overline{G} \ : \ G \in \varepsilon\} = \emptyset$, we could find $G \in \varepsilon$ so small that $\overline{G} \cap X = \emptyset$ which contradicts $\emptyset \neq G \subset X$. Even a noncompact connected Hausdorff space X may have no ends, for example an infinite collection of copies of $[0, 1]$ with all initial points identified.

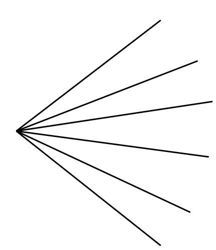


Figure 1:

However according to Theorem 1.6 below, every noncompact connected manifold (separable, topological) has at least one end. For example $\mathbb{R}$ has two ends and $\mathbb{R}^n$, $n \geqslant 2$, has one end. Also a compact manifold *minus* k connected boundary components $N_1, \ldots, N_k$ has exactly k ends $\varepsilon(N_1), \ldots, \varepsilon(N_k)$. The neighborhoods of $\varepsilon(N_i)$ are the sets

$$U - \bigcup_{j=1}^{k} N_j$$

where U is a neighborhood of N_i in M.

Theorem 1.6 (Freudenthal [5]) *A non-compact but σ–compact, connected Hausdorff space X that is locally compact and locally connected has at least one end.*

Remark Notice that the above example satisfies all conditions except local compactness.

Proof By a familiar argument one can produce a cover $U_1, U_2, U_3, \ldots$ so that $\overline{U_i}$ is compact, U_i is connected and meets only finitely many U_j, $j \neq i$. Then $\bigcap_n V_n = \emptyset$ where $V_n = U_n \cup U_{n+1} \cup \ldots$. Each component W of V_n apparently is the union of a certain subcollection of the connected open sets $U_n, U_{n+1}, \ldots$. In particular W is open and $\mathrm{b}W \cap V_n = \emptyset$. Then $\mathrm{b}W$ is compact since it must lie in $X - V_n \subset \overline{U_1} \cup \cdots \cup \overline{U_{n-1}}$ which is compact. Now $\mathrm{b}W \neq \emptyset$ or else W, being open and connected, is all of the connected space X. If $\mathrm{b}W \neq \emptyset$, some $U_j \subset W$ meets $U_1 \cup \cdots \cup U_n$. By construction this can happen for only finitely many U_j. Hence there can be only finitely many components W in V_n. It follows that at least one component of V_n – call it W – is unbounded (i.e. has non-compact closure).

Now $W \cap V_{n+1}$ is a union of some of the finitely many components of V_{n+1}. So, of these, at least one is unbounded. It is clear now that we can inductively define a sequence

$$\varepsilon\colon\ W_1 \supset W_2 \supset W_3 \supset \ldots, \tag{1}$$

where W_n is an unbounded component of V_n. Then ε satisfies (i), (ii), (iii) and determines an end of X. □

The above proof can be used to establish much more than Theorem 1.6. Very briefly we indicate some

Corollary 1.7 *It follows that an infinite sequence in X either has a cluster point in X, or else has a subsequence that converges to an end determined by a sequence (1). Also, an infinite sequence of end points always has a cluster point. Assuming now that X is* separable *we see $\widehat{X}$ is* compact. *One can see that every end of X is determined by a sequence (1). Then the end points $E = \widehat{X} - X$ are the inverse limit of a system of finite sets, namely the unbounded components of V_n, $n = 1, 2, \ldots$. From Eilenberg and Steenrod [6, p. 254, Ex. B.1] it follows that E is* compact and totally disconnected.

With X as in Theorem 1.6 let U range over all open subsets of X with $\overline{U}$ compact. Let $e(U)$ denote the number of non-compact components of $X - U$, and let e denote the number of ends of X. (We don't distinguish types of infinity.) Using Freudenthal's theorem it is not hard to show:

Lemma 1.8 *lub $e(U) = e$.*

Assume now that X is a topological manifold (always separable) or else a locally finite simplicial complex. Let $H^*_e(X)$ be the cohomology of singular cochains of X modulo cochains with compact support. Coefficients are in some field.

Theorem 1.9 *The dimension of the vector space $H^0_e(X)$ is equal to the number of ends of X or both are infinite.*

The proof uses the above lemma. (See Epstein [7, Theorem 1, p. 110]).

The universal covering of the figure 8 is contractible, but for manifolds, infinitely many ends implies infinitely generated homology.

Theorem 1.10 *If W^n is a connected combinatorial or smooth manifold with compact boundary and e ends,*

$$e \leqslant \operatorname{rank} H_{n-1}(W, \operatorname{Bd} W) + 1.$$

(Again we confuse types of infinity.)

Proof Let $\widehat{W}$ be W compactified by adding the end points E (c.f. 1.7). From the exact Čech cohomology sequence

$$\longrightarrow H^0(\widehat{W}) \longrightarrow H^0(E) \longrightarrow H^1(\widehat{W}, E) \longrightarrow$$

we deduce

$$e = \operatorname{rank} H^0(E) \leqslant \operatorname{rank} H^1(\widehat{W}, E) + 1,$$

since $\widehat{W}$ is connected and E is totally disconnected. By a form of Alexander-Lefschetz duality

$$H^1(\widehat{W}, E) \cong H_{n-1}(W, \mathrm{Bd}\, W) \tag{2}$$

with Čech cohomology and singular homology. This gives the desired result.

To verify this duality let $U_1 \subset U_2 \subset \dots$ be a sequence of compact n–submanifolds with $\mathrm{Bd}\, W \subset U_i$, $W = \bigcup_i U_i$. Let $\widehat{V_n}$ be $\widehat{W} - \mathrm{Int}\, U_n$. Then the following diagram commutes:

$$\begin{tikzcd}
H^1(\widehat{W}, \widehat{V}_i) \arrow[r, "e", "\cong"'] \arrow[d, "j^*"] & H^1(U_i, \mathrm{Bd}\, U_i) \arrow[r, "P", "\cong"'] & H_{n-1}(U_i, \mathrm{Bd}\, W) \arrow[d, "i_*"] \\
H^1(\widehat{W}, \widehat{V}_{i+1}) \arrow[r, "e", "\cong"'] & H^1(U_{i+1}, \mathrm{Bd}\, U_{i+1}) \arrow[r, "P", "\cong"'] & H_{n-1}(U_{i+1}, \mathrm{Bd}\, W)
\end{tikzcd}$$

where e is excision, P is Poincaré duality and i_*, j^* are induced by inclusion. Now $\varinjlim_i H^1(\widehat{W}, \widehat{V}_{i+1}) \cong H^1(\widehat{W}, E)$ by the continuity of Čech theory [6, p. 261]. Also $\varinjlim_i H_{n-1}(U_i, \mathrm{Bd}\, W) \cong H_{n-1}(W, \mathrm{Bd}\, W)$. Thus (2) is established. □

2 Completions, collars, and 0-neighborhoods

Suppose W is a smooth non-compact manifold with compact possibly empty boundary BdW.

Definition 2.1 A *completion* for W is a some imbedding $i\colon W \longrightarrow \overline{W}$ of W into a smooth compact manifold so that $\overline{W} - i(W)$ consists of some of the boundary components of $\overline{W}$.

Now let ε be an end of the manifold W above.

Definition 2.2 A *collar* for ε (or a *collar neighborhood* of ε) is a connected neighborhood V of ε which is a smooth submanifold of W with compact boundary so that $V \cong \mathrm{Bd}\, V \times [0, 1)$ ($\cong$ means "is diffeomorphic to").

The following proposition is evident from the collar neighborhood theorem, Milnor [4, p. 23]:

Proposition 2.3 *A smooth manifold W has a completion if and only if* $\mathrm{Bd}\, W$ *is compact and W has finitely many ends each of which has a collar.*

Thus the question whether a given smooth open manifold W is diffeomorphic to the interior of some smooth compact manifold is reduced to a question about the ends of W, namely, "When does a given end ε of W have a collar?" Our goal in §2 to §5 is to answer this question for dimension greater than 5. We remark immediately that the answer is determined by an arbitrarily small neighborhood of ε. Hence *it is no loss of generality to assume always that ε is an end of an open manifold* (rather than a non-compact manifold with compact boundary).

We will set up progressively stronger conditions which guarantee the existence of arbitrarily small neighborhoods of ε that share progressively more of the properties of a collar.

Remark "Arbitrarily small" means inside any prescribed neighborhood of ε, or equivalently, in the complement of any prescribed compact subset of W.

Definition 2.4 A 0–*neighborhood* of ε is a neighborhood V of ε which is a smooth connected manifold having a compact connected boundary and just one end.

Remark We will eventually define k–neighborhoods for any $k \geqslant 0$. Roughly, a k–neighborhood is a collar so far as k–dimensional homotopy type is concerned.

Theorem 2.5 *Every isolated end ε of a smooth open manifold W has arbitrarily small 0–neighborhoods.*

Proof Let K be a given compact set in W, and let $G \in \varepsilon$ be a member of no other end. Choose a proper Morse function $f\colon W \longrightarrow [0,\infty)$, Milnor [8, p. 36]. Since $\bigcup_n f^{-1}[0,n) = W$ there exists an integer n so large that $(K \cup \mathrm{b}G) \cap f^{-1}[n,\infty) = \emptyset$. As $f^{-1}[0,n)$ is compact, one of the components V_n of $f^{-1}[n,\infty)$ is a neighborhood of ε. As V_n is connected, necessarily $V_n \subset G$, and so V_n has just one end.

If $\mathrm{Bd}\, V_n$ is not connected, $\dim W > 1$ and we can join two of the components of $\mathrm{Bd}\, V_n$ by an arc D^1 smoothly imbedded in V_n that meets $\mathrm{Bd}\, V_n$ transversely. (In dimensions $\geqslant 3$, Whitney's imbedding theorem will apply. In dimension 2 one can use the Hopf-Rinow theorem, see Milnor [8, p. 62].) If we now excise from V_n an open tubular neighborhood T of D^1 in V_n and *round off the corners* (see the note below), the resulting manifold V_n' has one less boundary component, is still connected with compact boundary and satisfies $V_n' \cap K = \emptyset$. Hence after finitely many steps we obtain a 0–neighborhood V of ε with $V \cap K = \emptyset$. □

Note 2.1 *In the above situation, temporarily change the smoothness structure on $V_n - T$ smoothing the corners by the method of Milnor [9]. Then let $h\colon \mathrm{Bd}(V_n - T) \times [0,1) \longrightarrow (V_n - T)$ be a smooth collaring of the boundary. For any $\lambda \in (0,1)$, $h[\mathrm{Bd}(V_n - T) \times \lambda]$ is a smooth submanifold of $\mathrm{Int}(V_n - T \subset W$. We define $V_n' = (V_n - T) - h[\mathrm{Bd}(V_n - T) \times [0,\lambda)]$. Clearly V_n' is diffeomorphic to $V_n - T$ (smoothed). And the old $V_n - T \subset W$ is V_n' with a topological collar added in W.*

If one wishes to round off the corners of $V_n - T$ so that the difference of $V_n - T$ and V_n' lies in a given neighborhood N of the corners there is an obvious way to accomplish this with the collaring h and a smooth function $\lambda\colon \mathrm{Bd}(V_n - T) \longrightarrow [0,1)$ zero outside N and positive near the corner set.

Henceforth we assume that this sort of device is applied whenever rounding of corners is called for.

3 Stability of π_1 at an end

Definition 3.1 Two inverse sequences of groups

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$$

and

$$G_1 \xleftarrow{f_1'} G_2 \xleftarrow{f_2'} \cdots$$

are *conjugate* if there exist elements $g_i \in G_i$ so that $f_i'(x) = g_i^{-1} f_i(x) g_i$. (We say f_i' is conjugate to f_i.)

By a subsequence of

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$$

we mean a sequence

$$G_{n_1} \xleftarrow{f_1'} G_{n_2} \xleftarrow{f_2'} \cdots,$$

$n_1 < n_2 < \dots$, where f_i' is the composed map $G_{n_i} \longleftarrow G_{n_{i+1}}$ from the first sequence.

For two sequences

$$\mathcal{G}\colon \qquad G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$$

and

$$\mathcal{G}': \qquad G_1' \xleftarrow{f_1'} G_2' \xleftarrow{f_2'} \cdots$$

consider the following three possibilities. $\mathcal{G}$ and $\mathcal{G}'$ are isomorphic; they are conjugate; one is a subsequence of the other.

Definition 3.2 *Conjugate equivalence* of inverse sequences of groups is the equivalence relation generated by the above three relations. Thus $\mathcal{G}$ is conjugate equivalent to $\mathcal{G}'$ if and only if there exist a finite chain $\mathcal{G} = \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_k = \mathcal{G}'$ of inverse sequences so that adjacent sequences bear any one of the above three relations to each other.

Suppose X is a separable topological manifold and ε is an end of X. Let $X_1 \supset X_2 \supset \ldots$, $Y_1 \supset Y_2 \supset \ldots$ be two sequences of path-connected neighborhoods of ε so that $\bigcap_i \overline{X_i} = \emptyset = \bigcap_i \overline{Y_i}$. Choosing the base points $x_i \in X_i$ and base paths x_{i+1} to x_i in X_i we get an inverse sequence

$$\mathcal{G}: \qquad \pi_1(X_1, x_1) \longleftarrow \pi_1(X_2, x_2) \longleftarrow \cdots.$$

Similarly form

$$\mathcal{H}: \qquad \pi_1(Y_1, y_1) \longleftarrow \pi_1(Y_2, y_2) \longleftarrow \cdots.$$

Lemma 3.3 *$\mathcal{G}$ is conjugate equivalent to $\mathcal{H}$.*

Proof This is easy if $X_i = Y_i$, hence also if $\{Y_i\}$ is a subsequence of $\{X_i\}$. For the general case we can find a sequence

$$X_{r_1} \supset Y_{s_1} \supset X_{r_2} \supset Y_{s_2} \supset \ldots, \quad r_1 < r_2 < \ldots, \quad s_1 < s_2 < \ldots.$$

This sequence has the subsequence $\{X_{r_i}\}$ in common with $\{X_i\}$ and the subsequence $\{Y_{s_i}\}$ in common with $\{Y_i\}$. The result follows. $\square$

Definition 3.4 An inverse sequence

$$G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots$$

of groups is *stable* if there exists a subsequence

$$G_{r_1} \xleftarrow{f_1'} G_{r_2} \xleftarrow{f_2'} \cdots$$

so that isomorphisms

$$\mathrm{Im}(f_1') \xleftarrow{\cong} \mathrm{Im}(f_2') \xleftarrow{\cong} \cdots$$

are induced.

Remark If $G_1 \longleftarrow G_2 \longleftarrow \ldots$ is stable it is certainly conjugate equivalent to the constant sequence

$$\mathrm{Im}(f_1') \xleftarrow{\cong} \mathrm{Im}(f_2') \xleftarrow{\cong} \cdots .$$

The lemma below implies that conversely if $G_1 \longleftarrow G_2 \longleftarrow \ldots$ is conjugate equivalent to a constant sequence

$$G \xleftarrow{\mathrm{id}} G \xleftarrow{\mathrm{id}} \cdots$$

then $G_1 \longleftarrow G_2 \longleftarrow \ldots$ is stable.

Let

$$\mathcal{G}: \qquad G_1 \xleftarrow{f_1} G_2 \xleftarrow{f_2} \cdots ,$$

$$\mathcal{G}': \qquad G_1' \xleftarrow{f_1'} G_2' \xleftarrow{f_2'} \cdots$$

be two inverse sequences of groups.

Lemma 3.5 *Suppose $\mathcal{G}$ is conjugate equivalent to $\mathcal{G}'$. If $\mathcal{G}$ is stable so is $\mathcal{G}'$ and*

$$\varprojlim \mathcal{G} \cong \varprojlim \mathcal{G}'.$$

Proof If $\mathcal{G}$ is isomorphic to $\mathcal{G}'$ or $\mathcal{G}$ is a subsequence of $\mathcal{G}'$ or $\mathcal{G}'$ of $\mathcal{G}$, the proposition is obvious. So it will suffice to prove the lemma when $\mathcal{G}$ is conjugate to $\mathcal{G}'$. Taking subsequences we may assume that $\mathcal{G}$ induces isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \cdots .$$

And we still have $f_i'(x) = g_i f_i(x) g_i^{-1}$ for some $g_i \in G_i$ $(= G_i')$. Now $\mathrm{Im}(f_1') = g_1 \mathrm{Im}(f_1) g_1^{-1}$, and $\mathrm{Im}(f_2') = g_2 \mathrm{Im}(f_2) g_2^{-1}$. Clearly f_1 is (1-1) on $\mathrm{Im}(f_2')$; so f_1' is also. But $f_1(\mathrm{Im}(f_2')) = \mathrm{Im}(f_1)$ since $f_1(g_2) \in \mathrm{Im}(f_1)$. Thus $f_1'(\mathrm{Im}(f_2')) = g_1 \,\mathrm{Im}(f_1) g_1^{-1} = \mathrm{Im}(f_1')$. This establishes that f_1' induces

$$\mathrm{Im}(f_1') \xleftarrow{\cong} \mathrm{Im}(f_2') .$$

The same argument works for f_2', f_3', etc. Then $\mathcal{G}'$ is stable and

$$\varprojlim \mathcal{G} = \mathrm{Im}(f_1) = \mathrm{Im}(f_1') = \varprojlim \mathcal{G}'.$$

$\square$

Remark If $\mathcal{G}$ is conjugate equivalent to $\mathcal{G}'$, but not necessarily stable $\varprojlim \mathcal{G}$ will in general not be equal to $\varprojlim \mathcal{G}'$. Here is a simple example contributed by Professor Milnor. Consider the sequence

$$F_1 \supset F_2 \supset F_3 \supset \ldots$$

where F_n is free on generators $x_n, x_{n+1}, \ldots$ and y. The inverse limit (= intersection) is infinite cyclic. Now consider the conjugate sequence

$$F_1 \xleftarrow{f_1} F_2 \xleftarrow{f_2} F_3 \xleftarrow{f_3} \cdots$$

where $f_n(\xi) = x_n \xi x_n^{-1}$. Each map is an imbedding; consequently the inverse limit is $\bigcap_n f_1 f_2 \ldots f_n F_{n+1} \subset F_1$. Now an element $\eta \in F_1$ that lies in $f_1 f_2 \ldots f_n F_{n+1}$ has the form $x_1 x_2 \ldots x_n \xi x_n^{-1} \ldots x_2^{-1} x_1^{-1}$ where $\xi \in F_{n+1}$. As ξ does not involve $x_1, \ldots, x_n$ the (unique) reduced word for η certainly involves $x_1, \ldots, x_n$ or else is the identity. No reduced word can involve infinitely many generators. Thus the second inverse limit is the identity.

Again let ε be an end of the topological manifold X.

Definition 3.6 π_1 is *stable* at ε if there exists a sequence of path connected neighborhoods of ε, $X_1 \supset X_2 \supset \ldots$ with $\cap \overline{X_i} = \emptyset$ such that (with base points and base paths chosen) the sequence

$$\pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induces isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \cdots .$$

Lemma 3.3 and Lemma 3.5 show that if π_1 is stable at ε and $Y_1 \supset Y_2 \supset \ldots$ is *any* path connected sequence of neighborhoods of ε so that $\cap \overline{Y_i} = \emptyset$, then for *any* choice of base points and base paths, the inverse sequence

$$\mathcal{G}\colon \qquad \pi_1(Y_1) \xleftarrow{g_1} \pi_1(Y_2) \xleftarrow{g_2} \cdots$$

is stable. And conversely if $\mathcal{G}$ is stable π_1 is obviously stable at ε. Hence to measure stability of π_1 at ε we can look at any one sequence $\mathcal{G}$.

Definition 3.7 If π_1 is stable at ε, define $\pi_1(\varepsilon) = \varprojlim \mathcal{G}$ for some fixed system $\mathcal{G}$ as above.

By Lemma 3.3, $\pi_1(\varepsilon)$ is determined up to isomorphism. If $\mathcal{G}'$ is a similar system for ε, one can show that there is a preferred conjugacy class of isomorphisms $\varprojlim \mathcal{G} \longrightarrow \varprojlim \mathcal{G}'$ such that if V is any path connected neighborhood, the diagram

$$\begin{tikzcd} \varprojlim \mathcal{G} \arrow[rr] \arrow[dr, "j"'] & & \varprojlim \mathcal{G}' \arrow[dl, "j'"] \\ & \pi_1(V) & \end{tikzcd}$$

commutes for suitably chosen j, j' in the natural conjugacy classes determined by inclusions. This shows for example that the statement that $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ is an isomorphism (or onto, or 1-1) is independent of the particular choice of $\mathcal{G}$ to define $\pi_1(\varepsilon)$. The proof uses the ideas of Lemma 3.3 and Lemma 3.5 again. I omit it.

Example 3.1 If $\mathcal{G}$ is

$$\mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \mathbb{Z} \xleftarrow{\times 2} \cdots,$$

or

$$\mathbb{Z}_2 \xleftarrow{\text{onto}} \mathbb{Z}_4 \xleftarrow{\text{onto}} \mathbb{Z}_8 \xleftarrow{\text{onto}} \cdots,$$

π_1 cannot be stable at ε. The first sequence occurs naturally for the complement of the dyadic solenoid imbedded in S^n, $n \geqslant 4$.

Example 3.2 If W is formed by deleting a boundary component of M from a compact topological manifold $\overline{W}$, then π_1 is stable at the one end ε of W since $X_1, X_2, \dots$ can be a sequence of collars intersected with W. (See the collar theorem of M. Brown [15].) Further $\pi_1(\varepsilon) \cong \pi_1(M)$ is finitely presented. For M, being a compact absolute neighborhood retract (see [16]) that imbeds in Euclidean space, is dominated by a finite complex. Then $\pi_1(M)$ is at least a retract of a finitely presented group. But

Lemma 3.8 (Proved in Wall [2], Lemma 1.3) *A retract of a finitely presented group is finitely presented.*

Let W be a smooth open manifold and ε an end of W.

Definition 3.9 A *1–neighborhood* V of ε is a 0–neighborhood such that:

(1) The natural maps $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ are isomorphisms;

(2) $\operatorname{Bd} V \subset V$ gives an isomorphism $\pi_1(\operatorname{Bd} V) \longrightarrow \pi_1(V)$.

Here is the important result of this section.

Theorem 3.10 *Let W^n be a smooth open manifold, $n \geqslant 5$, and ε an isolated end of W. If π_1 is stable at ε and $\pi_1(\varepsilon)$ is finitely presented, then there exists arbitrarily small 1–neighborhoods of ε.*

Problem Is this theorem valid with $n = 3$ or $n = 4$?

Example 3.3 The condition that $\pi_1(\varepsilon)$ be finitely presented is not redundant. Given a countable presentation $\{x : r\}$ of a non-finitely presentable group G we can construct a smooth open manifold W of dimension $n \geqslant 5$ with one end so that for a suitable sequence of path connected neighborhoods $X_1 \supset X_2 \supset \dots$ of ∞ with $\cap \overline{X_i} = \emptyset$, the corresponding sequence of fundamental groups is

$$G \xleftarrow{\text{id}} G \xleftarrow{\text{id}} \cdots .$$

One simply takes the n–disk and attaches infinitely many 1–handles and 2–handles as the presentation $\{x : r\}$ demands, thickening at each step. (Keep the growing handlebody orientable so that product neighborhoods for attaching 1–spheres always exist.) If we let X_i be the complement of the ith handlebody, $\pi_1(X_i) \longrightarrow \pi_1(W) \cong G$ is an isomorphism because to obtain W from X_i we attach (dual) handles of dimension $(n-2)$, $(n-1)$, and one of dimension n.

Proof of Theorem 3.10 Let $V_1 \supset V_2 \supset \dots$ be a sequence of 0–neighborhoods of ε with $\cap V_i = \emptyset$ and $V_{i+1} \subset \operatorname{Int} V_i$. Since π_1 is stable at ε, after choosing a suitable subsequence we may assume

$$\pi_1(V_1) \xleftarrow{f_1} \pi_1(V_2) \xleftarrow{f_2} \cdots$$

is such that if $H_i = f_i \pi_1(V_{i+1}) \subset \pi_1(V_i)$ then the induced maps $H_1 \longleftarrow H_2 \longleftarrow \dots$ are isomorphisms.

Further if K is a prescribed compact set in W we may assume $K \cap V_1 = \emptyset$. We will produce a 1–neighborhood V of ε with $V \subset V_1$.

Assertion 1 There exists a 0–neighborhood $V' \subset V_3$ such that the image $\pi_1 \operatorname{Bd} V') \longrightarrow \pi_1(V_3)$ contains H_3 (equivalently, the image of $\pi_1(\operatorname{Bd} V') \longrightarrow \pi_1(V_2)$ contains H_2).

Proof V' will be V_4 modified by "trading 1–handles" along $\operatorname{Bd} V_4$. For convenience we may assume that the base points for $V_1, \dots, V_4$ are all the one point $* \in \operatorname{Bd} V_4$. By a nicely imbedded, based 1–disk in V_3 attached to $\operatorname{Bd} V_4$ we mean a triple (D, h, h') consisting of an orientable smoothly imbedded 1–disk D in $\operatorname{Int} V_3$ that meets $\operatorname{Bd} V_4$ in its two end points, transversely, and two paths h, h' in $\operatorname{Bd} V_4$ from $*$ to the negative and positive ends points of D.

Let $\{y_i\}$ be a finite set of generators for $H_3 \cong \pi_1(\varepsilon)$. Clearly each y_i can be represented by a disk (D_i, h_i, h_i') that is nicely imbedded except possibly that $\operatorname{Int} D_i$ meets $\operatorname{Bd} V_4$ in finitely many – say r_i – points, transversely. But then it is clear how to give $r_1 + 1$ nicely imbedded 1–disks representing elements $u_i^{(1)}, \ldots, u_i^{(r_i+1)}$ in $\pi_1(V_3)$ with $y_i = u_i^{(1)} \ldots u_i^{(r_i+1)}$.

In this way we obtain finitely many nicely imbedded based 1–disks in V_3 attached to $\operatorname{Bd} V_4$ representing elements in $\pi_1(V_3)$ which together generate a subgroup containing H_3. Arrange that the 1–disks are disjoint and then construct disjoint tubular neighborhoods $\{T_j\}$ for them, each T_j a tubular neighborhood in V_4 or in $V_3 - \operatorname{Int} V_4$. If T_j is in V_4 subtract the open tubular neighborhood $\overset{\circ}{T}_j$ from V_4. If T_j is in $V_3 - \operatorname{Int} V_4$ add T_j to V_4. Having done this for each T_j, smooth the resulting submanifold with corners (c.f. the note following Theorem 2.5) and call it V'. Apparently V' has the desired properties. □

Assertion 2 There exists a 0–neighborhood $V \subset \operatorname{Int} V_2$ such that $\pi_1(\operatorname{Bd} V) \longrightarrow \pi_1(V_2)$ is (1-1) onto H_2; and such V is a 1–neighborhood of ε.

Proof We begin with the last statement. Since

$$\pi_1(\operatorname{Bd} V) \longrightarrow H_2 \xrightarrow{\cong} H_1 \subset \pi_1(V_1)$$

is (1-1) onto H_1, $\pi_1(\operatorname{Bd} V) \longrightarrow \pi_1(V_1 - \operatorname{Int} V)$ and $\pi_1(\operatorname{Bd} V) \longrightarrow \pi_1(V)$ are both (1-1); so by Van Kampen's theorem $\pi_1(V) \longrightarrow \pi_1(V_1)$ is (1-1). But, since $\operatorname{Bd} V \subset V$, $\pi_1(V) \longrightarrow \pi_1(V_1)$ is onto H_1. This establishes

(1) $\pi_1(V) \longrightarrow \pi_1(V_1)$ is (1-1) onto H_1

(2) $\pi_1(\operatorname{Bd} V) \longrightarrow \pi_1(V)$ is an isomorphism.

Choose k so large that $V_k \subset V$. Then as $H_1 \xleftarrow{\cong} H_k$, we see $\pi_1(V_k) \longrightarrow \pi_1(V)$ sends H_k (1-1) onto $\pi_1(V)$ using (1). So

$(1')$ The map $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ is an isomorphism.

This establishes the second statement.

The neighborhood V will be obtained by trading 2–handles along $\operatorname{Bd} V'$, where V' is the neighborhood of Assertion 1. The following lemma shows that

$$\theta\colon\ \pi_1(\operatorname{Bd} V') \xrightarrow{\text{onto}} H_2 \subset \pi_1(V_2)$$

will become an isomorphism if we "kill" just finitely many elements $z_1, \ldots, z_k$ of the kernel.

Lemma 3.11 *Suppose $\theta\colon G \longrightarrow H$ is a homomorphism of a group G onto a group H. Let $\{x : r\}$ and $\{y : s\}$ be presentations for G and H with $|x|$ generators for G and $|s|$ relators for H. Then $\ker\theta$ can be expressed as the least normal subgroup containing (i.e. the normal closure of) a set of $|x| + |s|$ elements.*

Proof Let ξ be a (suitably indexed) set of words so that $\theta(x) = \xi(y)$ in H. Since θ is onto there exists a set of words η so that $y = \eta(\theta(x))$ in H. Then Tietze transformations give the following isomorphisms:

$$\begin{aligned}\{y : s\} &\cong \{x, y \ : \ x = \xi(y),\ s(y)\}\\ &\cong \{x, y \ : \ x = \xi(y),\ s(y),\ r(x),\ y = \eta(x)\}\\ &\cong \{x, y \ : \ x = \xi(\eta(x)),\ s(\eta(x)),\ r(x),\ y = \eta(x)\}\\ &\cong \{x \ : \ x = \xi(\eta(x)),\ s(\eta(x)),\ r(x)\}\,.\end{aligned}$$

Since θ is specified in terms of the last presentation by the correspondence $x \longrightarrow x$, it is clear that $\ker\theta$ is the normal closure of $|x| + |s|$ elements $\xi(\eta(x))$ and $s(\eta(x))$. □

Returning to the proof of Assertion 2 we represent z_1 by an oriented circle S (with base path) imbedded in $\operatorname{Bd} V'$. Since $\theta(z_1) = 0$ and $\operatorname{Bd} V'$ is 2–sided we can find a 2–disk D imbedded in V_1 so that D intersects $\operatorname{Bd} V'$ transversely, in $S = \operatorname{Bd} D$ and finitely many circles in $\operatorname{Int} D$.

If we are fortunate, $D \cap \operatorname{Bd} V' = \operatorname{Bd} D$. Then take a tubular neighborhood T of D in V' or in $V_2 - \operatorname{Int} V'$ depending on where D lies. If D is in V' subtract $\mathring{T}$ from V'. If T is in $V_2 - \operatorname{Int} V'$ add T to V'. Round off the corners and call the result V_1'. For short we say we have traded D along $\operatorname{Bd} V'$. Now we have the commutative diagram

$$\begin{tikzcd} \pi_1(\operatorname{Bd} V') \arrow[r, "j_*"] \arrow[dr, "i_*"'] & \pi_1(\operatorname{Bd} V')/(z_1) = \pi_1(\operatorname{Bd} V' \cup D) \arrow[d] & \pi_1(\operatorname{Bd} V') \arrow[l, "j_{1*}"'] \arrow[dl, "i_{1*}"] \\ & \pi_1(V_2) & \end{tikzcd}$$

where the maps are induced by inclusions and (z_1) denotes the normal closure of z_1. Since $n \geqslant 5$, j_{1*} is an isomorphism. Hence $\ker i_{1*}$ is the normal closure of $qz_2, \dots, qz_k$ in $\pi_1(\operatorname{Bd} V_1')$, where $q = j_{1*}^{-1} j_*$. Thus z_1 has been killed and we can start over again with V_1'.

If we are not fortunate, $\operatorname{Int} D$ meets $\operatorname{Bd} V'$ in circles $S_1, \ldots, S_\ell$ and some preliminary trading is required before z_1 can be killed. Let S_1 be an innermost circle in $\operatorname{Int} D$ so that S_1 bounds a disk $D_1 \subset \operatorname{Int} D$. Trade D_1 along $\operatorname{Bd} V'$. This kills an element which, happily, is in $\ker \theta$, and changes V' so that it meets D in one less circle. (D is unchanged.) After trading ℓ times we have again the more fortunate situation and D itself can finally be traded to kill z_1, or more exactly, the image of z_1 in the new $\pi_1(V')$.

When $z_1, \ldots, z_k$ have all been killed as above we have produced a manifold V so that

$$\pi_1(V) \longrightarrow \pi_1(V_2)$$

is (1-1) onto H_2. This completes the proof of Assertion 2 and Theorem 3.10. □

□

Here is a fact about 1–neighborhoods of the sort we will often accept without proof.

Lemma 3.12 *If V_1, V_2 are 1–neighborhoods of ε, $V_2 \subset \operatorname{Int} V_1$ then with $X = V_1 - \operatorname{Int} V_2$, all of the following inclusions give π_1–isomorphisms: $V_2 \hookrightarrow V_1$, $X \hookrightarrow V_1$, $\operatorname{Bd} V_1 \hookrightarrow X$, $\operatorname{Bd} V_2 \hookrightarrow X$.*

Proof The commutative diagram

$$\begin{tikzcd} \pi_1(V_1) & \pi_1(V_2) \arrow[l] \\ & \pi_1(\varepsilon) \arrow[ul, "\cong"] \arrow[u, "\cong"'] \end{tikzcd}$$

shows that $V_2 \hookrightarrow V_1$ gives a π_1–isomorphism. The rest follows easily. □

4 Finding small $(n-3)$-neighborhoods for a tame end

From this point we will always be working with spaces which are topological manifolds or CW complexes. So the usual theory of covering spaces will apply. $\widetilde{X}$ will regularly denote a *universal covering* of X with projection $p\colon \widetilde{X} \longrightarrow X$. If an inclusion $Y \hookrightarrow X$ is a 1–equivalence then $p^{-1}(Y)$ is a universal covering $\widetilde{Y}$ of Y. In this situation we say $Y \hookrightarrow X$ is *k–connected* ($k \geqslant 2$) if $H_i(\widetilde{X}, \widetilde{Y}) = 0$, $0 \leqslant i \leqslant k$, with integer coefficients. If $f\colon Y' \longrightarrow X$ is any 1–equivalence we say that f is *k–connected* ($k \geqslant 2$) if $Y' \hookrightarrow M(f)$ is k–connected where $M(f)$ is the mapping cylinder of f. Note that, if f is an inclusion, the definitions agree.

Remark Homology is more suitable for handlebody theory than homotopy. So we usually ignore higher homotopy groups.

Definition 4.1 A space X is *dominated by a finite complex* K if there are maps

$$K \underset{i}{\overset{r}{\rightleftarrows}} X$$

so that $r\cdot i$ is homotopic to the identity 1_X. $\mathcal{D}$ will denote the class of spaces of the homotopy type of a CW complex, that are dominated by a finite complex.

Let ε be an isolated end of a smooth open manifold W^n, $n \geqslant 5$, so that π_1 is stable at ε and $\pi_1(\varepsilon)$ is finitely presented. By Theorem 3.10 there exist arbitrarily small 1–neighborhoods of ε.

Definition 4.2 ε is called *tame* if, in addition, every 1–neighborhood of ε is in $\mathcal{D}$.

Remark It would be nice if tameness of the end ε were guaranteed by some restriction of the homotopy type of W. If $\pi_1(\varepsilon) = 1$, this is the case. The restriction is that $H_*(W)$ be finitely generated (see Theorem 5.9). However in §8 we construct contractible smooth manifolds W^m, $(m \geqslant 8)$ with one end ε so that π_1 is stable at ε and $\pi_1(\varepsilon)$ is finitely presented and nevertheless ε is not tame.

