Spacecraft Dynamics and Control

Matthew M. Peet
Arizona State University

Lecture 5: Hyperbolic Orbits

Introduction

In this Lecture, you will learn:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Given t, find r and v

For elliptic orbits:

1. Given time, t, solve for Mean Anomaly

$$M(t) = nt$$

2. Given Mean Anomaly, solve for Eccentric Anomaly

$$M(t) = E(t) - e \sin E(t)$$

3. Given eccentric anomaly, solve for true anomaly

$$\tan \frac{f(t)}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E(t)}{2}$$

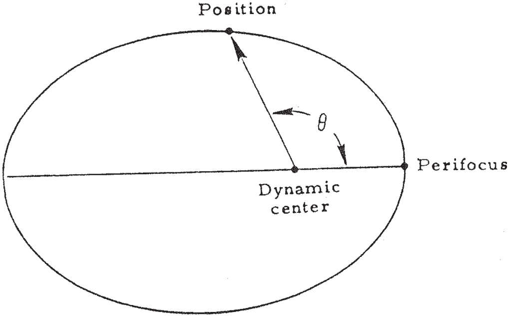
4. Given true anomaly, solve for r

$$r(t) = \frac{a(1-e^2)}{1 + e \cos f(t)}, \qquad v(t) = \sqrt{\mu \left(\frac{2}{r(t)} - \frac{1}{a} \right)}$$

Does this work for Hyperbolic Orbits? Lets recall the angles.

What are these Angles?

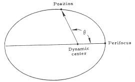
True Anomaly, f (θ)



- The angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$, measured COUNTERCLOCKWISE.
- The angle the position vector makes with periapse.

What are these Angles?

True Anomaly, f (θ)

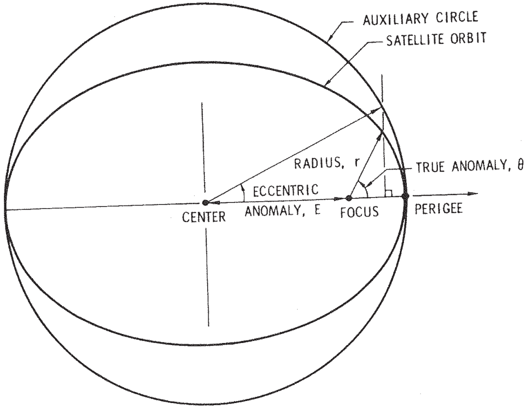


- The angle the position vector, $\vec{r}$ makes with the eccentricity vector, $\vec{e}$, measured COUNTERCLOCKWISE.
- The angle the position vector makes with periapse.

- In the figure, θ is used for true anomaly. We typically use f. Occasionally, ν is also used in the texts.
- True anomaly is always well-defined for hyperbolic orbits.

What are these Angles?

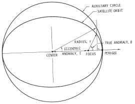
Eccentric Anomaly, E



- Measured from center of ellipse to a auxiliary reference circle.

What are these Angles?

Eccentric Anomaly, E



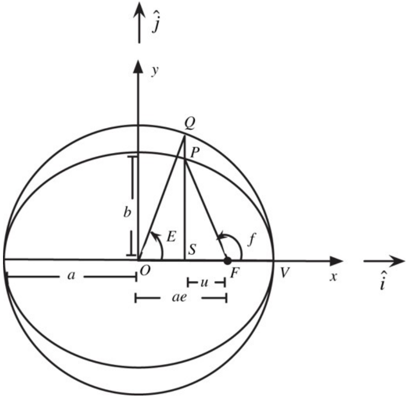
- Measured from center of ellipse to a auxiliary reference circle.

- Eccentric anomaly is not defined for hyperbolic orbits.
- For hyperbolic orbits, we use hyperbolic anomaly, which we will define shortly.

What are these Angles?

Mean Anomaly

$$M(t) = 2\pi \frac{t}{T} = 2\pi \frac{A_{PFV}}{A_{Ellipse}}$$

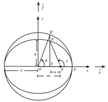


- The fraction of area of the ellipse which has been swept out, in radians.

