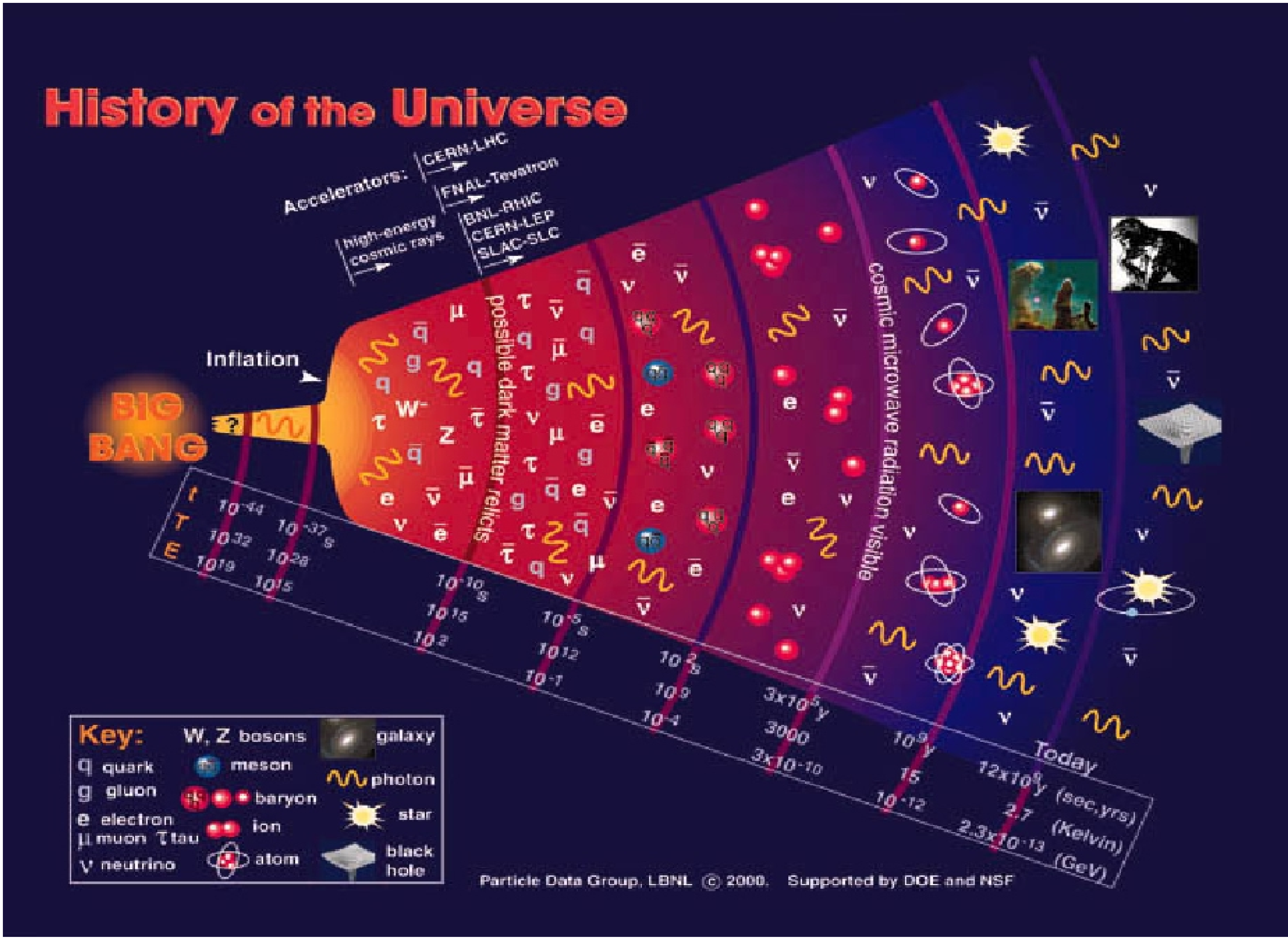
THERMAL HISTORY OF THE UNIVERSE AND EARLY GROWTH OF DENSITY FLUCTUATIONS



Evidence for the Hot Big Bang

1) Cosmic Expansion (covered in previous lectures)

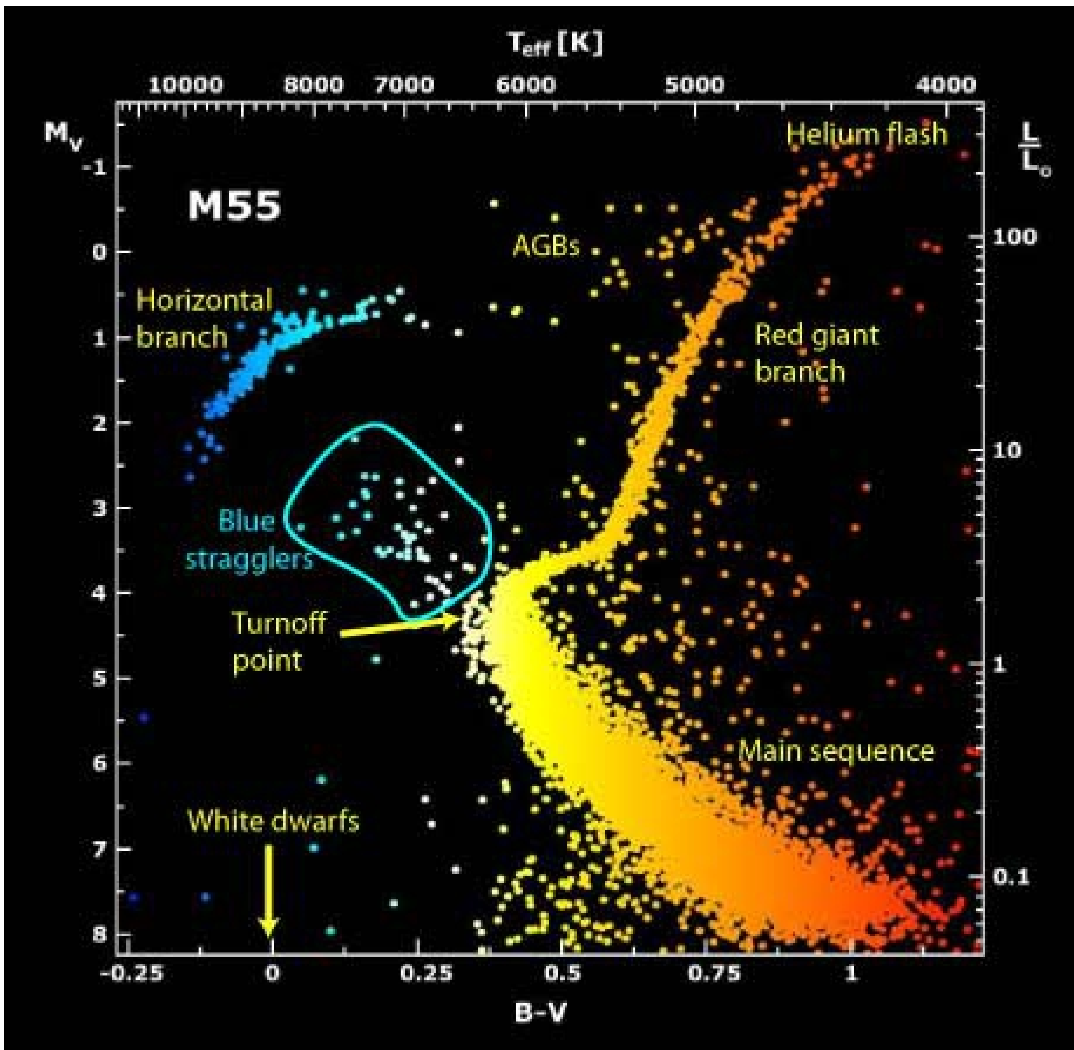
2) Big Bang theory can predict the 24% primordial helium abundance that is observed everywhere

3) Big Bang theory can explain the observed Cosmic Microwave Background (CMB) radiation

4) The distant Universe looks different to the present-day Universe (most of the content of this course)

5) We can measure the ages of the oldest stars, and get an answer that is close to the predicted age of the Universe

The Ages of the Oldest Stars in the Universe Measured from Globular Clusters



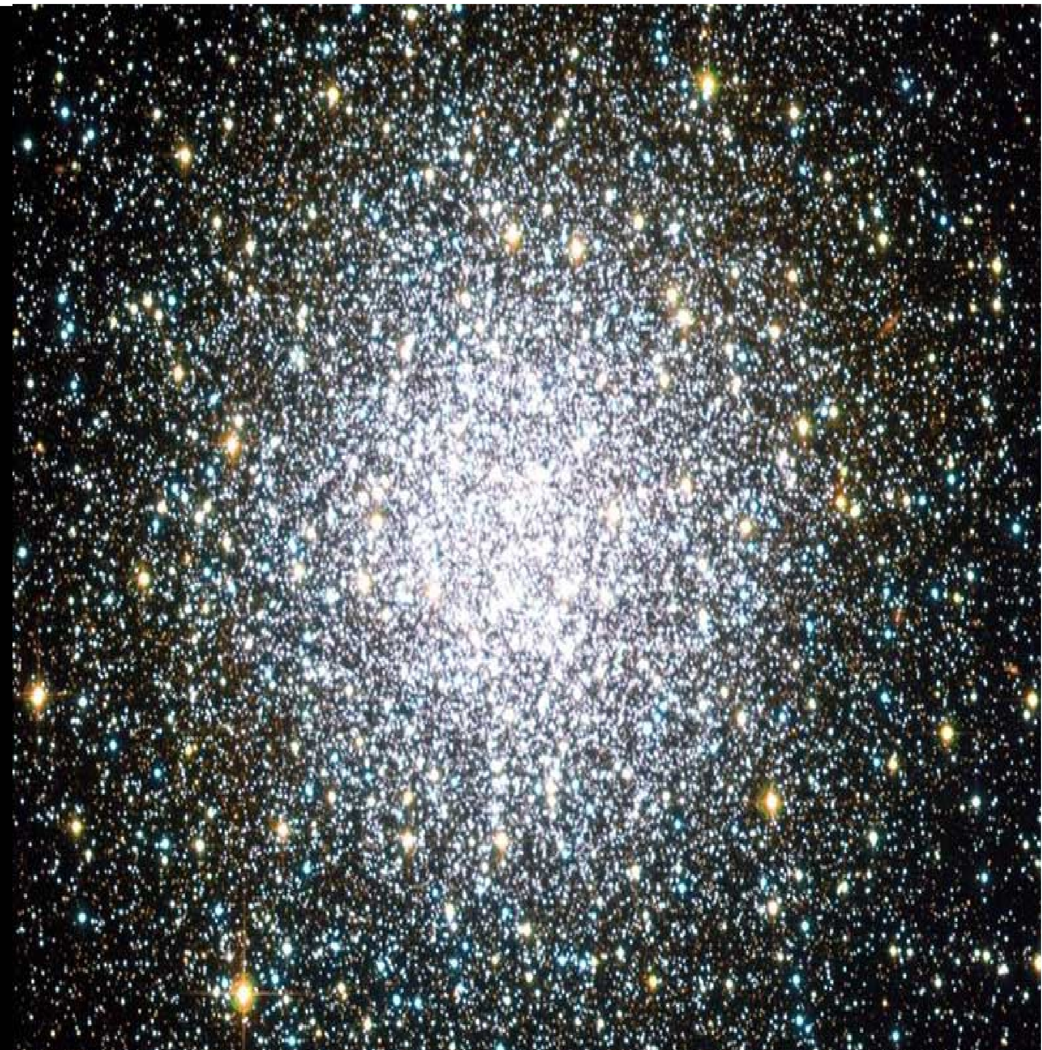


Table 1: **The particles in the Standard Model** *Particle Data Group, 2012* [1]