To clarify the notion of tameness one can prove, modulo a theorem of §6, the

Proposition 4.3 *With W and ε as introduced for the definition of tameness, there are implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) where (1),…, (4) are the statements: (the reverse implications are obvious.)*

(1) *There exists an open connected neighborhood U of ε in $\mathcal{D}$ such that the natural map $i\colon\ \pi_1(\varepsilon) \longrightarrow \pi_1(U)$ has a left inverse r, with $r\cdot i = 1$. (Since π_1 is stable at ε, r will exist whenever U is sufficiently small.)*

(2) *One 1–neighborhood of ε is in $\mathcal{D}$.*

(3) *Every 1–neighborhood of ε is in $\mathcal{D}$.*

(4) *Every 0–neighborhood of ε is in $\mathcal{D}$. More generally, if V is a neighborhood of ε which is a topological manifold so that $\mathrm{Bd}\, V$ is compact and V has one end, then $V \in \mathcal{D}$.*

Proof Apply the following theorem. In proving (3) $\Longrightarrow$ (4) use a triangulation of $\mathrm{Int}\, V \simeq V$ such that a 1–neighborhood $V' \subset \mathrm{Int}\, V$ is a subcomplex, and recall that every compact topological manifold is in $\mathcal{D}$ so that $(\mathrm{Int}\, V - \mathrm{Int}\, V') \in \mathcal{D}$. □

Theorem (Complement to the Sum Theorem 6.5). *Suppose a CW complex X is the union of two subcomplexes X_1, X_2 with intersection X_0.*

(a) $X_0, X_1, X_2 \in \mathcal{D} \Longrightarrow X \in \mathcal{D}$;

(b) $X_0, X \in \mathcal{D} \Longrightarrow X_1, X_2 \in \mathcal{D}$ *provided that* $\pi_1(X_1) \longrightarrow \pi_1(X)$ *and* $\pi_1(X_2) \longrightarrow \pi_1(X)$ *have left inverses (i.e.* $\pi_1(X_1), \pi_1(X_2)$ *are retracts of* $\pi_1(X)$*).*

After the above proposition we can give a concise definition of tameness, which we adopt for all dimensions.

Combined Definition 4.4 An end ε of a smooth open manifold W is tame if

(1) π_1 is stable at ε, viz. there is a sequence of connected open neighborhoods $X_1 \supset X_2 \supset \ldots$ of ε with $\bigcap_i \overline{X_i} = \emptyset$ so that (with some base points and base paths)

$$\pi_1(X_1) \xleftarrow{f_1} \pi_1(X_2) \xleftarrow{f_2} \cdots$$

induce isomorphisms

$$\mathrm{Im}(f_1) \xleftarrow{\cong} \mathrm{Im}(f_2) \xleftarrow{\cong} \cdots .$$

(2) There is a connected open neighborhood V of ε in $\mathcal{D}$ so small that $V \subset X_2$.

Notably, the hypothesis that $\pi_1(\varepsilon)$ be finitely presented is lacking. But as $V \subset X_2$, $\pi_1(\varepsilon) \cong \mathrm{Im}(f_1)$ is a retract of the finitely presented group $\pi_1(V)$ hence is necessarily finitely presented by Lemma 3.8. Also, by Theorem 1.10, $V \in \mathcal{D}$ has only finitely many ends. So ε must be an isolated end of W.

Suppose ε is an end of a smooth open manifold W, such that π_1 is stable at ε.

Definition 4.5 A neighborhood V of ε is a *k–neighborhood* ($k \geqslant 2$) if it is a 1–neighborhood and $H_i(\widetilde{V}, \mathrm{Bd}\, \widetilde{V}) = 0$, $0 \leqslant i \leqslant k$.

The main result of this section is:

Theorem 4.6 *If ε is a tame end of dimension $\geqslant 5$, there exist arbitrarily small $(n-3)$–neighborhoods of ε.*

Remark It turns out that a $(n-2)$–neighborhood V would be a collar neighborhood, i.e. $V \cong \mathrm{Bd}\, V \times [0,1)$. In the next section we show that, if V is an $(n-3)$–neighborhood, $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is a finitely generated projective module over $\pi_1(\varepsilon)$ and its class modulo free $\pi_1(\varepsilon)$–modules is the obstruction to finding a collar neighborhood of ε.

Lemma 4.7 *Let $f\colon K \longrightarrow X$ be a map from a finite complex to $X \in \mathcal{D}$ that is a 1–equivalence. Suppose f is $(k-1)$–connected, with $k \geqslant 2$. (This adds nothing if $k = 2$). Then $H_k(\widetilde{M(f)}, \widetilde{K})$ is a f.g. $\pi_1(X)$–module.*

Proof Let $L = K^{k-1}$ if $k \geqslant 3$ or K^2 if $k = 2$. Then $L \hookrightarrow K$ is a 1–equivalence and $(k-1)$–connected. Thus the composition $f \cdot i\colon\ L \subset K \xrightarrow{f} X$ is $(k-1)$–connected. Up to homotopy type we may assume f is an inclusion. According to Wall [2, Theorem A] $H_k(\widetilde{X}, \widetilde{L})$ is f.g. over $\pi_1(X)$. But for the triple $(\widetilde{X}, \widetilde{K}, \widetilde{L})$ we have

$$H_k(\widetilde{X}, \widetilde{L}) \longrightarrow H_k(\widetilde{X}, \widetilde{K}) \longrightarrow H_{k-1}(\widetilde{K}, \widetilde{L}) = 0$$

which implies $H_k(\widetilde{X}, \widetilde{K}) \cong H_k(\widetilde{M(f)}, \widetilde{K})$ is f.g. over $\pi_1(X)$. □

Proof of Theorem 4.6 Suppose inductively that the following proposition P_x holds with $x = k-1$, $2 \leqslant k \leqslant n-3$. (Notice that P_1 is Theorem 3.10).

P_x: There exist arbitrarily small x–neighborhoods of ε.

Given a compact set C we must construct a k–neighborhood that does not meet C. Choose a $(k-1)$–neighborhood V with $V \cap C = \emptyset$. By Lemma 4.7, $H_k(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is a f.g. $\pi_1(\varepsilon)$–module. So we can take a finite generating set $\{x_1, \ldots, x_m\}$ with the least possible number of elements. We will carve m thickened k–disks from V to produce a k–neighborhood.

Definition 4.8 A *nicely imbedded based k–disk representing* $x \in H_k(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is a pair (D, h) consisting of a smoothly imbedded oriented k–disk $D \subset V$ that intersects $\mathrm{Bd}\, V$ in $\mathrm{Bd}\, D$, transversely, and a path h from the base point to D, so that the lift $\widetilde{D} \subset \widetilde{V}$ of D by h represents x. (Since $\widetilde{D}$ is a smoothly imbedded oriented k–disk in $\widetilde{V}$ with $\mathrm{Bd}\, \widetilde{D} \subset \mathrm{Bd}\, \widetilde{V}$, this makes good sense.)

Fundamental Lemma 4.9 *If V is a $(k-1)$–neighborhood, $2 \leqslant k \leqslant n-3$, P_{k-1} implies that there is a nicely imbedded k–disk representing any given $x \in H_k(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$.*

Assuming Lemma 4.9, we complete the proof of Theorem 4.6. Let (D,h) represent x_1, take a tubular neighborhood of D in V, subtract the open tubular neighborhood from V rounding the corners, and call the result V'. We may suppose $V' \subset \operatorname{Int} V$ so that $V - \operatorname{Int} V' = U$ has $\operatorname{Bd} V \cup D$ as deformation retract.

First note that V' is at least a 1–neighborhood. For V has $V' \cup D'$ as deformation retract where D' is a $(n-k)$–disk of T transverse to D. Since $(n-k) \geqslant 3$ $\pi_1(V') \longrightarrow \pi_1(V)$ is an isomorphism so that $\pi_1(\varepsilon) \longrightarrow \pi_1(V')$ is too. Further $\operatorname{Bd} V' \hookrightarrow U$ and $\operatorname{Bd} V \hookrightarrow U$ gives π_1–isomorphisms. (When $k=2$ D is trivially attached). This easily implies $\pi_1(\operatorname{Bd} V') \longrightarrow \pi_1(V')$ is an isomorphism.

Next we establish that V' is really better than V. $H_*(\widetilde{U}, \operatorname{Bd} \widetilde{V}) = \mathbb{Z}\pi_1(\varepsilon)$, and (D,h) represents a generator $\overline{x}_1$ such that $i_*\overline{x}_1 = x_1$, $i\colon (\widetilde{U}, \operatorname{Bd}\widetilde{V}) \hookrightarrow (\widetilde{V}, \operatorname{Bd}\widetilde{V})$. From the sequence of $(\widetilde{V}, \widetilde{U}, \operatorname{Bd}\widetilde{V})$ we see that $H_*(\widetilde{V}, \widetilde{U}) \cong H_*(\widetilde{V'}, \operatorname{Bd}\widetilde{V'})$ is zero in dimensions $< k$ and in dimension k is generated by the $(m-1)$ images of $x_2, \ldots, x_m$ under j_*, $j\colon (\widetilde{V}, \operatorname{Bd}\widetilde{V}) \hookrightarrow (\widetilde{V}, \widetilde{U})$.

Thus V' is a $(k-1)$–neighborhood and $H_*(\widetilde{V'}, \operatorname{Bd}\widetilde{V'})$ has $(m-1)$ generators. After exactly m steps we obtain a k–neighborhood. This establishes P_k and completes the induction for Theorem 4.6 □

Proof of Fundamental Lemma 4.9 We begin with

Assertion There exists a $(k-1)$–neighborhood $V' \subset \operatorname{Int} V$ so small that x is represented by a cycle in $\widetilde{U}$ mod $\operatorname{Bd}\widetilde{V}$, where $U = V - \operatorname{Int} V'$.

Proof x is represented by a singular cycle and the singular simplices all map into a compact set $C \subset \widetilde{V}$. There exists a $(k-1)$–neighborhood $V' \subset \operatorname{Int} V$ so small that the projection of C lies in $U = V - \operatorname{Int} V'$. Then $\widetilde{U}$ contains C and the assertion follows. □

Now the exact sequence of $(\widetilde{V}, \widetilde{U}, \operatorname{Bd}\widetilde{V})$ shows that $\operatorname{Bd} V \hookrightarrow U$ is $(k-2)$–connected. Hence there exists a nice Morse function f and a gradient-like vector field ξ on the manifold triad $c = (U; \operatorname{Bd} V, \operatorname{Bd} V')$ with critical points of index λ, $\max(2, k-1) \leqslant \lambda \leqslant n-2$ only. See Wall [3, Theorem 5.5, p.24]. In other words $c = c_{k-1}c_k \ldots c_{n-3}c_{n-2}$ where $c_\lambda = (U_\lambda; B_\lambda, B_{\lambda+1})$ is a triad having critical points of index λ only and B_λ is a level manifold of f. Also c_{k-1} is a product if $k=2$.

We recall now some facts from handlebody theory using the language of Milnor [4]. For each critical point p of index λ a "left hand" λ–disk $D_L(p)$ in U_λ is formed by the ξ–trajectories going to p, and a "right hand" $(n-\lambda)$–disk $D_R(p)$ in U_λ is formed by the ξ–trajectories going from p. According to Milnor [4, p. 46] we may assume that in B_λ each left hand sphere $\operatorname{Bd} D_L^\lambda(p) = S_L(p)$ meets each right hand sphere $\operatorname{Bd} D_R^{\lambda-1}(q) = S_R(q)$ transversely, in a finite number of points.

Choose a lift $\widetilde{*} \in \widetilde{U}$ of the base point $* \in U$; choose base paths from $*$ to each critical point of f; and choose an orientation for each left hand disk. For $P \in S_L(p) \cap S_R(q)$ the *characteristic element* g_P is the class of the path formed by the base path $*$ to p, the trajectory p to q through P and the reversed base path q to $*$. (See Figure 2). With naturally defined orientations for the normal bundles of the right hand disks there is an *intersection number* $\varepsilon_P = \pm 1$ of $S_R(q)$ with $S_L(p)$ at P.

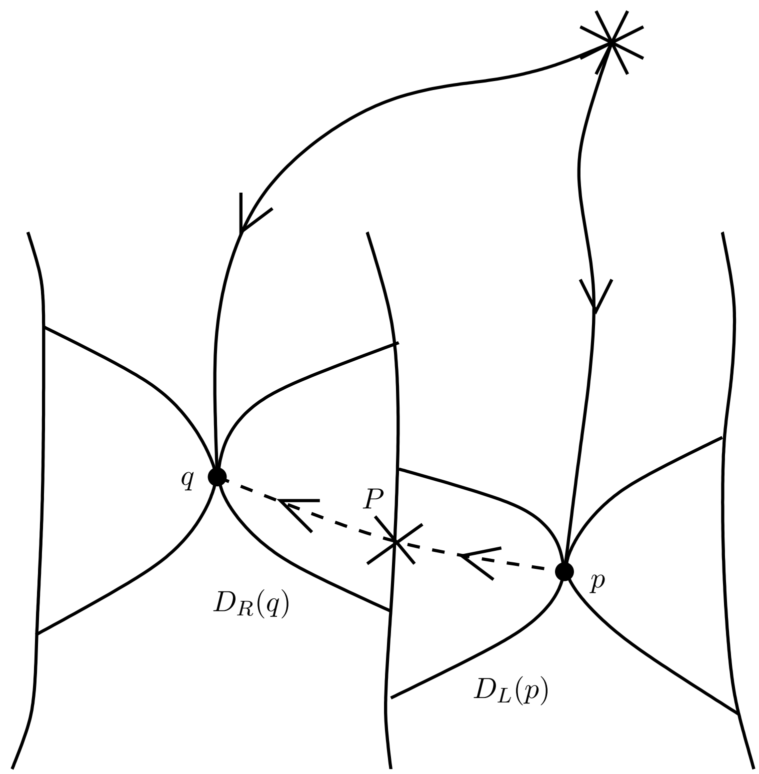


Figure 2:

Notice that $H_*(\widetilde{U_\lambda}, \widetilde{B_\lambda})$ is a free $\pi_1(U)$–module concentrated in dimension λ and has basis elements that correspond naturally to the based oriented disks

$\{D_L(p) \ : \ p$ critical of index $\lambda\}$. According to Milnor [4, p. 90], if we define $C_\lambda = H_\lambda(\widetilde{U_\lambda}, \widetilde{B_\lambda})$ and $\partial\colon\ C_\lambda \longrightarrow C_{\lambda-1}$ by

$$H_\lambda(\widetilde{U_\lambda}, \widetilde{B_\lambda}) \xrightarrow{d} H_{\lambda-1}(\widetilde{B_\lambda}) \xrightarrow{i_*} H_{\lambda-1}(\widetilde{U_{\lambda-1}}, \widetilde{B_{\lambda-1}})$$

then $H_*(\widetilde{U}, \operatorname{Bd}\widetilde{V}) \cong H_*(C)$. Further, by Wall [3, Theorem 5.1, p. 23] ∂ is expressed geometrically by the formula

$$\partial D_L^\lambda(p) = \sum_P \varepsilon_P g_P D_L^{\lambda-1}(q(P))$$

where $D_L^\lambda(p)$, $D_L^{\lambda-1}(q(P))$ stand for the basis elements represented by these based oriented disks and $P \in S_R(q(P)) \cap S_L(p)$ ranges over all intersection points of $S_L(p)$ with right hand spheres.

Here is a fact we will use later on. Suppose an orientation is specified at $* \in U$. Then using the base paths we can naturally orient all the right hand disks, and give normal bundles of the left hand disks corresponding orientations. With this system of orientations there is a new intersection number ε'_P determined for each $P \in S_R(q) \cap S_L(p)$. It is straightforward to verify that

$$\varepsilon'_P = (-1)^\lambda \operatorname{sign}(g_P)\varepsilon_P$$

where $\lambda =$ index p and $\sin(g_P)$ is $+1$ or -1 according as q_P is orientation preserving or orientation reversing. The new characteristic element for P is clearly $g'_P = g_P^{-1}$.

Let $\overline{x} \in H_k(\widetilde{U}, \operatorname{Bd}\widetilde{V})$ satisfy $i_*\overline{x} = x \in H_k(\widetilde{V}, \operatorname{Bd}\widetilde{V})$ and represent $\overline{x}$ by a chain

$$c = \sum_p r(p) D_L^k(p)$$

where p ranges over critical points of index k and $r(p) \in \mathbb{Z}\pi_1(U)$. Introduce a complementary (=auxiliary) pair p_0, q_0 of critical points of index k and $k{+}1$ using Milnor [4, p. 101] (c.f. Wall [3, p. 17]). The effect on C_* is to introduce two new basis elements $D_L(p_0) \in C_k$ and $D_L(q_0) \in C_{k+1}$ so that $\partial D_L(q_0) = D_L(p_0)$ (with suitable base paths and orientations) while ∂ is otherwise unchanged. In particular $\partial D_L(p_0) = 0$ so that $\overline{x}$ is represented by $D_L(p_0){+}c$. Now we can apply Wall's Handle Addition Theorem [3, p. 17] repeatedly changing the Morse function (or handle decomposition) to alter the basis of C_k so that the new based oriented left-hand disks represent $D_L(p_0) + c$ and the old basis element $D_L(p)$ with $p \neq p_0$. (We note that the proof, not the statement, of the "Basis Theorem" of Milnor [4, p. 92] can be strengthened to give this result.)

We now have a critical point p_1 so that $D_L(p_1)$ is a cycle representing $\overline{x}$. If $k = 2$, the rest of the argument is easy. (If $n = 5$, the only case in question is $2 = k = n - 3$.) From the outset, there are no critical points of index < 2. This means that the trajectories in U going to p_1 form a disk $D'_L(p_1)$ which is $D_L(p_1)$ with a collar added. It is easy to see that $D'_L(p_1)$ with the orientation and base path of $D_L(p_1)$ is a nicely imbedded 2–disk that represents $\overline{x} \in H_2(\widetilde{U}, \operatorname{Bd} \widetilde{V})$ and hence $x \in H_2(\widetilde{V}, \operatorname{Bd} \widetilde{V})$. (Actually, for $k = 2$, or even $2k + 1 \leqslant n$, one can imbed a suitable k–disk directly, without handlebody theory.)

If $3 \leqslant k \leqslant n - 3$ (and hence $n \geqslant 6$) argue as follows. Since $\partial D_L(p_1) = 0$ the points of intersection of $D_L(p_1)$ with any right hand sphere in B_k can be arranged in pairs (P, Q) so that $g_P = g_Q$ and $\varepsilon_P = -\varepsilon_Q$ (see the formula above). Take a loop L consisting of an arc P to Q in $S_L(p_1)$ then Q to P in $S_R(q)$. It is contractible in B_k because $g_P = g_Q$. So L can be spanned by a 2–disk and the device of Whitney permits us to eliminate the two intersection points P, Q by deforming $S_L(p_1)$. Theorem 6.6 of Milnor [4] explains all this in detail.

Then in finitely many steps we can arrange that $S_L(p_1)$ meets no right hand spheres (c.f. Milnor [4, §4.7]). Now observe that the trajectories in U going to p_1 form a disk $D'(p_1)$ which is $D_L(p_1)$ plus a collar. $D'(p_1)$ is a based oriented and nicely imbedded k–disk that apparently represents $\overline{x} \in H_k(\widetilde{U}, \operatorname{Bd} \widetilde{U})$ and hence $x \in H_k(\widetilde{V}, \operatorname{Bd} \widetilde{V})$. □

5 The Obstruction to Finding a Collar Neighborhood

This chapter brings us to the Main Theorem (5.7), which we have been working towards in §2, 3, and 4. What remains of the proof is broken into two parts. The first (Proposition 5.1) is an elementary observation that serves to isolate the obstruction. The second, (Proposition 5.6) proves that when the obstruction vanishes, one can find a collar. It is the heart of the theorem.

As usual ε is an end of a smooth open manifold W^n.

Proposition 5.1 *Suppose $n \geqslant 5$, π_1 is stable at ε, and $\pi_1(\varepsilon)$ is finitely presented. If V is a $(n-3)$–neighborhood of ε, then $H_i(\widetilde{V}, \operatorname{Bd} \widetilde{V}) = 0$, $i \neq n-2$ and $H_{n-2}(\widetilde{V}, \operatorname{Bd} \widetilde{V})$ is projective over $\pi_1(\varepsilon)$.*

Remark If ε is tame, by Lemma 4.7 $H_{n-2}(\widetilde{V}, \operatorname{Bd}\widetilde{V})$ is f.g. over $\pi_1(\varepsilon)$.

Corollary 5.2 *If V is a $(n-2)$–neighborhood of ε, $H_*(\widetilde{V}, \operatorname{Bd}\widetilde{V}) = 0$ so that $\operatorname{Bd} V \hookrightarrow V$ is a homotopy equivalence. If in addition there are arbitrarily small $(n-2)$–neighborhoods of ε, then V is a collar neighborhood.*

Proof of Corollary 5.2 The first statement is clear. The second statement follows from the invertibility of h–cobordisms. For $n = 5$ this seems to require the Engulfing Theorem (see Stallings [10]). □

Proof of Proposition 5.1 Since $\operatorname{Bd} V \hookrightarrow V$ is $(n-3)$–connected $H_i(\widetilde{V}, \operatorname{Bd}\widetilde{V}) = 0$, $i \leqslant n-3$. It remains to show that $H_i(\widetilde{V}, \operatorname{Bd}\widetilde{V}) = 0$ for $i \geqslant n-1$ and projective over $\pi_1(\varepsilon)$ for $i = n-2$.

By Theorem 3.10 we can find a sequence $V = V_0 \supset V_1 \supset V_2 \supset \cdots$ of 1–neighborhoods of ε with $\bigcap_n V_n = \emptyset$, and $V_{i+1} \subset \operatorname{Int} V_i$. If $U_i = V_i - \operatorname{Int} V_{i+1}$, $\operatorname{Bd} V_i \hookrightarrow U_i$ and $\operatorname{Bd} V_{i+1} \hookrightarrow U_i$ give π_1–isomorphisms. Put a Morse function $f_i \colon U_i \xrightarrow{\text{onto}} [i, i+1]$ on each triad $(U_i; \operatorname{Bd} V_i, \operatorname{Bd} V_{i+1})$.

Following the proof of Milnor [4, Theorem 8.1, p. 100] we can arrange that f_i has no critical points of index 0, 1, n, and $n-1$. (This is also the effect of Wall [3, Theorem 5.1].) Piece the Morse functions $f_0, f_1, f_2, \ldots$ together to give a proper Morse $f \colon V \xrightarrow{\text{onto}} [0, \infty)$ with $f^{-1}(0) = \operatorname{Bd} V$.

It follows from the well known lemma given below that $(V, \operatorname{Bd} V)$ is homotopy equivalent to $(K, \operatorname{Bd} V)$ where K is $\operatorname{Bd} V$ with cells of dimension λ, $2 \leqslant \lambda \leqslant n-2$ attached. Thus $H_i(\widetilde{V}, \operatorname{Bd}\widetilde{V}) = 0$, $i \geqslant n-1$. Further the cellular structure of $(K, \operatorname{Bd} V)$ gives a *free* $\pi_1(\varepsilon)$–complex for $H_*(\widetilde{K}, \operatorname{Bd}\widetilde{V})$

$$0 \longrightarrow C_{k-2}(\widetilde{K}, \operatorname{Bd}\widetilde{V}) \longrightarrow C_{k-3}(\widetilde{K}, \operatorname{Bd}\widetilde{V}) \longrightarrow \ldots \longrightarrow C_2(\widetilde{K}, \operatorname{Bd}\widetilde{V}) \longrightarrow 0.$$

Since the homology is isolated in dimension $(k-2)$ it follows easily that $H_{k-2}(\widetilde{K}, \operatorname{Bd}\widetilde{V})$ is projective. □

Lemma 5.3 *Suppose V is a smooth manifold and $f \colon V \longrightarrow [0, \infty)$ is a proper Morse function with $f^{-1}(0) = \operatorname{Bd} V$. Then there exists a CW complex K, consisting of $\operatorname{Bd} V$ (triangulated) with one cell of dimension λ in $K - \operatorname{Bd} V$ for each index λ critical point, and such that there is a homotopy equivalence $f \colon K \longrightarrow V$ fixing $\operatorname{Bd} V$.*

Proof Let $a_0 = 0 < a_1 < a_2 < \ldots$ be an unbounded sequence of noncritical points. Since f is proper $f^{-1}[a_i, a_{i+1}]$ is a smooth compact manifold and can contain only finitely many critical points. Adjusting f slightly (by Milnor [4, p. 17 or p. 37]) we may assume the critical levels in $[a_i, a_{i+1}]$ are distinct. Then there is a refinement $b_0 = 0 < b_1 < b_2 < \dots$ of $a_0 = 0 < a_1 < a_2 < \dots$ so that b_i is noncritical and $f^{-1}[b_i, b_{i+1}]$ contains at most one critical point.

We will construct a nested sequence of CW complexes $K_0 = \operatorname{Bd} V \subset K_1 \subset K_2 \subset \dots, K = \cup K_i$, and a sequence of homotopy equivalences $f_i\colon U_i \longrightarrow K_i$, $U_i = f^{-1}[0, b_i]$, $f_0 = 1_{\operatorname{Bd} V}$, so that $(f_{i+1})_{|U_i}$ agrees with f_i. Then $f_0, f_1, f_2, \dots$ define a continuous map $f\colon V \longrightarrow K$, which induces an isomorphism of all homotopy groups. By Whitehead's theorem [11] it will be the required homotopy equivalence.

Suppose inductively that f_i, K_i are defined. If $f[b_i, b_{i+1}]$ contains no critical point it is collar and no problem arises. Otherwise let $r\colon U_{i+1} \longrightarrow U_i \cup D_L$ be a deformation retraction where D_L is the left hand disk of the one critical point (c.f. Milnor [4, p. 28]). By Milnor [8, Lemma 3.7, p. 21] F_i extends to a homotopy equivalence

$$f_i\colon U_i \cup D_L \longrightarrow K_i \cup_\phi D'_L$$

where D'_L is a copy of D_L attached by the map $(f_i)_{|\operatorname{Bd} D_L} = \phi$. If $\phi' \cong \phi$ is a cellular approximation, by [4, Lemma 3.6, p. 20] the identity map of K_i extends to a homotopy equivalence

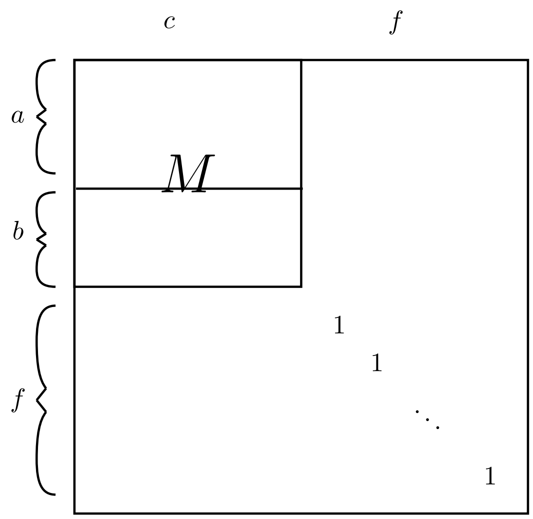
$$g\colon K_i \cup_\phi D'_L \longrightarrow K_i \cup_{\phi'} D'_L.$$

Define $K_{i+1} = K_i \cup_{\phi'} D'_L$ and let $f_{i+1} = g \circ f'_i \circ r$. Then F_i is a homotopy equivalence and $(f_{i+1})_{|K_i}$ agrees with f_i. □

Next we prove a lemma needed for the second main proposition. Let A, B, C be free f.g. modules over a group π with preferred bases a, b, c respectively. If $C = A \oplus B$ we ask whether there exists a basis $c' \sim c$ (i.e. c' is derived from c by repeatedly adding to one basis element a $\mathbb{Z}[\pi]$–multiple of a different basis element) so that some of the elements of c' generate A and the rest generate B. This is stably true. Let $B' = B \oplus F$ where F is a free π–module with preferred basis f. Let $C' = A \oplus B'$ and let the enlarged basis for B' and C' be $b' = bf$ and $c' = cf$.

Lemma 5.4 *If* $\operatorname{rank} F \geqslant \operatorname{rank} C$ *there exists a basis* $c'' \sim c'$ *for* C' *such that some of the elements of* c'' *generate* A *and the rest generate* B'*.*

Proof The matrix that expresses ab' in terms of the basis c' looks like Figure 3 below.



$$= \begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix}$$

Figure 3:

Notice that multiplication on the right by an 'elementary' matrix $(I+E)$ where E is zero but for one off-diagonal element in $\mathbb{Z}[\pi]$ and I is the identity, corresponds to adding to one basis element of c' a $\mathbb{Z}[\pi]$–multiple of a different basis element.

Suppose first that $\operatorname{rank} F = \operatorname{rank} C$. Then

$$\begin{pmatrix} M & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & M \end{pmatrix}.$$

But the right hand side of

$$\begin{pmatrix} M^{-1} & 0 \\ 0 & M \end{pmatrix} = \begin{pmatrix} I & M^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I-M & I \end{pmatrix} \begin{pmatrix} I & -I \\ 0 & I \end{pmatrix} \begin{pmatrix} I & 0 \\ I-M^{-1} & I \end{pmatrix}$$

is clearly a product of elementary matrices. So the Lemma is established in this case. In the general case, just ignore the last $[\operatorname{rank} F - \operatorname{rank} C]$ elements of f. □

Definition 5.5 Let G be a group. Two G–modules A, B are *stably isomorphic* (written $A \sim B$) if $A \oplus F \cong B \oplus F$ for some f.g. free G–module F. A f.g. G–module is called *stably free* if it is stably isomorphic to a free module.

Proposition 5.6 *Suppose ε is a tame end of dimension $\geqslant 5$. If V is a $(n-3)$–neighborhood of ε, the stable isomorphism class of $H_{n-2}(\widetilde{V}, \operatorname{Bd}\widetilde{V})$ (as* *a $\pi_1(\varepsilon)$–module) is an invariant of ε. If $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is stably free and $n \geqslant 6$, there exists a $(n-2)$–neighborhood $V_0 \subset \mathrm{Int}\, V$.*

Any f.g. projective module is a direct summand of a f.g. free module. Thus the stable isomorphism classes of f.g. G–modules form an abelian group. It is called the *projective class group* $\widetilde{K}_0(G)$. Apparently the class containing stably free modules is the zero element.

Combining Proposition 5.1, Corollary 5.2 and Proposition 5.6 we have our

Theorem 5.7 *If ε is a tame end of dimension $\geqslant 6$ there is an obstruction $\sigma(\varepsilon) \in \widetilde{K}_0(\pi_1(\varepsilon))$ that is zero if and only if ε has a collar.*

In §8 we construct examples where $\sigma(\varepsilon) \neq 0$. At the end of this section we draw a few conclusions from Theorem 5.7.

Proof of Proposition 5.6 The structure of $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ as a $\pi_1(\varepsilon)$–module is determined only up to conjugation by elements of $\pi_1(\varepsilon)$. Thus if one action is denoted by juxtaposition another equally good action is $g \cdot a = x^{-1} g x a$ where $x \in \pi_1(\varepsilon)$ is fixed and $g \in \pi_1(\varepsilon)$, $a \in H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ vary. Nevertheless the new $\pi_1(\varepsilon)$–module structure is isomorphic to the old under the mapping

$$a \longrightarrow x^{-1} a.$$

We conclude that the isomorphism class of $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ as a $\pi_1(\varepsilon)$–module is independent of the particular base point of V and covering base point of $\widetilde{V}$, and of the particular isomorphism $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ (in the preferred conjugacy class). We have to establish further that the stable isomorphism class of $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is an invariant of ε, i.e. does not depend on the particular $(n-3)$–neighborhood V This will become clear during the quest of a $(n-2)$–neighborhood $V' \subset \mathrm{Int}\, V$ that we now launch.

Since $H_{n-2}(\widetilde{V}, \mathrm{Bd}\, \widetilde{V})$ is f.g. over $\pi_1(\varepsilon)$, there exists a $(n-3)$–neighborhood $V' \subset \mathrm{Int}\, V$ so small that with $U = V - \mathrm{Int}\, V'$, $M = \mathrm{Bd}\, V$ and $N = \mathrm{Bd}\, V'$, the map

$$i_* \colon\ H_{n-2}(\widetilde{U}, \widetilde{M}) \longrightarrow H_{n-2}(\widetilde{V}, \widetilde{M})$$

is onto. By inspecting the exact sequence for $(\widetilde{V}, \widetilde{U}, \widetilde{M})$

$$0 \longrightarrow H_{n-2}(\widetilde{U}, \widetilde{M}) \xrightarrow{i_*} H_{n-2}(\widetilde{V}, \widetilde{M}) \longrightarrow H_{n-2}(\widetilde{V}, \widetilde{U}) \xrightarrow{d} H_{n-3}(\widetilde{U}, \widetilde{M}) \longrightarrow 0$$

we see that i_* and d are isomorphisms. Since the middle terms are f.g. projective $\pi_1(\varepsilon)$–modules so are $H_{n-2}(\widetilde{U}, \widetilde{M})$, $H_{n-3}(\widetilde{U}, \widetilde{M})$.

Since $M \subset U$ is $(n-4)$–connected and $N \hookrightarrow U$ gives a π_1–isomorphism we can put a self-indexing Morse function f with a gradient-like vector field ξ (see Milnor [4, p. 20, p. 44]) on the triad $c = (U; M, N)$ so that f has critical points of index $(n-3)$ and $(n-2)$ only (Wall [3, Theorem 5.5]).

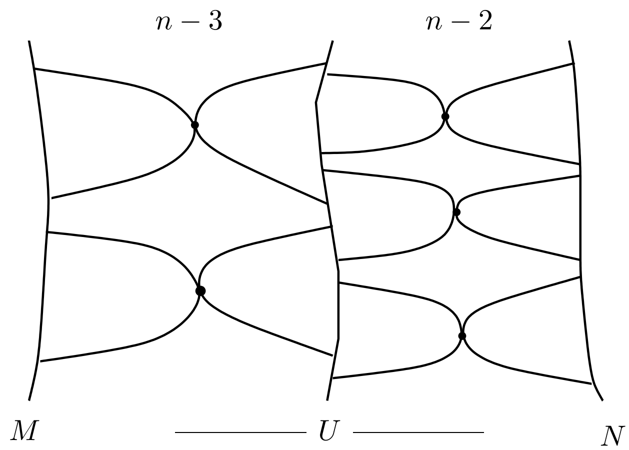


Figure 4:

We provide f with the usual equipment: base points $*$ for U and $\widetilde{*}$ over $*$ for $\widetilde{U}$; base paths from $*$ to the critical points; orientations for the left hand disks. And we can assume that left and right hand spheres intersect transversely [4, §4.6].

Now we have a well defined based, free $\pi_1(\varepsilon)$–complex C_* for $H_*(\widetilde{U}, \widetilde{M})$ ('based' means with distinguished basis over $\pi_1(\varepsilon)$). It may be written

$$\begin{tikzcd}
\cdots & 0 \arrow[l] & C_{n-3} \arrow[l] \arrow[dl] & & C_{n-2} \arrow[ll, "\partial"'] \arrow[dl] & 0 \arrow[l] & \cdots \arrow[l] \\
& H_{n-3} \arrow[dl] & & B_{n-3} \arrow[ul] \arrow[dl] & & H_{n-2} \arrow[ul] & \\
0 & & 0 & & 0 \arrow[ul] & & 0 \arrow[ul]
\end{tikzcd}$$

where we have inserted kernels and images.