What are these Angles?

Mean Anomaly

$$M(t) = 2\pi \frac{t}{T} = 2\pi \frac{A_{PFV}}{A_{Ellipse}}$$

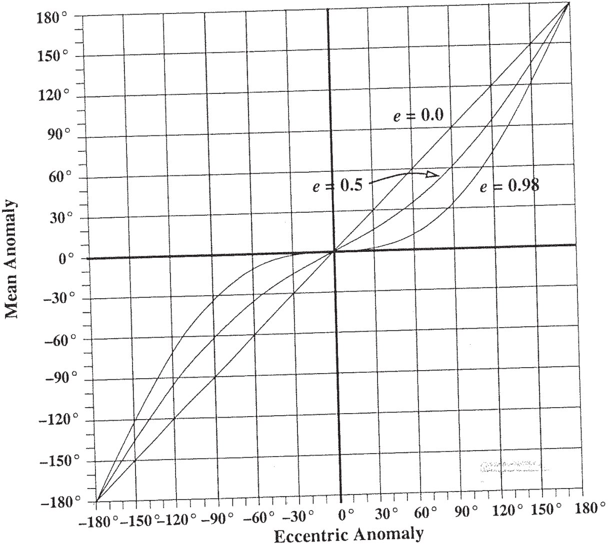


- The fraction of area of the ellipse which has been swept out, in radians.

- Mean anomaly is not defined for hyperbolic orbits, as these orbits do not have a period. Indeed, Kepler's third law is not relevant for hyperbolic orbits.

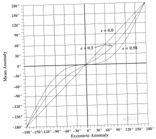
Relationships between M, E, and f

M vs. E



Relationships between M, E, and f

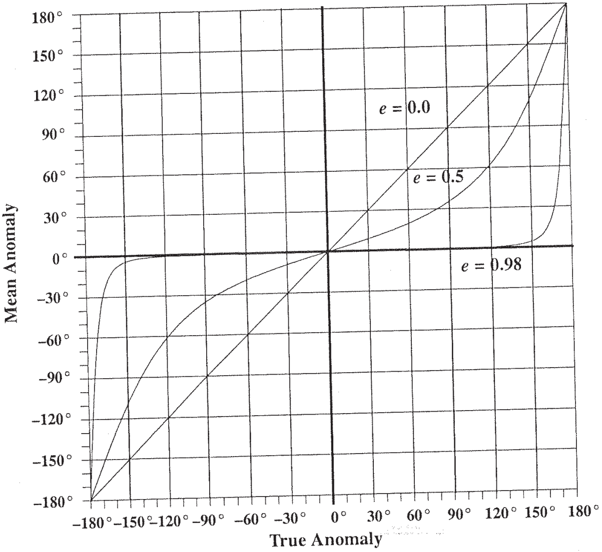
M vs. E



- These graphs are from Vallado's book.
- Graphical methods of relating mean and eccentric anomaly are difficult due to dependence on eccentricity.

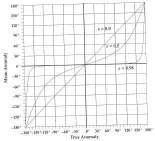
Relationships between M, E, and f

M vs. f



Relationships between M, E, and f

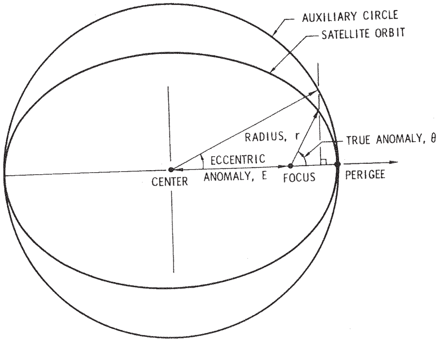
M vs. f



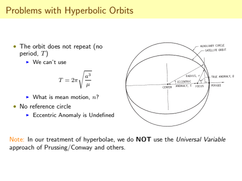
- These graphs are from Vallado's book.
- The difficulty in using graphical methods is exacerbated for true anomaly, especially for highly elliptic orbits.