Quarks	t	173.5 ± 1.0GeV	$\bar{t}$	spin=$\frac{1}{2}$	$g = 2 \cdot 2 \cdot 3 = 12$	
	b	4.18 ± 0.03GeV	$\bar{b}$	3 colors		
	c	1.275 ± 0.025GeV	$\bar{c}$			
	s	95 ± 5MeV	$\bar{s}$			
	d	4.5–5.5MeV	$\bar{d}$			
	u	1.8–3.0MeV	$\bar{u}$			
						72
Gluons	8 massless bosons			spin=1	$g = 2$	16
Leptons	τ^-	1776.82 ± 0.16MeV	τ^+	spin=$\frac{1}{2}$	$g = 2 \cdot 2 = 4$	
	μ^-	105.658MeV	μ^+			
	e^-	510.999keV	e^+			
						12
	ν_τ	< 2eV	$\bar{\nu}_\tau$	spin=$\frac{1}{2}$	$g = 2$	
	ν_μ	< 2eV	$\bar{\nu}_\mu$			
	ν_e	< 2 eV	$\bar{\nu}_e$			
						6
Electroweak gauge bosons	W^+	80.403 ± 0.029GeV		spin=1	$g = 3$	
	W^-	80.385 ± 0.015GeV				
	Z^0	91.1876 ± 0.0021GeV				
	γ	0 $(< 1 \times 10^{-18}$eV)			$g = 2$	
						11
Higgs boson (SM)	H^0	$115.5 - 127$GeV		spin=0	$g = 1$	1

$$g_f = 72 + 12 + 6 = 90$$
$$g_b = 16 + 11 + 1 = 28$$

COSMIC INFLATION

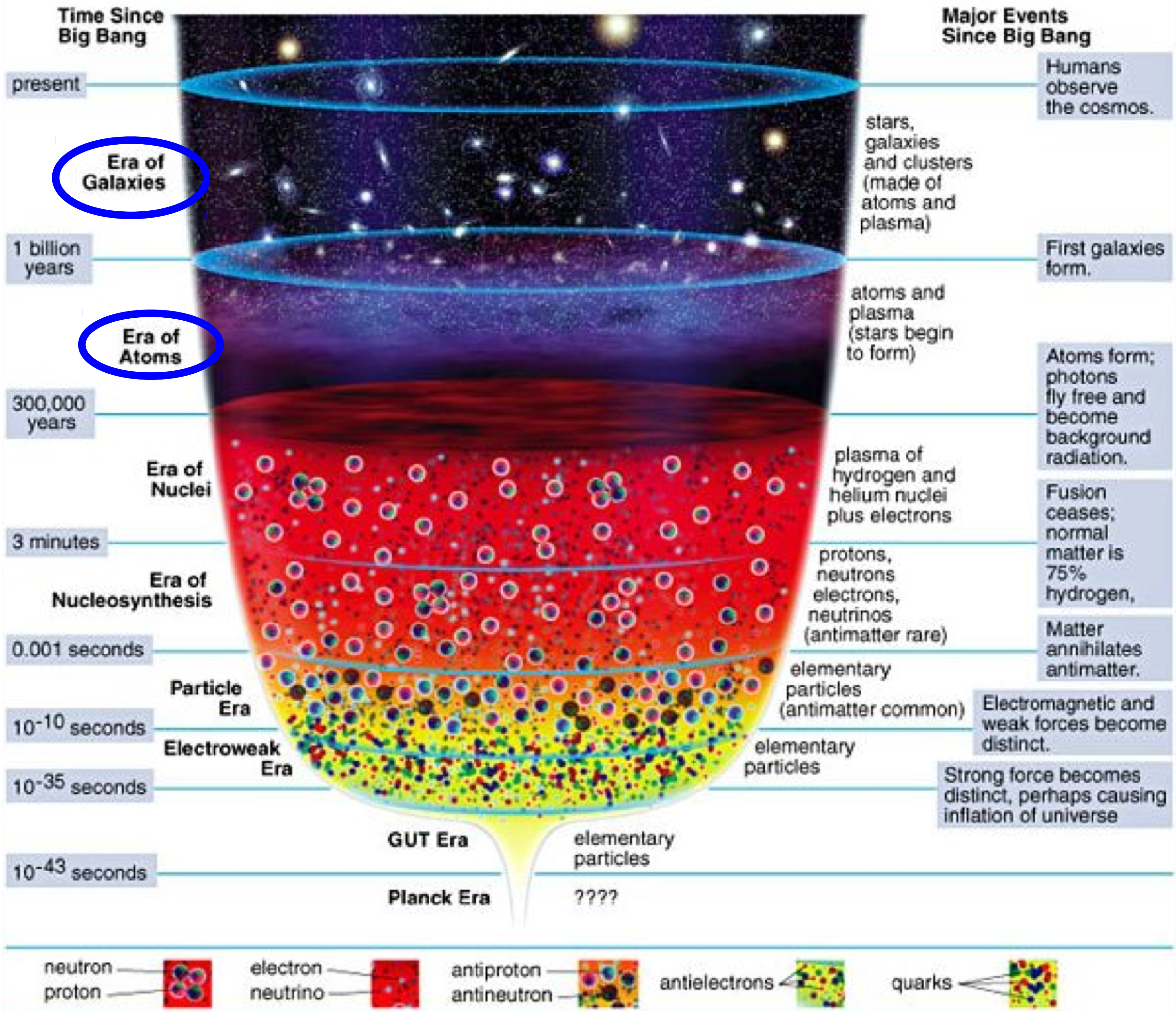
In physical cosmology, cosmic inflation, cosmological inflation, is the extremely rapid exponential expansion of the early universe by a factor of at least 1078 in volume, driven by a negative-pressure vacuum energy density.] The inflationary epoch comprises the first part of the electroweak epoch following the grand unification epoch. It lasted from 10^{-36} seconds after the Big Bang to sometime between 10^{-33} and 10^{-32} seconds. Following the inflationary period, the universe continued to expand, but at a slower rate.

Why was such a theory proposed?

1) THE HORIZON PROBLEM --In a big bang with only the matter and radiation known in the Standard Model, two widely separated regions of the observable universe cannot have equilibrated because they move apart from each other faster than the speed of light—thus have never come into causal contact.

2) THE FLATNESS PROBLEM -- a fine-tuning problem because the contribution of curvature to the universe is so small ((sixteen orders of magnitude less than the density of radiation at big bang nucleosynthesis)

3) MAGNETIC MONOPOLE PROBLEM (EXOTIC RELICS PROBLEM) --Monopoles are expected to be copiously produced in Grand Unified Theories at high temperature, and they should have persisted to the present day, to such an extent that they would become the primary constituent of the universe.



Time Since Big Bang
Major Events Since Big Bang
present
Humans observe the cosmos.
Era of Galaxies
stars, galaxies and clusters (made of atoms and plasma)
1 billion years
First galaxies form.
Era of Atoms
atoms and plasma (stars begin to form)
300,000 years
Atoms form; photons fly free and become background radiation.
Era of Nuclei
plasma of hydrogen and helium nuclei plus electrons
3 minutes
Fusion ceases; normal matter is 75% hydrogen,
Era of Nucleosynthesis
protons, neutrons electrons, neutrinos (antimatter rare)
0.001 seconds
Matter annihilates antimatter.
Particle Era
elementary particles (antimatter common)
10^{-10} seconds
Electromagnetic and weak forces become distinct.
Electroweak Era
elementary particles
10^{-35} seconds
Strong force becomes distinct, perhaps causing inflation of universe
GUT Era
elementary particles
10^{-43} seconds
Planck Era
????
neutron
proton
electron
neutrino
antiproton
antineutron
antielectrons
quarks

After cosmic inflation ends, the universe is filled with a quark–gluon plasma. From this point onwards the physics of the early universe is better understood, and less speculative.

Hadron epoch Between 10^{-6} seconds and 1 second after the Big Bang

Hadrons are particles that feel the strong nuclear force and are combinations of quarks. When the temperature cools below twice the nucleon rest mass, annihilations dominate creations, and the nucleon density drops dramatically, leading to the lepton era. 1 second after the Big Bang neutrinos decouple and begin traveling freely through space (cosmic neutrino background).

Lepton epoch Between 1 second and 10 seconds after the Big Bang

Leptons feel the weak force. This era ends when electon/anti-electron pair production stops.

Photon epoch Between 10 seconds and 380,000 years after the Big Bang
Lasts until photon recombine with atoms (epoch of matter-radiation equality)

What is the Physics of these Transitions?