We have shown H_{n-3} is projective, so B_{n-3} is too and $C_{n-3} = H_{n-3} \oplus B_{n-3}$, $C_{n-2} = B_{n-3} \oplus H_{n-2}$ (the second summands natural). It follows that $H_{n-2} \sim H_{n-3}$, hence $H_{n-2}(\widetilde{V}, \mathrm{Bd}\,\widetilde{V}) \sim H_{n-2}(\widetilde{V}', \mathrm{Bd}\,\widetilde{V}')$ ($\sim$ denotes stable isomorphism). This makes it clear that the stable isomorphism class of $H_{n-2}(\widetilde{V}, \mathrm{Bd}\,\widetilde{V})$

does not depend on the particular $(n-3)$–neighborhood V. So the first assertion of Proposition 5.6 is established.

Now suppose $H_{n-2}(\widetilde{V}, \operatorname{Bd}\widetilde{V}) \sim H_{n-2} \sim H_{n-3}$ is stably free. Then $B'_{n-3} \cong B_{n-3}$ is also stably free. For convenience identify B'_{n-3} with a fixed subgroup in C_{n-2} that maps isomorphically onto $B_{n-3} \subset C_{n-3}$, and define $H'_{n-3} \subset C_{n-3}$ similarly. Then C_* is

$$\ldots \longleftarrow 0 \longleftarrow H'_{n-3} \oplus B_{n-3} \longleftarrow B'_{n-3} \oplus H_{n-2} \longleftarrow 0 \longleftarrow \ldots .$$

Observe:

1) If we add an auxiliary (= complementary) pair of index $(n-3)$ and $(n-2)$ critical points, then a $\mathbb{Z}[\pi_1(\varepsilon)]$ summand is added to B_{n-3} and to B'_{n-3}. See Milnor [4, p. 101], Wall [3, p. 17].)
2) If we add an auxiliary pair as above and delete the auxiliary $(n-3)$–disk (thickened) from V, then a $\mathbb{Z}[\pi_1(\varepsilon)]$ summand has been added to H_{n-2}.

In the alteration 2) V changes. But $i_*\colon H_{n-2}(\widetilde{U}, \widetilde{M}) \longrightarrow H_{n-2}(\widetilde{V}, \widetilde{M})$ is still onto. For, as one easily verifies, the effect of 2) is to add a $\mathbb{Z}[\pi_1(\varepsilon)]$ summand to both of these modules and extend i_* by making generators correspond.

From 1) and 2) it follows that it is no loss of generality to assume that the stably free modules B'_{n-3} and H_{n-2} are actually free. What is more, Lemma 5.4, together with the Handle Addition Theorem (Wall [3, p. 17], c.f. Chapter IV, p. 30) shows that, after applying 1) sufficiently often, the Morse function can be altered so that some of the basis elements of C_{n-2} generate H_{n-2} and the rest generate B'_{n-3}.

We have reached the one point of the proof where we must have $n \geqslant 6$. Let D_L be an oriented left hand disk with base path, for one of those basis elements of C_{n-2} that lie in H_{n-2}. We want to say that, because $\partial D_L = 0$, it is possible to isotopically deform the left hand $(n-3)$–sphere $S_L = \operatorname{Bd} D_L$ to miss all the right hand 2–spheres in $f^{-1}(n-2\frac{1}{2})$. First try to proceed exactly as in §4. Notice that the intersection points of S_L with any one right hand 2–sphere can be arranged in pairs (P, Q) so that $g_P = g_Q$ and $\varepsilon_P = -\varepsilon_Q$. Form the loop L and attempt to apply Theorem 6.6 of Milnor [4] (which requires $(n-1) \geqslant 5$). This fails because the dimension restrictions are not quite satisfied. But fortunately they are satisfied after we replace f by $-f$ and correspondingly interchange tangent and normal orientations. We note that the new intersection numbers $\varepsilon'_P, \varepsilon'_Q$ are still opposite and that the new characteristic elements g'_P, g'_Q are still equal (see §4). For a device to show that the condition on the fundamental groups [4, Theorem 6.6] in is still satisfied, see Wall [3, p. 23-24]. After applying this argument sufficiently often we have a smooth isotopy that sweeps S_L clear of all right hand 2–spheres. Change ξ (and hence D_L) accordingly [4, §4.7].

Now D_L can be enlarged by adding the collar swept out by trajectories from M to S_L. This gives a nicely imbedded disk representing the class of D_L in $H_{n-2}(\widetilde{U}, \widetilde{M})$ (c.f. Definition 4.8). Now alter f according to Milnor [4, Lemma 4.1, p. 37] to reduce the level of the critical point of D_L to $(n - \frac{3}{4})$.

When this operation has been carried out for each basis element of C_{n-2} in H_{n-2}, the level diagram for f looks like:

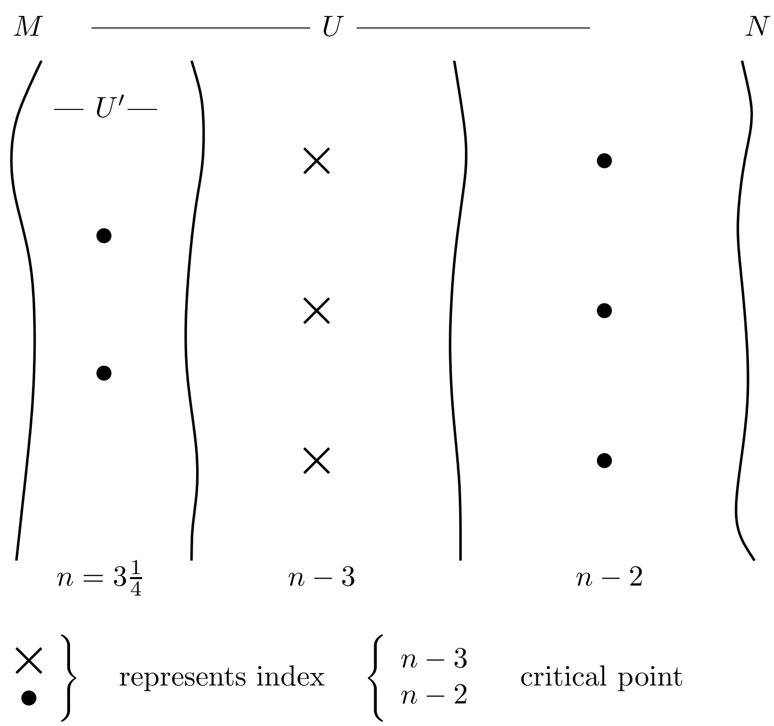


Figure 5:

Observe that $U' = f^{-1}[-\frac{1}{2}, n - \frac{3}{8}] \cap U$ can be deformed over itself onto $\operatorname{Bd} V = M$ with based $(n-2)$–disks attached which, in $\widetilde{U}$ and $\widetilde{M}$, give a basis for $H_{n-2}(\widetilde{U}, \widetilde{M})$ and so for $H_{n-2}(\widetilde{V}, \widetilde{M})$. Thus i_* is an isomorphism, in the sequence of $(\widetilde{V}, \widetilde{U}', \widetilde{M})$:

$$\cdots \longrightarrow 0 \longrightarrow H_{n-2}(\widetilde{U}', \widetilde{M}) \xrightarrow[\cong]{i_*} H_{n-2}(\widetilde{V}, \widetilde{M}) \longrightarrow H_{n-2}(\widetilde{V}, \widetilde{U}') \longrightarrow H_{n-3}(\widetilde{U}', \widetilde{M}) = 0.$$

It follows that $H_*(\widetilde{V}, \widetilde{U}') = 0$.

We assert that $V_0 = V - \operatorname{Int} U'$ is a $(n-2)$–neighborhood of ε. It will suffice to show that V_0 is a 1–neighborhood. For in that case excision shows $H_*(\widetilde{V}_0, \operatorname{Bd}\widetilde{V}_0) = H_*(\widetilde{V}, \widetilde{U}') = 0$. Now $\operatorname{Bd} V_0 \hookrightarrow V_0$ clearly gives a π_1–isomorphism.

Claim $\operatorname{Bd} V_0 \hookrightarrow V$ gives a π_1–isomorphism.

Granting that, we see that $V_0 \hookrightarrow V$ gives a π_1–isomorphism and hence $\pi_1(\varepsilon) \longrightarrow \pi_1(V_0)$ is an isomorphism – which proves that V_0 is a 1–neighborhood.

To prove the claim simply observe that $\operatorname{Bd} V_0 \hookrightarrow U - U'$ gives a π_1–isomorphism, that $(U - \operatorname{Int} U') \hookrightarrow U$ does too because $N \hookrightarrow U$ and $N \hookrightarrow (U - \operatorname{Int} U)$ do, and finally that $U \hookrightarrow V$ gives a π_1–isomorphism.

We have discovered a $(n-2)$–neighborhood V_0 in the interior of the original $(n-3)$–neighborhood. Thus Proposition 5.6 is established. □

To conclude this chapter we give some corollaries of Theorem 5.7. By Proposition 2.3 we have:

Theorem 5.8 *Suppose W is a smooth connected open manifold of dimension $\geqslant 6$. If W has finitely many ends $\varepsilon_1, \ldots, \varepsilon_k$, each tame, with invariant zero, then W is the interior of a smooth compact manifold $\overline{W}$. The converse is obvious.*

Assuming 1–connectedness at each end we get the main theorem of Browder, Levine, and Livesay [1] (slightly elaborated).

Theorem 5.9 *Suppose W is a smooth open manifold of dimension $\geqslant 6$ with $H_*(W)$ finite generated as an abelian group. Then W has finitely many ends $\varepsilon_1, \ldots, \varepsilon_k$. If π_1 is stable at each ε_i, and $\pi_1(\varepsilon_i) = 1$ then W is the interior of a smooth compact manifold.*

Proof By Theorem 1.10, there are only finite many ends. If V is a 1–neighborhood of ε_i, $\pi_1(V) = 1$, and $H_*(V)$ is finitely generate since $H_*(W)$ is. By an elementary argument, V has the type of a finite complex (c.f. Wall [2]). Thus ε_i is tame. The obstruction $\sigma(\varepsilon_i)$ is zero because every subgroup of a free abelian group is free. □

Theorem 5.10 *Let W^n, $n \geqslant 6$, be a smooth connected manifold with compact boundary and one end ε. Suppose*

1) $\mathrm{Bd}\, W \hookrightarrow W$ *is* $(n-2)$*–connected,*

2) π_1 *is stable at* ε *and* $\pi_1(\varepsilon) \longrightarrow \pi_1(W)$ *is an isomorphism.*

Then W^n *is diffeomorphic to* $\mathrm{Bd}\, X \times [0,1)$*.*

Proof By Corollary 5.2, $\mathrm{Bd}\, W \hookrightarrow W$ is a homotopy equivalence. Since W is a 1–neighborhood ε is tame. Since W is a $(n-2)$–neighborhood $\sigma(\varepsilon) = 0$ (Proposition 5.6). By Theorem 5.7, ε has a collar. Then Corollary 5.2 shows that W itself is diffeomorphic to $\mathrm{Bd}\, W \times [0,1)$. □

Remark The above theorem indicates some overlap of our result with Stallings' Engulfing Theorem, which would give the same conclusion for $n \geqslant 5$ with 1), 2) replaced by

1′) $\mathrm{Bd}\, W \hookrightarrow W$ is $(n-3)$–connected,

2′) for every compact $C \subset W$, there is a compact $D \supset \mathrm{Bd}\, W$ containing C so that $(W - D) \subset W$ is 2–connected.

See Stallings [12]. A smoothed version of the Engulfing Theorem appears in [13].

6 A Sum Theorem for Wall's Obstruction

For path-connected spaces X in the class $\mathcal{D}$ of spaces of the homotopy type of a CW complex that are dominated by a finite complex, C.T.C. Wall defines in [2] a certain obstruction $\sigma(X)$ lying in $\widetilde{K}_0(\pi_1(X))$, the group of stable isomorphism classes (see §5) of left $\pi_1(X)$–modules. The obstruction $\sigma(X)$ is an invariant of the homotopy type of X, and $\sigma(X) = 0$ if and only if X is homotopy equivalent to a finite complex. The obstruction of our Main Theorem 5.7 is, up to sign, $\sigma(V)$ for any 1–neighborhood V of the tame end ε. (See below. We will choose the sign for our obstruction $\sigma(\varepsilon)$ to agree with that of $\sigma(V)$.) The main result of this section is a sum formula for Wall's obstruction, and a complement that was useful already in §4.

Recall that $\widetilde{K}_0$ gives a covariant functor from the category of groups to the category of abelian groups. If $f\colon G \longrightarrow H$ is a group homomorphism, $f_*\colon \widetilde{K}_0(G) \longrightarrow \widetilde{K}_0(H)$ is defined as follows. Suppose given an element $[P] \in \widetilde{K}_0(G)$ represented be a f.g. projective left G–module P. Then $F_*[P]$ is represented by the left H–module $Q = \mathbb{Z}[h] \otimes_G P$ where the right action of G on $\mathbb{Z}[h]$ for the tensor product is given by $f\colon G \longrightarrow H$.

The following lemma justifies our omission of base point in writing $\widetilde{K}_0(\pi_1(X))$.

Lemma 6.1 *The composition of functors $\widetilde{K}\pi_1$ determines up to natural equivalence a covariant functor from the category of path-connected spaces* without base point *and continuous maps, to abelian groups (see below).*

Proof After an argument familiar for higher homotopy groups, it suffices to show that the automorphism θ_x of $\widetilde{K}_0(\pi_1(X,p))$ induced by the inner automorphism $g \longrightarrow x^{-1}gx$ of $\pi_1(X,p)$ is the identity for all x. If P is a f.g. projective, $\theta[P]$ is by definition represented by

$$P' = \mathbb{Z}[\pi_1(X,p)] \otimes_{\pi_1(X,p)} P$$

where the group ring has the right $\pi_1(X,p)$–action $r \cdot g = rx^{-1}gx$. From the definition of the tensor product

$$g \otimes p = 1 \otimes xgx^{-1}p.$$

Thus the map $\phi\colon\, P \longrightarrow P'$ given by $\phi(p) = 1 \otimes xp$, for $p \in P$, satisfies $\phi(gp) = 1 \otimes xgp = 1 \otimes xgx^{-1}xp = g \otimes xp = g\phi(p)$, for $g \in \pi_1(X,p)$ and $p \in P$. So ϕ gives a $\pi_1(X,p)$–module isomorphism $P \longrightarrow P'$ as required. □

If X has path components $\{X_i\}$ we define $\widetilde{K}_0(\pi_1(X)) = \sum_i \widetilde{K}_0(\pi_1(X_i))$. This clearly extends $\widetilde{K}_0\pi_1$ to a covariant functor from the category of all topological spaces and continuous maps to abelian groups. Thus for any $X \in \mathcal{D}$ with path components $X_1, \ldots, X_r$ we can define $\sigma(X) = (\sigma(X_1), \ldots, \sigma(X_r))$ in $\widetilde{K}_0(\pi_1(X)) = \widetilde{K}_0(\pi_1(X_1)) \times \cdots \times \widetilde{K}_0(\pi_1(X_r))$. And we notice that $\sigma(X)$ is, as it should be, the obstruction to X having the homotopy type of a finite complex.

For path-connected $X \in \mathcal{D}$ the invariant $\sigma(X)$ may be defined as follows (c.f. Wall [2]). It turns out that one can find a finite complex K^n for some $n \geqslant 2$, and a n–connected map $f\colon\, K^n \longrightarrow X$ that has a homotopy right inverse, i.e. a map $g\colon\, X \longrightarrow K^n$ so that $fg \simeq 1_X$. For any such map $H_i(\widetilde{M}(f), \widetilde{K}^n) = 0$, $i \neq n+1$, and $H_{n+1}(\widetilde{M}(f), \widetilde{K}^n)$ is f.g. projective over $\pi_1(X)$. The invariant $\sigma(X)$ is $(-1)^{n+1}$ the class of this module in $\widetilde{K}_0(\pi_1(X))$. (We have reversed the sign used by Wall.)

We will need the following notion of (cellular) surgery on a map $f\colon\, K \longrightarrow X$ where K is a CW complex and X has the homotopy type of one. If more than one path component of K maps into a given path component of X, one can join these components by attaching 1–cells to K, then extend f to a map $K \cup \{1-\text{cells}\} \longrightarrow X$. Suppose from now on that K and X are path-connected with fixed base points. If $\{x_i\}$ is a set of generators of $\pi_1(X)$ one can attach a wedge $\bigwedge_i S_i$ of circles to K and extend f in a natural way to a map $g\colon\ K \cup \{\bigwedge_i S_i\} \longrightarrow X$ that gives a π_1–epimorphism. If f gives a π_1–epimorphism from the outset an $\{y_i\}$ is a set in $\pi_1(K)$ whose normal closure is the kernel of $f_*\colon\ \pi_1(K) \longrightarrow \pi_1(X)$, then one can attach one 2–cell to K for each y_i and extend f to a 1–equivalence $K \cup \{2-\text{cells}\} \longrightarrow X$. Next suppose $f\colon\ K \longrightarrow X$ is $(n-1)$–connected, $n \geqslant 2$, and f is a 1–equivalence (in case $n=2$). If $\{z_i\}$ is a set of generators of $H_n(\widetilde{M}(f), \widetilde{K}) \cong \pi_n(\widetilde{M}(f), \widetilde{K}) \cong \pi_n(M(f), K)$ as a $\pi_1(X)$–module, then up to homotopy there is a natural way to attach one n–cell to K for each z_i and extend f to an n-connected map $K \cup \{n-\text{cells}\} \longrightarrow X$ (see Wall [2, p. 59]). Of course we always assume that the attaching maps are cellular so that $K \cup \{n-\text{cells}\}$ is a complex. Also, if X is a complex and f is cellular we can assume the extension of f to the enlarged complex is cellular. (See the cellular approximation theorem of Whitehead [11].)

Here is a lemma we will frequently use.

Lemma 6.2 *Suppose X is a connected CW complex and $f\colon\ K \longrightarrow X$ is a map of a finite complex to X that is a 1–equivalence. If $H_*(\widetilde{M}(f), \widetilde{K}) = P$ is a f.g. projective $\pi_1(X)$–module isolated in one dimension m, then $X \in \mathcal{D}$ and $\sigma(X) = (-1)^m[P]$.*

Proof Clearly it is enough to consider the case where K and X are connected. The argument for Theorem E of [2, p. 63] shows that X is homotopy equivalent to K with infinitely many cells of dimension m and $m+1$ attached. Hence X has the type of a complex of dimension $\max(\dim K, m+1)$.

Choose finitely many generators $x_i, \ldots, x_r$ for $H_m(\widetilde{M}(f), \widetilde{K})$. Perform the corresponding surgery, attaching r m–cells to K and extending f to a m–connected map

$$f'\colon\ K' = K \cup \{m-\text{cells}\} \longrightarrow X.$$

Up to homotopy we may assume that $K \subset K' \subset X$. Then the homology sequence of $\widetilde{K} \subset \widetilde{K}' \subset \widetilde{X}$ shows that $H_{m+1}(\widetilde{M}(f'), \widetilde{K}') = Q$ where $P \oplus Q = \Lambda^n$, $\Lambda = \mathbb{Z}[\pi_1(X)]$, and that $H_i(\widetilde{M}(f'), \widetilde{K}') = 0$, $i \neq m+1$. Notice that Q has class $-[P]$.

After finitely many such steps we get a finite complex L of dimension $n = \max(\dim K, m+1)$ and a n–connected map

$$g\colon\ L^n \longrightarrow X$$

such that $H_*(\widetilde{M}(g), \widetilde{L})$ is f.g. projective isolated in dimension $n+1$ and has class $(-1)^{n+1-m}[P]$. According to [2, Lemma 3.1] g has a homotopy right inverse. Thus $X \in \mathcal{D}$ and

$$\sigma(X) = (-1)^{n+1}(-1)^{n+1-m}[P] = (-1)^m[P].$$

□

The following are established by Wall in [2].

Lemma 6.3 *[2, Theorems E and F] Each $X \in \mathcal{D}$ is homotopy equivalent to a finite dimensional complex.*

Lemma 6.4 *[2, Lemma 2.1 and Theorem E] (c.f. Proposition 5.1) If X is homotopy equivalent to an m–complex and $f\colon L^{n-1} \longrightarrow X$, $n \geqslant 3$, $n \geqslant m$, is a $(n-1)$–connected map of an $(n-1)$–complex to X, then $H_*(\widetilde{M}(f), \widetilde{L})$ is a projective $\pi_1(X)$–module isolated in dimension n.*

Sum Theorem 6.5 *Suppose that a connected CW complex X is a union of two connected subcomplexes X_1 and X_2. If X, X_1, X_2, and $X_0 = X_1 \cap X_2$ are in $\mathcal{D}$, then*

$$\sigma(X) = j_{1*}\sigma(X_1) + j_{2*}\sigma(X_2) - j_{0*}\sigma(X_0)$$

where j_{k*} is induced by $X_k \hookrightarrow X$, $k = 0, 1, 2$.

Complement 6.6 *(a) $X_0, X_1, X_2 \in \mathcal{D}$ implies $X \in \mathcal{D}$.*

(b) $X_0, X \in \mathcal{D}$ implies $X_1, X_2 \in \mathcal{D}$ provided $\pi_1(X_i) \longrightarrow \pi_1(X)$ has a left inverse, $i = 1, 2$.

Remark 1 Notice that X_0 is not in general connected. Written out in full, the last term of the sum formula is

$$j_{0*}\sigma(X) = j_{0*}^{(1)}\sigma(Y_1) + \cdots + j_{0*}^{(s)}\sigma(Y_s)$$

where $Y_1, \ldots, Y_s$ are the components of X_0. We have assumed that X, X_1, and X_2 are connected. Notice that Complement 6.6 part (b) makes sense only when X, X_1, and X_2 are connected. But the assumption is unnecessary for Theorem 6.5 and Complement 6.6 (a). In fact by repeatedly applying the given versions one easily deduces the more general versions.

Remark 2 In the Complement, part (b), some restriction on fundamental groups is certainly necessary.

For a first example let X_1 be the complement of an infinite string in $\mathbb{R}^3$ that has an infinite sequence of knots tied in it. Let X_2 be a 2–disk cutting the string. Then $X_0 = X_1 \cap X_2 \cong S^1$, and $X = X_1 \cup X_2$ is contractible since $\pi_1(X) = 1$. Thus X_0 and X are in $\mathcal{D}$. However $X_1 \notin \mathcal{D}$ because $\pi_1(X_1)$ is not finitely generated. To see this observe that $\pi_1(X_1)$ is an infinite free product with amalgamation over $\mathbb{Z}$

$$\cdots *_{\mathbb{Z}} G_{-1} *_{\mathbb{Z}} G_0 *_{\mathbb{Z}} G_1 *_{\mathbb{Z}} G_2 *_{\mathbb{Z}} \cdots$$

($\mathbb{Z}$ corresponds to a small loop around the string and G_i is the group of the i–th knot.) Thus $\pi_1(X_1)$ has an infinite ascending sequence $H_1 \subsetneq H_2 \subsetneq H_3 \subsetneq \dots$ of subgroups. And this clearly shows that $\pi_1(X_1)$ is not finitely generated.

For examples where the fundamental groups are all finitely presented see the contractible manifolds constructed in §8.

Question *It is enough to assume in Complement 6.6 part* (*b*)*, that* $\pi_1(X_i) \longrightarrow \pi_1(X)$ *is* $(1-1)$*, for* $i = 1, 2$*?*

Proof of Theorem 6.5 To keep notation simple we assume for the proof that X_0 is connected. At the end of the proof we point out the changes necessary when X_0 is not connected.

By Lemma 6.3 we can suppose that X_0, X_1, X_2, X are all equivalent to complexes of dimension $\leqslant n$, $n \geqslant 3$. Using the surgery process with Lemma 3.8 and Lemma 4.7 we can find a $(n-1)$–connected cellular map $f_0 \colon K_0 \longrightarrow X_0$. Surgering the composed map $K_0 \longrightarrow X_0 \hookrightarrow X_i$ for $i = 1, 2$, we get finite $(n-1)$–complexes K_1, K_2 with $K_1 \cap K_2 = K_0$ and $(n-1)$–connected cellular maps $f_1 \colon K_1 \longrightarrow X_1$, $f_2 \colon K_2 \longrightarrow X_2$ that coincide with f_0 on K_0. Together they give a $(n-1)$–connected map $f \colon K = K_1 \cup K_2 \longrightarrow X = X_1 \cup X_2$. For f gives a π_1–isomorphism by Van Kampen's theorem; and f is $(n-1)$–connected according to the homology of the following short exact sequence.

$$0 \longrightarrow C_*(\overline{M}(f_0), \overline{K_0}) \xrightarrow{\phi} C_*(\overline{M}(f_1), \overline{K_1}) \oplus C_*(\overline{M}(f_2), \overline{K_2}) \xrightarrow{\psi} C_*(\widetilde{M}(f), \widetilde{K}) \longrightarrow 0$$

Here $\overline{S}$ denotes $p^{-1}(S)$, $p \colon \widetilde{M}(f) \longrightarrow M(f)$ being the universal cover of the mapping cylinder. Also $\phi(c) = (c, c)$ and $\psi(c_1, c_2) = c_1 - c_2$. To be specific let the chain complexes be for cellular theory. Each is a free $\pi_1(X)$–complex.

Let us take a closer look at the above exact sequence. For brevity we write it:

$$0 \longrightarrow \overline{C}(0) \longrightarrow \overline{C}(1) \oplus \overline{C}(2) \longrightarrow \widetilde{C} \longrightarrow 0. \tag{3}$$

We will establish below that

For $k = 0, 1, 2$, $H_i\overline{C}(k) = 0$, $i \neq n$, and $H_n\overline{C}(k)$ is f.g. projective of class $(-1)^n j_{k*}\sigma(X_k)$. (4)

From this the sum formula follows easily. Since we assumed X is equivalent to a complex of dimension $\leqslant n$, Lemmas Lemma 6.4, Lemma 4.7, Lemma 6.2 tell us that $H_i(\widetilde{C}) = H_i(\widetilde{M}(f), \widetilde{K}) = 0$ is $i \neq n$, and $H_n(\widetilde{C})$ is f.g. projective of class $(-1)^n \sigma(X)$. Thus the homology sequence of (3) is

$$0 \longrightarrow H_n\overline{C}(0) \longrightarrow H_n\overline{C}(1) \oplus H_n\overline{C}(2) \longrightarrow H_n\widetilde{C} \longrightarrow 0$$

and splits giving the desired formula. □

To prove (3) consider:

Lemma 6.7 *$\overline{C}(k) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(X_k)} \widetilde{C}(k)$, $k = 0, 1, 2$, where $\widetilde{C}(k) = C_*(\widetilde{M}(f_k), \widetilde{K}_k)$, and for the tensor product $\mathbb{Z}[\pi_1(X)]$ has the right $\pi_1(X_k)$–module structure given by $\pi_1(X_k) \longrightarrow \pi_1(X)$.*

Now recall that $H_*(\widetilde{C}(k))$ is f.g. projective of class $(-1)^n\sigma(X_k)$ and concentrated in dimension n. Then Lemma 6.7 shows that $H_i(\overline{C}(k)) = 0$, $i \neq n$, and $H_n(\overline{C}(k)) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(X_k)} H_n(\widetilde{C}(k))$, which is f.g. projective over $\pi_1(X)$ of class $(-1)^n j_{k*}\sigma(X_k)$. (Use the universal coefficient theorem [42, p. 113] or argue directly.) This establishes (3) and the Sum Theorem modulo a proof of Lemma 6.7.

Proof of Lemma 6.7 Fix k as 0, 1, 2. The map $j_k\colon \pi_1(X_k) \longrightarrow \pi_1(X)$ factors through $\mathrm{Im}(j_k) = G$. Then

$$\mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(X_k)} \widetilde{C}(k) = \mathbb{Z}[\pi_1(X)] \otimes_G G \otimes_{\pi_1(X_k)} \widetilde{C}(k).$$

Step 1. Let $(\widehat{M}(f_k), \widehat{X}_k)$ be the component of $(\overline{M}(f_k), \overline{X}_k)$ containing the base point. Apparently it is the G–fold regular covering corresponding to $\pi_1(X_k) \longrightarrow G$. Then $\widehat{C}(k) = C_*(\widehat{M}(f_k), \widehat{X}_k)$ is a free G–module with one generator for each cell e of $M(f_k)$ outside X_k. Choosing one covering cell $\widehat{e}$ for each, we get a preferred basis for $\widehat{C}(k)$. Now the universal covering $(\widetilde{M}(f_k), \widetilde{X}_k)$ is naturally a cover of $(\widehat{M}(f_k), \widehat{X}_k)$. So in choosing a free $\pi_1(X_k)$–basis for $\widetilde{C}(k)$ we can choose the cell $\widetilde{e}$ over e to lie above $\widehat{e}$. Suppose the boundary formula for $\widetilde{C}(k)$ reads $\partial\widetilde{e}_i^n = \sum_j r_{ij}\widetilde{e}_j^{n-1}$ where $r_{ij} \in \mathbb{Z}[\pi_1(X_k)]$. Then one can verify that the boundary formula for $\widehat{C}(k)$ reads $\partial\widehat{e}_i^n = \sum_j \theta(r_{ij})e_j^{n-1}$ where θ is the map of $\mathbb{Z}[\pi_1(X_k)]$ onto $\mathbb{Z}[G]$. By inspecting the definitions we see that this means $\widehat{C}(k) = G \otimes_{\pi_1(X_k)} \widetilde{C}(k)$.

Step 2. We claim $\overline{C}(k) = \mathbb{Z}[\pi_1(X)] \otimes_G \widehat{C}(k)$.

$\overline{C}(k)$ is a free $\pi_1(X)$–module and we may assume that the distinguished cell $\widetilde{e}$ over any cell e in $M(f_k)$ coincides with $\widehat{e}$ in $\widehat{M}(f_k) \subset \widetilde{M}(f_k)$. Then if the boundary formula for $\widehat{C}(k)$ reads

$$\partial \widehat{e}_i^n = \sum_j s_{ij} \widehat{e}_j^{n-1}, s_{ij} \in \mathbb{Z}[G]$$

the boundary formula for $\overline{C}(k)$ is exactly the same except that s_{ij} is to be regarded as an element of the larger ring $\mathbb{Z}[\pi_1(X)]$. Going back to the definitions again we see this verifies our claim. This completes the proof of Lemma 6.7. $\square$

Remark *On the general case of Theorem 6.5 where X_0 is not connected* Let X_0 have components $Y_1, \ldots, Y_s$. We pick base points $p_i \in Y_i$, $i = 1, \ldots, s$ and let p_1 be the common basepoint for X_1, X_2, and X. Choose a path γ_i from p_i to p_1 to define homomorphisms $j_0^{(i)} \colon \pi_1(Y_i) \longrightarrow \pi_1(X)$, $i = 1, \ldots, s$. (By Lemma 6.1 the homomorphism $j_{0*}^{(i)} \colon \widetilde{K}_0(\pi_1(Y_i)) \longrightarrow \widetilde{K}_0(\pi_1(X))$ does not depend on the choice of γ_i.) Now consider again the proof of Theorem 6.5. Everything said up to Lemma 6.7 remains valid. Notice that

$$\overline{C}(0) = C_*(\overline{M}(f_0), \overline{K}_0) = \bigoplus_{i=1}^{s} C_*(\overline{M}(g_i), \overline{L}_i)$$

where L_i is the component of K_0 corresponding to the component Y_i of X_0 under f_0, and $g_i \colon L_i \longrightarrow Y_i$ is the map given by f_0. For short we write this $\overline{C}(0) = \oplus_{i=1}^{s} \overline{C}(0,i)$. For $k = 0$, the assertion of Lemma 6.7 should be changed to

$$\overline{C}(0,i) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(Y_i)} \widetilde{C}(0,i), i = 1, \ldots, s \tag{5}$$

where $\widetilde{C}(0,i) = C_*(\widetilde{M}(g_i), \widetilde{L}_i)$ and for the i–th tensor product, $\mathbb{Z}[\pi_1(X)]$ has the right $\pi_1(Y_i)$–action given by the map $\pi_1(Y_i) \longrightarrow \pi_1(X)$. Granting this, an obvious adjustment of the original argument will establish (4). The argument given for Lemma 6.7 establishes (5) with slight change. Here is the beginning. We fix i, $0 \leqslant i \leqslant s$, and let H be the image of $\pi_1(Y_i) \longrightarrow \pi_1(X)$. Then

$$\mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(Y_i)} \widetilde{C}(0,i) = \mathbb{Z}[\pi_1(X)] \otimes_H H \otimes_{\pi_1(Y_i)} \widetilde{C}(0,i).$$

Step 1. Let $(\widehat{M}(g_i), \widehat{Y}_i)$ be the component of $(\overline{M}(g_i), \overline{Y}_i)$ containing the lift $\widehat{p}_i$ in $M(f)$ of p_i by the path γ_i^{-1} from p_1 to p_i. Apparently it is the $H-$fold regular covering corresponding to $\pi_1(Y_i) \xrightarrow{\text{onto}} H$. The rest of Step 1 and Step 2 give no new difficulties. They prove respectively that $C(\widehat{M}(g_i), \widehat{Y}_i) = H \otimes_{\pi_1(Y_i)} \widetilde{C}(0,i)$ and $\overline{C}(0,i) = \mathbb{Z}[\pi_1(X)] \otimes_H C(\widehat{M}(g_i), \widehat{Y}_i)$, and thus establish (5). This completes the exposition of the Sum Theorem 6.5.

Proof of Complement 6.6 part (a) We must show $X_0, X_1, X_2 \in \mathcal{D}$ implies $X \in \mathcal{D}$. The proof is based on:

Lemma 6.8 *Suppose X_0 has the type of a complex of dimension $\leqslant n-1$, $n \geqslant 3$; and X_1, X_2 have the type of a complex of dimension $\leqslant n$. Then X has the homotopy type of a complex of dimension $\leqslant n$.*

Proof Let K_0 be a complex of dimension $\leqslant n-1$ so that there is a homotopy equivalence $f_0\colon K_0 \longrightarrow X_0$. Surgering f_0 we can enlarge K_0 and extend f_0 to an $(n-1)$–connected map of a $(n-1)$–complex $F_1\colon K_1 \longrightarrow X_1$. Similarly form $f_2\colon K_2 \longrightarrow X_2$. According to Lemma 6.3 the groups $H_n(\widetilde{M}(f_i), \widetilde{K}_i)$, $i = 1, 2$, are projective $\pi_1(X_i)$–modules. Then surgering f_1, f_2, we can add $(n-1)$–cells and n–cells to K_1, K_2 and extend f_1, f_2 to homotopy equivalences $g_1\colon L_1^n \longrightarrow X_1$, $g_2\colon L_2^n \longrightarrow X_2$. (See the proof of Theorem E on p. 63 of Wall [2]). Since $L_1 \cap L_2 = K_0$ and g_1, g_2 coincide with f_0 on K_0, we have a map $g\colon L = L_1 \cup L_2 \longrightarrow X = X_1 \cup X_2$ which is apparently a homotopy equivalence of the n–complex L with X. □

For the proof of Complement 6.6 part (a), we simply look back at the proof of the Sum Theorem and omit the assumption that $X \in \mathcal{D}$. By the above Lemma we can still assume X_0, X_1, X_2, X are equivalent to complexes of dimension $\leqslant n$, $(n \geqslant 3)$. Lemma 6.4 says that $H_*(\widetilde{C})$ is projective and isolated in dimension n. The exact homology sequence shows that $H_n(\widetilde{C})$ is f.g. Then Lemma 6.2 says $X \in \mathcal{D}$. □

Proof of Complement 6.6 part (b) We must show that $X_0, X \in \mathcal{D}$ implies $X_1, X_2 \in \mathcal{D}$ provided $\pi_1(X_j)$ is a retract of $\pi_1(X)$, $j = 1, 2$. Let X_0 have components $Y_1, \ldots, Y_s$ and use the notations of this section.