Problems with Hyperbolic Orbits

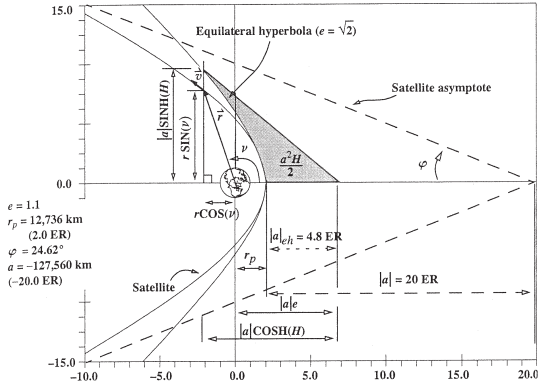
- The orbit does not repeat (no period, T)
  - We can't use
    $$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$
  - What is mean motion, n?
- No reference circle
  - Eccentric Anomaly is Undefined



Note: In our treatment of hyperbolae, we do **NOT** use the *Universal Variable* approach of Prussing/Conway and others.



- The universal variable approach redefines the Kepler equation to be valid for both eccentric and hyperbolic orbits.
- Does not require us to know what type of orbit we have apriori.
- Useful for computer algorithms as it avoids case logic. Occasionally, student try and use Kepler's equation to solve hyperbolic orbit problems.
- No useful geometric interpretation, however.



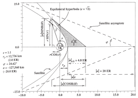
Eccentricity is still

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

and $e = \|\vec{e}\|$.

Solutions for Hyperbolic Orbits

Eccentricity

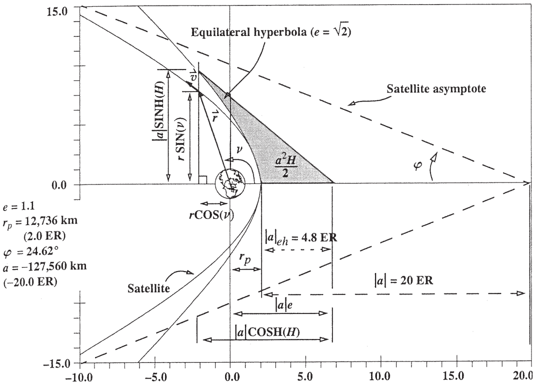


Eccentricity is still

$$\vec{e} = \frac{1}{\mu}\left(\dot{\vec{r}} \times \vec{h} - \mu \frac{\vec{r}}{\|\vec{r}\|}\right)$$

and $e = \|\vec{e}\|$.

- Image is from Vallado. Note the use of ν for representing true anomaly.
- eccentricity vector still points toward periapse for hyperbolic orbits. Still yields scalar value of eccentricity.



Semimajor axis can still be defined by energy as

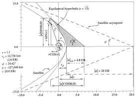
$$E = -\frac{\mu}{2a} = \frac{1}{2}v_{excess}^2$$

The periapse is still

$$r_p = a(1-e)$$

Solutions for Hyperbolic Orbits

Semimajor axis



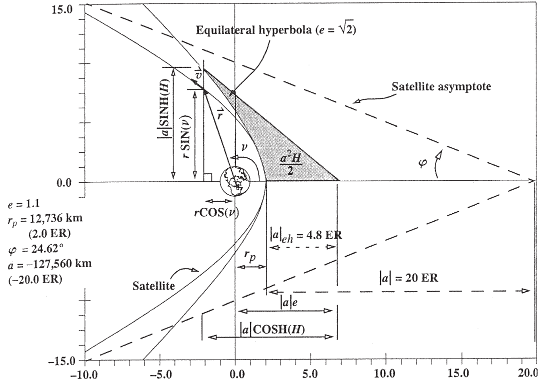
Semimajor axis can still be defined by energy as

$$E = -\frac{\mu}{2a} = \frac{1}{2}v_{excess}^2$$

The periapse is still

$$r_p = a(1-e)$$

- Energy can still be used to calculate a.
- a and e can still be used to calculate r_p
- Of course, r_a is undefined