Interaction Equilibrium

Calculation of reaction rates from first principles, allow us to calculate the time when equilibrium is broken, and forward (building) processes dominate. Reaction time-scale depends on the number density n of the source component, the cross-section σ for the process, and velocity v of the of the source

$$\Gamma = n \langle \sigma v \rangle$$

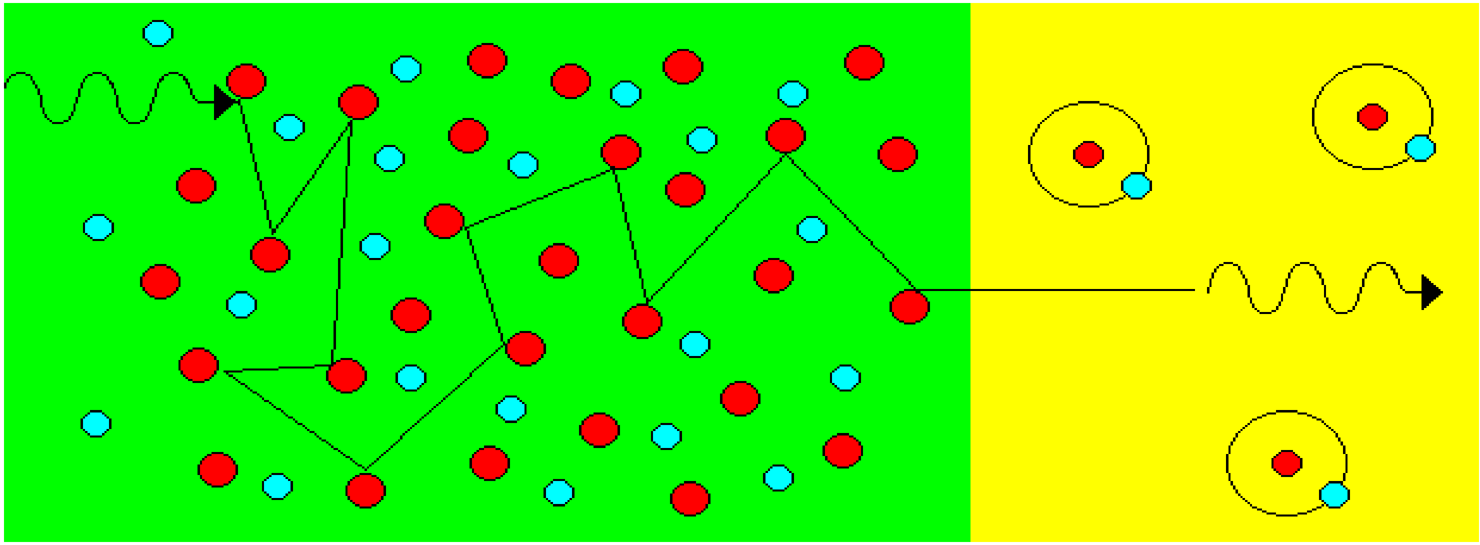
(angle brackets indicate averaging over the source thermal (momentum) distribution at a given energy). $\langle \sigma v \rangle$ is calculated from quantum physics, or accelerator experiments of the proper energy. The density n evolves In proportion to $a^{-3.}$

For example, electron-positron freeze-out depends on the reaction rate of electron+positron --> γγ. Interaction rates often scale with energy (i.e. T) as $\Gamma = T^n$. In the radiation epoch, $T \sim t^{-1/2}$ and $H=1/(2t)$ so that the number of future interactions $N_{int} = \int dt' \ \Gamma(t')$ reduces to $(n-2)^{-1} (\Gamma/H)_{t.}$
For n>2, the number of future interactions is greater than 1 if $\Gamma > H_t$

Electroweak Phase Transition	$T \sim 100$ GeV	$t \sim 20$ ps
QCD Phase Transition	$T \sim 150$ MeV	$t \sim 20\mu$s
Neutrino Decoupling	$T \sim 1$ MeV	$t \sim 1$ s
Electron-Positron Annihilation	$T < m_e \sim 0.5$ MeV	$t \sim 10$ s
Big Bang Nucleosynthesis	$T \sim 50$–100 keV	$t \sim 10$ min
Matter-Radiation Equality	$T \sim 0.8$ eV ~ 9000 K	$t \sim 60000$ yr
Recombination + Photon Decoupling	$T \sim 0.3$ eV ~ 3000 K	$t \sim 380000$ yr

Last Scattering Epoch

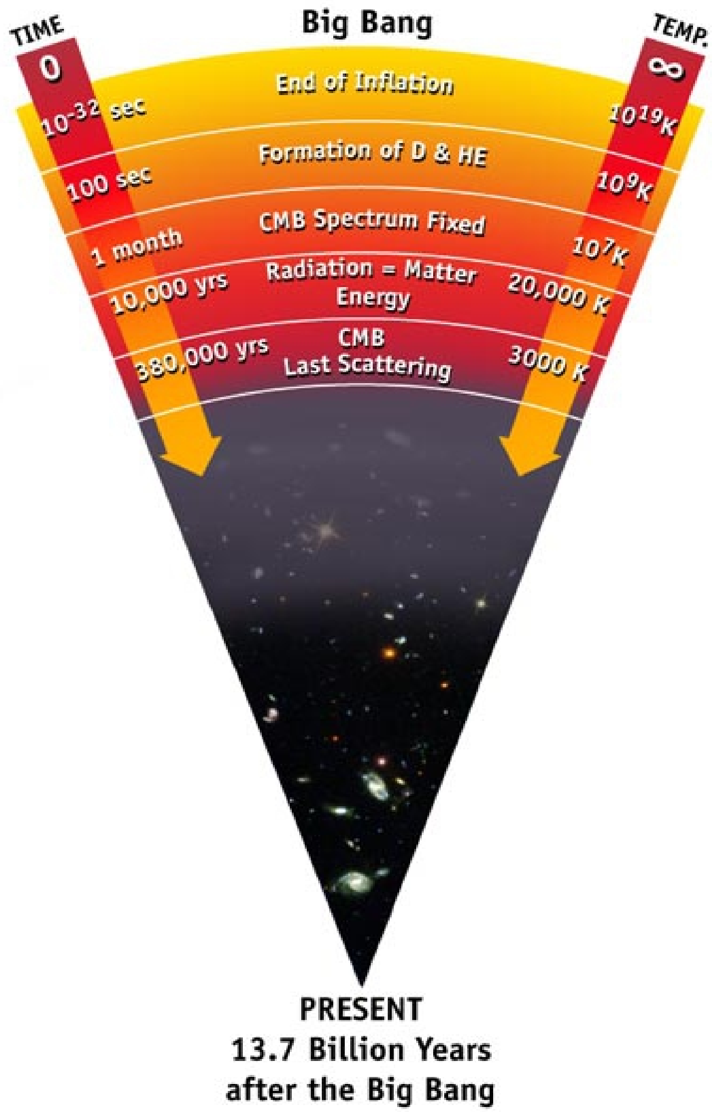
As the Universe cooled, the free electrons and protons could finally bond togther to form hydrogen atoms. At the same time, the Universe went from a rich plasma to a gas of neutral hydrogen.



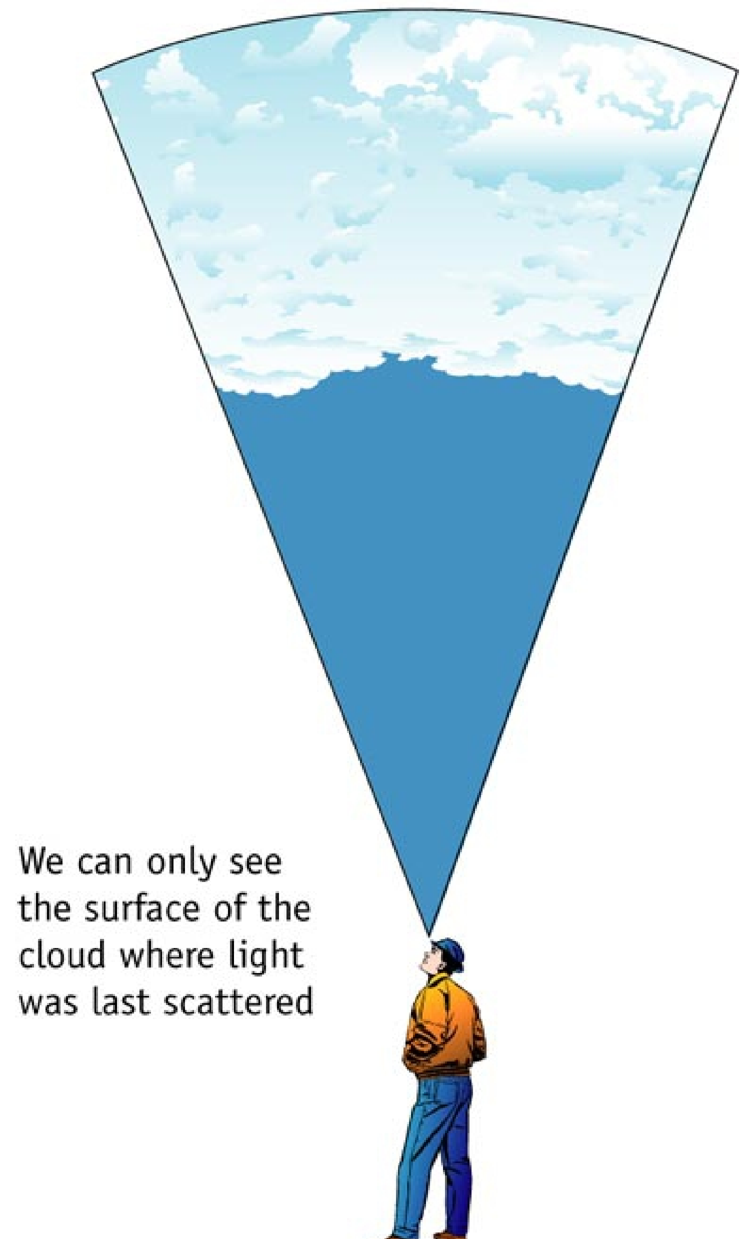
hydrogen plasma | atomic hydrogen

In a plasma, the mean free path of a photon is very short. In a gas of atomic hydrogen, the mean free path is very long, as long as the size of the Universe. Thus, the transition from the early plasma to atomic hydrogen is the epoch of last scattering, the point in time when the photons became free to travel without hindrance.

Surface of Last Scattering of the Cosmic Microwave Background Radiation



The cosmic microwave background Radiation's "surface of last scatter" is analogous to the light coming through the clouds to our eye on a cloudy day.