Since $\pi_1(X)$ is finitely presented so are $\pi_1(X_1)$, $\pi_1(X_2)$ by Lemma 3.8. This shows that the following proposition P_x holds with $x = 1$.

(P_x) : There exists a finite complex K^x (or K^2 is $x = 1$) that is a union of subcomplexes K_1, K_2 with intersection K_0, and a map $F\colon K \longrightarrow X$ so that, restricted to K_k, f gives a map $f_k\colon K_k \longrightarrow X_k$, $k = 0, 1, 2$, which is x–connected and a 1–equivalence if $x = 1$.

Suppose for induction that P_{n-1} holds, $n \geqslant 2$, and consider the exact sequence

$$0 \longrightarrow C_*(\overline{M}(f_0), \overline{K}_0) \longrightarrow C_*(\overline{M}(f_1), \overline{K}_1) \oplus C_*(\overline{M}(f_2), \overline{K}_2) \longrightarrow C_*(\widetilde{M}(f), \widetilde{K}) \longrightarrow 0$$

where $\overline{S} = p^{-1}(S)$, $p\colon \widetilde{M}(f) \longrightarrow M(f)$ being the universal cover. For short we write

$$0 \longrightarrow \overline{C}(0) \xrightarrow{\phi} \overline{C}(1) \oplus \overline{C}(2) \xrightarrow{\psi} \widetilde{C} \longrightarrow 0.$$

Part of the associated homology sequence is

$$H_n\overline{C}(0) \xrightarrow{\phi_*} H_n\overline{C}(1) \oplus H_n\overline{C}(2) \xrightarrow{\psi_*} H_n\widetilde{C} \longrightarrow H_{n-1}\overline{C}(0) = 0. \qquad (6)$$

Now $H_n(\widetilde{C})$ is f.g. over $\pi_1(X)$ by Lemma 4.7. Similarly, for each component Y_i of X_0, the corresponding summand $H_n(\widetilde{C}(0,i))$ of $H_n(\widetilde{C}(0)) = H_n(\widetilde{M}(f_0), \widetilde{K}_0)$ is f.g. over $\pi_1(Y_i)$. Since

$$\overline{C}(0,i) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(Y_i)} \widetilde{C}(0,i) \quad \text{(this is (5))}$$

and since $\widetilde{C}(0,i)$ is acyclic below dimension n, the right exactness of $\otimes$ shows that $H_n(\overline{C}(0,i)) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(Y_i)} H_n(\widetilde{C}(0,i))$. Hence $H_n\overline{C}(0) = \oplus_{i=1}^{s} H_n(\overline{C}(0,i))$ is finitely generated over $\pi_1(X)$. Thus (6) shows that $H_n(\overline{C}(j))$ is f.g. over $\pi_1(X)$, $j = 1,2$. (This uses the fact that ψ_* is onto!)

We would like to conclude that $H_n(\widetilde{C}(j))$ is f.g., $j = 1,2$. In fact we have

$$H_n(\widetilde{C}(j)) = \mathbb{Z}[\pi_1(X_j)] \otimes_{\pi_1(X)} H_n(\overline{C}(j)) \qquad (7)$$

where a retraction $\pi_1(X) \longrightarrow \pi_1(X_j)$ makes $\mathbb{Z}[\pi_1(X_j)]$ a $\pi_1(X)$–module. For $H_n(\overline{C}(j)) = \mathbb{Z}[\pi_1(X)] \otimes_{\pi_1(X_j)} H_n(\widetilde{C}(j))$ and $\mathbb{Z}[\pi_1(X_j)] \otimes_{\pi_1(X)} \mathbb{Z}[\pi_1(X)] = \mathbb{Z}[\pi_1(X_j)]$. So (7) is verified by substituting for $H_n(\overline{C}(j))$.

Since $H_n(\widetilde{C}(k))$ are f.g., $k = 0,1,2$, we can surger f to establish P_n. This completes the induction. The proof that $X_1, X_2 \in \mathcal{D}$ is completed as follows. We can suppose that X_0 and X have the homotopy type of an n–dimensional complex (Lemma 6.3), and that $f\colon K \longrightarrow X$ is a $(n-1)$–connected map as in P_{n-1}. Then in the exact sequence (6), $H_*(\overline{C}(0)$ and $H_*(\widetilde{C})$ are f.g. projective and concentrated in dimension n for $j = 1,2$. Then by the argument of the previous paragraph $H_*(\widetilde{C}(j))$ is f.g. projective over $\pi_1(X_j)$ are concentrated in dimension n, $j = 1,2$. By Lemma 6.2 $X_j \in \mathcal{D}$, $j = 1,2$. This completes the proof of the Complement to Complement 6.6. □

In passing we point out the analogous sum theorem for Whitehead torsion.

Theorem 6.9 *Let X, X' be two finite connected complexes each the union of two connected subcomplexes $X = X_1 \cup X_2$, $X' = X'_1 \cup X'_2$. Let $f\colon X \longrightarrow X'$ be a map that restricts to give maps $f_1\colon X_1 \longrightarrow X'_1$, $f_2\colon X_2 \longrightarrow X'_2$ and* *so* $f_0\colon\, X_0 = X_1 \cap X_2 \longrightarrow X_0' = X_1' \cap X_2'$. *If* f_0, f_1, f_2, f *are all homotopy equivalences then*

$$\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) - \sum_{i=1}^{s} j_{0*}^{(i)}(f_0^{(i)})$$

where j_{k*} *is induced by* $X_k \hookrightarrow X$, $k = 1, 2$, $X_0^{(1)}, \dots, X_0^{(s)}$ *are the components of* X_0 *and* $j_{0*}^{(i)}$ *is induced by* $X_0^{(i)} \hookrightarrow X$, $i = 1, \dots, s$.

Complement 6.10 *If* f_0, f_1, f_2 *are homotopy equivalences so is* f. *If* f_0 *and* f *are homotopy equivalences so are* f_1 *and* f_2 *provided that* $\pi_1(X_i) \longrightarrow \pi_1(X)$ *has a left inverse,* $i = 1, 2$.

We leave the proof on one side. It is similar to and rather easier than for Wall's obstruction. A special case is proved by Kwun and Szczarba [19].

With the Sum Theorem 6.5 established we are in a position to relate our invariant for tame ends to Wall's obstruction. Lemma 6.2 and Proposition 5.6 together show that if ε is a tame end of dimension $\geqslant 5$ and V is a $(n-3)$–neighborhood of ε, then up to sign (which we never actually specified), $\sigma(\varepsilon)$ corresponds to $\sigma(V)$ under the natural identification of $\widetilde{K}_0(\pi_1(\varepsilon))$ with $\widetilde{K}_0(\pi_1(V))$. Let us agree that $\sigma(\varepsilon)$ is to be the class $(-1)^{n-2}[H_{n-2}(\widetilde{V}, \mathrm{Bd}\,\widetilde{V})] \in \widetilde{K}_0(\pi_1(\varepsilon))$ (compare Proposition 5.6). Then signs correspond.

Here is a definition of $\sigma(\varepsilon)$ in terms of Wall's obstruction. ε is a tame end of dimension $\geqslant 5$. Suppose V is a closed neighborhood of ε that is a *smooth* submanifold with compact frontier and one end, so small that

$$\pi_1(\varepsilon) \longrightarrow \pi_1(V)$$

has a left inverse r.

Proposition 6.11 $\sigma(\varepsilon) = r_*\sigma(V)$.

Proof Take a $(n-3)$–neighborhood $V' \subset \mathrm{Int}\,V$. Then $V - \mathrm{Int}\,V'$ is a compact smooth manifold. So the Sum Theorem says $\sigma(V) = i_*\sigma(V')$ where i is the map $\pi_1(V') = \pi_1(\varepsilon) \longrightarrow \pi_1(V)$. Since $r_*i_*\sigma(V') = \sigma(V')$ we get $r_*\sigma(V) = \sigma(V) = \sigma(\varepsilon)$. □

A direct consequence of the Sum Theorem is that if W^n $n \geqslant 5$ is a smooth manifold with $\mathrm{Bd}\,W$ compact that has finitely many ends $\varepsilon_1, \dots, \varepsilon_k$, all tame, then

$$\sigma(W) = j_{1*}\sigma(\varepsilon_1) + \ldots + j_{k*}\sigma(\varepsilon_k)$$

where $j_s\colon \pi_1(\varepsilon_s) \longrightarrow \pi_1(W)$ is the natural map, $s = 1,\ldots,k$. Notice that $\sigma(W)$ may be zero while some of $\sigma(\varepsilon_1),\ldots,\sigma(\varepsilon_k)$ are nonzero. One can use the constructions of §8 to give examples. On the other hand, if there is just one end, ε_1, $\sigma(W) = j_{1*}\sigma(\varepsilon_1)$; so if j_{1*} is an isomorphism $\sigma(W)$ determines $\sigma(\varepsilon_1)$. In this situation $\sigma(\varepsilon_1)$ is a topological invariant of W since $\sigma(W)$ and j_{1*} are. Theorem 6.12 below points out a large class of examples. In general I am unable to decide whether the invariant of a tame end depends on the smoothness structure as well as the topological structure (See §11).

Theorem 6.12 *Suppose W is a smooth open manifold of dimension $\geqslant 5$ that is homeomorphic to $X \times \mathbb{R}^2$ where X is an open topological manifold in $\mathcal{D}$. Then W has one end ε and ε is tame. Further $j\colon \pi_1(\varepsilon) \longrightarrow \pi_1(W)$ is an isomorphism.*

Proof Identify W with $X \times \mathbb{R}^2$ and consider complements of sets $K \times D$ where $K \subset X$ is compact and D is a closed disk in $\mathbb{R}^2$. The complement is a connected smooth open neighborhood of ∞ that is the union of $W \times (\mathbb{R}^n - D)$ and $(W-K)\times\mathbb{R}^2$. Applying Van Kampen's theorem one finds that $\pi_1((W-K)\times D) \longrightarrow \pi_1(W)$ is an isomorphism. We conclude that W has one end ε, π_1 is stable at ε, and $j\colon\ \pi_1(\varepsilon) \longrightarrow \pi_1(W)$ is an isomorphism. Since $W \in \mathcal{D}$ $\pi_1(W)$ is finitely presented (c.f. Lemma 3.8). Thus ε has small 1–neighborhoods by Theorem 3.10. By Complement 6.6 part (b) each is in $\mathcal{D}$. Hence ε is tame. □

7 A Product Theorem for Wall's Obstruction

The Product Theorem 7.2 takes the wonderfully simple form $\rho(X_1 \times X_2) = \rho(X_1)\otimes\rho(X_2)$ if for path-connected X in $\mathcal{D}$ we define the composite invariant $\rho(X) = \sigma(X)\oplus\chi(X)$ in the Grothendieck group $K_0(\pi_1(X)) \cong \widetilde{K}_0(\pi_1(X))\oplus\mathbb{Z}$. I introduce ρ for aesthetic reasons. We could get by with fewer words using σ and $\widetilde{K}_0$ alone.

The Grothendieck group $K_0(G)$ of finitely generated (f.g) projective modules over a group G may be defined as follows. Let $\mathcal{P}(G)$ be the abelian monoid of isomorphism classes of f.g. projective G–modules with addition given by direct sum. We write $(P',P) \sim (Q',Q)$ for elements of $\mathcal{P}(G)\times\mathcal{P}(G)$ if there exists free $R \in \mathcal{P}(G)$ so that $P'+Q+R = P+Q'+R$. This is an equivalence relation, and $\mathcal{P}(G)\times\mathcal{P}(G)/\sim$ is the abelian group $K_0(G)$.

Let $\phi\colon\ \mathcal{P}(G) \longrightarrow K_0(G)$ has the following natural homomorphism given by $P \longrightarrow (0,P)$. It is apparent that $\phi(P) = \phi(Q)$ if and only if $P+F = Q+F$

for some f.g. free module F. For convenience we will write $\phi(P) = \overline{P}$; we will even write $\overline{P}_0$ for ϕ applied to the isomorphism class of a given f.g. projective module P_0.

$\phi\colon\ \mathcal{P}(G) \longrightarrow K_0(G)$ has the following universal property. If $f\colon\ \mathcal{P}(G) \longrightarrow A$ is any homomorphism there is a unique homomorphism $g\colon\ K(G) \longrightarrow A$ so that $f = g\phi$. As an application suppose $\theta\colon\ G \longrightarrow H$ is any group homomorphism. There is a unique induced homomorphism $\mathcal{P}(G) \longrightarrow \mathcal{P}(H)$ (c.f. §6). By the universal property of ϕ there is a unique homomorphism θ_* that makes the diagram on the next page commute. In this way K_0 gives a covariant functor from groups to abelian groups.

$$\begin{tikzcd}
\mathcal{P}(G) \arrow[r] \arrow[d, "\phi"'] & \mathcal{P}(H) \arrow[d, "\phi"] \\
K_0(G) \arrow[r, "\theta_*"'] & K_0(H)
\end{tikzcd}$$

The diagram

$$\begin{tikzcd}
G \arrow[r, shift left, "r"] & 1 \arrow[l, shift left, "i"]
\end{tikzcd}$$

shows that $K_0(G) \cong \ker(r_*) \oplus K_0(1)$. Now 1–modules are just abelian groups; so $K_0(1) \cong \mathbb{Z}$. Notice that $r_*\colon\ K_0(G) \longrightarrow \mathbb{Z}$ is induced by assigning to $P \in \mathcal{P}(G)$ the rank of P, i.e. the rank of $\mathbb{Z} \otimes_G P$ as an abelian group (here $\mathbb{Z}$ has the trivial action of G on the right). Next observe that by associating to a class $[P] \in \widetilde{K}_0(G)$ the element $\overline{P} - \overline{F}_P \in \ker(r_*)$, where F_P is free on $p = \mathrm{rank}(\mathbb{Z} \otimes_G P)$ generators, one gets a natural isomorphism $\widetilde{K}_0(G) \cong \ker(r_*)$. Thus we have

$$K_0(G) \cong \widetilde{K}_0(G) \oplus \mathbb{Z}$$

and for convenience we regard $\widetilde{K}_0(G)$ and $\mathbb{Z}$ as subgroups.

The commutative diagram

$$\begin{tikzcd}
G \arrow[rr, "\theta"] \arrow[dr, "r"] & & H \arrow[dl, "r"] \\
& 1 & 1
\end{tikzcd}$$

shows that the map $\theta_*\colon\ K_0(G) \longrightarrow K_0(H)$ induces a map $\theta_*\colon\ \widetilde{K}_0(G) \longrightarrow \widetilde{K}_0(H)$; and the latter determines the former because the $\mathbb{Z}$ summand is mapped by a natural isomorphism. The latter is of course the map described in §6.

If G and H are two groups, a pairing

$$\text{`}\otimes\text{'}\colon\ K_0(G) \times K_0(H) \longrightarrow K_0(G \times H)$$

is induced by tensoring projectives. (Recall that if $A \otimes B$ is a tensor product of abelian groups and A has a left G–action while B has a left H–action, then $A \otimes B$ inherits a left $G \times H$–action.) This pairing carries $\ker(r_*) \times \ker(r_*)$ into $\ker(r_*)$ and so a pairing

$$\text{`}\cdot\text{'}\colon\ \widetilde{K}_0(G) \times \widetilde{K}_0(H) \longrightarrow \widetilde{K}_0(G \times H)$$

is induced. Thus if $P \in \mathcal{P}(G)$, $Q \in \mathcal{P}(H)$ the class $[P] \cdot [Q] \in \widetilde{K}_0(G \times H)$ is $(\overline{P} - \overline{F}_p) \otimes (\overline{Q} - \overline{F}_q) = \overline{P} \otimes \overline{Q} - \overline{F}_p \otimes \overline{Q} - \overline{P} \otimes \overline{F}_q + \overline{F}_p \otimes \overline{F}_q$, where F_p is free over G on $p = r_*(\overline{P})$ generators and F_q is free over H on $q = r_*(\overline{Q})$ generators.

Since an inner automorphism of G gives the identity map of $\mathcal{P}$ (c.f. Lemma 6.1) and so of $K_0(G)$ (and $\widetilde{K}_0(G)$), it follows that the composition of functors $K_0(\pi_1)$ (or $\widetilde{K}_0(\pi_1)$) determines a covariant functor from path-connected topological spaces to abelian groups. More precisely we must fix some base point for each path-connected space X to define $K_0(\pi_1(X))$ (or $\widetilde{K}_0(\pi_1(X))$), but a different choice of base points leads to a naturally equivalent functor. This is the precise meaning of Lemma 6.1 for $\widetilde{K}_0(\pi_1)$.

Definition 7.1 If $X \in \mathcal{D}$ is path-connected, define

$$\rho(X) \in K_0(\pi_1(X)) \cong \widetilde{K}_0(\pi_1(X)) \oplus \mathbb{Z}$$

to be $\sigma(X) \oplus \chi(X)$ where $\chi(X) = \sum_i (-1)^i \operatorname{rank}\ H_i(X)$ is the Euler characteristic of X (it is well defined since $X \in \mathcal{D}$).

If X is a space with path-components $\{X_i\}$ we define $K_0(\pi_1(X)) = \oplus_i K_0(\pi_1(X_i))$. This extends $K_0(\pi_1)$ to a functor on all topological spaces. Then if $X \in \mathcal{D}$ has path components $X_i, \ldots, X_s$ we define $\rho(X) = (\rho(X_1), \ldots, \rho(X_s))$ in $K_0(\pi_1(X)) = K_0(\pi_1(X_1)) \oplus \cdots \oplus K_0(\pi_1(X_s))$.

Suppose X_1 and X_2 are path-connected. Then $X_1 \times X_2$ is path-connected and $\pi_1(X_1 \times X_2) = \pi_1(X_1) \times \pi_1(X_2)$. Hence we have a pairing

$$\text{`}\otimes\text{'}\colon\ K_0(\pi_1(X_1)) \times K_0(\pi_1(X_2)) \longrightarrow K_0(\pi_1(X_1 \times X_2)).$$

This pairing extends naturally to the situation where X_1 and X_2 are not path-connected.

Product Theorem 7.2 *Let X_1, X_2, and $X_1 \times X_2$ be connected CW complexes. If X_1, X_2, and $X_1 \times X_2$ are in $\mathcal{D}$, then*

$$\rho(X_1 \times X_2) = \rho(X_1) \otimes \rho(X_2). \tag{8}$$

In terms of the obstruction σ this says

$$\sigma(X_1 \times X_2) = \sigma(X_1) \cdot \sigma(X_2) + \{\chi(X_2) j_{1*}\sigma(X_1) + \chi(X_1) j_{2*}\sigma(X_2)\}. \qquad (9)$$

Complement 7.3 *If X_1, X_2 are any spaces,*

$$X_1, X_2 \in \mathcal{D} \Leftrightarrow X_1 \times X_2 \in \mathcal{D}.$$

Remark We can immediately weaken the assumptions of Theorem 7.2 in two ways:

(a) Since σ and ρ are invariants of homotopy type, it is enough to assume that X_1, X_2, and (hence) $X_1 \times X_2$ are path-connected spaces in $\mathcal{D}$ in order to get (8) and (9).

(b) Further, if X_1, X_2 are any spaces in $\mathcal{D}$ (8) continues to hold with the extended pairing $\otimes$ (because of the way $\otimes$ is extended). But note that (9) has to be revised since $K_0(\pi_1(X)) \ncong \widetilde{K}_0(\pi_1(X)) \oplus \mathbb{Z}$ when X is not connected.

Remark The idea for the product formula comes from Kwun and Szczarba [19] (January 1965) who proved a product formula for the Whitehead torsion of $f \times 1_{X_2}$ where $f\colon X_1 \longrightarrow X_1'$ is a homotopy equivalence of finite connected complexes and X_2 is any finite connected complex; namely

$$\tau(f \times 1_{X_2}) = \chi(X_2) j_{1*}\tau(f) \qquad (10)$$

where j_{1*} is induced by $X_1 \hookrightarrow X_1 \times X_2$. This corresponds to the basic case of (8) with $\sigma(X_2) = 0$; namely

$$\sigma(X_1 \times X_2) = \chi(X_1) j_{1*}\sigma(X_2). \qquad (11)$$

Steven Gersten [20] has independently derived (11). His proof is purely algebraic so does not use the Sum Theorem. It was Professor Milnor who proposed the correct general form of the product formula and the use of ρ. Already in 1964, M.R. Mather had a (purely geometrical) proof that for $X \in \mathcal{D}$, $X \times S^1$ is homotopy equivalent to a finite complex.

Proof of Complement 7.3 Fortunately the proof of the Complement 7.3 is trivial (unlike Complement 6.6). If K_i, $i = 1, 2$, are finite complexes and $r_i\colon K_i \longrightarrow X_i$ are maps with left homotopy inverses s_i, $i = 1, 2$, $r_1 \times r_2\colon K_1 \times K_2 \longrightarrow X_1 \times X_2$ has left homotopy inverse $s_1 \times s_2$. This gives the implication $\Rightarrow$. For the reverse implication note that $X_1 \times X_2 \in \mathcal{D}$ dominates X_i, which implies $X \in \mathcal{D}$, $i = 1, 2$. $\square$

Proof of Theorem 7.2 The proof is based on the Sum Theorem 6.5 and divides naturally into there steps. Since $\chi(X_1 \times X_2) = \chi(X_2)\chi(X_2)$, it will suffice to establish the second formula (9).

Case 1. $X_2 = S^n$, $n = 1, 2, 3, \ldots$.

Suppose inductively that (9) holds for $X_2 = S^k$, $1 \leq s < n$. Let $S^n = D^n_- \cup D^n_+$ be the usual decomposition of S^n into closed northern and southern hemispheres with intersection S^{n-1}. Then apply the sum theorem to the partition $X_1 \times S^n = X_1 \times D^n_- \cup X_1 \times D^n_+$.

Case 2. $X_2 =$ a finite complex.

Since X_2 is connected, we can assume it has a single 0–cell. We assume inductively that (9) has been verified for such X_2 having $< n$ cells. Consider X_2 with exactly n cells. Then $X_2 = Y \cup_f D^k$, $k \geqslant 1$, where $f\colon S^{k-1} \longrightarrow Y$ is an attaching map and Y has $n-1$ cells. Up to homotopy type we assume f is an imbedding and $Y \cap D^k$ is a $(k-1)$–sphere. Now apply the sum theorem to the partition $X_1 \times X_2 = X_1 \times Y \cup X_1 \times D^k$. The inductive assumption and (for $k \geqslant 2$) the case I) complete the induction.

Case 3. The general case.

We insert a Lemma needed for the proof.

Lemma 7.4 *Suppose that (X, Y) is a connected CW pair with X and Y in $\mathcal{D}$. Suppose that $Y \hookrightarrow X$ gives a π_1–isomorphism and $H_*(\widetilde{X}, \widetilde{Y})$ is $\pi_1(X)$–projective and isolated in dimension n. Then*

$$\chi(X) - \chi(Y) = (-1)^n \operatorname{rank} \left\{ \mathbb{Z} \otimes_{\pi_1(X)} H_n(\widetilde{X}, \widetilde{Y}) \right\}.$$

Proof Since $C_*(X, Y) = \mathbb{Z} \otimes_{\pi_1(X)} C_*(\widetilde{X}, \widetilde{Y})$, the universal coefficient theorem shows that $H_*(X, Y) = \mathbb{Z} \otimes_{\pi_1(X)} H_*(\widetilde{X}, \widetilde{Y})$. The lemma now follows from the exact sequence of (X, Y). $\square$

[Proof of Case 3] Replacing X_1, X_2 by homotopy equivalent complexes we may assume that X_1, X_2 have finite dimension $\leqslant n$ say, and that there are finite $(n-1)$–subcomplexes $K_i \subset X_i$, $i = 1, 2$, such that the inclusions give isomorphisms of fundamental groups and $H_*(\widetilde{X}_i, \widetilde{K}_i)$ are f.g. projective $\pi_1(X_i)$–modules P_i concentrated in dimension n. Let $X = X_1 \times X_2$. See Figure 6.

Since the complex $Y = X_1 \times K_2 \cup K_1 \times X_2$ has dimension $\leqslant 2n$, there exists a finite $(2n-1)$-complex K and a map $f\colon K \longrightarrow Y$, giving a π_1–isomorphism,

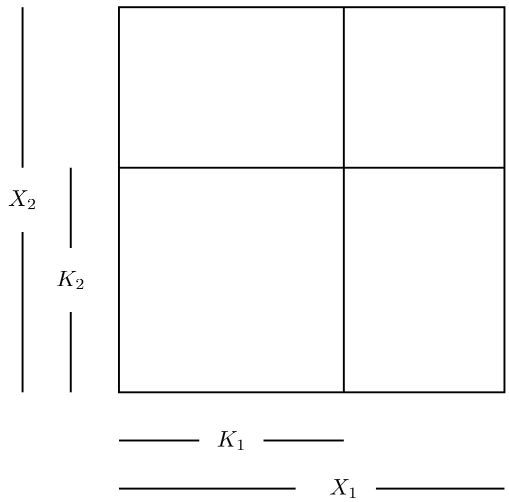


Figure 6:

such that $H_*(\widetilde{M}(f), \widetilde{K})$ is a f.g. projective $\pi_1(X)$–module P concentrated in dimension $2n$. Replacing Y by $M(f)$ we may assume that $K \subset Y \subset X = X_1 \times X_2$. Now the exact sequence of the triple $\widetilde{K} \subset \widetilde{Y} \subset \widetilde{X}$ is

$$\begin{tikzcd} 0 \arrow[r] & H_{2n}(\widetilde{Y}, \widetilde{K}) \arrow[r] \arrow[d, "\cong"] & H_{2n}(\widetilde{X}, \widetilde{K}) \arrow[r] & H_{2n}(\widetilde{X}, \widetilde{Y}) \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \cdots \\ & P & & P_1 \otimes P_2 & & \end{tikzcd}$$

Hence $H_*(\widetilde{X}, \widetilde{K})$ is $P \oplus (P_1 \otimes P_2)$ concentrated in dimension $2n$. But

$$\sigma(Y) = \chi(K_2) j_{1*}\sigma(X_1) + \chi(K_1) j_{2*}\sigma(X_2)$$

by Case 2 and the Sum Theorem. Hence

$$\sigma(X) = [P] + [P_1 \otimes P_2] = [P_1 \otimes P_2] + \{\chi(K_2) j_{1*}\sigma(X_1) + \chi(K_1) j_{2*}\sigma(X_2)\}.$$

As an equation in $K_0(\pi_1(X)) = \widetilde{K}_0(\pi_1(X)) \oplus \mathbb{Z}$ this says

$$\sigma(X) = \{\overline{P}_1 \otimes \overline{P}_2 - \overline{F}_1 \otimes \overline{F}_2\} \underset{1}{+} \{\sigma(X_1) \otimes \chi(K_2) + \chi(K_1) \otimes \sigma(X_2)\} \qquad (12)$$

where F_1, F_2 are free modules over $\pi_1(X_1), \pi_1(X_2)$ of the same rank as P_1, P_2. Notice that the first bracket can be rewritten

$$(\overline{P}_1 - \overline{F}_1) \otimes (\overline{P}_2 - \overline{F}_2) + (\overline{P}_1 - \overline{F}_1) \otimes \overline{F}_2 + \overline{F}_1 \otimes (\overline{P}_2 - \overline{F}_2).$$

But according to Lemma 7.4, $(-1)^n\overline{F}_i = \chi(X_i) - \chi(K_i)$, $i = 1, 2$. Also $(-1)^n(\overline{P}_i - \overline{F}_i) = (-1)^n[P_i] = \sigma(X_i)$. Hence on substituting in (12) we get

$$\sigma(X) = (\overline{P}_1 - \overline{F}_1) \otimes (\overline{P}_2 - \overline{F}_2) + \sigma(X_1) \otimes \chi(X_2) + \chi(X_1) \otimes \sigma(X_2)$$

which is the formula (9). This completes the proof of Case 3 and hence of the Product Theorem. □

Here is an attractive corollary to the Product Theorem 7.2 and Complement 7.3. Let M^n be a fixed closed smooth manifold with $\chi(M) = 0$. (The circle is the simplest example.) Let ε be an end of a smooth open manifold.

Theorem 7.5 *Suppose* $\dim(W \times M) \geqslant 6$. *The end* ε *is tame if and only if the end* $\varepsilon \times M$ *of* $W \times M$ *has a collar.*

Our definition of tameness (Definition 4.4) makes sense for any dimension. But so far we have had no theorems that apply to a tame end of dimension 3 or 4. (A tame end of dimension 2 always has a collar – c.f. Kerékjártó [26, p. 171].) Now we know that the tameness conditions for such an end are equivalent, for example, to $\varepsilon \times S^3$ having a collar.

It is perhaps worth pointing out now that the invariant σ can be written for a tame end ε of any dimension. Since ε is isolated there exist arbitrary small closed neighborhoods V of ε that are smooth submanifolds with compact boundary and one end. Since π_1 is stable at ε, we can find such a V so small that $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ has a left inverse r.

Proposition 7.6 $V \in \mathcal{D}$ *and* $r_*\sigma(V) \in \widetilde{K}_0(\pi_1(\varepsilon))$ *is an invariant of* ε.

Definition 7.7 $\sigma(\varepsilon) = r_*\sigma(V)$.

Notice that, by Theorem 6.9, this agrees with our original definition of $\sigma(\varepsilon)$ for dimension $\geqslant 5$.

Proof of Proposition 7.6 We begin by showing that $V \in \mathcal{D}$. Since we do not know that ε has arbitrarily small 1–neighborhoods we employ an interesting device. Consider the end $\varepsilon \times M$ where M is a connected smooth closed manifold so that $\dim(\varepsilon \times M) \geqslant 5$. ($S^5$ should always do.) By Complement 7.3 we know that $V \in \mathcal{D}$ if and only if $V \times M \in \mathcal{D}$. Also $\varepsilon \times M$ is a tame end of dimension $\geqslant 5$ and so has arbitrarily small 1–neighborhoods. Notice that $r' = r \times \mathrm{id}(\pi_1(M))$ gives a right inverse for $\pi_1(\varepsilon \times M) \longrightarrow \pi_1(V \times M)$. Applying Proposition 4.3 we see that $V \times M \in \mathcal{D}$. So $V \in \mathcal{D}$ by Complement 7.3.

To prove that $r_*\sigma(V)$ is independent of the choice of V and of r use Theorem 6.5 and the existence of neighborhoods $V' \subset V$ with the properties of V and so small that $j\colon \pi_1(V') \longrightarrow \pi_1(V)$ has image $\pi_1\varepsilon \subset \pi_1(V)$ (– whence $r \cdot j$ is independent of the choice of r). □

Remark In §8 we construct tame ends of dimension $\geqslant 5$ with prescribed invariant. I do not know any tame end ε of dimension 3 or 4 with $\sigma(\varepsilon) \neq 0$. Such an end would be very surprising in dimension 3.

As an exercise with the Product Theorem one can calculate the invariant for the end of the product of two open manifolds. Notice that if ε is a tame end of a smooth manifold W^n, $n \geqslant 5$, there is a natural way to define

$$\rho(\varepsilon) = \rho(\varepsilon) \oplus \chi(\varepsilon) \in \widetilde{K}_0(\pi_1(\varepsilon)) \oplus \mathbb{Z} = K_0(\pi_1(\varepsilon)).$$

In fact let $\chi(\varepsilon)$ be $\chi(\operatorname{Bd} V)$ where V is any 0–neighborhood of ε. Notice that $\chi(\operatorname{Bd} V) = 0$ for n even and that $\chi(\operatorname{Bd} V)$ is independent of V for n odd. Also observe that as $n \geqslant 5$, there are arbitrarily small 1–neighborhoods V of ε so that $\chi(V, \operatorname{Bd} V) = 0$, i.e. $\chi(\varepsilon) = \chi(\operatorname{Bd} V) = \chi(V)$.

Theorem 7.8 *Suppose W and W' are smooth connected open manifolds of dimension $\geqslant 5$ with tame ends ε and ε' respectively. Then $W \times W'$ has a single, tame end $\overline{\varepsilon}$ and*

$$\rho(\overline{\varepsilon}) = i_{1*}\{\rho(\varepsilon) \otimes \rho(W')\} + i_{2*}\{\rho(W) \otimes \rho(\varepsilon')\} = i_{0*}\{\rho(\varepsilon) \otimes \rho(\varepsilon')\}$$

for naturally defined homomorphisms i_{0*}, i_{1*}, i_{2*}.

Proof Consider the complement of $U \times U'$ in $W \times W'$ where $V = W - U, V' = W' - U'$ are 1–neighborhoods of ε and ε' with $\chi(\varepsilon) = \chi(V), \chi(\varepsilon') = \chi(V')$. Then apply the Sum Theorem and Product Theorem. (The sum formula looks the same for σ and ρ.) The reader can check the details. □

Remark If W has several ends, all tame $\varepsilon = \{\varepsilon_1, \ldots, \varepsilon_r\}$, and W' has tame ends $\varepsilon' = \{\varepsilon'_1, \ldots, \varepsilon'_s\}$ then $W \times W'$ still has just one tame end. And if we define $\rho(\varepsilon) = (\rho(\varepsilon_1), \ldots, \rho(\varepsilon_r))$ in $K_0(\pi_1(\varepsilon_1)) \oplus \cdots \oplus K_0(\pi_1(\varepsilon_r))$ and $\rho(\varepsilon')$ similarly, the above formula remains valid. Also, with the help of Definition 7.7 one can eliminate the assumption that dimension $\geqslant 5$.

8 The Construction of Strange Ends

The first task is to produce tame ends ε of dimension $\geqslant 5$ with $\sigma(\varepsilon) \neq 0$. Such ends deserve the epithet strange because $\varepsilon \times S^1$ has a collar while ε itself does not (Theorem 7.5). At the end of this section we construct the contractible manifolds promised in §4.

We begin with a crude but simple construction for strange ends. Let a closed smooth manifold M^{n-1}, $n \geqslant 6$, be given together with a f.g. projective $\pi_1(M)$–module P that is not stably free. Such a P exists if $\pi_1(M) = \mathbb{Z}_{23}$ since $\widetilde{K}_0(\mathbb{Z}_{23}) \neq 0$. (For a resume of what is known about $\widetilde{K}_0(G)$ for various G see Wall [2, p. 67].) Build up a smooth manifold W^n with $\operatorname{Bd} W = M$ by attaching infinitely many (trivial) 2–handles and (nontrivial) 3–handles so that the corresponding free $\pi_1(M)$–complex C_* for $H_*(\widetilde{W}, \widetilde{M})$ has the form

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] & C_3 \arrow[r, "\partial"] \arrow[d, "\cong"] & C_2 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[r] & \cdots \\
 & & F & F & &
\end{tikzcd}$$

where F is a free $\pi_1(M)$–module on infinitely many generators, and ∂ is onto with kernel P. For example, if $P \oplus Q$ is f.g. and free, ∂ can be the natural projection $F \cong P \oplus Q \oplus P \oplus Q \oplus \cdots \longrightarrow 0 \oplus Q \oplus P \oplus Q \oplus \cdots \cong F$. The analogous construction for h–cobordisms of dimension $\geqslant 6$ with prescribed torsion is explained in Milnor [17, § 9]. The problem of suitably attaching handles is the same here. Of course, we must add infinitely many handles. But we can add them one at a time thickening at each stage. Before adding a 3–handle we add all the 2–handles involved in its boundary. W is then an infinite union of finite handlebodies on M. Proposition 8.2 below can be used to show rigorously that $H_*(\widetilde{W}, \widetilde{M}) = H_*(C)$.