Hyperbolic Orbits still satisfy the polar equation

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

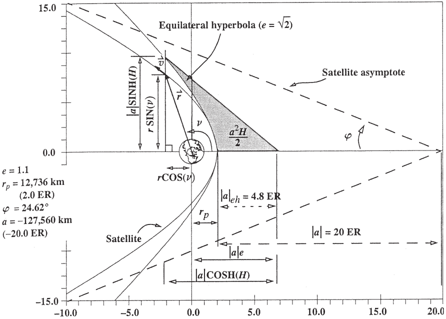
Solutions for Hyperbolic Orbits

Velocity

Velocity can still be calculated from the vis - viva equation

$$v = \sqrt{\mu \left(\frac{2}{r} - \frac{1}{a} \right)}$$

Hyperbolic Anomaly is defined by the projection onto a reference hyperbola.



- defined using the reference hyperbola, tangent at perigee. Equation for reference hyperbola:

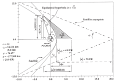
$$x^2 - y^2 = a^2$$

Hyperbolic anomaly (H) is the hyperbolic angle using the area enclosed by the center of the hyperbola, the point of perifocus and the point on the reference hyperbola directly above the position vector.

Solutions for Hyperbolic Orbits

Reference Hyperbola

Hyperbolic Anomaly is defined by the projection onto a reference hyperbola.



- defined using the reference hyperbola, tangent at perigee. Equation for reference hyperbola:

$$x^2 - y^2 = a^2$$

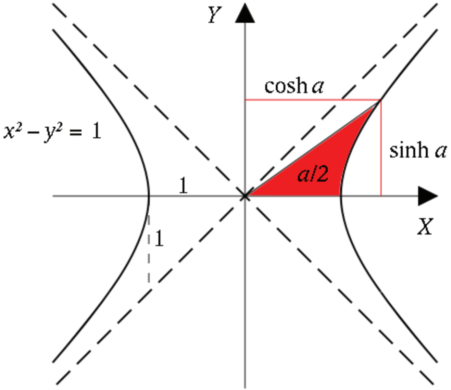
Hyperbolic anomaly (H) is the hyperbolic angle using the area enclosed by the center of the hyperbola, the point of perifocus and the point on the reference hyperbola directly above the position vector.

- The reference hyperbola is the hyperbola with an eccentricity of $\sqrt{2}$ whose periapse is the same as the periapse of the actual orbit.

Recall your Hyperbolic Trig.

Cosh and Sinh

Consider $x^2 - y^2 = 1$

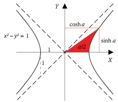


$cosh$ and $sinh$ relate area swept out by the reference hyperbola to lengths.

- Yet another branch of mathematics developed for solving orbits (Lambert).

Recall your Hyperbolic Trig.
Cosh and Sinh

Consider $x^2 - y^2 = 1$

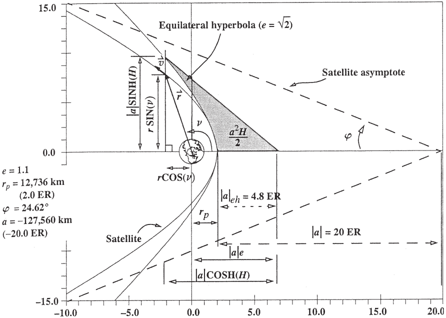


cosh and *sinh* relate area swept out by the reference hyperbola to lengths.

- Yet another branch of mathematics developed for solving orbits (Lambert).

- Defined using the normalized reference hyperbola.
- Lambert invented hyperbolic functions in the 18th century to compute the area of a hyperbolic triangle. We will meet Lambert again in a later lecture.
- See `en.wikipedia.org/wiki/Hyperbolic_function` for a thorough treatment of hyperbolic functions

Hyperbolic Anomaly



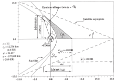
- Hyperbolic Trig (which I won't get into) gives a relationship to true anomaly, which is

$$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f}{2}\right)$$

- Alternatively,

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

Hyperbolic Anomaly



- Hyperbolic Trig (which I won't get into) gives a relationship to true anomaly, which is

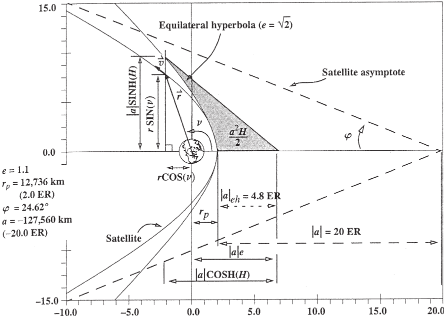
$$\tanh\left(\frac{H}{2}\right) = \sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f}{2}\right)$$

- Alternatively,

$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}}\tanh\left(\frac{H}{2}\right)$$

- Derivation similar to that for Eccentric anomaly.