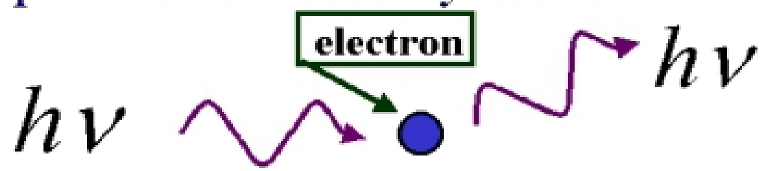
Recombination Process

As the temperature drops electrons and protons combine to form atoms: e- + p+ <--> H + γ

High photon number density delays recombination through process of Thomson scattering:

Thomson Scattering

low-E photon scattered by electron -



Thomson cross-section is given by -

$$\sigma = \frac{8}{3}\pi r_e^2, \text{ where } r_e = 2.82\times10^{-15}m$$

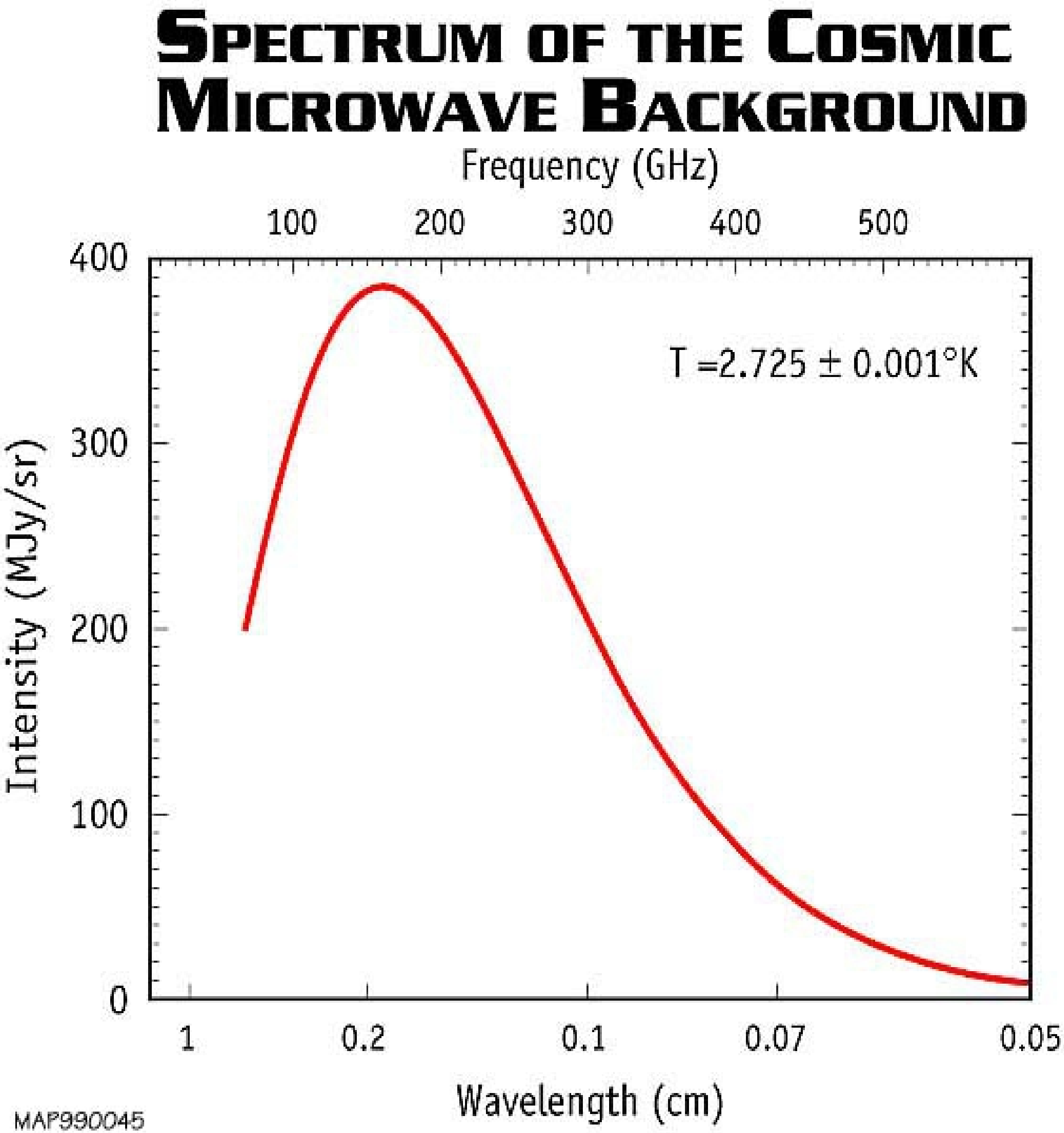
$$\boxed{\Rightarrow \sigma_e = 6.65\times10^{-29}m^2}$$

Eventually the number density of sufficiently energetic photons drops too low to undo recombinations (electron captures). Thus, final electron decoupling occurs about $10^{5.5}$ years after the recombination epoch.

Microwave background photons arrive from a recombination “shell” rather than a surface of optical depth

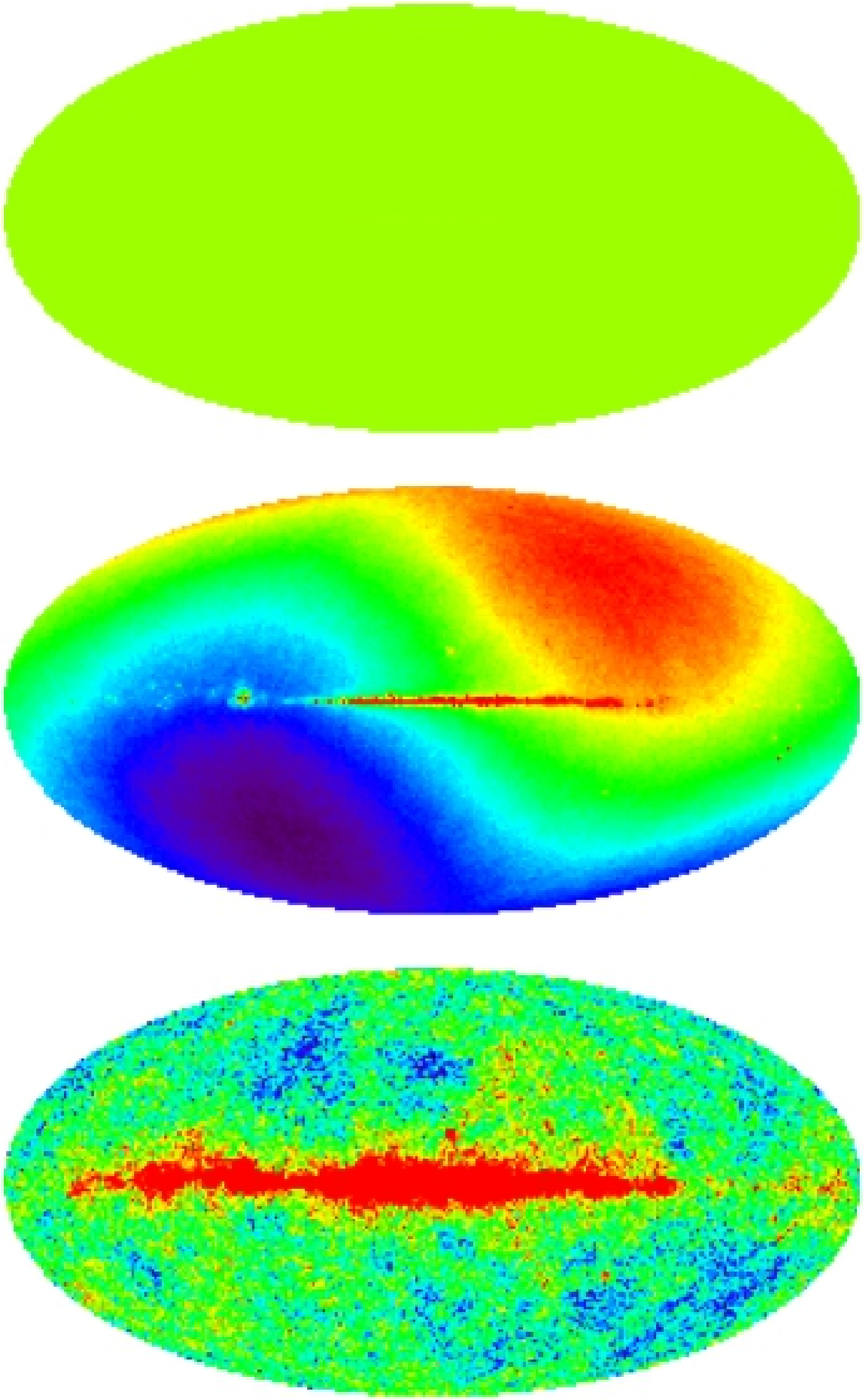
$$\tau = \int n_e \sigma_T \, dr$$

Note that the cosmic plasma cooled during recombination, but photons also are redshifted by different amounts. These two factors cancel , so the resulting spectrum is very close to a Planck spectrum of single temperature.