We proceed to give a more delicate construction for strange ends which has three attractive features:

(a) It proves that strange ends exist in dimension 5.

(b) The manifold W itself can provide a $(n-4)$–neighborhood of ε.

(c) W is an open subset of $M \times [0, 1)$.

The construction is best motivated by an analogous construction for h–cobordisms. Given M^{n-1}, $n \geqslant 6$, and a $d \times d$ matrix T over $\mathbb{Z}[\pi_1(X)]$ we are to find an h–cobordism $c = (V; M, M')$ with torsion $\tau \in \operatorname{Wh}(\pi_1(M))$ represented by T. Take the product cobordism $M \times [0, 1]$ and insert $2d$ complementary (= auxiliary) pairs of critical points of index 2 and 3 in the projection to $[0, 1]$ (c.f.

[4, p. 101]). If the resulting Morse function f is suitably equipped, in the corresponding complex

$$\cdots \longrightarrow 0 \longrightarrow C_3 \xrightarrow{\partial} C_2 \longrightarrow 0 \longrightarrow 0$$

∂ is given by the $2d \times 2d$ identity matrix I. By [17, p. 2] elementary row of column operations serve to change I to $\left(\begin{smallmatrix} T & 0 \\ 0 & T^{-1} \end{smallmatrix}\right)$. Each elementary operation can be realized by a change of f (c.f. [3, p. 17]). After using Whitney's device as in §4 we can lower the level of the first d critical points of index 3 and raise the level of the last d critical points of index 2 so that $M \times [0,1]$ is split as the product of two h–cobordisms c, c' with torsions $\tau(c) = [T] = \tau$ and $\tau(c') = [T^{-1}] = -\tau$. The corresponding construction for strange ends succeeds even in dimension 5 because Whitney's device is not used.

Before giving this delicate construction we introduce some necessary geometry and algebra.

Let $f\colon W \longrightarrow [0,\infty)$ be a proper Morse function with gradient-like vector field ξ, on a smooth manifold W having $\mathrm{Bd}\, W = f^{-1}(0)$. Suppose that a base point $* \in \mathrm{Bd}\, W$ has been chosen together with base paths from $*$ to each critical point. At each critical point p we fix an orientation for the index(p)–dimensional subspace of the tangent space TW_p to W at p that is defined by trajectories of ξ converging to p from below. Now f is called an *equipped proper Morse function*. The equipment consists of ξ, $*$, base paths, and orientations.

When f has infinitely many critical points we cannot hope to make f nice in the sense that the level of a critical point is an increasing function of its index. But we can still put conditions on f which guarantee that it determines a free $\pi_1(W)$–complex for $H_*(\widetilde{W}, \mathrm{Bd}\, \widetilde{M})$.

Definition 8.1 We say that f is *nicely equipped* (or that ξ is *nice*) if the following two conditions on ξ hold:

(1) If p and q are critical points and $f(p) < f(q)$, but $\mathrm{index}(p) > \mathrm{index}(q)$, then no ξ–trajectory goes from p to q. This guarantees that if for any non-critical level a, f restricted to $f^{-1}[0,a]$ can be adjusted without changing ξ to a nice Morse function g (see [4, § 4.1]).

(2) Any such $g\colon f^{-1}[0,a] \longrightarrow [0,a]$ has the property that for every index λ and for every level between index λ and index $\lambda+1$, the left hand λ–spheres in $g^{-1}(b)$ intersect the right hand $(n-\lambda-1)$–spheres transversely, in a finite number of points. In fact (2) is a property of ξ alone, for it is equivalent to the following property (2′). Note that for every (open) trajectory T from a critical point p of index λ to a critical point q

of index $\lambda+1$ and for every $x \in T$, the trajectories from p determine a $(n-\lambda)$–subspace $V_x^{n-\lambda}(p)$ of the tangent space TW_x and the trajectories to q determine a $(\lambda+1)$–subspace $V_x^{\lambda+1}(q)$ of TW_x.

$(2')$ For every such T and for one (and hence all) points x in T, $V_x^{n-\lambda}(p) \cap V_x^{\lambda+1}(q)$ is the line in TW_x determined by T.

Remark Any gradient-like vector field for f can be approximated by a nice one (c.f. Milnor [4, § 4.4, § 5.2]). We will not use this fact.

We say that a Morse function f on a compact triad $(W; V, V')$ is *nicely equipped* if it is nicely equipped on $W - V'$ in the sense of Definition 8.1. This simply means that f can be made nice without changing the gradient-like vector field and that when this is done left hand λ–spheres meet right hand $(n-\lambda+1)$–spheres transversely in any level between index λ and $\lambda+1$.

Suppose that $f\colon W \xrightarrow{\text{onto}} [0,\infty)$ is a nicely equipped proper Morse function on the noncompact smooth manifold with $\mathrm{Bd}\, W = f^{-1}(0)$. We explain now how f gives a free $\pi_1(W)$–complex for $H_*(\widetilde{W}, \mathrm{Bd}\, \widetilde{W})$. Let a be a noncritical level and adjust f to a nice Morse function f' on $f^{-1}[0,a]$ without changing ξ. From the discussion in §4 one can see that the equipment for f completely determines a based, free $\pi_1(W)$–complex $C_*(a)$ for f' with homology $H_*(p^{-1}f^{-1}[0,a], \mathrm{Bd}\, \widetilde{W})$, where $p\colon \widetilde{W} \longrightarrow W$ is the universal cover. Then it is clear that $C_*(b)$ is independent of the particular choice of f', and that if $a < b$ is another non-critical level, there is a natural inclusion $C_*(a) \hookrightarrow C_*(b)$ of based $\pi_1(W)$–complexes. Let $0 = a_0 < a_1 < a_2 < a_3 < \ldots$ be an unbounded sequence of non-critical levels of f. Then $C_* = \cup_i C_*(a_i)$ is defined, and from its structure we see that it depends only on the equipment of f, i.e. it the same for any other proper Morse function with the same equipment. There is one generator for each critical point, and the boundary operator is given in terms of the geometrically defined characteristic elements and intersection numbers as in §4.

Proposition 8.2 *In the above situation* $H_*(C_*) = H_*(\widetilde{W}, \mathrm{Bd}\, \widetilde{W})$.

Proof There is no problem when f has only finitely many critical points. For if a is very large $C_* = C_*(a)$ and $H_*(C_*(a)) \cong H_*(p^{-1}f^{-1}[0,a], \mathrm{Bd}\, \widetilde{W}) \cong H_*(\widetilde{W}, \mathrm{Bd}\, \widetilde{W})$ where the last isomorphism holds because W is $f^{-1}[0,a]$ with an open collar attached. Thus we can assume from this point that f has infinitely many critical points.

We can adjust f without changing ξ so that at most one critical point lies at a given level; so we may assume that for the sequence $a_0 < a_1 < \dots$ above $f^{-1}[a_i, a_{i+1}]$ always contains exactly one critical point. Also, arrange that $a_1 = n+1$.

Notice that $H_*(C_*) \cong H_*(\cup_n C(a_n)) \cong \varinjlim_n H_*(C(a_n))$. We will show that the limit on the right is isomorphic to $H_*(\widetilde{W}, \operatorname{Bd}\widetilde{W})$.

We define a sequence $f_0, f_1, f_2, \dots$ of proper Morse function each with the same equipment as f. Let $f_0 = f$. Suppose inductively that we have defined a Morse function f_n having the same equipment of f so that f_n is nice on $f^{-1}[0, a_n]$ and coincides with f elsewhere. Suppose also that the level of f_n for index λ in $f^{-1}[0, a_n]$ is $\lambda + \dfrac{1}{2}$. Define f_{n+1} by adjusting f_n on $f^{-1}[0, a_n]$ without changing ξ, so as to lower the level of the critical point p in $f^{-1}[a_n, a_{n+1}]$ to the level $\operatorname{index}(p) + \dfrac{1}{2}$. (See Milnor [4, § 4.1].) By induction the sequence $f_0, f_1, f_2, \dots$ is now well defined.

There is a filtration of $f^{-1}[0, a_n]$ determined by f_n: $\operatorname{Bd} W = X^{(n)}_{-1} \subset X^{(n)}_0 \subset \dots \subset X^{(n)}_w$, $w = \dim W$, where $X^{(n)}_\lambda = f_n^{-1}[0, \lambda+1]$. The chain complex for the 'lifted' filtration $p^{-1}X^{(n)}_{-1} \subset p^{-1}X^{(n)}_0 \subset \dots \subset p^{-1}X^n_w$ of $p^{-1}f^{-1}[0, a_n] \subset \widetilde{W}$ is naturally isomorphic with the complex $C_*(a_n)$. And the homology for the filtration complex is $H_*(p^{-1}f^{-1}[0, a_n], \operatorname{Bd}\widetilde{W})$. Now we notice that the inclusion $j\colon f^{-1}[0, a_n] \hookrightarrow f^{-1}[0, a_{n+1}]$ respects filtrations. In fact, if the new critical point has index λ, $X^{(n+1)}_i = X^{(n)}_i$ for $i < \lambda$, and for $i \geqslant \lambda$, $X^{(n+1)}_i \supset X^{(n)}_i$ is up to homotopy $X^{(n)}_i$ with a λ–handle attached. One can verify in a straightforward way that the induced map $j_\#\colon C_*(a_n) \longrightarrow C_*(a_{n+1})$ of filtration complexes is just the natural inclusion $C(a_n) \hookrightarrow C_*(a_{n+1})$ noted above. Thus the commutativity of

$$\begin{tikzcd}
H_*(C(a_n)) \arrow[r, "j_{\#*}"] \arrow[d, "\cong"] & H_*(C(a_{n+1})) \arrow[d, "\cong"] \\
H_*(p^{-1}f^{-1}[0, a_n], \operatorname{Bd}\widetilde{W}) \arrow[r, "j_*"] & H_*(p^{-1}f^{-1}[0, a_{n+1}], \operatorname{Bd}\widetilde{W})
\end{tikzcd}$$

where the vertical arrows are the natural isomorphisms) tells us that

$$\varinjlim H_*(C(a_n)) = \varinjlim H_*(p^{-1}f^{-1}[0, a_n], \operatorname{Bd}\widetilde{W}) = H_*(\widetilde{W}, \operatorname{Bd}\widetilde{W})$$

as required. □

Next come some algebraic preparations. Let Λ be a group ring $\mathbb{Z}[G]$ and consider infinite 'elementary' matrices $E = E(r;i,j)$ in $\mathrm{GL}(\Lambda,\infty) = \varinjlim_n \mathrm{GL}(\Lambda,n)$ that have 1's on the diagonal, the element $r \in \Gamma$ in the i,j position $(i \neq j)$ and zeros elsewhere. Suppose F is a free Λ–module with a given basis $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ indexed on the natural numbers $< N$ where N may be finite or ∞. Then provided i,j are less than N, $E(r;i,j)$ determines the *elementary operation* on α that adds to the j–th basis element of α, r times the i–th basis element – i.e. $E(r;i,j)\alpha = \{\alpha_1, \alpha_2, \ldots \alpha_{j-1}, \alpha_j + r\alpha_i, \alpha_{j+1}, \ldots\}$. In this way elementary matrices are identified with elementary operations.

Suppose now that F is an infinitely generated free Λ–module and let $\alpha = \{a_1, \alpha_2, \ldots\}$ and $\beta = \{\beta_1, \beta_2, \ldots\}$ be two bases. It is convenient to write the submodule of F generated by the elements $\gamma_1, \gamma_2, \ldots$ as $(\gamma_1, \gamma_2, \ldots)$ – with round brackets.

Lemma 8.3 *There exists an infinite sequence of elementary operations $E_1, E_2, E_3, \ldots$ and a sequence of integers $0 = N_0 < N_1 < N_2 < \ldots$ so that for each integer k, the following statement holds:*

(*) $n \geqslant N_k$ implies that $E_n E_{n-1} \cdots E_1 \alpha$ coincides with β for at least the first k elements.

Remark 1 *(*) implies that for $n \geqslant N_k$, $E_n = E(r;i,j)$ with $j > k$ (or $r = 0$). But $i < k$ is certainly allowable.*

Proof Suppose inductively that $N_0, N_1, \ldots, N_{x-1}$ and $E_1, E_2, \ldots, E_{N_{x-1}}$ have been defined so that $(*)$ holds for $k \leqslant x-1$. (The induction begins with $N_0 = 0$ and no E's.) Then $E_{N_{x-1}} \cdots E_1 \alpha = \{\beta_1, \ldots, \beta_{x-1}, \gamma_x, \gamma_{x+1}, \ldots\}$ for some $\gamma_x, \gamma_{x+1}, \ldots$. Set $\gamma_i = \beta_i$, $i = 1, \ldots, x-1$.

Suppose that β_x is expressed in terms of the basis $E_{N_{x-1}} \cdots E_1 \alpha = \gamma$ by $\beta_x = b_1\gamma_1 + \ldots + b_y\gamma_y$, $b_i \in \Lambda$, $y > x$. Then the composed map $p\colon (\gamma_1, \ldots, \gamma_y) \hookrightarrow F \xrightarrow[p_1]{} (\beta_1, \ldots, \beta_{x-1}, \beta_x)$, where p_1 is natural projection determined by the basis β, is certainly onto. Hence $(\gamma_1, \ldots, \gamma_y)$ is the direct sum of two submodules:

$$(\gamma_1, \ldots, \gamma_y) = (\beta_1, \ldots, \beta_x) \oplus \ker(p).$$

This says that $\ker(p)$ is stably free. One can verify that the result of increasing y by one is to add a free summand to $\ker(p)$. Thus, after making y sufficiently large we can assume $\ker(p)$ is free. Choose a basis $(\gamma'_{x+1}, \ldots, \gamma'_y)$ for $\ker(p)$. (Note that this basis necessarily has $\mathrm{rank}\{\mathbb{Z} \otimes_\Lambda \ker(p)\} = y - x$ elements.)

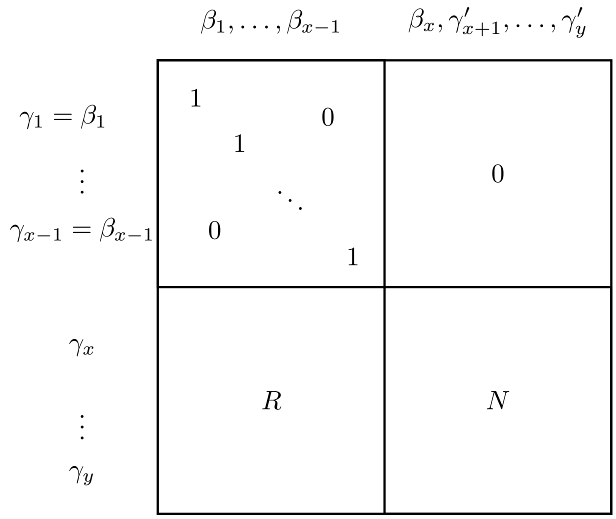


Figure 7:

Now consider the matrix whose rows express $\gamma_1, \ldots, \gamma_y$ in terms of $\beta_1, \ldots, \beta_{x-1}, \beta_x, \gamma'_{x+1}, \ldots, \gamma'_y$.

The upper right rectangle clearly contains only zeros. Notice that elementary row operations correspond to elementary operations on the basis $\gamma_1, \ldots, \gamma_y$ – and hence on γ.

Reduce the lower left rectangle to zeros by adding suitable multiples of the first $x-1$ rows to the last $y-x+1$. Now adjoin to each basis the elements $\gamma_{y+1}, \gamma_{y+2}, \ldots, \gamma_{2y-x+1}$ so that the lower right box has the form $\left(\begin{smallmatrix} N & 0 \\ 0 & I \end{smallmatrix}\right)$ where I is an identity matrix of the same dimension as N. By the proof of Lemma 5.4 there are further row operations that change this box to $\left(\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right)$ (and don't involve the first $x-1$ rows). Clearly we have produced a finite sequence of elementary operations on γ, $E_{N_{x-1}+1}, \ldots, E_{N_x}$, so that $(*)$ now holds for $k \leqslant x$. This completes the induction. □

What we actually need is a mild generalization of Lemma 8.3. Suppose that $F \cong G \oplus H$ where G, like F, is a copy of Λ^∞. Regard G and H as submodules of F and let $\alpha = \{\alpha_1, \alpha_2, \ldots\}$, $\beta = \{\beta_1, \beta_2, \ldots\}$ be bases for F and G respectively.

Lemma 8.4 *In this situation too, the assertion of Lemma 8.3 is true.*

Proof Again suppose inductively that $N_0, \ldots, N_{x-1}$ and $E_1, \ldots, E_{N_{x-1}}$ have been defined so that (*) holds for $k \leqslant n-1$. Since $(\beta_{x+1}, \beta_{x+2}, \ldots) \oplus H \cong$

$G \oplus H \cong \Lambda^\infty$ there is a basis $\beta' = \{\beta_1, \ldots, \beta_{x-1}, \beta_x, \beta'_{x+1}, \beta'_{x+2}, \ldots\}$ for F. Now we can repeat the argument of Lemma 8.3 with β' in place of β to complete the induction. □

We need a carefully stated version of the Handle Addition Theorem [3, p. 17]. Suppose $(W; V, V')$ is a compact smooth triad with a nicely equipped Morse function f that has critical points $p_1, \ldots p_m$ all of index λ, $3 \leqslant \lambda \leqslant n-2$. The complex C_* for f has the form

$$\cdots \longrightarrow 0 \longrightarrow C_\lambda \longrightarrow 0 \longrightarrow \cdots$$

where $C_\lambda \cong H_\lambda(\widetilde{W}, \widetilde{V})$ is free over $\pi_1(W)$ with one generator $e(p_i)$ for each critical point p_i. Suppose $f(p_1) > f(p_2)$. Let $g \in \pi_1(W)$ be prescribed, together with a real number $\epsilon > 0$ and a sign ± 1.

Proposition 8.5 *By altering the gradient-like vector field on $f^{-1}[f(p_1) - \epsilon, f(p_1) - \frac{\epsilon}{2}]$ only, it is possible to give C_λ the basis $e(p_1), e(p_2) \pm g e(p_1), e(p_3), \ldots, e(p_m)$.*

Remark A composition of such operations gives any elementary operation $E(r; 1, 2)$, $r \in \mathbb{Z}[\pi_1(W)]$. And by permuting indices we see that $e(p_i)$ and $e(p_j)$ could replace $e(p_1)$ and $e(p_2)$ if $f(p_i) > f(p_j)$.

Proof The construction is essentially the same as for the Basis Theorem [4, § 7.6]. We point out that the choice of $g \in \pi_1(W)$ demands a special choice of the imbedding "$\phi_1 \colon (0,3) \longrightarrow V_0$" on p. 96 of [4]. Also, f is never changed during our construction. The proof in [4, § 7.6] that the construction accomplishes what one intends is not difficult to generalize to this situation. □

Finally we are ready to establish

Existence Theorem 8.6 *Suppose given*

(1) M^{w-1}, $w \geqslant 5$, *a smooth closed manifold*

(2) k, *an integer with* $2 \leqslant k \leqslant w-3$

(3) P, *a f.g. projective* $\pi_1(M)$*–module.*

Then there exists a smooth manifold W^w, with one tame end ε, which is an open subset of $M \times [0,1)$ with $\operatorname{Bd} W = M \times 0$, such that

(a) *Inclusions induce isomorphisms*

$$\pi_1(M \times 0) \xrightarrow{\cong} \pi_1(W) \xleftarrow{\cong} \pi_1(\varepsilon)$$

(b) $\sigma(\varepsilon) = (-1)^k[P] \in \widetilde{K}_0(\pi_1(M \times 0))$

(c) $M \times 0 \hookrightarrow W$ *is a* $(k-1)$*–equivalence. Further* $H_i(\widetilde{W}, \mathrm{Bd}\,\widetilde{W}) \cong P$ *and* $H_i(\widetilde{W}, \mathrm{Bd}\,\widetilde{W}) = 0$*,* $i \neq k$*.*

Remark After an adequate existence theorem there follows logically the question of classifying strange ends. It is surely one that should have some interesting answers. I ignore it simply because I have only begun to consider it.

Proof By construction W will be an open subset of $M \times [0,1)$ that admits a nicely equipped proper Morse function $f\colon W \xrightarrow{\text{onto}} [0, \frac{1}{2})$ with $f^{-1}(0) = M \times 0$. Only index k and index $k+1$ critical points will occur. Then according to Theorem 1.10 W can have just one end ε. The left-hand sphere of each critical point of index k will be contractible in $M \times 0$. Thus $M \times 0 \hookrightarrow W$ will be a $(k-1)$–equivalence. If $k < n-3$, π_1 is automatically tame at ε, and $\pi_1(\varepsilon) \longrightarrow \pi_1(W)$ is an isomorphism. If $k = n-3$ we will have to check this during the construction. The complex C'_* for f will be chosen that $H_*(C'_*) = H_*(\widetilde{W}, \mathrm{Bd}\,\widetilde{W})$ is isomorphic to P and connected in dimension k. Thus (c) will follow. Then the tameness of ε and condition (b) will follow from (c) and Lemma 6.2.

With this much introduction we begin the proof in serious. Consider the free $\Lambda = \mathbb{Z}[\pi_1(M \times 0)]$–complex

$$\begin{array}{cccccccccc} C_* & \cdots \longrightarrow & 0 & \longrightarrow & C_{k+1} & \xrightarrow{\partial} & C_k & \longrightarrow & 0 & \longrightarrow \cdots \\ & & & & \downarrow\cong & & \downarrow\cong & & & \\ & & & & F & & F & & & \end{array}$$

where $F \cong \Lambda^\infty$ and ∂ corresponds to the identity map of F. There exists an integer r and a Λ–module Q so that $P \oplus Q \cong \Lambda^r$. Then

$$F \cong (P \oplus Q) \oplus (P \oplus Q) \oplus \ldots \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \ldots \cong P \oplus F.$$

So we have $F \cong G \oplus P$ where $G \cong \Lambda^\infty$. Regard G and P as submodules of F and choose bases $\alpha = \{\alpha_1, \alpha_2, \ldots\}$ and $\beta = \{\beta_1, \beta_2, \ldots\}$ for F and G respectively.

Consider the subcomplex of C_*:

$$\begin{array}{cccccccc} C'_*\colon & \cdots \longrightarrow & C'_{k+1} & \xrightarrow{\partial} & C'_k = C_k & \longrightarrow & 0 & \longrightarrow \cdots \\ & & \downarrow\cong & & \downarrow\cong & & & \\ & & G & & G \oplus P \cong F & & & \end{array} \qquad (13)$$

where ∂ corresponds to the inclusion $G \hookrightarrow F$. Let α give the basis for C_{k+1}, C_k, C'_k; and let β give the basis for C'_{k+1}. We will denote the based complexes by C and C' (without $*$). By a segment of C we will mean the based subcomplex of C corresponding to a segment $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ of α.

By Lemma 8.4 there exists a sequence $E_1, E_2, E_3, \ldots$ of elementary operations and a sequence $0 = N_0 < N_1 < N_2 \ldots$ of integers so that, for $n \geqslant N_s$, the first s basis elements of $E_n E_{n-1} \ldots E_1 \alpha$ coincide with $\beta_1, \ldots, \beta_s$. We let $E_1, E_2, E_3, \ldots$ act on α as a basis of C_{k+1} and in this way on the based complex C. For each integer s choose a segment $C(s)$ of C so large that $\beta_1, \ldots, \beta_s \in C_{k+1}(s)$ and $E_1, \ldots, E_{N_s}$ act on $C(s)$.

Let $C'(s)$ be the based subcomplex of $E_{N_s} \ldots E_1 C(s)$ consisting of $C_k(s)$ and the span of $\beta_1, \ldots, \beta_s$ in $C_{k+1}(s)$ (with basis $\beta_1, \ldots, \beta_s$). Notice that $C'(s)$ is a based subcomplex of C' and $C' = \cup_s C'(s)$.

Choose any sequence $0 = a_0 < a_1 < a_2 < \ldots$ of real numbers converging to $\dfrac{1}{2}$. We will construct a sequence $f_1, f_2, f_3, \ldots$ of nicely equipped Morse functions

$$M \times [0,1] \xrightarrow{\text{onto}} [0,1]$$

so that, for $n \geqslant m$, f_n coincides with f_m on $f_m^{-1}[0, a_m]$. The based free f.g. $\pi_1(M \times 0)$–complex for f_n is to be $E_{N_n} \cdots E_1 C(n)$ and for f_n restricted to $f_n^{-1}[0, a_n]$ it is to be $C'(n)$.

Notice that such a sequence $f_1, f_2, f_3, \ldots$ determines a nicely equipped proper Morse function f on $W = \bigcup_n f_n^{-1}[0, a_n]$ mapping onto $[0, \dfrac{1}{2})$; and its complex is $C' = \bigcup_n C'(n)$. The left hand sphere S_L^{k-1} in $M \times 0$ of any critical point of index k is contractible since it is contractible in $M \times [0,1]$ and $M \times 0 \hookrightarrow M \times [0,1]$ is a homotopy equivalence. In case $k = n - 3$ we now verify that $f_n^{-1}(a_n) \hookrightarrow M \times [0,1]$ and $f_n^{-1}[0, a_n] \hookrightarrow M \times [0,1]$ give π_1–isomorphisms. For this easily implies that π_1 is stable at the end ε of W and that $\pi_1(\varepsilon) \longrightarrow \pi_1(W)$ is an isomorphism. Now $f_n^{-1}[0, a_n]$ contains all critical points of f_n of index k so, even when $k = 2$, $f_n^{-1}[0, a_n] \hookrightarrow M \times [0,1]$ and $f^{-1}(a_n) \hookrightarrow f^{-1}[a_n, 1]$ give π_1-isomorphisms. Also $f^{-1}[a_n, 1] \hookrightarrow M \times [0,1]$ gives a π_1–isomorphism since $M \times 1 \hookrightarrow M \times [0,1]$ and $M \times 1 \hookrightarrow f^{-1}[a_n, 1]$ do. Thus $f^{-1}(a_n) \hookrightarrow M \times [0,1]$ does too, and our verification is complete.

In view of our introductory remarks in §3 it now remains only to construct the sequence $f_1, f_2, f_3, \ldots$ as advertised in the second to last paragraph above. Here are the details. Insert enough complementary pairs of index k and $k+1$ critical points (c.f. [4, § 8.2]) in the projection $M \times [0,1] \xrightarrow{\text{onto}} [0,1]$ to get a Morse function that, when suitably equipped, realizes the segment $C(1)$ of C. Apply the elementary operations $E_1, \dots, E_{N_1}$ to $C(1)$ and alter the Morse function accordingly using the Handle Addition Theorem of Wall [3, p. 17, p. 19]. Now lower the critical points represented in $C'(1) \subset E_{N_1} \cdots E_1 C(1)$ to levels $< a_1$ without changing the gradient-like vector field or the rest of the equipment. This is possible because all the critical points of index k are in $C'(1)$. Call the resulting Morse function f_1. Adjust the gradient-like vector field ξ [4, § 4.4, § 5.2] so that f_1 is nicely equipped.

Next, suppose inductively that a nicely equipped Morse function f_n has been defined realizing $E_{N_n} \cdots E_1 C(n)$ on $M \times [0,1]$ and $C'(n)$ on $f_n^{-1}[0, a_n]$. Enlarge $E_{N_n} \cdots E_1 C(n)$ to $E_{N_n} \cdots E_1 C(n+1)$ and insert corresponding complementary pairs in $f_n^{-1}[a_n, 1]$. Now apply $E_{N_n+1}, \dots, E_{N_{n+1}}$. We assert that the equipped Morse function can be adjusted correspondingly. At first sight this just requires the Handle Addition Theorem again. But we must leave f_n (and its equipment) unchanged on $f_n^{-1}[0, a_n]$; so we apply Proposition 8.5. Any elementary operation we have to realize is of the form $E(r; i, j)$ where $j > n$, which means that r times the i–th basis element $e(p_i)$ is to be added to the j–th basis element $e(p_j)$ where p_j lies in $f_n^{-1}[a_n, 1]$. Change the present Morse function f'_n on $f_n^{-1}[a_n, 1]$ increasing the level of p_j so that $f'_n(p_j) - \epsilon = d$, $(\epsilon > 0)$ exceeds $f'(p_i)$, a_n, and the levels of the index k critical points. Temporarily change f'_n on $(f'_n)^{-1}[0, d]$ to a nice Morse function and let c be a level between index k and $k+1$. Applying Proposition 8.5 on $f'^{-1}[c, 1]$ we can now make the required change of basis merely be altering ξ on ${f'_n}^{-1}[d, d + \frac{\epsilon}{2}]$. By [4, § 4.4, § 5.2] we cab assume that ξ is still nice. Next we can let f'_n return to its original form on $f'_n[0, d]$ without changing ξ. (This shows that we didn't really have to change f'_n on $(f'_n)^{-1}[0, d]$ in the first place.) After repeating this performance often enough we get a nicely equipped Morse function – still called f'_n – that realizes $E_{n+1} \cdots E_1 C(n+1) \supset C'(n+1)$ and coincides with f_n on $f_n^{-1}[0, a_n]$. Changing f'_n on $f_n^{-1}[a_n, 1]$ adjust to values in (a_n, a_{n+1}) the levels of critical points of f'_n that lie in $C'(n+1)$ but do not lie in $C'(n)$ (i.e. do not lie in $f_n^{-1}[0, a_n]$). Since all index k critical points of f'_n are included in $C'(n+1)$ this is certainly possible. We call the resulting nicely equipped Morse function f_{n+1}.

Apparently f_{n+1} realizes the complex $E_{N_n} \cdots E_1 C(n+1)$ on $M \times [0,1]$ and realizes $C'(n+1)$ when restricted to $f_{n+1}^{-1}[0, a_{n+1}]$. The inductive definition of the desired Morse functions $f_1, f_2, f_3, \dots$ is now complete. Thus Theorem 8.6 is established. $\square$

In the last part of this chapter we construct the contractible manifolds promised in §4. That the reader may keep in mind just what we want to accomplish we state

Proposition 8.7 *Let π be a finitely presented perfect group that has a finite nontrivial quotient group. Then for $w \geqslant 8$ there exists a contractible open manifold W such that π_1 is stable at the one end ε of W and $\pi_1(\varepsilon) = \pi$, but ε is nevertheless not tame.*

Remark Such examples should exist with $w \geqslant 5$ at least for suitable π.

Let $\{x : r\}$ be a finite presentation for a perfect group π, and form a 2–complex K^2 realizing $\{x : r\}$. Since $H_2(K^2)$ must be free abelian, one can attach finitely many 3–cells to K^2 to form a complex L^3 with $H_i(L) = 0$, $i \geqslant 2$. Since $H_1(L) = H_1(K) = \pi/[\pi,\pi] = 1$, L has the homology of a point. If we imbed L in S^w, $w \geqslant 7$, or rather imbed a smooth handlebody $H \simeq L$ that has one handle for each cell of L, then $M^w = S^w - \operatorname{Int} H$ is a smooth compact contractible manifold with $\pi_1(\operatorname{Bd} M) = \pi$. The construction is due to M.H.A. Newman [27].

Remark If one uses a homologically trivial presentation, $H_*(K^2) = H_*(\text{point})$ and one can get by with $w \geqslant 5$. Some examples are $\{a, b\,;\, a^5 = (ab)^2 = b^3\}$, which gives the binary icosahedral group of 120 elements, and $P_n = \{a, b\,;\, a^{n-2} = (ab)^{n-1}, b^3 = (ba^{-2}ba^2)^2\}$ with n any integer. The presentations P_n are given by Curtis and Kwun [24]. For n even $\geqslant 6$ there is a homomorphism of the group P_n onto the alternating group A_n on n letters. (See Coxeter-Moser [21, p. 67].) Unfortunately we will actually need $w \geqslant 8$ for different reasons.

Let π be a group and $\theta\colon \pi \longrightarrow \pi_0$ a homomorphism of π onto a finite group π_0 of order $p \neq 1$. Let $\Sigma \in \mathbb{Z}[\pi]$ be $(g_1 + \cdots + g_p)$ where $g_1, \ldots, g_p$ are some elements so that $\theta g_1, \ldots, \theta g_p$ are the p distinct elements of π_0. Consider the following free complex C over $\mathbb{Z}[\pi]$

$$C\colon \quad 0 \longrightarrow C_4 \xrightarrow{\partial} C_3 \xrightarrow{\partial} C_2 \longrightarrow 0$$

where C_2 has one free generator a, C_3 has two free generators b_1 and b_2 with

$$\partial b_1 = ma \ (m \text{ an integer})$$
$$\partial b_2 = \Sigma a$$

and C_4 has one free generator c with

$$\partial c = \Sigma b_1 - m b_2.$$

Lemma 8.8 *Suppose m is prime to p. Then $\mathbb{Z} \otimes_\pi C$ is a cyclic, but $H_2(C)$ is nonzero.*

Proof Tensoring C with the trivial right π–module $\mathbb{Z}$ has the effect of replacing each group element in $\mathbb{Z}[\pi]$ by 1. If we let $\overline{a} = 1 \otimes a$ and define $\overline{b}_1$, $\overline{b}_2$, and $\overline{c}$ similarly, then

$$\begin{aligned} \partial\overline{b}_1 &= m\overline{a} \\ \partial\overline{b}_2 &= p\overline{a} \\ \partial\overline{c} &= p\overline{b}_1 - m\overline{b}_2. \end{aligned}$$

So we easily see that $\mathbb{Z} \otimes_\pi C$ is acyclic.

To show that $H_2(C) \neq 0$ is to show that the ideal in $\mathbb{Z}[\pi]$ generated by m and Σ is not the whole ring. If it were, there would be $r, s \in \mathbb{Z}[\pi]$ so that $rm + s\Sigma = 1$. Letting primes denote images under $\theta\colon \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\pi_1]$ we would have

$$mr' + s'\Sigma' = 1 \in \mathbb{Z}[\pi_0].$$

Now $s'\Sigma' = k\Sigma'$ for some integer k since $g\Sigma' = \Sigma'$ for each $g \in \pi_0$. Thus we have

$$mr' = 1 - k\Sigma'$$

which is impossible because m $(\neq 1)$ cannot divide both $1-k$ (the coefficient of 1 in $1-k\Sigma$) and also $-k$ (the coefficient of other elements of π_0 in $1-k\Sigma$). This contradiction completes the proof. $\square$

Now we are ready to construct the contractible manifold W. Let π be the perfect group given in Proposition 8.7. Take a complex C provided by Lemma 8.8 and let C' be the direct sum of infinitely many copies of C. Then $\mathbb{Z} \otimes_\pi C'$ is cyclic but $H_2(C)$ is infinitely generated over $\mathbb{Z}[\pi]$. Let M^w, $w \geqslant 8$ be a contractible manifold with $\pi_1(\mathrm{Bd}\, M) = \pi$. To form W we attach one at a time infinitely many 2, 3, and 4–handles to M thickening after each step. The attaching 1–sphere of each 2–handle is to be contractible. Then W has one end and π_1 is stable at ε with $\pi_1(\varepsilon) \longrightarrow \pi_1(W - M) = \pi$ an isomorphism. The handles are to be so arranged that there is a nicely equipped Morse function (see §8) $f\colon V = W - \mathrm{Int}\, M \longrightarrow [0, \infty)$ with $f^{-1}(0) = \mathrm{Bd}\, M$ having associated free $\pi_1(\mathrm{Bd}\, M) = \pi$–complex precisely C'. By Proposition 8.2 $H_2(\widetilde{V}, \mathrm{Bd}\, \widetilde{M}) = H_2(C')$. But $H_2(C')$ is infinitely generated over π and V is a 1–neighborhood of ε. So Definition 4.4 and Lemma 4.7 say that ε cannot be tame. However $H_*(W, M) = H_*(V, \mathrm{Bd}\, M) = H_*(\mathbb{Z} \otimes_\pi C') = 0$, and the exact sequence of (W, M) then shows that W has the homology of a point. Since $\pi_1(W) = 1$, W is a contractible by Hilton [23, p. 98].