Hyperbolic Anomaly



Using hyperbolic anomaly, we can give a simpler form of the polar equation.

$$r(t) = a(1 - e\cosh H(t))$$

Hyperbolic Kepler's Equation

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

$$n = \sqrt{\frac{\mu}{-a^3}}$$

Definition 1 (Hyperbolic Kepler's Equation).

$$nt = \sqrt{\frac{\mu}{-a^3}}t = M = e\sinh(H) - H$$

If we want to solve this for H, we get a different Newton iteration.
Newton Iteration for Hyperbolic Anomaly:

$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

with starting guess $H_1 = M$.

Hyperbolic Kepler's Equation

To solve for position, we redefine mean motion, n, and mean anomaly, M, to get

$$n = \sqrt{\frac{\mu}{-a^3}}$$

Definition 1 (Hyperbolic Kepler's Equation).

$$nt = \sqrt{\frac{\mu}{-a^3}}t = M = e\sinh(H) - H$$

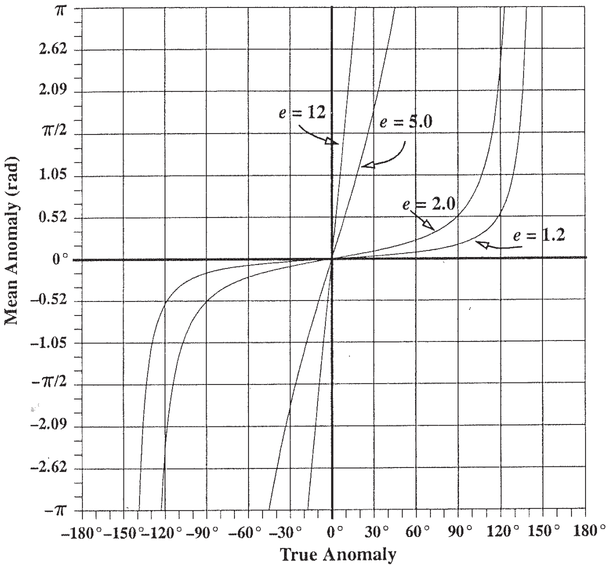
If we want to solve this for H, we get a different Newton iteration.
Newton Iteration for Hyperbolic Anomaly:

$$H_{k+1} = H_k + \frac{M - e\sinh(H_k) + H_k}{e\cosh(H_k) - 1}$$

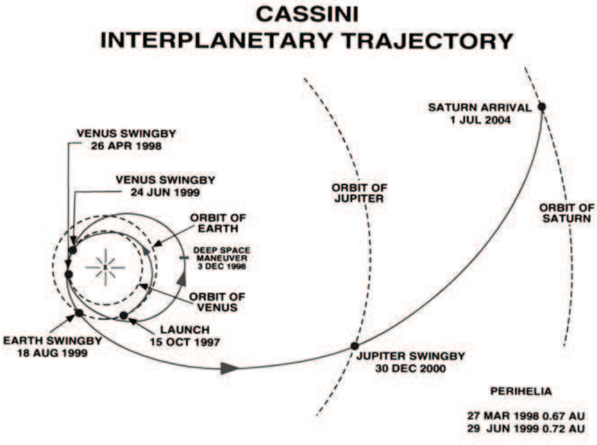
with starting guess $H_1 = M$.

- Very similar to Kepler's equation. But don't confuse them! I have given hyperbolic orbits on exams in the past.