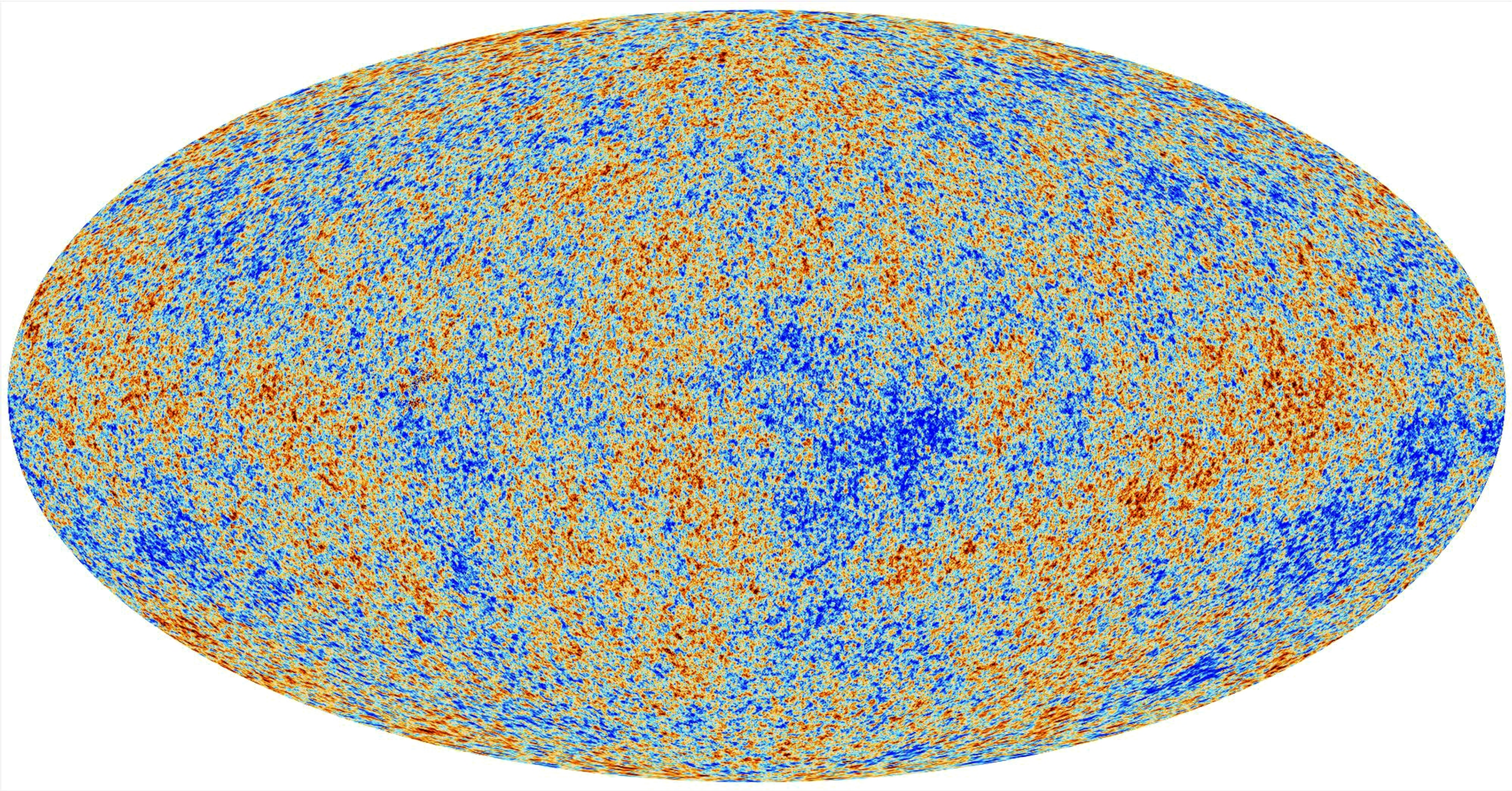
$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{k_B T}} - 1}$$

Planck's law for radiation in thermodynamic equilibrium at a definite temperature.

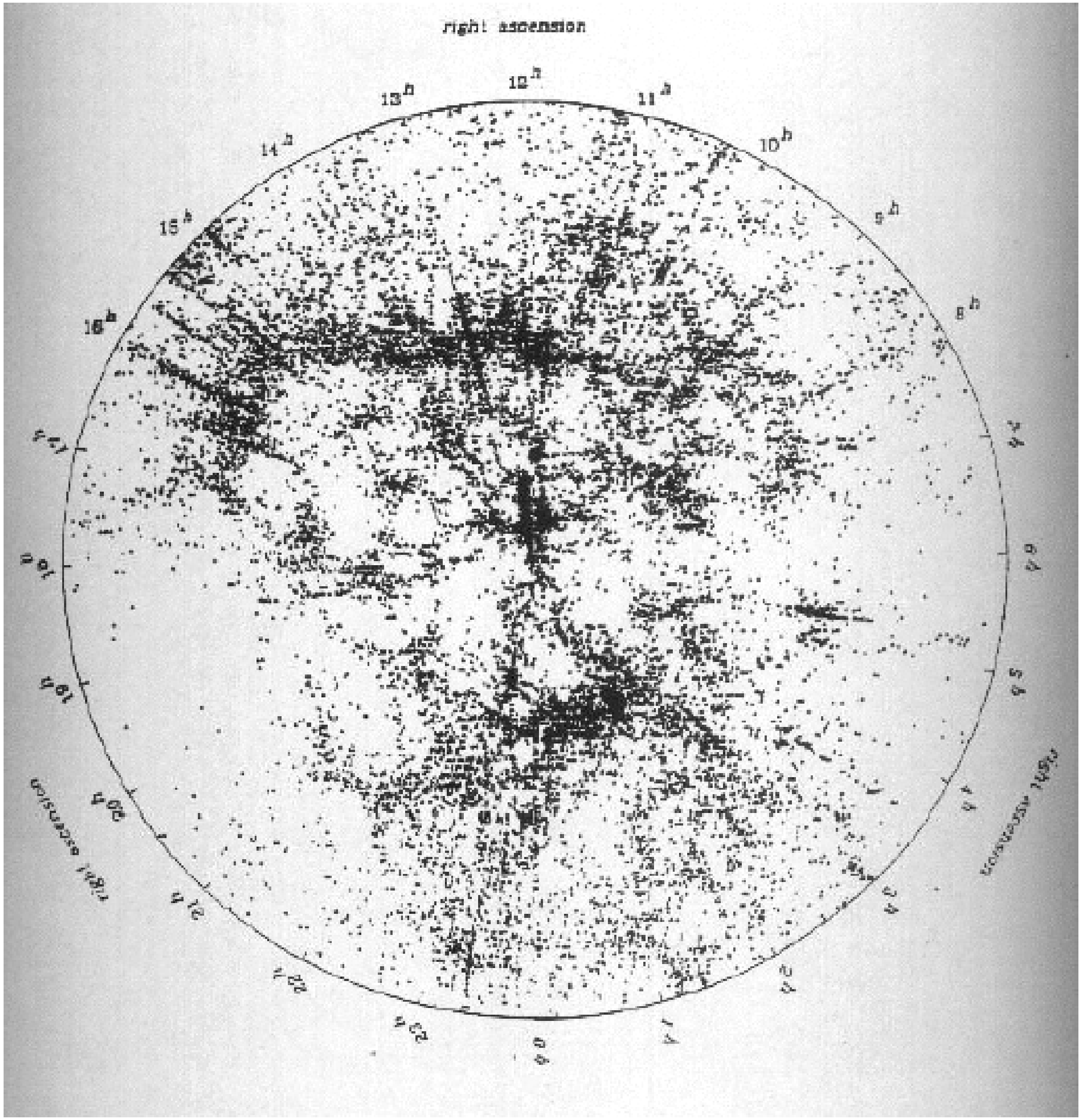


The cosmic microwave background radiation is isotropic to one part in 100,000

Most recent CMB map from the Planck satellite



THEORY OF THE GROWTH OF PERTURBATIONS



We now develop the theory of how the tiny (1/100,000) density fluctuations that we see imprinted on the CMB radiation at z=1300-1100 grow to become the galaxies we see today.

Primary Sources of CMB Anisotropies

1) Gravitational (Sachs-Wolfe) perturbations. These arise because a photon moving away from a slightly overdense region on the last scattering surface loses energy and is red-shifted.

2) Density (adiabatic) perturbations: the coupling of radiation and matter results in the compression of the radiation and an increase in temperature.

3) Velocity (Doppler) perturbations.: the ionized plasma has a non-zero velocity. This velocity results in a Doppler shift in the observed frequency of the radiation and a corresponding change in temperature.

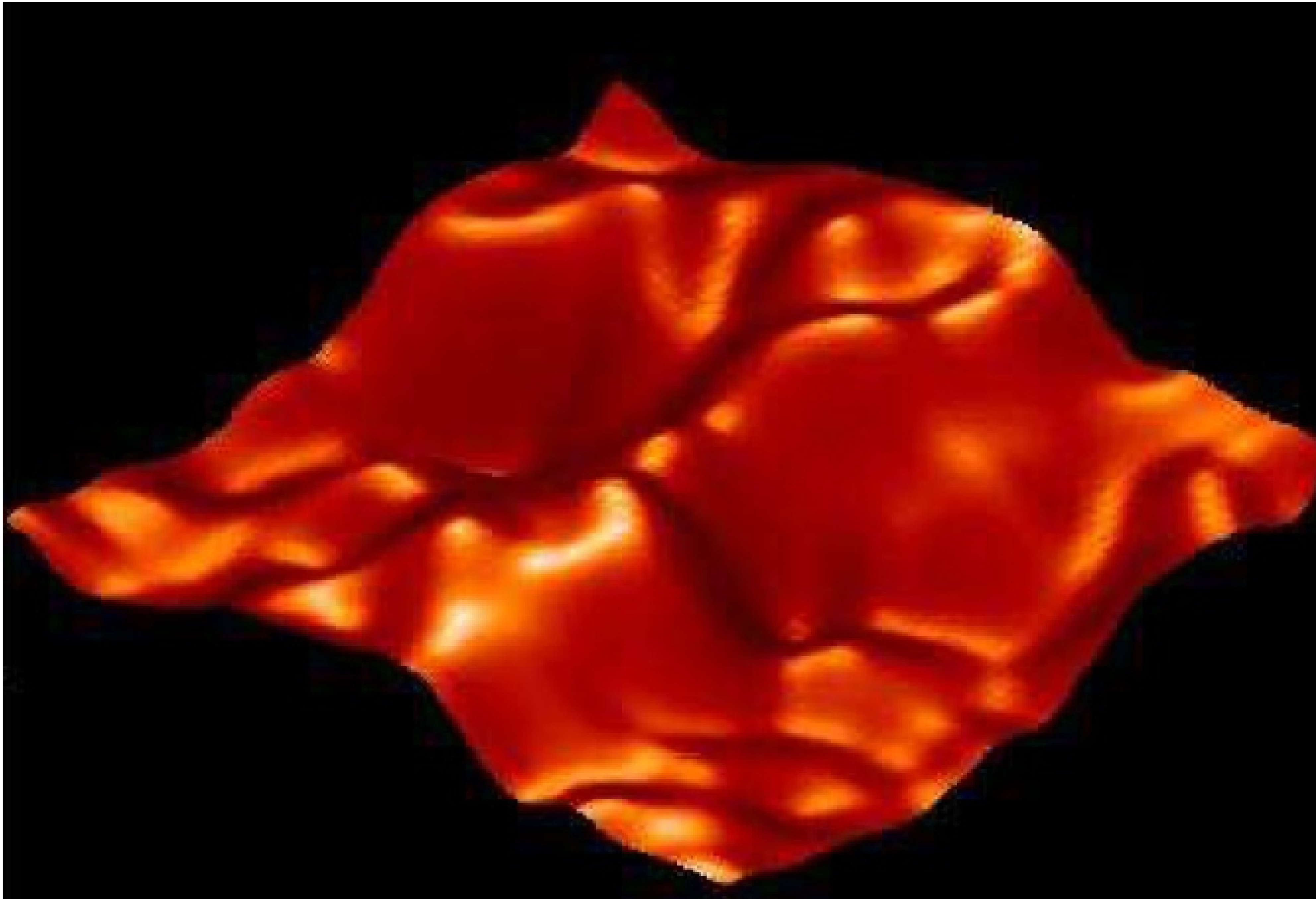
Candidates for what might have initially seeded the structure are:

1. Amplification of quantum zero-point fluctuations during an inflationary era. (leads to adiabatic perturbations)
2. "Topological defects" formed in a cosmological phase transition (leads to isocurvature perturbations)

Consider an initial density perturbation defined as follows:

$$1+ \delta(\mathbf{x}) = \rho(\mathbf{x})/ <p>$$

Adiabatic perturbations: compress a region containing matter and radiation adiabatically. This would change the matter density and the photon number density by the same factor.
The **energy density** of matter and radiation respond differently to a change in scale factor: $\rho_m \propto 1/a^3$ and $\rho_r \propto 1/a^4$. Thus $\delta_{rad} = 4/3\ \delta_m$



Isocurvature perturbations: perturb the entropy density but not the energy density. Since the total energy density remains homogeneous, there is no perturbation to the spatial curvature and

$$\rho_{rad}\delta_{rad} = -\rho_m\delta_m$$

Examples of iso-curvature perturbations include variations in the relative fraction of baryons to photons. This will be a very small effect in the radiation-dominated epoch.