It remains now to add handles to M realizing C' as claimed. Each λ–handle added is, to be precise, an elementary cobordism of index λ. It is equipped with Morse function, gradient field, orientation for the left hand disk, and base path to the critical point. It contributes one generator to the complex for f. We order the free generators $z_1, z_2, \ldots$ of C' so that z_i involves z_j with $j < i$, then add corresponding handles in this order.

Suppose inductively that we have constructed a finite handlebody W' on M and formed a nicely equipped Morse function on $W' - \operatorname{Int} M$ that realizes the subcomplex of C' generated by $z_1, \ldots, z_{n-1}$. We suppose also that W' is parallelizable, that the attaching 1–spheres for all 2–handles are spanned by disks in $\operatorname{Bd} M$ and that the 3–handles all have a certain desirable property that we state precisely below.

Since we are building a contractible (hence parallelizable) manifold we must certainly keep each handlebody parallelizable. Now in the proof of Theorem 2 in Milnor [14, p. 47] it is shown how to take a given homotopy class in $\pi_k(\operatorname{Bd} W')$, $k < \frac{w}{2}$, and paste on a handle with attaching sphere in the given class so that $W' \cup \{\text{handle}\}$ is still parallelizable. We agree that handles are all to be attached in this way.

Without changing the gradient-like vector field ξ, temporarily make the Morse function nice so that $W' - \operatorname{Int} M$ is a product $c_2 c_3 c_4$ of cobordisms $c_\lambda = (X_\lambda; B_{\lambda-1}, B_\lambda)$, $\lambda = 2, 3, 4$, with critical points of one index λ only.

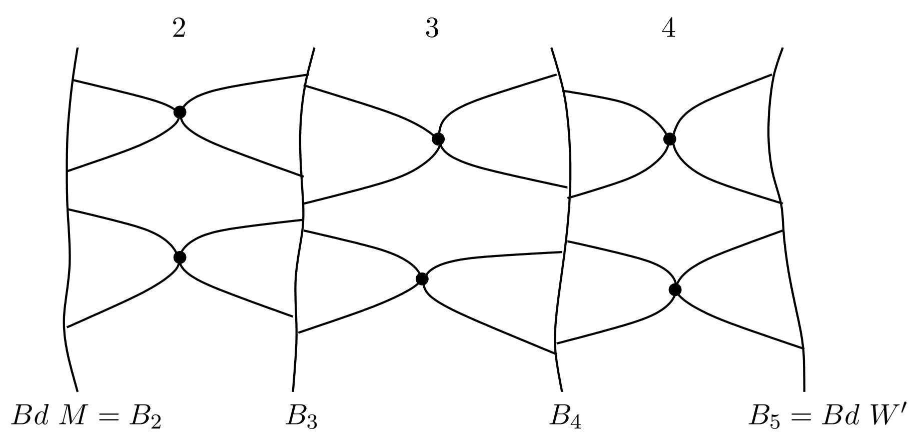


Figure 8:

If z_n is in dimension 2 we add a small trivial handle at $\operatorname{Bd} W'$ so that the (contractible) attaching sphere spans a 2–disk in $\operatorname{Bd} W'$ which translates along ξ–trajectories to $\operatorname{Bd} M$.

If z_n is in dimension 3, ∂z_n determines a unique element of $H_2(\widetilde{X}_2, \widetilde{B}_1) \cong \pi_2(X_2, B_1)$, hence a unique element of $\pi_2(B_2) \cong \pi_2(X_2) \cong \pi_2(X_2, B_1) \oplus \pi_2(B_1)$. (The last isomorphism holds because the 2–handles are capped by disks in $B_1 = \operatorname{Bd} M$.) Realize this element of $\pi_2(B_2)$ by an imbedded oriented 2–sphere S with base path in B_2. Slide S to general position in B_2; translate it along ξ–trajectories to $\operatorname{Bd} W'$ and add a suitable 3–handle with this attaching 2–sphere.

We assume inductively that for *each* 3–handle the attaching 2–sphere in B_2 gives a class in the summand $\pi_2(X_2, B_1)$ of $\pi_2(B_2)$. This is the desirable feature we mentioned above. Notice that the new 3–handle has this property. We will need this property presently.

If the dimension of z_n is 4, ∂z_n gives a unique class in $H_3(\widetilde{X_3}, \widetilde{B_2})$. We want an imbedded oriented 3–sphere S with base path in B_3 so that the class of S in $H_3(X_3)$ goes to $\partial z_n \in H_3(\widetilde{X_3}, \widetilde{B_2})$. Now ∂z_n is in the kernel of the composed map to $H_2(\widetilde{X_2}, \widetilde{B_1})$

$$:\qquad \begin{tikzcd} H_3(\widetilde{X_3}, \widetilde{B_2}) \arrow[r, "d"] & H_2(\widetilde{B_2}) \arrow[r, "\cong"] \arrow[d, "\cong"] & H_2(\widetilde{X_2}) \arrow[r] \arrow[d, "\cong"] & H_2(\widetilde{X_2}, \widetilde{B_2}) \arrow[d, "\cong"] \\ & \pi_2(B_2) \arrow[r] & \pi_2(X_2) \arrow[r] & \pi_2(X_2, B_1) \end{tikzcd}$$

The property assumed for 3–handles guarantees that $\operatorname{Im}(d)$ lies in the summand $H_2(\widetilde{X_2}, \widetilde{B_1})$ of $H_2(\widetilde{B_2})$, i.e. $\operatorname{Im}(d)$ goes (1-1) into $H_2(\widetilde{X_2}, \widetilde{B_1})$. Thus $\partial(\partial z_n) = 0$ implies $d(\partial z_n) = 0$. From the exact sequence of $(\widetilde{X}_3, \widetilde{B}_2)$ we see that ∂z_n is in the image of an element in $H_3(\widetilde{X_3})$. Now the Hurewicz map $\pi_3(B_3) \cong \pi_3(X_3) \cong \pi_3(\widetilde{X_3}) \longrightarrow H_3(\widetilde{X_3})$ is onto. (See [23, p. 167]). So there is a homotopy class s in $\pi_3(B_3)$ that goes to $\partial z_n \in H_3(\widetilde{X_3}, \widetilde{B}_2)$. Since $\dim(B_3) = w - 1 \geqslant 7$ we can represent s by an imbedded oriented 3–sphere S in B_3 with base path. This is the desired attaching sphere. We slide it to general position, translate it to $\operatorname{Bd} W'$ and add the desired 4–handle with this attaching sphere.

We conclude that with any dimension 2, 3, or 4 for z_n we can add a handle at $\operatorname{Bd} W'$ and extend the Morse function and its equipment to the handle so the subcomplex of C' generated by $z_1, \ldots, z_n$ is realized, and all inductive assumptions still hold. Thus the required construction has been defined to establish Proposition 8.7. $\square$

Remark 2 *M^w was a smooth compact submanifold of S^w. It is easy to add all the required 2, 3 and 4–handles to M* inside *S^w. Then W will be a contractible open subset of S^w.*

9 Classifying Completions

Recall that a completion of a smooth open manifold W is a smooth imbedding i of W onto the interior of a smooth compact manifold $\overline{W}$. Our Main Theorem 5.7 gives necessary and sufficient conditions for the existence of a completion when $\dim W \geqslant 6$. If a completion does exist one would like to classify the different ways of completing W. We give two classifications by Whitehead torsion corresponding to two notions of equivalence between completions – isotopy equivalence and pseudo-isotopy equivalence. As a corollary we find that there exist diffeomorphisms of contractible open subsets of Euclidean space that are pseudo-isotopic but not isotopic. According to J. Cerf this cannot happen for diffeomorphisms of closed 2–connected smooth manifolds of dimension $\geqslant 6$.

For the arguments of this chapter we will frequently need the following

Collaring Uniqueness Theorem 9.1 *Let V be a smooth manifold with compact boundary M. Suppose h and h' are collarings of M in V – viz. smooth imbeddings of $M \times [0,1]$ into V so that $h(x,0) = h'(x,0) = x$ for $x \in M$. Then there exists a diffeomorphism f of V onto itself, fixing M and points outside some compact neighborhood of M, so that $h' = f \circ h$.*

The proof follows directly from the proof of the tubular neighborhood uniqueness theorem in Milnor [25, p. 22]. To apply the latter directly one can extend h and h' to bicollars (= tubular neighborhoods) of M in the double V.

Definition 9.2 Two collars V, V' of a smooth end ε are called *parallel* if there exists a third collar neighborhood $V'' \subset \operatorname{Int} V \cap \operatorname{Int} V'$ such that the cobordisms $V - \operatorname{Int} V''$ are diffeomorphic to $\operatorname{Bd} V'' \times [0,1]$.

Lemma 9.3 *If V and V' are parallel collars and $V' \subset \operatorname{Int} V$, then $V - \operatorname{Int} V' \cong$* $\operatorname{Bd} V \times [0,1]$.

Proof Let $V'' \subset \operatorname{Int} V \cap \operatorname{Int} V'$ be as in Definition 9.2. Then $V' - \operatorname{Int} V''$ is a collar neighborhood of $\operatorname{Bd} V''$ in $V - \operatorname{Int} V''$. By the Collaring Uniqueness Theorem 9.1 there is a diffeomorphism of $V - \operatorname{Int} V''$ onto itself that carries

$V' - V''$ onto a small standard collar of $\mathrm{Bd}\, V''$ and hence $V - \mathrm{Int}\, V'$ onto the complement of the small standard collar. Since the 'standard' collar can be so chosen that its complement is diffeomorphic to $\mathrm{Bd}\, V \times [0,1]$, the Lemma is established. $\square$

Definition 9.4 If V and V' are any two collars of ε, the *difference torsion* $\tau(V, V') \in \mathrm{Wh}(\pi_1 \varepsilon)$ is determined as follows. Let V'' be a collar parallel to V' so small that $V' \subset \mathrm{Int}\, V$. Then $(V - \mathrm{Int}\, V''; \mathrm{Bd}\, V, \mathrm{Bd}\, V'')$ is easily seen to be an h–cobordism. Its torsion is $\tau(V, V')$.

It is a trivial matter to verify that $\tau(V, V')$ is well defined and depends only on the parallel classes of V and V'. Notice that $\tau(V', V) = -\tau(V, V')$ and $\tau(V, V'') = \tau(V, V') + \tau(V', V'')$ if V'' is a third collar. (See Milnor [17, §11].)

An immediate consequence of Stallings' classification of h–cobordisms (Milnor [17]) is

Theorem 9.5 *If* $\dim W \geqslant 6$ *and one collar* V_0 *of* ε *is given, then the difference torsions* $\tau(V_0, V)$, *for collars* V *of* ε *put the classes of parallel collars of* ε *in 1-1 correspondence with the elements of* $\mathrm{Wh}(\pi_1 \varepsilon)$.

If W is an open manifold that has a completion and V is a closed neighborhood of ∞ that is a smooth submanifold with $V \cong \mathrm{Bd}\, V \times [0,1)$ we call V a *collar of* ∞. Apparently the components of V give one collar for each end of W. Thus there is a natural notion of parallelism for collars of ∞ and Theorem 9.1 holds good.

Observe that a completion $i\colon W \longrightarrow \overline{W}$ of a smooth open manifold W determines a unique parallel class of collars of each end ε of W. (This uses collaring uniqueness again.) Conversely if a collar V of ∞ is specified in W, form $\overline{W}$ from the disjoint union of W and $\mathrm{Bd}\, V \times [0,1]$ by identifying $V \subset W$ with $\mathrm{Bd}\, V \times [0,1)$ under a diffeomorphism. Then $i\colon W \hookrightarrow \overline{W}$ is a completion and the parallel class of collars it determines certainly includes V.

Let $i\colon W \longrightarrow \overline{W}$ and $i'\colon W \longrightarrow \overline{W'}$ be two completions of the smooth open manifold W. If $f\colon \overline{W} \longrightarrow \overline{W'}$ is a diffeomorphism the *induced diffeomorphism* $f'\colon W \longrightarrow W$ is defined by $f'(x) = (i')^{-1}(f \circ i(x))$.

Proposition 9.6 *The completions* i *and* i' *determine the same class of parallel collars of* ∞ *if and only if for any prescribed compact set* $K \subset W$ *there exists a diffeomorphism* $f\colon \overline{W} \longrightarrow \overline{W'}$ *so that the induced diffeomorphism of* W *fixes* K.

Proof Let $K \subset W$ be a given compact set. Let $\overline{V}$ be a collar of $\mathrm{Bd}\,\overline{W}$ so small that $V = i^{-1}(\overline{V})$ does not meet K. If i and i' determine the same class of collars at each end of W, the closure $\overline{V'}$ of $i'(V)$ in $\overline{W'}$ is a collar of $\mathrm{Bd}\,\overline{W'}$. Let $f_0\colon\ \mathrm{Int}\,\overline{W} \longrightarrow \mathrm{Int}\,\overline{W'}$ be the diffeomorphism given by $f_0(x) = i' \circ i^{-1}(x)$. Let C be a collar of $i(\mathrm{Bd}\,V)$ in $\overline{V}$. The Collaring Uniqueness Theorem 9.1 shows that the map $(f_0)_{|C}$ extends to a diffeomorphism $f_1\colon\ \overline{V} \longrightarrow \overline{V'}$. Now define $f\colon\ \overline{W} \longrightarrow \overline{W'}$ to be f_0 on $(\overline{W} - \overline{V}) \cup C$ and f_1 on $\overline{V}$. Since f_0 coincides with f_1 on C, f is a diffeomorphism. The induced map $f'\colon\ W \longrightarrow W$ fixes $W - V$ and hence K.

The reverse implication is easy. If V is a collar of ∞ in the class determined by i, choose a diffeomorphism $f\colon\ \overline{W} \longrightarrow \overline{W'}$ so that the induced map $f'\colon\ W \longrightarrow W$ fixes $W - \mathrm{Int}\,V$. Then $f'(V) = V'$ is a collar in the class for i'. □

Let $i\colon\ W \longrightarrow \overline{W}$ and $i'\colon\ W \longrightarrow \overline{W'}$ be two completions of the smooth open manifold W. By Definition 9.4 and the discussion preceding Proposition 9.6 there is a natural way to define a *difference torsion* $\tau(i,i') \in \mathrm{Wh}(\pi_1\varepsilon_1) \times \ldots \times \mathrm{Wh}(\pi_1\varepsilon_k)$ where $\varepsilon_1, \ldots \varepsilon_k$ are the ends of W. Combining Theorem 9.5 and Proposition 9.6 we get

Theorem 9.7 *If* $\dim W \geqslant 6$, $\tau(i,i') = 0$ *if and only if given any compact* $K \subset W$ *there exists a diffeomorphism* $f\colon\ \overline{W} \longrightarrow \overline{W'}$ *so that the induced diffeomorphism* $f'\colon\ W \longrightarrow W$ *fixes* K. *Further, if* i *is fixed, every possible torsion occurs as* i' *varies.*

Recall that two diffeomorphism f and g of a smooth manifold W onto itself are called (smoothly) *isotopic* [respectively *pseudo-isotopic*] if there exists a level preserving [respectively not necessarily level preserving] diffeomorphism $F\colon\ W \times [0,1] \longrightarrow W \times [0,1]$ so that $F_{|W \times 0}$ gives $f \times 0$ and $F_{|W \times 1}$ gives $g \times 1$.

Definition 9.8 Let $i\colon\ W \longrightarrow \overline{W}$ and $i'\colon\ W \longrightarrow \overline{W'}$ be two completions of the smooth open manifold W. We say i is *isotopy equivalent* [resp. *pseudo-isotopy equivalent*] to i' if there exists a diffeomorphism $f\colon\ \overline{W} \longrightarrow \overline{W'}$ so that the induced diffeomorphism $f'\colon\ W \longrightarrow W$ is isotopic [resp. pseudo-isotopic] to the identity. Also, we say i and i' are *perfectly equivalent* if there exists a diffeomorphism $f\colon\ \overline{W} \longrightarrow \overline{W'}$ so that the induced diffeomorphism $f'\colon\ W \longrightarrow W$ is the identity – or equivalently so that $i' = f \circ i$.

We examine perfect equivalence first. The completions i and i' are apparently perfectly equivalent if and only if the map $f_0\colon\ \mathrm{Int}\,\overline{W} \longrightarrow \mathrm{Int}\,\overline{W'}$ given by

$f_0(x) = i'(i^{-1}(x))$ extends to a diffeomorphism $\overline{W} \longrightarrow \overline{W'}$. Notice that if the map f_0 extends to a continuous map $f_1\colon \overline{W} \longrightarrow \overline{W'}$, this map is unique. Thus i and i' are perfectly equivalent precisely when f_1 exists and turns out to be smooth, 1-1 and smoothly invertible.

Although perfect equivalence is perhaps the most natural of the three above it is unreasonable stringent at least from the point of view of algebraic topology. For example we easily form uncountable many completions of $\operatorname{Int} D^2$ (or $\operatorname{Int} D^n$, $n \geqslant 2$) as follows. If S is a segment on $\operatorname{Bd} D^2$ let $i\colon \operatorname{Int} D^2 \longrightarrow D^2$ be any completion which is the restriction of a smooth map $g\colon D^2 \overset{\text{onto}}{\longrightarrow} D^2$ that collapses S to a point but maps $D^2 - S$ diffeomorphically. (Such a map is easy to construct.)

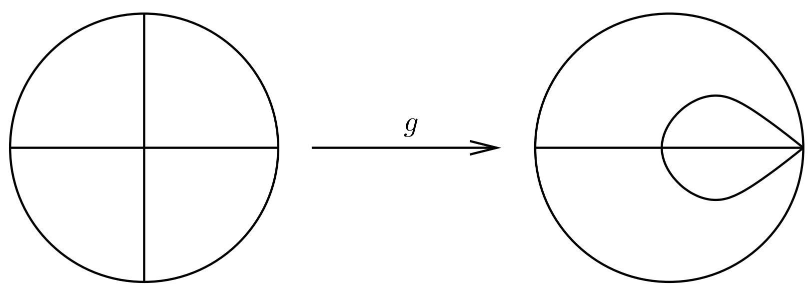


Figure 9:

Let r_θ be the rotation of $\operatorname{Int} D^2$ through an angle θ. Then for distinct angles θ_1, θ_2 the completions $i \circ r_{\theta_1}$, $i \circ r_{\theta_2}$ are distinct. In fact the induced map $\operatorname{Int} D^2 \longrightarrow \operatorname{Int} D^2$ does not extend to a continuous map $D^2 \longrightarrow D^2$. Apparently these completions would not even be perfectly equivalent in the topological category.

For a somewhat less obvious reason, there are uncountable many completions of $\operatorname{Int} D^1 = (-1,1)$ no two of which are perfectly equivalent. If i and i' are two completions $(-1,1) \longrightarrow [-1,1]$ there is certainly an induced *homeomorphism* f_1 of $[-1,1]$ onto itself that extends the monotone smooth function $f'(t) = i'(i^{-1}(t))$. Up to a perfect equivalence we can assume that $i(t) \longrightarrow 1$ and $i'(t) \longrightarrow 1$ as $t \longrightarrow 1$. Let $h\colon (-1,1) \longrightarrow (0,\infty)$ be the map $h(t) = (1+t)/(1-t)$ and form the functions $g(t) = h \circ i(t)$, $g'(t) = h \circ i'(t)$. In case i and i' are perfectly equivalent f_1 is a diffeomorphism and one can verify that $g(t)/g'(t)$ has limit $Df_1(1)$ as $t \longrightarrow 1$ and limit $1/Df_1(-1)$ as $t \longrightarrow -1$. (Hint: $Df'(t) = (Di'(i^{-1}(t)))/(Di(i')^{-1}(t))$ is shown to have the same limit as $\{g(t)/g'(t)\}^{\pm 1}$ when $t \longrightarrow \pm 1$ by applying l'Hôpital's rule.) For any positive real number α consider the completion $i_\alpha(t) = h^{-1}(h(t)^\alpha)$ and the map $g_\alpha(t) = h(i_\alpha(t)) = h(t)^\alpha$. When α and β are distinct positive real numbers

$$\frac{g_\alpha(t)}{g_\beta(t)} = \frac{h(t)^\alpha}{h(t)^\beta} = h(t)^{\alpha-\beta}$$

does not converge to a finite non-zero value as $t \longrightarrow \pm 1$. Thus the above discussion shows that i_α and i_β cannot be perfectly equivalent.

Using the idea of our first example one can show that if a smooth open manifold W $(\neq D^1)$ has one completion, then it has $2^{\aleph_0}$ completions no two of which are perfectly equivalent. In fact up to perfect equivalence there are exactly $2^{\aleph_0}$ completions. To show that there are no more observe that

(a) If $\overline{W}$ is fixed there are at most $2^{\aleph_0}$ completions $i\colon W \longrightarrow \overline{W}$, since there are only $2^{\aleph_0}$ continuous maps $W \longrightarrow \overline{W}$.

(b) There are only $2^{\aleph_0}$ diffeomorphism classes of smooth manifolds since each smooth manifold is imbeddible as a closed smooth submanifold of a Euclidean space.

We have already studied isotopy equivalence in another guise.

Proposition 9.9 *The classification of completions up to isotopy equivalence is just classification according to the corresponding families of parallel collars of ∞.*

Proof Let $i\colon W \longrightarrow \overline{W}$, $i'\colon W \longrightarrow \overline{W'}$ be two completions and $f\colon \overline{W} \longrightarrow \overline{W'}$ a diffeomorphism so that the induced diffeomorphism $f'\colon W \longrightarrow W$ is isotopic to the identity. We show that collars V, V' of ∞ corresponding to i, i' are necessarily parallel. We know that $f'(V)$ is parallel to V'. Consider the isotopic deformation of $j\colon \operatorname{Bd} V \hookrightarrow W$ induced by the isotopy of f' to 1_W. Using Thom's Isotopy Extension Theorem [25] we can extend this to an isotopy h_t, $0 \leqslant t \leqslant 1$ of 1_W that fixes points outside some compact set K. If we choose V' so small that $V' \cap K = \emptyset$, h_t fixes V'. Now $h_1(V) = f'(V)$ and $h_1(V') = V'$ so $V - \operatorname{Int} V' \cong h_1(V - \operatorname{Int} V') = h_1(V) - \operatorname{Int} V' = f'(V) - \operatorname{Int} V' \cong \operatorname{Bd} V' \times [0,1]$ which means V and V' are parallel.

To prove the opposite implication suppose $V \subset W$ is a collar of ∞ for both i and i'. Thus there are diffeomorphisms:

$$h\colon\ i(V) \cup \operatorname{Bd} \overline{W} \longrightarrow \operatorname{Bd} V \times [0,1]$$
$$h'\colon\ i'(V) \cup \operatorname{Bd} \overline{W'} \longrightarrow \operatorname{Bd} V \times [0,1].$$

Using the Collaring Uniqueness Theorem 9.1 we see that h can be altered so that $h' \circ h^{-1}$ fixes points near $\mathrm{Bd}\,V \times 0$. Define $f\colon \overline{W} \longrightarrow \overline{W'}$ by

$$f(x) = \begin{cases} i'(i^{-1}(x)) & \text{for } x \notin i(\mathrm{Int}\,V) \\ (h')^{-1}h(x) & \text{for } x \in i(V) \cup \mathrm{Bd}\,\overline{W} \end{cases}$$

Then f is a diffeomorphism such that the induced diffeomorphism $f'\colon W \longrightarrow W$ fixes a neighborhood of $W - \mathrm{Int}\,V$. The following lemma provides a smooth isotopy of f' to 1_W that actually fixes a neighborhood of $W - \mathrm{Int}\,V$. □

Lemma 9.10 *Let M be a closed smooth manifold and g be a diffeomorphism of $M \times [0,1)$ that fixes a neighborhood of $M \times 0$. Then there exists an isotopy g_t, $0 \leqslant t \leqslant 1$, of the identity of $M \times [0,1)$ to g that fixes a neighborhood of $M \times 0$.*

Proof The isotopy is

$$g_t(m,x) = \begin{cases} (m,x) & \text{if } t = 0 \\ tg(m, \dfrac{x}{t}) & \text{if } t \neq 0 \end{cases}$$

where $(m,x) \in M \times [0,1)$. □

We now discuss the looser pseudo-isotopy equivalence between completions. For simplicity we initially suppose that the smooth open manifold W^n has just one end ε. Then if $i\colon W \longrightarrow \overline{W}$ and $i'\colon W \longrightarrow \overline{W'}$ are two completions there is by Theorem 9.5 and Proposition 9.9 a difference torsion $\tau(i,i') \in \mathrm{Wh}(\pi_1\varepsilon)$ that is an invariant of isotopy equivalence, and, provided $n \geqslant 6$, classifies completions i' as i' varies while i remains fixed. Here $\tau(i,i') = \tau(V,V')$ where V and V' are collars corresponding to i and i'.

Theorem 9.11 *Suppose the manifold W^n above has dimension $n \geqslant 5$. If the completion i is pseudo-isotopy equivalent to i', then $\tau(i,i') = \tau_0 + (-1)^{n-1}\overline{\tau}_0$ where $\tau_0 \in \mathrm{Wh}(\pi_1\varepsilon)$ is an element so that $j_*(\tau_0) = 0 \in \mathrm{Wh}(\pi_1 W)$. If $n \geqslant 6$ the converse is true. (Here j_* is the inclusion induced map $\mathrm{Wh}(\pi_1\varepsilon) \longrightarrow \mathrm{Wh}(\pi_1 W)$ and $\overline{\tau_0}$ is the conjugate of τ_0 under the involution of $\mathrm{Wh}(\pi_1\varepsilon)$ discussed by Milnor in [17, p. 49 and pp. 55-56].)*

Proof First we explain the construction that gives the key to the proof. Given a smooth closed manifold M^m, $m \geqslant 4$, we form the unique (relative) h–cobordism X with left end $M \times [0,1]$ that has torsion $\tau \in \mathrm{Wh}(\pi_1 M)$. It is understood that X is to give product cobordisms X_0 and X_1 over $M \times 0$ and $M \times 1$.

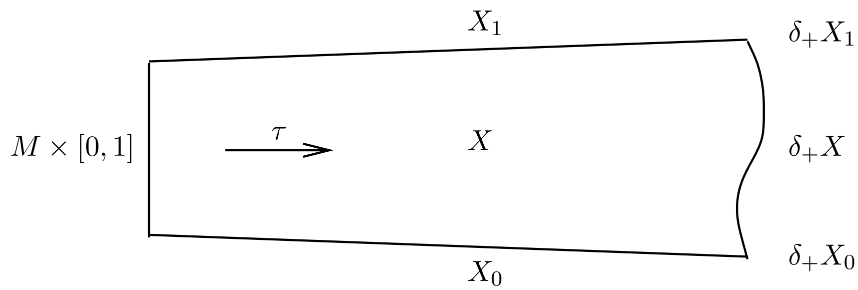


Figure 10:

The construction in Milnor [17, p. 58] applies with only obvious changes needed because $M \times [0,1]$ has a boundary. We will call X the *wedge* over $M \times [0,1]$ with torsion τ.

Notice that the right hand end ∂_+X of X gives an h–cobordism between the right hand ends ∂_+X_0 and ∂_+X_1 of X_0 and X_1. The torsion of $\partial_+X_0 \hookrightarrow X$ is τ and the torsion of $\partial_+X \hookrightarrow X$ is $(-1)^{m+1}\overline{\tau}$ by the duality theorem of Milnor [17]. It follows that the torsion of $\partial_+X_0 \subset \partial_+X$ is $\tau - (-1)^{m+1}\overline{\tau} = \tau + (-1)^m\overline{\tau}$ by [17, p. 35].

Observe also that, as a cobordism X_0 to X_1, X has a two-sided inverse, namely the wedge over $M \times [0,1]$ with torsion $-\tau$. Then the infinite product argument of Stallings [10] shows that $X - X_0 \cong X_1 \times [0,1)$.

We now prove the first statement of the theorem. Suppose that there exists a diffeomorphism $f\colon \overline{W} \longrightarrow \overline{W'}$ so that there is a pseudo-isotopy F of the induced map $f'\colon W \longrightarrow W$ to the identity. The pseudo-isotopy F is a diffeomorphism of $W \times [0,1]$ that gives the identity on $W \times 0$ and $f' \times 1$ on $W \times 1$. It will be convenient to identify W with $i(W) \subset \overline{W}$.

If V is a collar neighborhood for i, the closure $\overline{V}$ of V in $\overline{W}$, is a collar of $\mathrm{Bd}\,\overline{W}$, and the closure $F(V \times [0,1]) \subset \mathrm{Bd}\,\overline{W} \times [0,1]$ of $F(V \times [0,1])$ in $\overline{W} \times [0,1]$ is a wedge over $\overline{V} \times 0$ with torsion τ_0 say. Now $f'(V)$ is a collar of V' corresponding to i'. So the end of the wedge $f'(V) \times 1 \cup \mathrm{Bd}\,\overline{W} \times 1 \subset \overline{W} \times 1$ gives an h–cobordism with torsion $-\tau(i,i') = \tau_0 + (-1)^{n-1}\overline{\tau_0}$. Since the product cobordism $\overline{W} \times [0,1]$ is the union of the wedge over $\overline{V} \times 0$ with torsion τ_0 and another product, the Sum Theorem 6.9 for Whitehead Torsion says that $j_*(\tau_0) = 0 \in \mathrm{Wh}(\pi_1 W)$. This completes the proof of the first statement.

To prove the converse assertion suppose that $-\tau(i,i')$ has the form $\tau_0 + (-1)^{n-1}\overline{\tau_0}$, where $j_*(\tau_0) = 0 \in \mathrm{Wh}(\pi W)$. As above W is identified with $i(W) = \mathrm{Int}\,\overline{W}$,

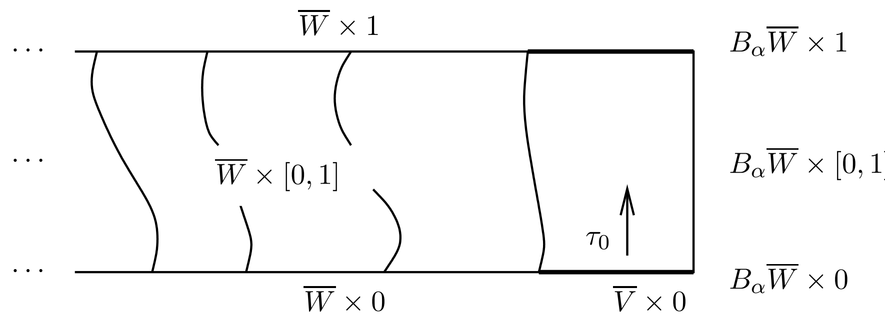


Figure 11:

$\overline{V}$ is a collar of $\operatorname{Bd}\overline{W}$ and $V=\overline{V}-\operatorname{Bd}\overline{W}$ is a collar of ε. For the wedge over $\overline{V}$ with torsion τ_0, choosing X_0 over $\operatorname{Bd}V$ (not $\operatorname{Bd}\overline{W}$). From X and $W\times[0,1]$ form a completion Z of $W\times[0,1]$ (in the sense of Definition 10.2) by identifying $X-X_1\cong X_0\times[0,1)$ with $V\times[0,1]\cong\operatorname{Bd}V\times[0,1]\times[0,1)$ under a diffeomorphism that is the identity on the last factor $[0,1)$, and matches X_0 with $\operatorname{Bd}V\times[0,1]$ in the natural way.

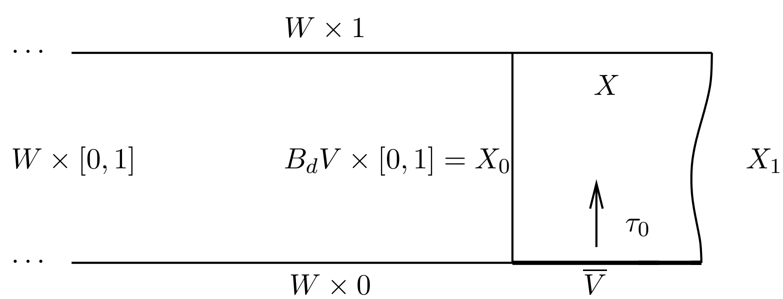


Figure 12:

Now Z is a compact h–cobordism from a manifold we can identify with $\overline{W}$ to a manifold we call $\overline{W}^n$. We claim that the completion $i''\colon W\xrightarrow{\mathrm{id}\times 1}W\times 1\hookrightarrow\overline{W}^n$ is isotopy equivalent to i'. For $\partial_+X=V\times 1\cup\operatorname{Bd}\overline{W}^n$ is an h–cobordism with torsion $\tau_0+(-1)^m\overline{\tau}_0$. So $-\tau(i,i'')=\tau_0+(-1)^{n-1}\overline{\tau_0}=-\tau(i,i')$. Thus $\tau(i',i'')=0$. Since $n\geqslant 6$, our claim is verified.

Also $(Z;W,W'')=0$, since Z is the union of a product cobordism and the wedge X with torsion τ_0 satisfying $j_*\tau_0=0$ (c.f. Theorem 6.9). By the s–cobordism theorem (Wall [2]), $Z\cong\overline{W}\times[0,1]$. Any such product structure gives a diffeomorphism $\overline{W}\longrightarrow\overline{W}''$ and a pseudo-isotopy to the identity of the induced map $W\longrightarrow W$ (since $Z-X_1$ is by construction $W\times[0,1]$). As i

and i'' are isotopy equivalent there is a diffeomorphism $\overline{W''} \longrightarrow \overline{W'}$ and an isotopy to the identity of the induced map $W \longrightarrow W$. Thus the composed diffeomorphism $\overline{W} \longrightarrow \overline{W}'' \longrightarrow \overline{W'}$ induces a map which is pseudo-isotopic to the identity. This completes the proof. □

Remark If instead of one end ε, W has a finite set of ends $\varepsilon = \{\varepsilon_1, \dots, \varepsilon_k\}$, Theorem 9.11 generalizes almost word for word. In the statement, $\mathrm{Wh}(\pi_1\varepsilon)$ is $\mathrm{Wh}(\pi_1\varepsilon_1) \times \cdots \times \mathrm{Wh}(\pi_1\varepsilon_k)$ and j_* is induced by the maps $\pi_i(\varepsilon_i) \longrightarrow \pi_1(W)$, $i = 1, \dots, k$.

Remark As a further generalization one can consider the problem of completing only a subset ε of all the ends of W while leaving the other ends open. Thus a completion for ε is a smooth imbedding of W onto the interior of a smooth manifold W' so that the components of a collar for $\mathrm{Bd}\, W'$ give collars for ends in ε (and no others.) With the obvious definition of pseudo-isotopy equivalence Theorem 9.11 is generalized by substituting a quotient $\mathrm{Wh}(\pi_1 W)/N$ for $\mathrm{Wh}(\pi_1 W)$. Here N is the subgroup generated by the images of the maps $\mathrm{Wh}(\pi_1\varepsilon_i) \longrightarrow \mathrm{Wh}(\pi_1 W)$ where ε_i ranges over the ends *not* in the set ε. This is justified by the following theorem.