Relationship between M and f for Hyperbolic Orbits



Example: Jupiter Flyby

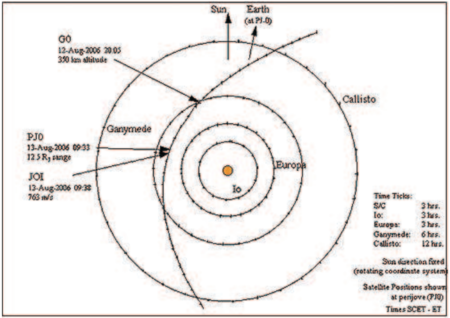


Problem: Suppose we want to make a flyby of Jupiter. The relative velocity at approach is $v_\infty = 10 km/s$. To achieve the proper turning angle, we need an eccentricity of $e = 1.07$. Radiation limits our time within radius $r = 100,000km$ to 1 hour (radius of Jupiter is $71,000km$). Will the spacecraft survive the flyby?

Example: Jupiter Flyby

Example Continued

Solution: First solve for a and p. $\mu = 1.267E8$.



- The total energy of the orbit is given by

$$E_{tot} = \frac{1}{2} v_\infty^2$$

- The total energy is expressed as

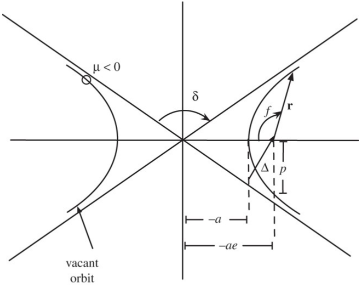
$$E = -\frac{\mu}{2a} = \frac{1}{2} v_\infty^2$$

which yields

$$a = -\frac{\mu}{v_\infty^2} = -1.267E6$$

- The parameter is $p = a(1 - e^2) = 1.8359E5$

Example Continued



We need to find the time between $r_1 = 100,000km$ and $r_2 = 100,000km$. Find f at each of these points.

- Start with the polar equation:

$$r(t) = \frac{p}{1 + e\cos f(t)}$$

- Solving for f,

$$f_{1,2} = \cos^{-1}\left(\frac{1}{e} - \frac{r}{ep}\right) = \pm 64.8\,\mathrm{deg}$$

Example Continued

Given the true anomalies, $f_{1,2}$, we want to find the associated times, $t_{1,2}$.

- Only solve for t_2, get t_1 by symmetry.
- First find Hyperbolic Anomaly,

$$H_2 = \tanh^{-1}\left(\sqrt{\frac{e-1}{e+1}}\tan\left(\frac{f_2}{2}\right)\right) = .1173$$

- Now use Hyperbolic anomaly to find mean anomaly

$$M_2 = e\sinh(H_2) - H_2 = .0085$$

  - This is the "easy" direction.
  - No Newton iteration required.
- t_2 is now easy to find

$$t_2 = M_2\sqrt{\frac{-a^3}{\mu}} = 1076.6$$

Finally, we conclude $\Delta t = 2 * t_2 = 2153s = 35min$.

So the spacecraft survives.

The Method for Hyperbolic Orbits

Given t, find r and v

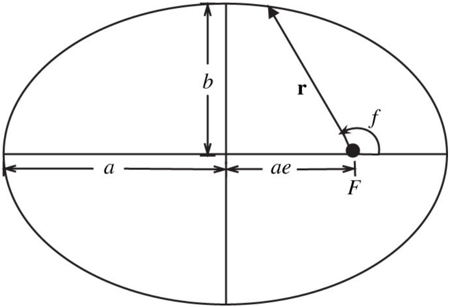
For elliptic orbits:

1. Given time, t, solve for Hyperbolic Mean Anomaly
$$M(t) = \sqrt{\frac{\mu}{-a^3}}t$$
2. Given Mean Anomaly, solve for hyperbolic anomaly
$$M(t) = e \sinh H - H$$
3. Given hyperbolic anomaly, solve for true anomaly
$$\tan\left(\frac{f}{2}\right) = \sqrt{\frac{e+1}{e-1}} \tanh\left(\frac{H}{2}\right)$$
4. Given true anomaly, solve for r
$$r(t) = \frac{a(1-e^2)}{1+e\cos f(t)}, \qquad v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)}$$