Perturbations and causality: let's imagine that we wanted to create a non-uniform density field at some given time in the early Universe.

It is impossible to change the mean density on scales larger than the horizon at that time, so iso-curvature fluctuations can be argued to be more natural. However, inflation vastly expands the horizon scale, allowing fluctuations to be produced on scales larger than c/H_0, so adiabatic perturbations can dominate if generated early enough

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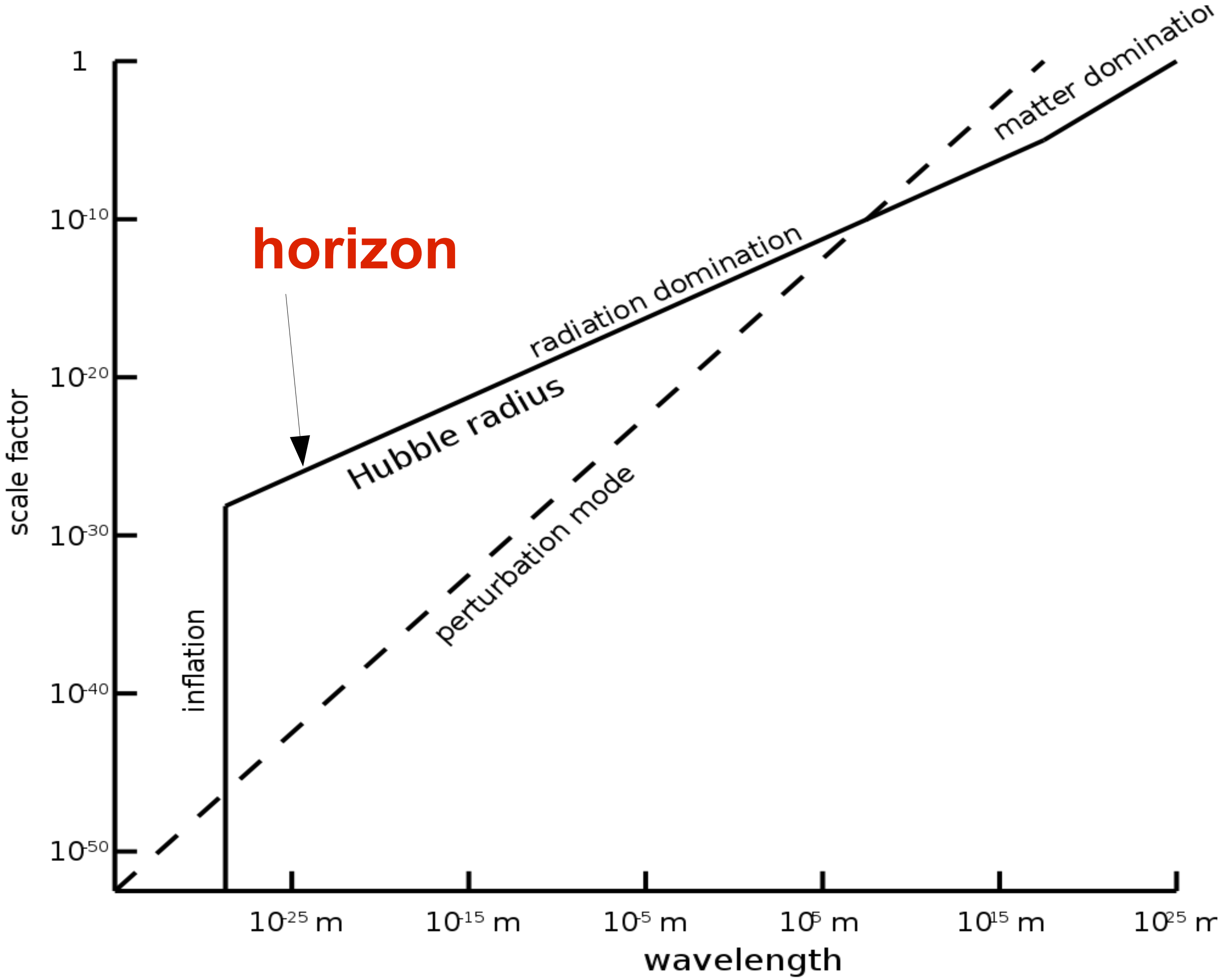


The physical size of the Hubble radius (solid line) as a function of the linear expansion (scale factor) of the universe. During cosmological inflation, the Hubble radius is constant. The physical wavelength of a perturbation mode (dashed line) is also shown. The plot illustrates how the perturbation mode grows larger than the horizon during cosmological inflation before coming back inside the horizon, which grows rapidly during radiation domination.

Primordial density perturbations will subsequently be modified by:
a) growth by self-gravitation, b) the effect of pressure,
c) dissipative processes.

Preamble: Fourier analysis of density fluctuations:

It is often convenient to consider building up a general field by the superposition of many modes. The natural tool for achieving this is via Fourier analysis. In three dimensions, the forward and inverse Fourier transforms of a field F are:

$$\delta(\mathbf{x}) = V \int \frac{d^3k}{(2\pi)^3}\, \delta(\mathbf{k})\, e^{-i\mathbf{k}\cdot\mathbf{x}}$$

$$\delta(\mathbf{k}) = \frac{1}{V} \int d^3x\, \delta(\mathbf{x})\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

Primordial density perturbations will subsequently be modified by a variety of physical processes: growth under self-gravitation, the effect of pressure, and dissipative processes. The effect is summarized in the transfer function:

$$T_k \equiv \frac{\delta_k(z=0)}{\delta_k(z)D(z)}$$

D(z) is the growth factor between redshift z and present

The form of the transfer function depends on the type of fluctuation (adiabatic or iso-curvature) and on the matter content of the universe. Consider a flat (k = 0) FRW model, where $\rho=\rho_{crit}$ and

$$H^2= 8\pi G/3\ \rho$$

A spherical region of enhanced density ρ' > ρ within this flat universe will also expand with the same Hubble law if its size exceeds the particle horizon. The dynamics of the spherical density fluctuation depend only on the mass contained within (recall Birkhoff's theorem), so it evolves as a separate entity, like a miniature closed Universe according to:

$$H^2 = \frac{8\pi G}{3}\rho' - \frac{kc^2}{a^2},$$

with k>0. Subtracting the two equations yields:

$$\delta \equiv \frac{\rho' - \rho}{\rho} = \frac{3}{8\pi G}\frac{kc^2}{a^2}\frac{1}{\rho}$$

In a matter-dominated Universe, $\rho \propto a^{-3}$ and in a radiation-dominated Universe $\rho \propto a^{-4}$, so that

$$\delta \propto \frac{a^{-2}}{\rho} \propto \begin{cases} a^2 \\ a \end{cases} \quad \text{and} \quad \delta = \delta_i \cdot \begin{cases} t/t_i & \text{in the radiation dominated era} \\ (t/t_i)^{2/3} & \text{in the matter dominated era} \end{cases}$$

Note that the growth of adiabatic perturbations is just due to the difference in the rates at which the density decreased inside and outside the fluctuation as the universe expanded.

In the adiabatic case, gravity causes the mode amplitude to increase; in the iso-curvature case the evolution acts to preserve the initial uniform density.

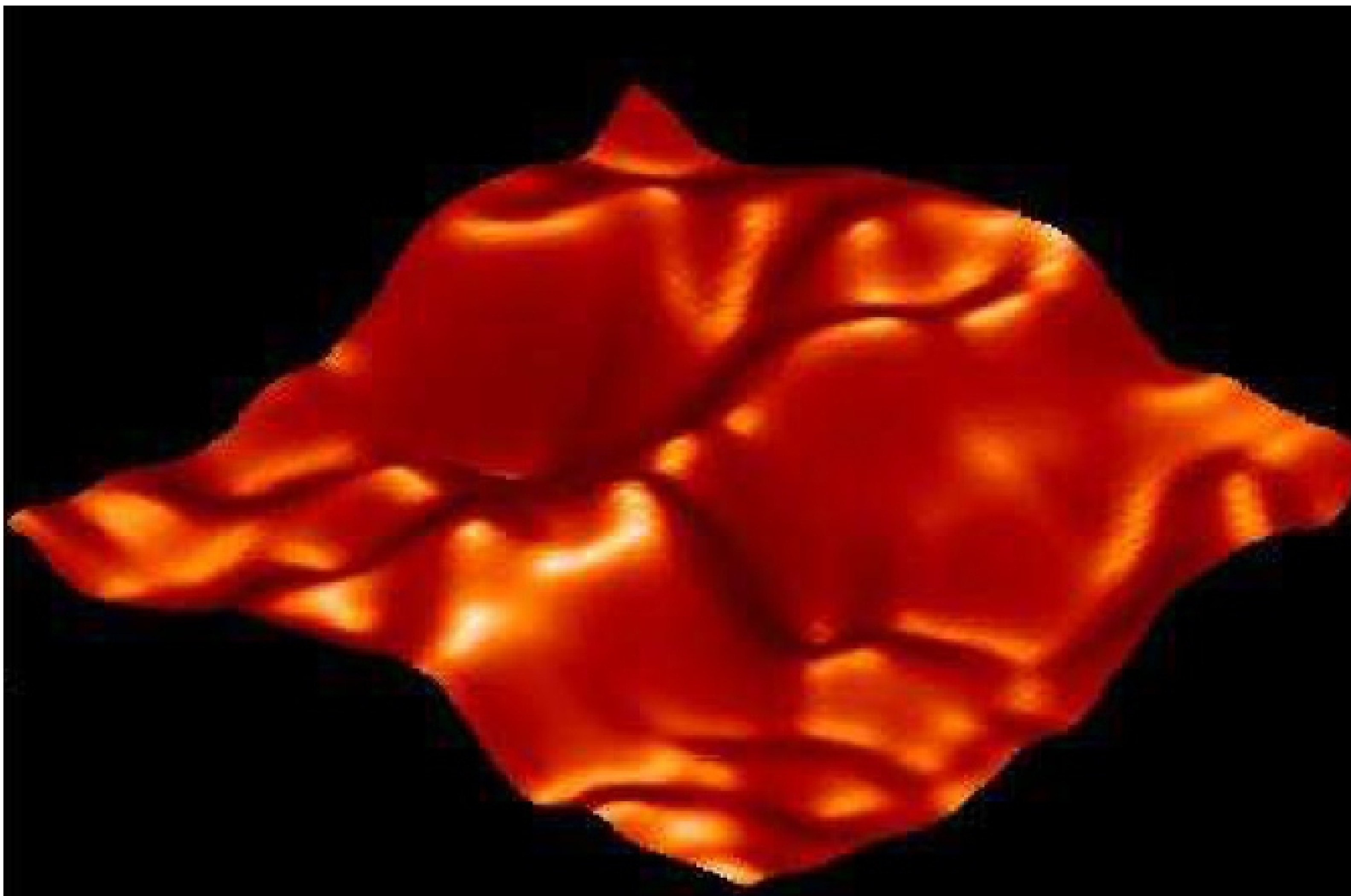
$$\delta_{\mathrm{m}} \propto \begin{cases} \text{constant} & \text{in the radiation dominated era} \\ a^{-1} & \text{in the matter dominated era} \end{cases}$$