Let W' be a smooth manifold with $\mathrm{Bd}\, W'$ compact so that W' admits a completion. An *h–cobordism* on W' is by definition a relative (non-compact) cobordism $(V; W', W'')$ so that V has a completion $\overline{V}$ (in the sense of Definition 10.2) which gives a compact relative h–cobordism $(\overline{V}; \overline{W'}, \overline{W''})$ between a completion $\overline{W'}$ of W' and a completion $\overline{W''}$ of W''. The h–cobordism is understood to be a product over $\mathrm{Bd}\, \overline{W'}$. Let N be the subgroup of $\mathrm{Wh}(\pi_1 W')$ generated by the images of the maps $\mathrm{Wh}(\pi_1\varepsilon') \longrightarrow \mathrm{Wh}(\pi_1 W)$ as ε' ranges over the ends of W'.

Theorem 9.12 *If* $\dim W' \geqslant 5$, *the h–cobordisms on W' are classified up to diffeomorphism fixing W' by the elements of* $\mathrm{Wh}(\pi_1 W')/N$.

I omit the proof. It is not difficult to derive from Stallings' classification of (relative) h–cobordisms (c.f. [17, p. 58].) with the help of the wedges. The torsion for $(V; W', W'')$ above is the coset $\tau(\overline{V}; \overline{W'}, \overline{W}'') + N$.

Jean Cerf has recently established that pseudo-isotopy implies isotopy on smooth closed n–manifolds, $n \geqslant 6$, that are 2–connected (c.f. [28]). Theorem 9.13 shows this is false for open manifolds – even contractible open subsets of Euclidean space.

Theorem 9.13 *For $n \geqslant 2$ there exists a contractible smooth open manifold W^{2n+1} that is the interior of a smooth compact manifold and an infinite sequence $f_1, f_2, f_3, \ldots$ of diffeomorphisms of W onto itself such that all are pseudo-isotopic to 1_W but no two are smoothly isotopic. Further for each $n \geqslant 2$ this occurs with infinitely many topologically distinct contractible manifolds like W, each of which is an open subset of $\mathbb{R}^{2n+1}$.*

Remark 1) The maps $f_k \times 1_{\mathbb{R}}\colon W \times \mathbb{R} \longrightarrow W \times \mathbb{R}$, $k = 1, 2, \ldots$ are all smoothly isotopic.

Proof of Remark If $W \cong \operatorname{Int}\overline{W}$, $W \times \mathbb{R} \cong \operatorname{Int}(\overline{W} \times [0,1])$. But $W \times [0,1]$ (with corners smoothed) is a contractible smooth manifold with simply connected boundary – hence is a smooth $(2n+1)$–disk by [4, §9.1]. Thus $W \times \mathbb{R} \cong \mathbb{R}^{2n+1}$ and it is well known that any two orientation preserving diffeomorphisms of $\mathbb{R}^{2n+1}$ are isotopic (see [4, p. 60]).

$\square$

Remark 2) To extend Theorem 9.13 to allow even dimensions ($\geqslant 6$) for W, I would need a torsion τ with $\overline{\tau} \neq \tau$, (for the standard involution), and none is known for any group. However using the example $\mathrm{Wh}(\mathbb{Z}_8)$ with $\tau^* = -\tau$ [17, p. 56] one can distinguish isotopy and pseudo-isotopy on a suitable *non-orientable* $W = \operatorname{Int}\overline{W}^{2n}$, $n \geqslant 3$, where $\overline{W}^{2n}$ is smooth and compact with $\pi_1(W) = \mathbb{Z}_2$, $\pi_1 \operatorname{Bd}\overline{W} = \mathbb{Z}_8$.

Remark 3) I do not know whether pseudo-isotopy implies isotopy for diffeomorphisms of open manifolds that are interiors of compact manifolds with 1–connected boundary. Also it seems important to decide this for diffeomorphisms of closed smooth manifolds that are not 2–connected.

Proof of Theorem 9.13 We suppose first that n is $\geqslant 3$. Form a contractible smooth compact manifold $\overline{W}^{2n+1} \subset S^{2n+1}$ with $\pi_1 \operatorname{Bd}\overline{W} = \pi$ the binary icosahedral group $\{a, b\,;\, a^5 = b^3 = (ab)^2\}$ (see §8), and let $W = \operatorname{Int}\overline{W}$. In Lemma 9.14 below we show that there is a mapping $\phi\colon \mathbb{Z}_5 \longrightarrow \pi$ so that $\phi_*\colon \mathrm{Wh}(\mathbb{Z}_5) \longrightarrow \mathrm{Wh}(\pi)$ is 1-1. By Milnor [17, p. 26] $\mathrm{Wh}(\mathbb{Z}_5) = \mathbb{Z}$ and $\tau = \overline{\tau}$ for all $\tau \in \mathrm{Wh}(\mathbb{Z}_5)$ – hence for all elements of $\phi_* \mathrm{Wh}(\mathbb{Z}_5)$. Let β be a generator of $\phi_* \mathrm{Wh}(\mathbb{Z}_5)$ and form completions $i_k\colon W \longrightarrow \overline{W}_k$ of W, $k = 1, 2, \ldots$ such that $\tau(i, i_k) = k\beta + (-1)^{2n}\overline{k\beta} = 2k\beta$ where $i\colon W \hookrightarrow \overline{W}$. Since $\pi_1 W = 1$, Theorem 9.11 says that i and i_k are pseudo-isotopy equivalent i.e. there exists a diffeomorphism $g_k\colon W \longrightarrow \overline{W}_k$ so that the induced diffeomorphism $f_k\colon W \longrightarrow W$ is pseudo-isotopic to 1_W, $k = 1, 2, \ldots$. If f_j were isotopic to f_k, $j \neq k$, $f_k \circ f_j^{-1}\colon W \longrightarrow W$ would be isotopic to 1_W. But $f_k^{-1} \circ f_j$ is induced by $g_k \circ g_j^{-1}\colon W_j \longrightarrow W_k$. Hence i_j and i_k would be isotopy equivalent in contradiction to $\tau(i_j, i_k) = 2(k-j)\beta \neq 0$.

When $n = 2$, i.e. $\dim W = 5$, the above argument breaks down in two spots. It is not apparent that i_k exists with $\tau(i, i_k) = 2k\beta$. And when i_k is constructed is not clear that it is pseudo-isotopy equivalent to i. Repair the argument as follows. If V is a collar corresponding to i, let $V_k \subset \operatorname{Int} V$ be a collar such that the h–cobordism $V - \operatorname{Int} V_k$ is diffeomorphic to the right end of the wedge over $\operatorname{Bd} V \times [0,1]$ with torsion k. Then $\tau(V, V_k) = k\beta + (-1)^4\overline{k\beta} = 2k\beta$. So $\tau(i, i_k) = 2k\beta$ if we let $i_k\colon W \longrightarrow \overline{W}_k$ be a completion for which V_k is a collar. To show that this particular i_k is pseudo-isotopy equivalent to i we try to follow the proof for the second statement of Theorem 9.11 taking $i' = i_k$ and $\tau_0 = k\beta$. What needs to be adjusted is the proof that i'' and $i' (= i_k)$ are isotopy equivalent. Now, if $V'' \subset \operatorname{Int} V$ is a collar for i'', it is clear that $V - \operatorname{Int} V''$ is diffeomorphic to $\partial_+ X$, the right hand end of the wedge over $\operatorname{Bd} V \times [0,1]$ with torsion $\tau_0 = k\beta$. But in our situation $V - \operatorname{Int} V_k$ is by construction diffeomorphic to $\partial_+ X$. Because $\partial_+ X$ is an invertible h–cobordism, V'' and V_k are parallel collars. Thus Proposition 9.9 says that i'' and $i' = i_k$ are isotopy equivalent. The rest of the argument in the proof of Theorem 9.11 establishes that i and $i' = i_k$ are pseudo-isotopy equivalent.

Finally we give infinitely many topologically distinct contractible manifolds like $W \subset \mathbb{R}^{2n+1}$. Let W_s be the interior of the connected sum along the boundary of s copies of $\overline{W}$, $s = 1, 2, \ldots$. Now $W^{2n+1} \subset \mathbb{R}^{2n+1} \subset S^{2n+1} - \{\text{point}\}$, and the connected sum can clearly be formed inside $\mathbb{R}^{2n+1}$. Hence we can suppose $W_s \subset \mathbb{R}^{2n+1}$. W_s is distinguished topologically from W_r, $r \neq s$, by the fundamental group of the end which is the s–fold free product of π. As Wh is a functor $\operatorname{Wh}\pi$ is a natural summand of $\operatorname{Wh}(\pi * \cdots * \pi)$. Hence the argument for W will also work for W_s. This completes the proof of Theorem 9.13 modulo Lemma 9.14. □

Lemma 9.14 *There is a homomorphism* $\phi\colon \mathbb{Z}_5 \longrightarrow \pi = \bigl\{a, b\,;\, a^5 = b^3 = (ab)^2\bigr\}$ *so that* $\phi_*\colon \operatorname{Wh}(\mathbb{Z}_5) \longrightarrow \operatorname{Wh}(\pi)$ *is 1-1.*

Proof By [17, p. 26] $\operatorname{Wh}(\mathbb{Z}_5)$ is infinite cyclic with generator α represented by the unit $(t + t^{-1} - 1) \in \mathbb{Z}[\mathbb{Z}_5]$ where t is a generator of $\mathbb{Z}_5$. The quotient $\{a, b\,;\, a^5 = b^3 = (ab)^2 = 1\}$ of π is the rotation group A_5 of the icosahedron (see [21, pp. 67-69]). π has order 120 and A_5 has order 60 so $a^{10} = 1$ in π. Thus we can define $\phi(t) = a^2 \in \pi$.

To show that ϕ_* is 1-1 in $\mathrm{Wh}(\pi)$ it will suffice to give a homomorphism

$$h\colon\ \pi \longrightarrow \mathrm{O}(3)$$

so that if we apply h to $\phi(t+t^{-1}-1) = a^2 + a^{-2} - 1$ we get a matrix M with determinant not equal to ± 1. For by Milnor [17, p. 36-40] h determines a homomorphism h_* from $\mathrm{Wh}(\pi)$ to the multiplicative group of positive real numbers, and $h_*\phi_*(\alpha) = |\det M|$.

The homomorphism we choose is the composite

$$\pi \longrightarrow A_5 \longrightarrow \mathrm{O}(3)$$

where the second map is an inclusion so chosen that $a \in A_5$ is a rotation about the $\mathbb{Z}$–axis through angle $\theta = 72^\circ$. Thus

$$h(a) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$h(a^2 + a^{-2} - 1) = \begin{pmatrix} 2\cos 2\theta - 1 & 0 & 0 \\ 0 & 2\cos 2\theta - 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has determinant $\neq \pm 1$. $\square$

10 The Main Theorem Relativized and Applications to Manifold Pairs

We consider smooth manifolds W^n such that $\mathrm{Bd}\, W$ is a manifold without boundary that is diffeomorphic to the interior of a smooth compact manifold. A simple example is the closed upper half plane. An end ε of W is *tame* if it is isolated and satisfies conditions 1) and 2) of Definition 4.4. In defining a k*–neighborhood* V of an end ε of W, $k = 0, 1, 2, \ldots$, we must insist that V be a closed submanifold of W so that $V' = V \cap \mathrm{Bd}\, W$ is a smooth, possibly empty, submanifold of $\mathrm{Bd}\, W$ with $V' \cong \mathrm{Bd}\, V' \times [0, \infty)$. The frontier $\mathrm{b}V$ of V in W must be a smooth compact submanifold of W that meets $\mathrm{Bd}\, W$ transversely, in $\mathrm{Bd}(\mathrm{b}V) = \mathrm{Bd}\, V'$. Otherwise the definition of k–neighborhood is that given in Definition 2.4, Definition 3.9, and Definition 4.5, with frontier substituted for boundary. To show that an isolated end ε of W has arbitrarily small 0–neighborhoods form a proper smooth map

$$f\colon\ W \longrightarrow [0, \infty)$$

so that

(1) $f_{|\,\mathrm{Bd}\,W}$ is a proper Morse function with only finitely many critical points.

(2) f is the restriction of a proper Morse function f' on the double DW.

(To do this one first fixes $f_{|\,\mathrm{Bd}\,W}$, then constructs f' by the methods of Milnor [4, §2].) Then follow the argument of Theorem 2.5 to the desired conclusion remembering that frontier should replace boundaries. If ε is an isolated end of W so that π_1 is stable at ε and $\pi_1(\varepsilon)$ is finitely presented, then ε has arbitrarily small 1–neighborhoods. (The proof of Definition 3.9 is easily adapted.) Thus we can give the following definition of the *invariant* $\sigma(\varepsilon)$ *of a tame end* ε. Consider a connected neighborhood V of ε that is a smooth submanifold (possibly with corners) having compact frontier and one end. If V is so small that $\pi_1(\varepsilon) \longrightarrow \pi_1(V)$ has a left inverse r then $V \in \mathcal{D}$ and

$$r_*\sigma(V) \in \widetilde{K_0}(\pi_1\varepsilon)$$

is an invariant of ε (see Proposition 7.6). Define $\sigma(\varepsilon) = r_*\sigma(V)$. A *collar* for an end ε of W is a connected neighborhood V of ε that is a closed submanifold W such that the frontier $\mathrm{b}V$ of V is a compact smooth submanifold of W (possibly with boundary), and V is diffeomorphic to $\mathrm{b}V \times [0,\infty)$.

Relativized Main Theorem 10.1 *Suppose* W^n, $n \geqslant 6$, *is a smooth manifold such that* $\mathrm{Bd}\,W$ *is diffeomorphic to the interior of a compact manifold. If* ε *is a tame end of* W *the invariant* $\sigma(\varepsilon) \in \widetilde{K_0}(\pi_1\varepsilon)$ *is zero if and only if* ε *has a collar neighborhood.*

Proof We have already observed that ε has arbitrarily small 1–neighborhoods. To complete the proof one has to go back and generalize the argument of §4 and 5. There is no difficulty in doing this; one has only to keep in mind that frontiers of k–neighborhoods are now to replace boundaries, and that all handle operations are to be performed away from $\mathrm{Bd}\,W$. This should be sufficient proof. □

Suppose again that W is a smooth manifold such that $\mathrm{Bd}\,W$ is diffeomorphic to the interior of a compact smooth manifold.

Definition 10.2 A *completion* of W is a smooth imbedding $i\colon W \longrightarrow \overline{W}$ of W into a compact smooth manifold so that $i(\mathrm{Int}\,W) = \mathrm{Int}\,\overline{W}$ and the closure of $i(\mathrm{Bd}\,W)$ is a compact smooth manifold with interior $i(\mathrm{Bd}\,W)$. If N is a properly imbedded submanifold so that $\mathrm{Bd}\,N$ is compact and N meets $\mathrm{Bd}\,W$ in $\mathrm{Bd}\,N$, transversely, we say i gives a *completion* of (W,N) if the closure of $i(N)$ in $\overline{W}$ is a compact submanifold $\overline{N}$ that meets $\mathrm{Bd}\,W$ in $\mathrm{Bd}\,N$, transversely.

When W has a completion a *collar of* ∞ is a neighborhood V of ∞ so that the frontier $\mathrm{b}V$ is a smooth compact submanifold and $V \cong \mathrm{b}V \times [0,1)$. Notice that W has a completion if (and only if) it has finitely many ends, each with a collar. The natural construction for $\overline{W}$ (c.f. §9) yields a manifold $\overline{W} \supset W$ that has corners at the frontier of $\mathrm{Bd}\, W$. Of course they can be smoothed as in Milnor [9].

For the purposes of the theorem below observe that if the end ε of the Relativized Main Theorem has one collar, then one can easily find another collar V of ε so that $V \cap \mathrm{Bd}\, W$ is a *prescribed* collar of the ends of $\mathrm{Bd}\, W$ contained by ε.

The following theorem is a partial generalization of unknotting theorems for $\mathbb{R}^k$ in $\mathbb{R}^n$, $n-k \neq 2$. (See Theorem 10.7.) It might be called a 'peripheral unknotting theorem'. The notion of tameness and the invariant σ are essential in the proof but obligingly disappear in the statement.

Theorem 10.3 *Let W be a smooth open manifold of dimension $n \geqslant 6$ and N a smooth properly imbedded submanifold (without boundary). Suppose W and N separately admit a boundary. If N has codimension $\geqslant 3$ or else has codimension one and is 1–connected at each end, then there exists a compact pair $(\overline{W}, \overline{N})$ such that $W = \mathrm{Int}\, \overline{W}$, $N = \mathrm{Int}\, \overline{N}$.*

Complement 10.4 *It is a corollary of the proof we give and of the observation above that $\overline{N}$ can be chosen to determine a prescribed collar of ∞ in N.*

Remark A counterexample for codimension 2 is provided by an infinite string K in $\mathbb{R}^3$ that has evenly spaced trefoil knots.

Figure 13:

($\mathbb{R}^3 - K$ has non-finitely generated fundamental group – see §6). The boundary of a tubular neighborhood of K gives an example for codimension 1 showing that a restriction on the ends of N is necessary. To get examples in any dimension $\geqslant 3$ consider $(\mathbb{R}^3, K) \times \mathbb{R}^k$, $k = 0, 1, 2, \ldots$.

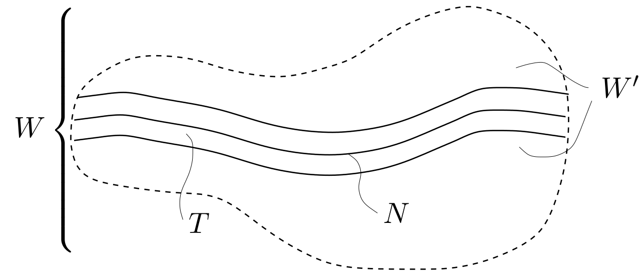


Figure 14:

Proof of Theorem 10.3 Let W' be W with the interior of a tubular neighborhood T of N removed. Apparently it will suffice to show that (T, N) and W' both have completions.

Let $U \cong \operatorname{Bd} U \times [0, \infty)$ be a collar of ∞ in N. Then the part $T_{|U}$ of the smooth disk bundle T over U is smoothly equivalent to the bundle $(T_{|\operatorname{Bd} U}) \times [0, \infty)$ over $\operatorname{Bd} U \times [0, \infty) \cong U$. One can deduce this from a smooth version of Theorem 11.4 in Steenrod [29]. It follows that (T, N) has a completion.

By the method suggested above, form a proper Morse function $f\colon W \longrightarrow [0, \infty)$ so that $f|N$ has no critical point on a collar $U = N \cap f^{-1}[a, \infty) \cong \operatorname{Bd} U \times [a, \infty)$, and so that, when restricted to $T_{|U} \cong (T_{|\operatorname{Bd} U}) \times [a, \infty)$, f gives the obvious map to $[a, \infty)$. Then for $b > a$, $V_b = f^{-1}[b, \infty)$ meets T in a collar T_b of ∞ in T. Consider $V'_b = V_b - \operatorname{Int} T = V_b \cap W'$ for any b noncritical, $b > a$. If N has codimension $\geqslant 3$, $i\colon V'_b \hookrightarrow V_b$ is a 1–equivalence by a general position argument. Since V_b and $V'_b \cap T_b$ are in $\mathcal{D}$, so is V'_b by Complement 6.6 and $0 = \sigma(V_b) = i_* \sigma(V'_b)$ by the Sum Theorem 6.5; as i_* is an isomorphism $\sigma(V'_b) = 0$. This shows that for each end ε of W there is a unique contained end ε' of W' and that ε' (like ε) is tame with $\sigma(\varepsilon') = 0$. Thus the Relativized Main Theorem says that W' has a completion. This completes the proof if N has codimension $\geqslant 3$.

For codimension 1 we will reduce the proof that (W, N) has a completion to

Proposition 10.5 *Let W be a smooth manifold of dimension $\geqslant 6$ so that $\operatorname{Bd} W$ is diffeomorphic to the interior of a compact manifold, and let N be a smooth properly imbedded submanifold of codimension 1 so that $\operatorname{Bd} N$ is compact and N meets $\operatorname{Bd} W$ in $\operatorname{Bd} N$, transversely. Suppose that W and N both have one end and separately admit a completion. If $\pi_1(\varepsilon_N) = 1$, then the pair (W, N) admits a completion.*

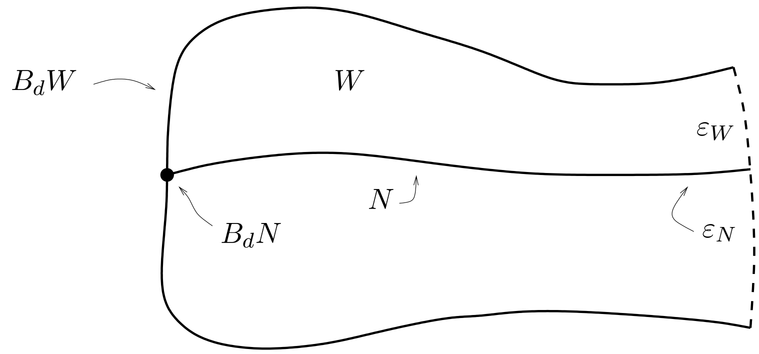


Figure 15:

The proof appears below. Observe that Proposition 10.5 continues to hold if N is replaced by several disjoint submanifolds $N_1, \ldots, N_k$ each of which enjoys the properties postulated for N. For we can apply Proposition 10.5 with $N = N_1$, then replace W by W minus a small open tubular neighborhood of N_1 (with resulting corners smoothed), and apply Proposition 10.5 again with $N = N_2$. Eventually we deduce that W minus small open tubular neighborhoods of $N_1, \ldots, N_k$ (with resulting corners smoothed) admits a completion – which implies that $(W, N_1 \cup \ldots \cup N_k)$ admits a completion as required.

Applying Proposition 10.5 thus extended, to the pair $(V_b, N \cap V_b)$, we see immediately that the pair (W, N) of Theorem 10.3 has a completion when N has codimension 1. □

Proof of Proposition 10.5 If T is a tubular neighborhood of N in W we know that (T, N) admits a completion. With the help of Lemma 1.8 one sees that $W' = W - \overset{\circ}{T}$ has at most two ends, (where $\overset{\circ}{T}$ denotes the *open* 1–disk bundle of T). Consider a sequence $V_1, V_2, \ldots$ of 0–neighborhoods of ∞ in W (constructed with the help of a suitable proper Morse function, as above) so that

(1) $V_{i+1} \subset \operatorname{Int} V_i$ and $\bigcap V_i = \emptyset$.

(2) $T_i = V_i \cap T$ is $T|N_i$ where N_i is a collar of ∞ in N.

After replacing $V_1, V_2, \ldots$ by a subsequence we may assume

(i) $\pi_1(\varepsilon_W) \longrightarrow \pi_1(V_i)$ is an imbedding and $\pi_1(V_{i+1}) \longrightarrow \pi_1(V_i)$ has image $\pi_1(\varepsilon_W) \subset \pi_1(V_i)$ for all i (c.f. Definition 4.4).

(ii) If W' has two ends ε_1 and ε_2, then $V_i' = V_i - \overset{\circ}{T}$ has two components A_i and B_i that are, respectively, neighborhoods of ε_1 and ε_2, $i = 1, 2, \ldots$. If W' has one end then V_i' is connected.

Case A) *W' has two ends ε_1, ε_2.*

Since $\pi_1(T_i) = \pi_1(\varepsilon_N) = 1$, $\pi_1(V_i) = \pi_1(A_i) * \pi_1(B_i)$. Thus with suitably chosen base points and base paths the system $\mathcal{V}\colon\ \pi_1(V_1) \xleftarrow{V_1} \pi_1(V_2) \xleftarrow{V_2} \ldots$ is the free product of $\mathcal{A}\colon\ \pi_1(A_1) \xleftarrow{a_1} \pi_1(A_2) \xleftarrow{a_2}$ with $\mathcal{B}\colon\ \pi_1(B_1) \xleftarrow{b_1} \pi_1(B_2) \xleftarrow{b_2} \ldots$. Observe that Image$(V_i)$ intersects $\pi_1(A_i)$ in Image(a_i) and intersects $\pi_1(B_i)$ in Image(b_i). Thus if $\mathcal{A}$ or $\mathcal{B}$ were not stable $\mathcal{V}$ would not be stable. As $\mathcal{V}$ is stable both $\mathcal{A}$ and $\mathcal{B}$ must be. Now $\pi_1(\varepsilon_1)$ is a retract of $\pi_1(B_i)$ for i large and $\pi_1(B_i)$ is a retract of $\pi_1(V_i)$, which is finitely presented. Hence $\pi_1(\varepsilon_1)$ and similarly $\pi_1(\varepsilon_2)$ is finitely presented by Lemma 3.8. By Theorem 3.10 (relativized) we can assume that A_i, B_i are 1–neighborhoods of ε_1, ε_2, so that $\pi_1(\varepsilon_W) \cong \pi_1(V_i) \cong \pi_1(A_i) * \pi_1(B_i) \cong \pi_1(\varepsilon_1) * \pi_1(\varepsilon_2)$.

Now $V_i, T_i \in \mathcal{D}$ implies $A_i, B_i \in \mathcal{D}$ by Complement 6.6, and $0 = \sigma(\varepsilon_W) = i_{1*}\sigma(\varepsilon_1) + i_{2*}\sigma(\varepsilon_2)$. Since $\widetilde{K_0}$ is functorial, i_{1*}, i_{2*} imbed $\widetilde{K_0}(\pi_1\varepsilon_1)$, $\widetilde{K_0}(\pi_1\varepsilon_2)$ as summands of $\widetilde{K_0}(\pi_1\varepsilon_W)$. We conclude that $\sigma(\varepsilon_1) = 0$, $\sigma(\varepsilon_2) = 0$. Thus W' admits a completion. As (W, N) does too Proposition 10.5 is established in Case A).

Case B) *W' has just one end ε'.*

There exists a smooth loop γ_1 in V_1 that intersects N just once, transversely. Since $\pi_1(T_1) = 1$, $\pi_1(V_1) = \pi_1(V_1') * \mathbb{Z}$ where $1 \in \mathbb{Z}$ is represented by γ_1. Since γ_1 could lie in V_2 we may assume $[\gamma_1] \in \mathrm{Image}\pi_1(V_2) = \pi_1(\varepsilon_W) \subset \pi_1(V_1)$. Then γ_1 can be deformed to a sequence of loops $\gamma_2, \gamma_3, \ldots$ so that $[\gamma_i] \in \pi_1(\varepsilon_W) \subset \pi_1(V_i)$ and γ_i cuts N just once. Thus with suitable base points and paths $\mathcal{V}\colon\ \pi_1(V_1) \longleftarrow \pi_1(V_2) \longleftarrow \ldots$ is the free product of $\mathcal{V}'\colon\ \pi_1(V_1') \longleftarrow \pi_1(V_2') \longleftarrow \ldots$ with the trivial system $\mathbb{Z} \xleftarrow{\times 1} \mathbb{Z} \xleftarrow{\times 1} \ldots$. The remainder of the proof is similar to Case A) but easier, as the reader can verify. This completes the proof of Proposition 10.5, and hence of Theorem 10.3. □

The analogue of Theorem 10.3 in the theory of h–cobordisms is

Theorem 10.6 *Let M and V be smooth closed manifolds and suppose $N = M \times [0,1]$ is smoothly imbedded in $W = V \times [0,1]$ so that N meets $\mathrm{Bd}\,W$ in $M \times 0 \subset V \times 0$ and $M \times 1 \subset V \times 1$, transversely. If W has dimension $\geqslant 6$ and N has codimension $\geqslant 3$, then (W, N) is diffeomorphic to $(V \times 0, M \times 0) \times [0,1]$.*

The same is true if N has codimension 1, provided each component of V is simply connected.

Proof Let W' be W with an open tubular neighborhood $\overset{\circ}{T}$ of N in W deleted. One shows that W' gives a product cobordism from $V \times 0 - \overset{\circ}{T}$ to $V \times 1 - \overset{\circ}{T}$ using the s–cobordism theorem. For codimension $\geqslant 3$ see Wall [3, p. 27]. For codimension 1, the argument is somewhat similar to that for Theorem 10.3 but more straightforward. $\square$

The canonically simple application of Theorem 10.3 and Theorem 10.6 is the proof that $\mathbb{R}^k$ unknots in $\mathbb{R}^n$, $n \geqslant 6$, $n - k \neq 2$. This is already well known. In fact it is true for any n except for the single case $n = 3$, $k = 2$ where the result is false! See Connell, Montgomery and Yang [13], and Stallings [10].

Theorem 10.7 *If $(\mathbb{R}^n, N)$ is a pair consisting of a copy N of $\mathbb{R}^k$ smoothly and properly imbedded in $\mathbb{R}^n$, then $(\mathbb{R}^n, N)$ is diffeomorphic to the standard pair $(\mathbb{R}^n, \mathbb{R}^k)$ provided $n \geqslant 6$ and $n - k \neq 2$.*

Proof By Theorem 10.3 and its Complement 10.4 we know that $(\mathbb{R}^n, N)$ is the interior of a compact pair $(\overline{R}, \overline{N})$, where $\overline{N}$ is a copy of D^k. We establish the theorem by showing $(\overline{R}, \overline{N})$ is diffeomorphic to the standard pair (D^n, D^k). Choose a small ball pair (D_0^n, D_0^k) in $\mathbb{R}^n$ so that $D_0^k = D_0^n \cap N$ is concentric with $N \approx D^k$. By the h–cobordism theorem $\overline{R} - \operatorname{Int} D_0^n$ is an annulus. Thus, applying Theorem 10.6, we find that $(\overline{R}, \overline{N})$ is (D_0^n, D_0^k) with a (relative) product cobordism attached at the boundary. This completes the proof. $\square$

The Isotopy Extension Theorem of Thom (Milnor [25]) shows that if N is a smoothly properly imbedded submanifold of an open manifold W and h_t, $0 \leqslant t \leqslant 1$, is a smooth isotopy of the inclusion map $N \hookrightarrow W$ then h_t extends to an ambient isotopy of W provided h_t fixes points outside some compact set. The standard example to show that this proviso is necessary involves a knot in a string that moves to ∞ like a wave disturbance. N can be the center of the string (codimension 2) or its surface (codimension 1).

Figure 16:

Do counterexamples occur only in codimension 2 or 1? Here is an attempt to say yes.

Theorem 10.8 *Suppose N^k is a smooth open manifold smoothly and properly imbedded in a smooth open manifold W^n, $n \geqslant 6$, $n-k \neq 2$. Suppose that N and W both admit a completion, and if $n-k=1$, suppose N is 1–connected at each end. Let H be a smooth proper isotopy of the inclusion $N \hookrightarrow W$, i.e. a smooth level preserving proper imbedding $H\colon N \times [0,1] \longrightarrow W \times [0,1]$, that fixes $N \times 0$. Then H extends to an ambient pseudo-isotopy – i.e. to a diffeomorphism $H'\colon W \times [0,1] \longrightarrow W \times [0,1]$ that is the identity on $W \times 0$.*

Corollary 10.9 *The pair (W,N) is diffeomorphic to the pair (W,N_1) if N_1 is the deformed image of N – i.e. $N_1 = h_1(N)$ where h_t, $0 \leqslant t \leqslant 1$, is defined by $H(t,x) = (t, h_t(x))$, $t \in [0,1]$, $x \in N$.*

Proof (in outline): Observe that $N' = H(N \times [0,1])$ and $W' = W \times [0,1]$ both admit completions that are products with $[0,1]$. By Theorem 10.3 (relativized) there exists a compact pair $(\overline{W'}, \overline{N'})$ with $W' = \operatorname{Int} \overline{W'}$, $N' = \operatorname{Int} \overline{N'}$. By the Complement 10.4 (relativized), we can assume $\overline{N'}$ is a product $\overline{N} \times [0,1]$, the product structure agreeing on N' with that given by H. Furthermore, after attaching a suitable (relative) h–cobordism at the boundary of $(\overline{W'}, \overline{N'})$ we may assume $\overline{W'}$ is also a product with $[0,1]$.

Applying Theorem 10.6 we find $(\operatorname{Bd} \overline{W'}, \operatorname{Bd} \overline{N'})$ is a product with $[0,1]$. Applying Theorem 10.6 again (now in a relativized form) we find $(\overline{W'}, \overline{N'})$ is a product. What is more, if we now go back and apply the relativized s–cobordism theorem we see that the given product structure $\overline{N'} \approx \overline{N} \times [0,1]$ can be extended to a product structure on $\overline{W'}$ (Wall [3, Theorem 6.2]). Restricted to W' this product structure gives the required diffeomorphism H'. □

For amusement we unknot a whole forest of $\mathbb{R}^k$'s in $\mathbb{R}^n$, $n-k \neq 2$.

Theorem 10.10 *Suppose N is a union of s disjoint copies of $\mathbb{R}^k$, smoothly and properly imbedded in $\mathbb{R}^n$, $n \geqslant 6$, $n-k \neq 2$. Then $(\mathbb{R}^n, N^k)$ is diffeomorphic to a standard pair consisting of the cosets $\mathbb{R}^k + (0, \ldots, 0, i) \subset \mathbb{R}^n$, $i = 1, 2, \ldots, s$.*

Proof There always exists a smoothly, properly imbedded copy of $\mathbb{R}^1$ that meets each component of N in a single point, transversely. Thus after a diffeomorphism of $\mathbb{R}^n$ we can assume that the component N_i of N meets the last co-ordinate axis in $(0,\dots,i)$, transversely, $i=1,\dots,s$. Using [4, §5.6] we see that after another diffeomorphism of $\mathbb{R}^n$ we can assume that N_i coincides with $\mathbb{R}^k+(0,\dots,0,i)$ near $(0,\dots,0,i)$.A smooth proper isotopy of N in $\mathbb{R}^n$ makes N coincide with the standard cosets. Now apply Corollary 10.9. □

11 A Duality Theorem and the Question of Topological Invariance for $\sigma(\varepsilon)$.

We give here a brief exposition of a duality between the two ends ε_- and ε_+ of a smooth manifold W^n homeomorphic to $M\times(0,1)$ where M is a closed topological manifold. The ends ε_- and ε_+ are necessarily tame and the duality reads $\sigma(\varepsilon_+)=(-1)^{n-1}\overline{\sigma(\varepsilon_-)}$ where the bar denotes a certain involution of $\widetilde{K_0}(\pi_1 W)$ that is the analogue of the involution of $\mathrm{Wh}(\pi_1 W)$ defined by Milnor in [17]. Keep in mind that, by the Sum Theorem, $\sigma(\varepsilon_+)+\sigma(\varepsilon_-)=\sigma(W)=\sigma(M)$, which is zero if M is equivalent to a finite complex. I unfortunately do not know any example where $\sigma(\varepsilon_+)\neq 0$. If I did some compact topological manifold (with boundary) would certainly be non-triangulable – namely the closure $\overline{V}$ in $M\times[0,1]=W$ of a 1–neighborhood V of ε_+ in W. When W is orientable the involution 'bar' depends on the group $\pi_1(W)$ alone. Prof. Milnor has established that this standard involution is in general non-trivial. There exists non-zero $x,y\in\widetilde{K_0}(\mathbb{Z}_{257})$ so that $\bar{x}=x$ and $\bar{y}=-y\neq y$. The appendix explains this (page 95).