The Orbital Elements

So far, all orbits are parameterized by 3 parameters

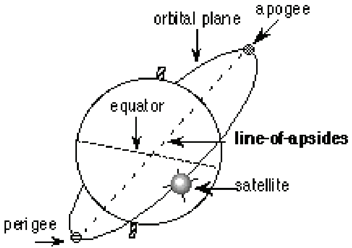
- semimajor axis, a
- eccentricity, e
- true anomaly, f



- a and e define the geometry of the orbit.
- f describes the position within the orbit (a proxy for time).

The Orbital Elements

Note: We have shown how to use a, e and f to find the scalars r and v.



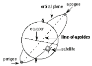
Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.

The Orbital Elements

Note: We have shown how to use a, e and f to find the scalars r and v.



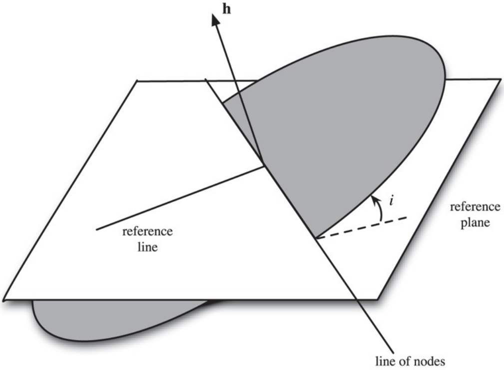
Question: How do we find the vectors $\vec{r}$ and $\vec{v}$?

Answer: We have to determine how the orbit is oriented in space.

- Orientation is determined by vectors $\vec{e}$ and $\vec{h}$.
- We need 3 new orbital elements
  - Orientation can be determined by 3 rotations.

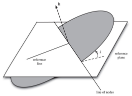
- Although $\vec{e}$ and $\vec{h}$ represent components, we only actually need three. The $\vec{h}$ represents orientation of the orbital plane, and so we don't care about the roll axis in the classic 1-2-3 rotation matrices. That is, this orientation has symmetry about the angular momentum vector. The eccentricity vector is always perpendicular to the angular momentum vector, which gives one constraint. The second is that its length equals the eccentricity of the orbit. This leaves a single degree of freedom.

Inclination, i



Angle the orbital plane makes with the reference plane.

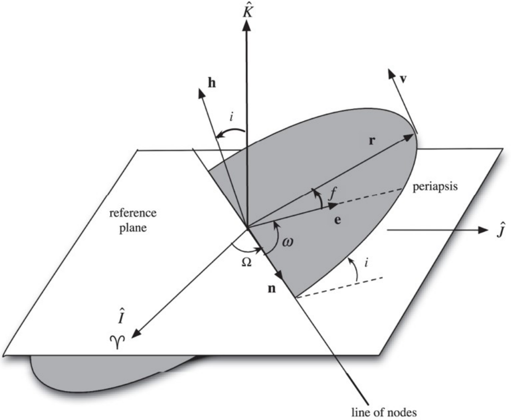
Inclination, i



Angle the orbital plane makes with the reference plane.

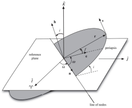
- Think of the 2D orbit in space. Z-axis is out of the ecliptic plane. X-axis is line of nodes.
- first rotation is about the line of nodes.

Right Ascension of Ascending Node, Ω



Angle measured from reference direction in the reference plane to intersection with orbital plane.

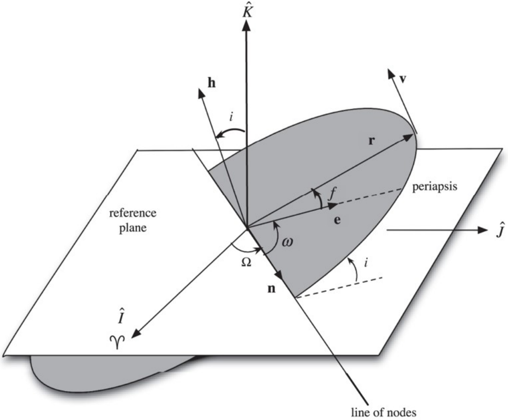
Right Ascension of Ascending Node, Ω



Angle measured from reference direction in the reference plane to intersection with orbital plane.