In the standard cosmological model the density fluctuation field is assumed (with good observational support) to be a Gaussian fluctuation field, i.e., its m-point joint probability distribution obeys the multi-variate Gaussian

$$P(\delta_1, \delta_2, \ldots, \delta_m) d\delta_1 d\delta_2 \ldots d\delta_m = \frac{1}{\sqrt{(2\pi)^m \det(M)}} \exp\left[-\sum_{i,j=1}^{m} \frac{1}{2} \delta_i (M^{-1})_{ij} \delta_j\right] d\delta_1 d\delta_2 \ldots d\delta_m,$$

Here $M_{ij} = \langle \delta_i \delta_j \rangle$ is the covariance matrix and M^{-1} is its inverse.

The Gaussian nature of the primordial density field is preserved in its linear evolution stage, but this is not the case in the nonlinear stage. In the real density field, δ_i cannot be less than −1. This assumption does not make any practical difference as long as the fluctuations are small, but it is invalid in the nonlinear regime where the typical amplitude of the fluctuations exceeds unity.



On small scales and at early times, a variety of non-gravitational processes also affect the way perturbations grow:

1. Pressure

Hydrostatic equilibrium:

$$\frac{dp}{dr} = -\frac{G\rho(r) M_{enc}(r)}{r^2},$$

where $M_{enc}(r)$ is the enclosed mass, p is the pressure, and ρ(r) is the density of the gas. The equilibrium is stable if small perturbations are damped and unstable if they are amplified.

Simple argument: Consider a spherical gaseous region of radius R, mass M, and with a gaseous sound speed c_s. Imagine that we compress the region slightly. It takes a time, $t_{sound} = R/c_s$ for sound waves to cross the region and re-establish pressure balance, Gravity will attempt to contract the system even further, and will do so on a free-fall time $T_{ff} = 1/(G\rho)^{1/2}$. The condition for gravitational collapse is $t_{ff} < t_{sound}$, and one can express the length scale condition , called the Jeans length as

$$\lambda_J = c_s \sqrt{\frac{\pi}{G\rho}}$$

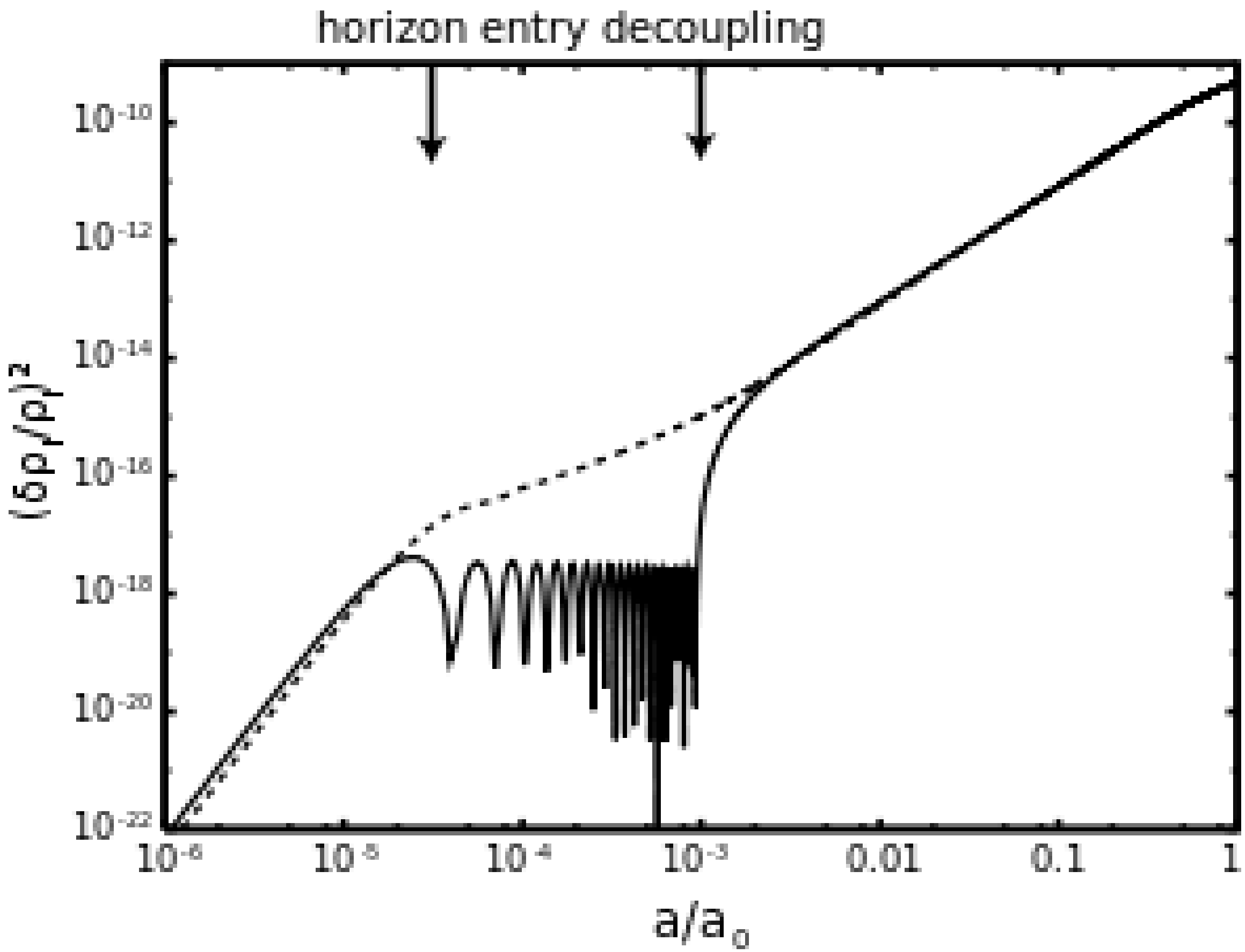
In the radiation era, the Jeans length is always close to the size of the horizon. It reaches a maximum value at the matter-radiation equality. This defines an important scale, the co-moving horizon size at z_{eq}:

$$r_H(z_{\rm eq}) \simeq \frac{16}{\Omega_{\rm m,0}h^2}\,{\rm Mpc}$$

Beyond this scale, perturbations should be affected by gravity only and we would then expect to see a bend in the spectrum of perturbations where pressure starts to become important.

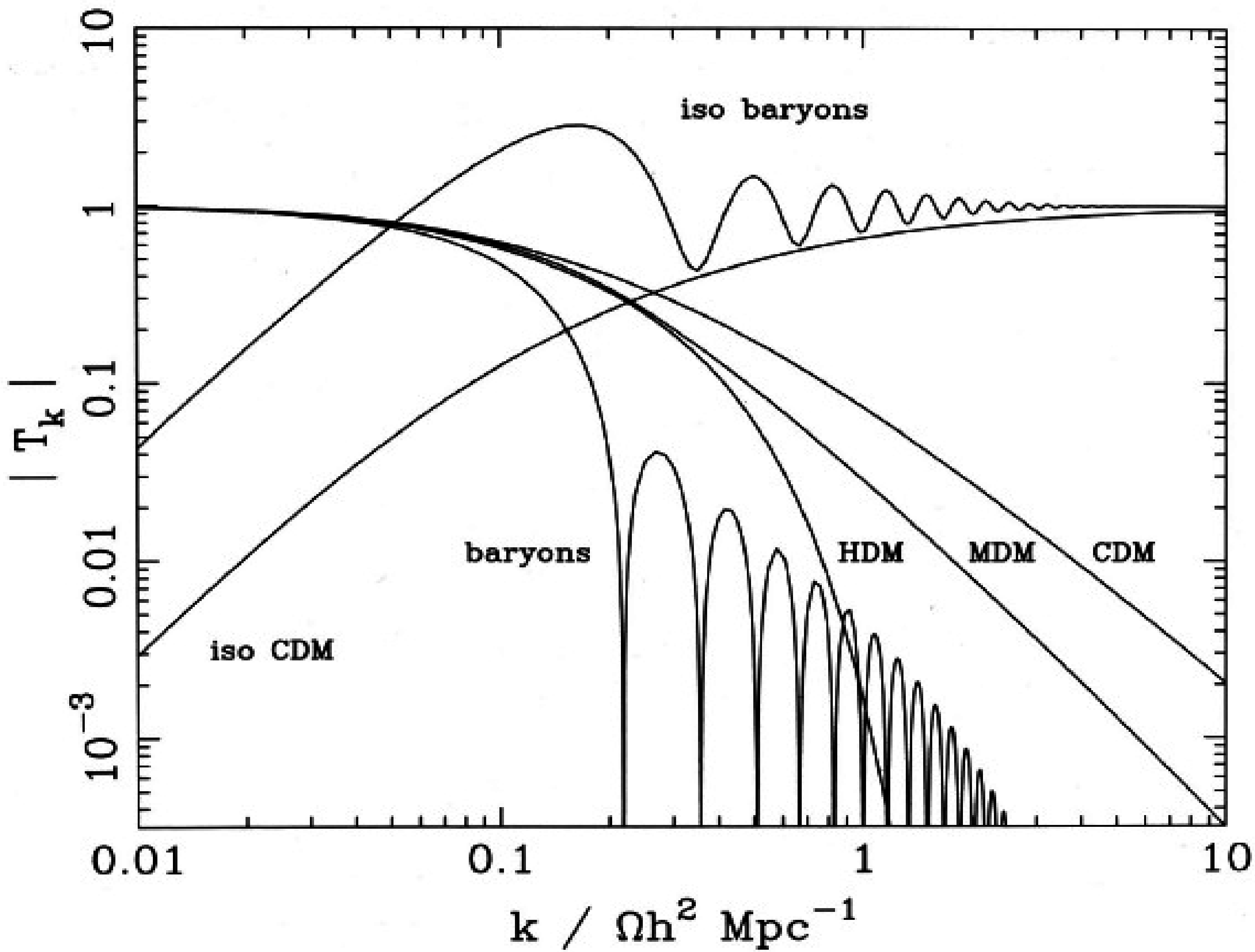
The Jeans analysis assumes a tightly coupled baryon-photon plasma. There are two situations where this is inappropriate:
1) DARK MATTER may undergo "free streaming" At early times, dark matter particles will undergo free streaming at the speed of light, and so erase all scales up to the horizon, a process which only stops when the particles go non-relativistic. This is important for neutrino-dominated models.