Suppose $h\colon W\longrightarrow W'$ is a homeomorphism of a smooth open manifold W onto a smooth open manifold W' that carries an end ε of W to the end ε' of W'. From Definition 4.4 it follows that ε is tame if and only if ε' is. For tame ends we ask whether

$$h_*\sigma(\varepsilon)=\sigma(\varepsilon').$$

The duality theorem shows that the difference $h_*\sigma(\varepsilon)-\sigma(\varepsilon')=\sigma_0$ satisfies the restriction

$$\sigma_0+(-1)^{n-1}\bar{\sigma}_0=0,\ \ n=\dim W.$$

This is far from the answer that $\sigma_0=0$. An example with $\sigma_0\neq 0$ would again involve a non-triangulable manifold.

A related question is "Does every tame end have a topological collar neighborhood?" This may be just as difficult to answer as "Is every smooth h–cobordism topologically a product cobordism?" It seems a safe guess that the answer to both these questions is no. But proof is lacking.

The same duality $\sigma(\varepsilon_+) = (-1)^{n-1}\overline{\sigma(\varepsilon_-)}$ holds for the ends ε_+ and ε_- of a manifold W^n that is an infinite cyclic covering of a smooth compact manifold – provided these two ends are tame. The proof is like that for $M \times \mathbb{R}$. It can safely be left to the reader.

Question *Let ε be a tame end of dimension $\geqslant 5$ with $\sigma(\varepsilon) \neq 0$, and let M be the boundary of a collar for $\varepsilon \times S^1$. Does the infinite cyclic cover of M corresponding to the cokernel $\pi_1(M) \longrightarrow \mathbb{Z}$ of the natural map $\pi_1(\varepsilon) \longrightarrow \pi_1(M) \cong \pi_1(\varepsilon \times S^1)$ provide a non-trivial example of this duality?*

To explain duality we need some algebra. Let R be an associative ring with one-element 1 and a given anti-automorphism 'bar': $R \longrightarrow R$ of period two. Thus $\overline{r+s} = \overline{r} + \overline{s}$, $\overline{rs} = \overline{sr}$ and $\overline{\overline{r}} = r$ for $r, s \in R$. Modules are understood to be left R–modules. For any module A, the anti-homomorphisms from A to R – denoted $\bar{A}$ or $\overline{\mathrm{Hom}}_R(A,R)$ – form a left R–module. (Note that $\mathrm{Hom}_R(A,R)$ would be a right R–module.) Thus $\alpha \in \bar{A}$ is an additive map $A \longrightarrow R$ so that $\alpha(ra) = \alpha(a)\bar{r}$ for $a \in A$, $r \in R$. And $(s\alpha)(a) = s(\alpha(a))$ for $a \in A$, $s \in R$. I leave it to the reader to verify that $P \longrightarrow \bar{P}$ gives an additive involution on the isomorphism classes $\mathcal{P}(R)$ of f.g. projective R–modules and hence additive involutions (that we also call 'bar') on $K_0(R)$ and $\widetilde{K_0}(R)$.

If $C\colon \ldots \longrightarrow C_\lambda \xrightarrow{\partial} C_{\lambda-1} \longrightarrow \ldots$ is a chain complex we define $\overline{C}$ to be the cochain complex

$$\ldots \longleftarrow \overline{C}_\lambda \xleftarrow{\bar{\partial}} \overline{C}_{\lambda-1} \longleftarrow \ldots$$

where $\bar{\partial}$ is defined by the rule

$$(\bar{\partial}\bar{c})(e) = (-1)^\lambda \bar{c}(\partial e)$$

for $e \in C_\lambda$ and $\bar{c} \in \overline{C}_{\lambda-1}$.

For our purposes R will be a group ring $\mathbb{Z}[G]$ where G is a fundamental group of a manifold and the anti-automorphism 'bar' is that induced by sending g to $\theta(g)g^{-1}$ in $\mathbb{Z}[G]$, where $\theta(g) = \pm 1$ according as g gives an orientation preserving or orientation reversing homeomorphism of the universal cover. If the manifold is orientable $\theta(g)$ is always $+1$ and 'bar' then depends on G alone and is called the standard involution.

Let $(W^n; V, V')$ be a smooth manifold triad with self-indexing Morse function f. Provide the usual equipment; base point p for W; base paths to the critical points of f; gradient-like vector field for f; orientations for the left hand disks. Then a based free $\pi_1 W$ complex C_* for $H_*(\widetilde{W}, \widetilde{V})$ is well defined (see §4).

When we specify an orientation at p, geometrically dual equipment is determined for the Morse function $-f$ and hence a geometrically dual complex C'_* for $H_*(\widetilde{W},\widetilde{V}')$. With the help of the formula $\varepsilon'_p = (-1)^\lambda \operatorname{sign}(g_p)\varepsilon_p$ in §4, one shows that

$$C'_* = \overline{C}_{n-*}$$

i.e. C'_* is the cochain complex $\overline{C}_*$ with the grading suitably reversed.

Duality Theorem for $M \times \mathbb{R}$ 11.1 *Suppose that W is a smooth open manifold of dimension $n \geqslant 5$ that is homeomorphic to $M \times \mathbb{R}$ for some connected closed topological manifold M. Then W has two ends ε_- and ε_+, both tame, and when we identify $\widetilde{K_0}\pi_1\varepsilon_-$ and $\widetilde{K_0}\pi_1\varepsilon_+$ with $\widetilde{K_0}\pi_1 W$ under the natural isomorphisms,*

$$\sigma(\varepsilon_+) = (-1)^{n-1}\overline{\sigma(\varepsilon_-)}.$$

The proof begins after Theorem 11.4 below.

Corollary 11.2 *The above theorem holds without restriction on n.*

Proof of Corollary 11.2 Form the cartesian product of W with a closed smooth manifold N^6 having $\chi(N) = 1$, e.g. real projective space $P^6(\mathbb{R})$. Then we have maps

$$\pi_1(W) \underset{r}{\overset{i}{\rightleftarrows}} \pi_1(W \times N)$$

so that $r \cdot i = 1$. Using Definition 7.7 and the Product Theorem 7.2 one easily shows that $\sigma(\varepsilon_+ \times N) = \chi(N) i_*\sigma(\varepsilon_+)$ and hence $r_*\sigma(\varepsilon_+ \times N) = \sigma(\varepsilon_+)$. The same holds for ε_-. Since r_* commutes with 'bar', duality for $W \times N$ implies duality for W. □

Corollary 11.3 *Without restriction on n,*

$$\sigma(M) = \sigma(\varepsilon_+) + (-1)^{n-1}\overline{\sigma(\varepsilon_+)}$$

and, consequently, $\sigma(M) = (-1)^{n-1}\overline{\sigma(M)}$.

Proof of Corollary 11.3 By Theorem 6.5, $\sigma(M) = \sigma(W) = \sigma(\varepsilon_+) + \sigma(\varepsilon_-)$. □

Remark It is a conjecture of Professor Milnor that if M^m is any closed topological manifold, then $\sigma(M) = (-1)^m\overline{\sigma(M)}$ or equivalently

$$\rho(M) = (-1)^m\overline{\rho(M)}.$$

Of course the conjecture vanishes if all closed manifolds are triangulable. Theorem 11.1 shows at least that

Theorem 11.4 *If M^m is a closed topological manifold such that for some k, $M \times \mathbb{R}^k$ has a smoothness structure then*

$$\sigma(M) = (-1)^m \overline{\sigma(M)}.$$

Proof of Theorem 11.4 We can assume k is even and $k > 2$. We will be able to identify all fundamental groups naturally with $\pi_1(M)$. By Theorem 6.12 the end ε of $M \times \mathbb{R}^k$ is tame and $\sigma(\varepsilon) = \sigma(M)$. The open submanifold $W = M \times \mathbb{R}^k - M \times 0$ is homeomorphic to $M \times S^{k-1} \times \mathbb{R}$ and $\sigma(W) = 0$ by the Product Theorem since $k-1$ is odd. From Corollary 11.2 and the Sum Theorem we get

$$0 = \sigma(W) = \sigma(\varepsilon) + (-1)^{m+k-1}\overline{\sigma(\varepsilon)}$$

or

$$0 = \sigma(M) + (-1)^{m-1}\overline{\sigma(M)} \text{ as required.}$$

□

Remark It is known that not every closed topological manifold M is stably smoothable (see the Footnote below). However it is conceivable that, for sufficiently large k, $M \times \mathbb{R}^k$ can always be triangulated as a combinatorial manifold. Then the piecewise linear version of Theorem 11.4 (see the introduction) would prove Professor Milnor's conjecture.

Proof of Duality Theorem 11.1 For convenience identify the underlying topological manifold W with $M \times \mathbb{R}$. By Theorem 4.6 we can find a $(n-3)$–neighborhood V of ε_+ so small that it lies in $M \times (0, \infty)$. After adding suitable (trivially attached) 2–handles to V *in* $M \times (0, \infty)$, we can assume that $U = W - \operatorname{Int} V$ is a 2–neighborhood of ε_-. Next find a $(n-3)$–neighborhood of the positive end of $M \times (-\infty, 0)$. Adding $M \times [0, \infty)$ to it we get a $(n-3)$–neighborhood V' of ε_+ that contains $M \times [0, \infty)$. After adding 2–handles to V' we can assume that $U' = W - \operatorname{Int} V'$ is a 2–neighborhood of ε_-.

By Proposition 5.1 we know that $H_*(\widetilde{V}, \operatorname{Bd} \widetilde{V})$ and $H_*(\widetilde{V'}, \operatorname{Bd} \widetilde{V'})$ are f.g. projective $\pi_1(W)$–modules P_+ and P'_+ concentrated in dimension $n-2$ and both of class $(-1)^{n-2}\sigma(\varepsilon_+)$. By an argument similar to that for Proposition 5.1 one shows that U admits a proper Morse function $f \colon U \longrightarrow [0, \infty)$ with $f^{-1}(0) = \operatorname{Bd} U$ so that f has critical points of index 2 and 3 only. (The strong handle cancellation theorem in Wall [3, Theorem 5.5] is needed.) The same is true for U'. It follows that $H_*(\widetilde{U}, \operatorname{Bd} \widetilde{V})$ and $H_*(\widetilde{U'}, \operatorname{Bd} \widetilde{V'})$ are f.g. projective modules P_- and P'_- concentrated in dimension 3 and both of class $(-1)^3\sigma(\varepsilon_-)$ by Lemma 6.2 and Proposition 6.11.

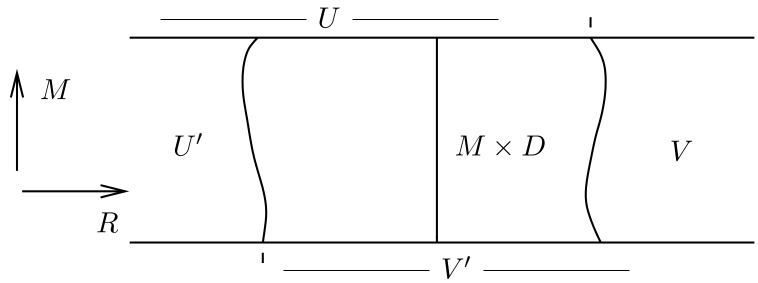


Figure 17:

Let $X = V' - \operatorname{Int} V$. Since the composition $V' - M \times (0,\infty) \hookrightarrow X \hookrightarrow V'$ is a homotopy equivalence $H_*(\widetilde{X}, \operatorname{Bd}\widetilde{V'}) \longrightarrow H_*(\widetilde{V'}, \operatorname{Bd}\widetilde{V'})$ is onto. Thus from the exact sequence of $(\widetilde{V'}, \widetilde{X}, \widetilde{M})$ we deduce (c.f. §5) that $H_{n-2}(\widetilde{X}, \operatorname{Bd}\widetilde{V'}) \cong H_{n-2}(\widetilde{V'}, \operatorname{Bd}\widetilde{V'}) \cong P'_+$ and $H_{n-3}(\widetilde{X}, \operatorname{Bd}\widetilde{V'}) \cong H_{n-2}(\widetilde{V'}, \widetilde{X}) \cong H_{n-2}(\widetilde{V}, \operatorname{Bd}\widetilde{V}) \cong P_+$.

Similarly one shows that $H_*(\widetilde{X}, \operatorname{Bd}\widetilde{V}) \longrightarrow H_*(\widetilde{U}, \operatorname{Bd}\widetilde{V})$ is onto. As a consequence $H_3(\widetilde{X}, \operatorname{Bd}\widetilde{V}) \cong H_3(\widetilde{U}, \operatorname{Bd}\widetilde{V}) = P_-$.

Now $\operatorname{Bd} V \hookrightarrow X$ gives a π_1–isomorphism. Also $\operatorname{Bd} V' \hookrightarrow X$ is $(n-4)$–connected (and gives a π_1–isomorphism when $n = 5$). It follows from Wall [3, Theorem 5.5] that the triad $(X; \operatorname{Bd} V', \operatorname{Bd} V)$ admits a nice Morse function f with critical points of index $n-3$ and $n-2$ only.

Let f be suitably equipped and consider the free $\pi_1(W)$–complex C_* for $H_*(\widetilde{X}, \operatorname{Bd}\widetilde{V'})$. It has the form (c.f. §5)

$$0 \longrightarrow H_{n-2} \oplus B'_{n-3} \xrightarrow{\partial} B_{n-3} \oplus H_{n-3} \longrightarrow 0$$

where ∂ is an isomorphism of B'_{n-3} onto B_{n-3} and $H_{n-2} \cong P'_+$ and $H_{n-3} \cong P_+$. Then the complex $\overline{C}_*$ is

$$0 \longleftarrow \bar{H}_{n-2} \oplus \bar{B}'_{n-3} \xleftarrow{\bar{\partial}} \bar{B}_{n-3} \oplus H_{n-3} \longleftarrow 0$$

where $\overline{\partial}$ gives an isomorphism of $\bar{B}_{n-3}$ onto $\bar{B}'_{n-3}$. But we have observed that $\overline{C}_{n-*}$ is the complex C'_* for $H_*(\widetilde{X}, \operatorname{Bd}\widetilde{V})$ that is geometrically dual to C_*. Hence we have

$$H_3(C') \cong \bar{H}_{n-3}.$$

But $H_3(C') \cong H_3(\widetilde{X}, \operatorname{Bd}\widetilde{V}) \cong P_-$ has class $(-1)^3\sigma(\varepsilon_-)$ and $\bar{H}_{n-3} \cong \bar{P}_+$ has class $(-1)^{n-2}\overline{\sigma(\varepsilon_+)}$. So the duality relation is established. ◻

Suppose W^n is an open topological manifold and ε an end of W. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two smoothness structures for W and denote the smooth ends corresponding to ε by $\varepsilon_1, \varepsilon_2$. Notice that ε_1 is tame if and only if ε_2 is, since the definition of tameness does not mention the smoothness structure.

Theorem 11.5 *Suppose $n \geqslant 5$. If ε_1 is tame, so is ε_2, and the difference $\sigma(\varepsilon_1) - \sigma(\varepsilon_2) = \sigma_0 \in \widetilde{K_0}\pi_1\varepsilon$ satisfies the relation*

$$\sigma_0 + (-1)^{n-1}\overline{\sigma_0} = 0.$$

Further σ_0 is always zero if and only if the following statement (S) is true.

(S) If M^{n-1} is a closed smooth manifold and W^n is a smooth manifold homeomorphic to $M \times \mathbb{R}$ then both ends of W^n have invariant zero.

Corollary 11.6 *The first assertion of Theorem 11.5 is valid for any dimension n.*

Proof Let N^6 be a closed smooth manifold with $\chi(N) = 1$, and consider the smoothings $\varepsilon_1 \times N$, $\varepsilon_2 \times N$ of $\varepsilon \times N$. Now follow the proof of Corollary 11.2. □

Proof Let V_1 be a 1–neighborhood of ε_1. With smoothness from $\mathcal{S}_2$, $\operatorname{Int} V_1$ has two ends – viz. ε_2, and the end ε_0 whose neighborhoods are those of $\operatorname{Bd} V_1$ intersected with $\operatorname{Int} V_1$. Since ε_0 has a neighborhood homeomorphic to $\operatorname{Bd} V_1 \times \mathbb{R}$, ε_0 is tame and $\sigma(\varepsilon_0) + (-1)^{n-1}\overline{\sigma(\varepsilon_0)} = 0$ by Corollary 11.3 to the duality theorem. Let U be a 1–neighborhood of ε_0. Then $V_2 = \operatorname{Int} V_1 - \operatorname{Int} U$ is clearly a 1–neighborhood of ε_2. But $V_1 \cong \operatorname{Int} V_1 = U \cup V_2$ and $U \cap V_2$ is a finite complex. Thus, by the Sum Theorem 6.5, $\sigma(\varepsilon_1) = \sigma(V_1) = \sigma(V_2) + \sigma(U) = \sigma(\varepsilon_2) + \sigma(\varepsilon_0)$. Thus the first assertion holds with $\sigma_0 = \sigma(\varepsilon_0)$.

Now if (S) holds, $\sigma_0 = \sigma(\varepsilon_0) = 0$ because ε_0 is an end of a smooth manifold homeomorphic to $\operatorname{Bd} V_1 \times \mathbb{R}$. Conversely, if σ_0 is always zero, i.e. $\sigma(\varepsilon_2) = \sigma(\varepsilon_1)$, then σ does not depend on the smoothness structure. Thus (S) clearly holds. This completes the proof. □

Footnote To justify the above assertion here is a folklore example, due to Professor Milnor, of a closed topological manifold which is not stably smoothable. It is shown in Milnor [32, 9.4, 9.5] that there is a finite complex K and a topological microbundle ξ^n over K which is stably distinct (as microbundle) from any vector bundle. Further one can arrange that K is a compact k-submanifold with boundary, of $\mathbb{R}^k$ for some k. By Kister [33] the induced microbundle $D\xi^n$ over the double DK of K contains a locally trivial bundle with fibre $\mathbb{R}^n$. If one suitably compactifies the total space adding a point-at-infinity to each fibre, a closed topological manifold results which cannot be stably smoothable since its tangent microbundle restricts to $\xi^n \otimes \{\text{trivial bundle}\}$ over K.

Appendix

This appendix explains Professor Milnor's proof that there exist nonzero x and y in $\widetilde{K_0}(\mathbb{Z}_{257})$ so that $\bar{x} = x$ and $\bar{y} = -y \neq y$ where the bar denotes the standard involution (see §11). Theorems A.6 and A.7 below actually tell a good deal about the standard involution on the projective class group $\widetilde{K_0}(\mathbb{Z}_p)$ of the cyclic group $\mathbb{Z}_p$ of prime order p.

Suppose A and B are rings with identity each equipped with anti-automorphisms 'bar' of period 2. If $\theta\colon A \longrightarrow B$ is a ring homomorphism so that $\theta(\bar{a}) = \overline{\theta(a)}$, then one can show that the diagram

$$\begin{array}{ccc} \widetilde{K_0}(A) & \xrightarrow{'\mathrm{bar}'} & \widetilde{K_0}(A) \\ \Big\downarrow{\scriptstyle \theta_*} & & \Big\downarrow{\scriptstyle \theta_*} \\ \widetilde{K_0}(B) & \xrightarrow{'\mathrm{bar}'} & \widetilde{K_0}(B) \end{array}$$

commutes where 'bar' is the additive involution of the projective class group defined in §11. Now specialize. Let $A = \mathbb{Z}[\pi]$ where $\pi = \{t; t^p = 1\}$ is cyclic of prime order. Define $\overline{a(t)} = a(t^{-1})$ for $a(t) \in \mathbb{Z}[\pi]$ so that

$$\mathrm{bar}\colon K_0\mathbb{Z}[\pi] \longrightarrow K_0\mathbb{Z}[\pi]$$

is the standard involution of $\widetilde{K_0}\mathbb{Z}[\pi] \equiv \widetilde{K_0}(\pi)$. Let $B = \mathbb{Z}[\xi]$ where ξ is a primitive p–th root of 1, and let $\theta(t) = \xi$ define $\theta\colon \mathbb{Z}[\pi] \longrightarrow \mathbb{Z}[\xi]$. (Notice that $\ker\theta$ is the principal ideal generated by $\Sigma = 1 + t + \ldots + t^{p-1}$.) Since ξ^{-1} is the complex conjugate $\bar{\xi}$ of ξ, $\theta(\bar{a}) = \overline{\theta(a)}$ where the second bar is complex conjugation.

The following is due to Rim [38, pp. 708-711].

Theorem A.1 $\theta_*\colon \widetilde{K_0}\mathbb{Z}[\pi] \longrightarrow \widetilde{K_0}\mathbb{Z}[\xi]$ *is an isomorphism.*

Remark Rim assigns to a f.g. projective P over $\mathbb{Z}[\pi]$ the subobject

$${}_{\Sigma}P = \{x \in P \mid \Sigma x = 0\},\ \Sigma = 1 + t + \ldots + t^{p-1}\ ,$$

with the obvious action of $\mathbb{Z}[\pi]/(\Sigma) \cong \mathbb{Z}[\xi]$. But there is an exact sequence

$$0 \longrightarrow \Sigma P \xrightarrow{\alpha} P \xrightarrow{\beta} {}_{\Sigma}P \longrightarrow 0$$

where α is inclusion and β is multiplication by $1-t$. Hence ${}_{\Sigma}P \cong P/(\Sigma P)$ as $\mathbb{Z}[\xi]$–modules. But $P/(\Sigma P)$ is easily seen to be isomorphic with $\mathbb{Z}[\xi] \otimes_{\mathbb{Z}[\pi]} P$. Thus Rim's isomorphism is in fact θ_*.

We now have a commutative diagram

$$\begin{array}{ccc} \widetilde{K_0}\mathbb{Z}[\pi] & \xrightarrow{\text{'bar'}} & \widetilde{K_0}\mathbb{Z}[\pi] \\ \cong\Big\downarrow\theta_* & & \cong\Big\downarrow\theta_* \\ \widetilde{K_0}\mathbb{Z}[\xi] & \xrightarrow{\text{'bar'}} & \widetilde{K_0}\mathbb{Z}[\xi] \end{array}$$

So it is enough to study 'bar' on $\widetilde{K_0}\mathbb{Z}[\xi]$. To do this we go one more step to the ideal class group of $\mathbb{Z}[\xi]$.

Now $\mathbb{Z}[\xi]$ is known to be the ring of all algebraic integers in the cyclotomic field $\mathbb{Q}(\xi)$ of p-th roots of unity [39, p. 70]. Hence $\mathbb{Z}[\xi]$ is a Dedekind domain [40, p. 281]. A *Dedekind domain* may be defined as an integral domain R with 1–element in which the (equivalent) conditions A) and B) hold. [40, p. 275][41, Chap. 7, pp. 29-33].

A) The fractional ideals from a group under multiplication. (A *fractional ideal* is an R–module $\mathcal{A}$ imbedded in the quotient field K of R such that for some $r \in R$, $r\mathcal{A} \subset R$.)

B) Every ideal in R is a f.g. projective R–module.

The *ideal class group* $C(R)$ of R is by definition the group of fractional ideals modulo the subgroup generated by principal ideals. B) implies that any f.g. projective P over R is a direct sum $\mathcal{A}_1 \oplus \ldots \oplus \mathcal{A}_r$ of ideals in R [42, p. 13]. According to [38, Theorem 6.19] the ideal class of the product $\mathcal{A}_1 \ldots \mathcal{A}_r$ depends only on P and the correspondence $P \longrightarrow \mathcal{A}_1 \ldots \mathcal{A}_r$ gives an isomorphism $\varphi\colon \widetilde{K_0}(R) \longrightarrow C(R)$.

Let us define 'bar': $C\mathbb{Z}[\xi] \longrightarrow C\mathbb{Z}[\xi]$ by sending a fractional ideal $\mathcal{A}$ to the fractional ideal $\sigma(\mathcal{A}^{-1})$ where σ denotes complex conjugation in $\mathbb{Q}(\xi)$. The following two lemmas show that the diagram

$$\begin{array}{ccc} \widetilde{K_0}\mathbb{Z}[\xi] & \xrightarrow{\text{'bar'}} & \widetilde{K_0}\mathbb{Z}[\xi] \\ \cong\Big\downarrow\varphi & & \cong\Big\downarrow\varphi \\ C\mathbb{Z}[\xi] & \xrightarrow{\text{'bar'}} & C\mathbb{Z}[\xi] \end{array}$$

commutes.

Lemma A.2 *In any Dedekind domain R, $Hom_R(\mathcal{A}, R) \cong \mathcal{A}^{-1}$ for any fractional ideal $\mathcal{A}$.*

Lemma A.3 *Let $\mathcal{A}$ be any fractional ideal in $\mathbb{Z}[\xi]$. Then $\sigma(\mathcal{A})$ is naturally isomorphic as $\mathbb{Z}[\xi]$–module to $\mathcal{A}$ with a new action of $\mathbb{Z}[\xi]$ given by $r \cdot a = \overline{ra}$ for $r \in \mathbb{Z}[\xi]$, $a \in \mathcal{A}$.*

The second lemma is obvious. The first is proved below. To see that these lemmas imply that the diagram above commutes notice that for a ring R equipped with anti-automorphism 'bar', the left R–module $\bar{P} = \overline{\mathrm{Hom}}_R(P,R)$ used in §11 to define 'bar': $\widetilde{K_0}(R) \longrightarrow \widetilde{K_0}(R)$, is naturally isomorphic to $P^* = \mathrm{Hom}_R(P,R)$ provided with a left action of R by the rule $(r \cdot f)(x) = f(x)\bar{r}$ for $r \in R$, $f \in P^*$ and $x \in P$.

Proof of Lemma A.2 We know $\mathcal{A}^{-1} = \{y \in K \mid y\mathcal{A} \subset R\}$ where K is the quotient field of R [40, p. 272]. So there is a natural imbedding

$$\alpha\colon \mathcal{A}^{-1} \longrightarrow \mathrm{Hom}_R(\mathcal{A}, R)$$

which we prove is onto. Take $f \in \mathrm{Hom}_R(\mathcal{A}, R)$ and $x \in \mathcal{A} \cap R$. Let $b = f(x)/x$ and consider the map f_b defined by $f_b(x) = bx$. For $a \in \mathcal{A}$

$$\begin{aligned} 0 &= (f - f_b)(x) = a(f - f_b)(x) = (f - f_b)ax = \\ &= x(f - f_b)(a) = (f - f_b)(a). \end{aligned}$$

hence $f(a) = f_b(a) = ba$. Thus $b \in \mathcal{A}^{-1}$ and α is onto as required.

□

Let A be a Dedekind domain, K its quotient field, L a finite Galois extension of K with degree d and group G. Then the integral closure B of A in L is a Dedekind domain. [40, p.281]. Each element $\sigma \in G$ maps integers to integers and so gives an automorphism of B fixing A. Then σ clearly gives an automorphism of the group of fractional ideals of B that sends principal ideals to principal ideals. Thus σ induces an automorphism σ_* of $C(B)$. Let us write $C(A)$ and $C(B)$ as additive groups.

Theorem A.4 *There exist homomorphisms $j\colon C(A) \longrightarrow C(B)$ and $N\colon C(B) \longrightarrow C(A)$ so that $N \circ j$ is multiplication by $d = [L;K]$ and $j \circ N = \sum_{\sigma \in G} \sigma_*$.*

Proof j is induced by sending each fractional ideal $\mathcal{A} \in A$ to the fractional ideal $\mathcal{A}B$ of B. N comes from the norm homomorphism defined in Lang [43, p. 18–19]. It is Proposition 22 on p '1 of [43] that shows N is well defined. That $N \circ j = d$ and $j \circ N = \sum_{\sigma \in G} \sigma_*$ follows immediately from Corollary 1 and Corollary 3 on pp. 20–21 of [43]. □

Since $\xi, \xi^2, \ldots, \xi^{p-1}$ form a $\mathbb{Z}$–basis for the algebraic integers in $\mathbb{Q}(\xi)$ [39, p. 70], $\xi+\bar{\xi}, \ldots, \xi^{\frac{p-1}{2}}+\bar{\xi}^{\frac{p-1}{2}}$ form a $\mathbb{Z}$–basis for the self-conjugate integers in $\mathbb{Q}(\xi)$, i.e. the algebraic integers in $\mathbb{Q}(\xi) \cap \mathbb{R} = \mathbb{Q}(\xi+\bar{\xi})$. But $\mathbb{Z}[\xi+\bar{\xi}]$ is the span of $\xi+\bar{\xi}, \ldots, \xi^{\frac{p-1}{2}}+\bar{\xi}^{\frac{p-1}{2}}$. Hence $\mathbb{Z}[\xi+\bar{\xi}]$ is the full ring of algebraic integers in $\mathbb{Q}(\xi+\bar{\xi})$ and so is a Dedekind domain [40, p. 281]. It is now easy to check that we have a situation as described above with $A = \mathbb{Z}[\xi+\bar{\xi}]$, $B = \mathbb{Z}[\xi]$, $d = 2$ and $G = \{1, \sigma\}$ where σ is complex conjugation. Observe that with the ideal class group $C\mathbb{Z}[\xi]$ written additively $\bar{x} = \sigma_*(-x) = -\sigma_* x$, for $x \in C\mathbb{Z}[\xi]$ (see above). As a direct application of the theorem above we have

Theorem A.5 *There exist homomorphisms N and j*

$$\widetilde{K_0}(\mathbb{Z}_p) \cong C\mathbb{Z}[\xi] \underset{j}{\overset{N}{\rightleftarrows}} C\mathbb{Z}[\xi+\bar{\xi}]$$

so that $j \circ N = 1 + \sigma_*$ and $N \circ j = 2$.

Now the order $h = h(p)$ of $C\mathbb{Z}[\xi]$ is the so-called class number of the cyclotomic field $\mathbb{Q}(\xi)$ of p–th roots of unity. It can be expressed as a product $h_1 h_2$ of positive integral factors, where the first is given by a closed formula of Kummer [44] 1850, and the second is the order of $C\mathbb{Z}[\xi+\bar{\xi}]$, Vandiver [45, p. 571]. (In fact j is $1-1$ and N is onto, Kummer [50], Hasse [46, p.13 footnote 3, p.49 footnote 2]). Write $h_2 = h_2' 2^s$ where h_2' is odd. Recall that p is a prime number and $\widetilde{K_0}(\mathbb{Z}_p)$ is the group of stable isomorphism classes of f.g. projective over the group $\mathbb{Z}_p$. Bar denotes the standard involution of $\widetilde{K_0}(\mathbb{Z}_p)$ (see §11).

Theorem A.6

1) *The subgroup in $\widetilde{K_0}(\mathbb{Z}_p)$ of all x with $\bar{x} = x$ has order at least h_1;*
2) *There is a summand S in $\widetilde{K_0}(\mathbb{Z}_p)$ of order h_2' so that $\bar{y} = -y$ for all $y \in S$.*

Proof For $x \in \text{Kernel}(N)$, $(1+\sigma_*)x = j \circ Nx = 0$ implies $\bar{x} = x$. But Kernel(N) has order at least h_1; so 1) is established. The component of $C\mathbb{Z}[\xi+\bar{\xi}]$ prime to 2 is a subgroup S of order h_2'. Since multiplication by 2 is an automorphism of S, $N \circ j = 2$ says that j maps S $1-1$ into a summand of $\widetilde{K_0}(\mathbb{Z}_p)$. For $y = j(x)$, $x \in S$, we have $j \circ N(y) = j(2x) = 2y$. Thus $y + \sigma_* y = 2y$ or $\bar{y} = -y$. This proves 2). □

In case h_2 is odd $h_2 = h_2'$, and the proof of Theorem A.6 gives the clear-cut result:

Theorem A.7 *If the second factor h_2 of the class number for the cyclotomic field of p-th roots of unity is odd, then*

$$\widetilde{K_0}(\mathbb{Z}_p) \cong Kernel(N) \oplus C\mathbb{Z}[\xi + \bar{\xi}]$$

and $x = \bar{x}$ for $x \in Kernel(N)$ while $\bar{y} = -y$ for $y \in C\mathbb{Z}[\xi + \bar{\xi}]$.

In [47] 1870, Kummer proved that $2|h_2$ implies $2|h_1$ (c.f. [46], p. 119). He shows that, although h_1 is even for $p = 29$ and $p = 113$, h_2 is odd. Then he shows that both h_1 and h_2 are even for $p = 163$ and states that the same is true for $p = 937$. Kummer computed h_1 for all $p < 100$ in [44] 1850 (see [50, p. 199] for the correction $h_1(71 = 7^2 \times 79241)$, and for $101 \leqslant p \leqslant 163$ in [49] 1874. In [49], h_1 is incorrectly listed as odd for $p = 163$. Supposing that the other computations are correct, one observes that for $p < 163$, h_1 is odd except when $p = 29$ or $p = 113$. We conclude that $p = 163$ is the least prime so that h_2 is even. Thus $p = 163$ is the least prime where have to fall back from Theorem A.7 to the weaker Theorem A.6.

Elements x in $K_0(\mathbb{Z}_p)$, so that $x = \bar{x}$, are plentiful. After a slow start the factor h_1 grows rapidly: $h_1(p) = 1$ for primes $p < 23$, $h_1(23) = 3$, $h_1(29) = 8$, $h_1(31) = 9$, $h_1(37) = 37$, $h_1(41) = 11 \cdot 11 = 121$, $h_1(47) = 5 \cdot 139$, $h_1(53) = 4889, \ldots, h_1(101) = 5^5 \cdot 101 \cdot 11239301$, etc. Kummer [44] 1850 gives (without proof) the asymptotic formula

$$h_1(p) \sim p^{\frac{p+3}{4}} / 2^{\frac{p-3}{2}} \pi^{\frac{p-1}{2}}.$$

But it seems no one has show that $h_1(p) > 1$ for *all* $p > 23$.

On the other hand elements with $\bar{x} = -x$ are hard to get hold of, for information about h_2 is scanty. It has been established that $h_2(p) = 0$ for primes $p < 23$ (see Minkowski [48, p. 296]). In [47] 1870, Kummer shows that h_2 is divisible by 3 for $p = 229$, and he asserts the same for $p = 257$.

Vandiver [45, p. 571] has used a criterion of Kummer to show that $p|h_1(p)$ for $p = 257$ (but $p \not| h_1(p)$ for $p = 229$). Since $3|h_2(257)$, Theorem A6 shows for example that there is in $\widetilde{K_0}(\mathbb{Z}_{257})$ an element x of order 257 with $x = \bar{x}$ and another element y of order 3 with $\bar{y} = -y$. Notice that $\overline{(x+y)} \neq \pm(x+y)$.

Remark 3 It is not to be thought that $\widetilde{K_0}(\mathbb{Z}_p)$ is a cyclic group in general. In [50] 1853, Kummer discussed the structure of the subgroup G_p of all elements for which $\bar{x} = x$, i.e. the subgroup corresponding to the ideals $\mathcal{A}$ in $\mathbb{Z}[\xi]$ such that $\mathcal{A}\sigma(\mathcal{A})$ is principal. For $p < 100$, h_2 is odd so that this subgroup is a summand of order h_1 by Theorem A.7. He found that

$$\begin{aligned} G_{29} &= \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\ G_{31} &= \mathbb{Z}_9 \\ G_{41} &= \mathbb{Z}_{11} \oplus \mathbb{Z}_{11} \\ G_{71} &= \mathbb{Z}_{49} \oplus \mathbb{Z}_{79241} \end{aligned}$$

For other $p < 100$ there are no repeated factors in h_1 hence no structure problem exists.

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