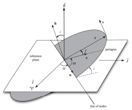
- Second rotation is about the Z-axis.

Argument of Periapse, ω



Angle measured from reference plane to point of periapse.

Argument of Periapse, ω



Angle measured from reference plane to point of periapse.

- Third rotation is about angular momentum vector.

Summary

This Lecture you have learned:

Hyperbolic orbits

- Hyperbolic Anomaly
- Kepler's Equation, Again.
- How to find r and v

A Mission Design Example

Introduction to the Orbital Plane

Summary

Properties of Keplerian Orbits

Quantity	Circle	Ellipse	Parabola	Hyperbola
Defining Parameters	a = semimajor axis = radius	a = semimajor axis b = semiminor axis	p = semi-latus rectum q = perifocal distance	a = semi-transverse axis $a < 0$ b = semi-conjugate axis
Parametric Equation	$x^2 + y^2 = a^2$	$x^2/a^2 + y^2/b^2 = 1$	$x^2 = 4qy$	$x^2/a^2 - y^2/b^2 = 1$
Eccentricity, e	$e = 0$	$e = \sqrt{a^2 - b^2}\big/a \quad 0 < e < 1$	$e = 1$	$e = \sqrt{a^2 + b^2 / a^2} \quad e > 1$
Perifocal Distance, q	$q = a$	$q = a\ (1 - e)$	$q = p/2$	$q = a\ (1 - e)$
Velocity, V, at distance, r, from Focus	$V^2 = \mu / r$	$V^2 = \mu\ (2/r - 1/a)$	$V^2 = 2\mu / r$	$V^2 = \mu(2/r - 1/a)$
Total Energy Per Unit Mass, $\mathcal{E}$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = -\mu/2a < 0$	$\mathcal{E} = 0$	$\mathcal{E} = -\mu/2a > 0$
Mean Angular Motion, n	$n = \sqrt{\mu / a^3}$	$n = \sqrt{\mu / a^3}$	$n = \sqrt{\mu}$	$n = \sqrt{\mu / (-a)^3}$
Period, P	$P = 2\pi / n$	$P = 2\pi / n$	$P = \infty$	$P = \infty$
Anomaly	$\nu = M = E$	Eccentric anomaly, E $\tan\frac{\nu}{2} = \left(\frac{1+e}{1-e}\right)^{1/2} \tan\left(\frac{E}{2}\right)$	Parabolic anomaly, D $\tan\frac{\nu}{2} = D\big/\sqrt{2q}$	Hyperbolic anomaly, F $\tan\frac{\nu}{2} = \left(\frac{e+1}{e-1}\right)^{1/2} \tanh\left(\frac{F}{2}\right)$
Mean Anomaly, M	$M = M_0 + nt$	$M = E - e \sin E$	$M = qD + (D^3/6)$	$M = (e \sinh F) - F$
Distance from Focus, $r = q\,(1 + e) / (1 + e \cos \nu)$	$r = a$	$r = a\ (1 - e \cos E)$	$r = q + (D^2/2)$	$r = a\ (1 - e \cosh F)$
$r\ dr / dt \equiv r\dot{r}$	0	$r\dot{r} = e\sqrt{a\mu} \sin E$	$r\dot{r} = \sqrt{\mu}\, D$	$r\dot{r} = e\sqrt{(-a)\mu} \sinh F$
Areal Velocity, $\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\nu}{dt}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$	$\frac{dA}{dt} = \sqrt{\frac{\mu q}{2}}$	$\frac{dA}{dt} = \frac{1}{2}\sqrt{a\mu\,(1-e^2)}$

$\mu = GM$ is the gravitational constant of the central body; ν is the true anomaly, and $M \equiv n\ (t - T)$ is the mean anomaly, where t is the time of observation, T is the time of perifocal passage, and n is the mean angular motion.