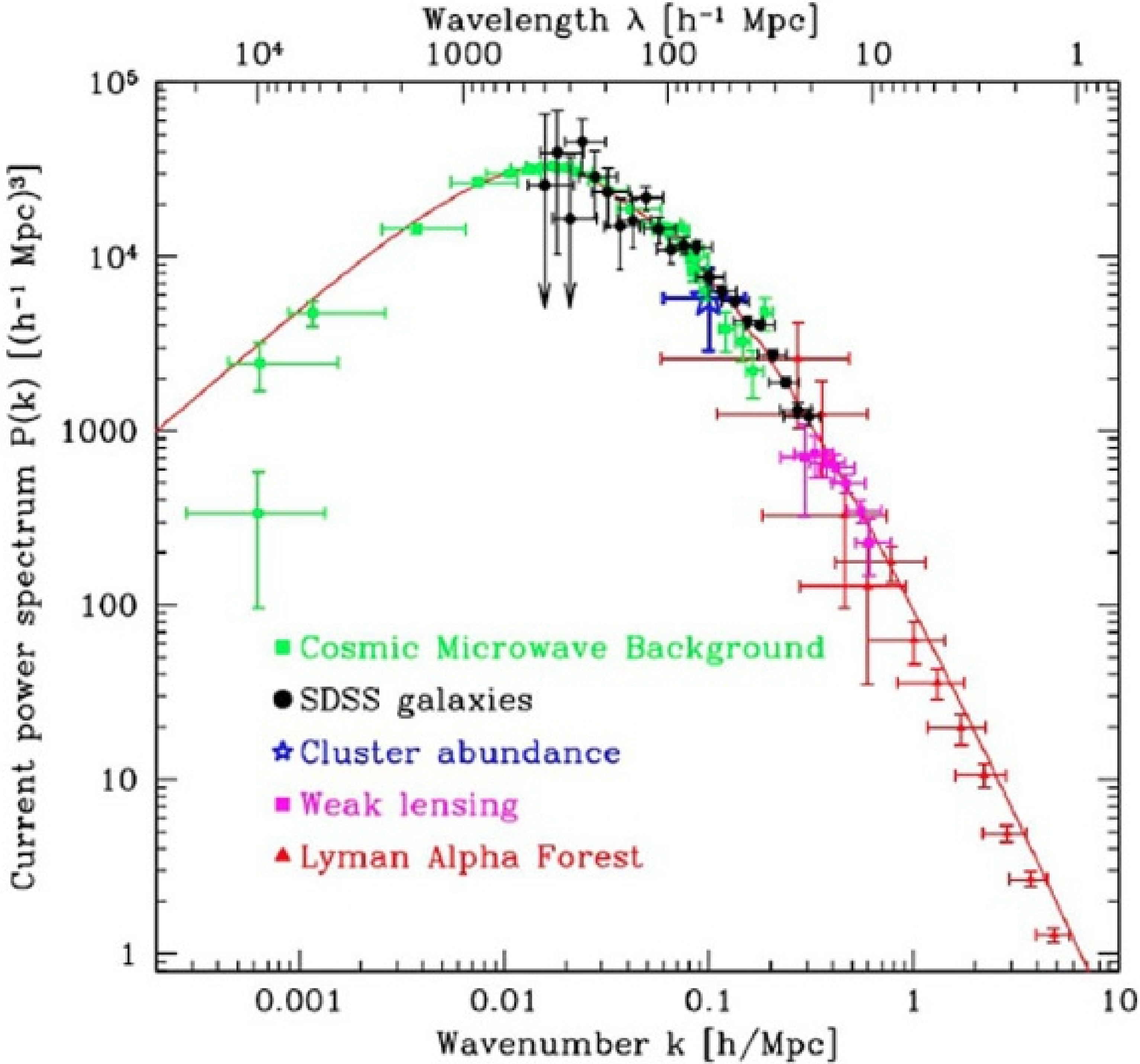
2) At early times the plasma is very optically thick, but as the universe expands the mean free path for photons increases and the photons tend to leak out of the sound waves and then damp out, smoothing the inhomogeneities in the photon-baryon fluid. This process is termed Silk damping (not important in DM-dominates models)



Evolution of two perturbations to the ΛCDM homogeneous big bang model. Between entering the horizon and decoupling, the dark matter perturbation (dashed line) grows logarithmically, before the growth accelerates in matter domination. On the other hand, between entering the horizon and decoupling, the perturbation in the baryon-photon fluid (solid line) oscillates rapidly. After decoupling, it grows rapidly to match the dominant matter perturbation, the dark matter mode.

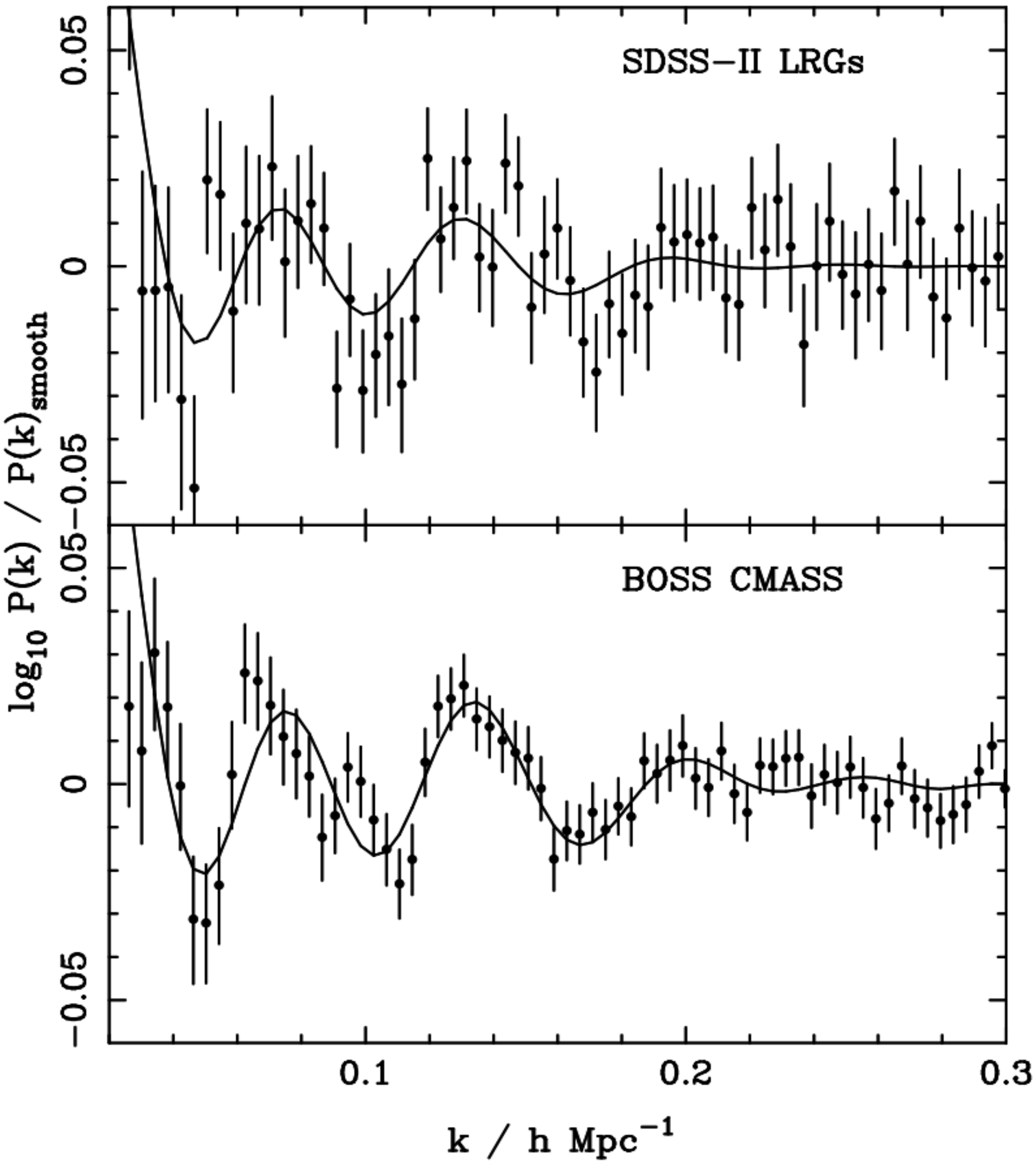
Transfer function for different models





Wavelength λ [h⁻¹ Mpc]
10⁴
1000
100
10
1
10⁵
10⁴
1000
100
10
1
Current power spectrum P(k) [(h⁻¹ Mpc)³]
Cosmic Microwave Background
SDSS galaxies
Cluster abundance
Weak lensing
Lyman Alpha Forest
0.001
0.01
0.1
1
10
Wavenumber k [h/Mpc]

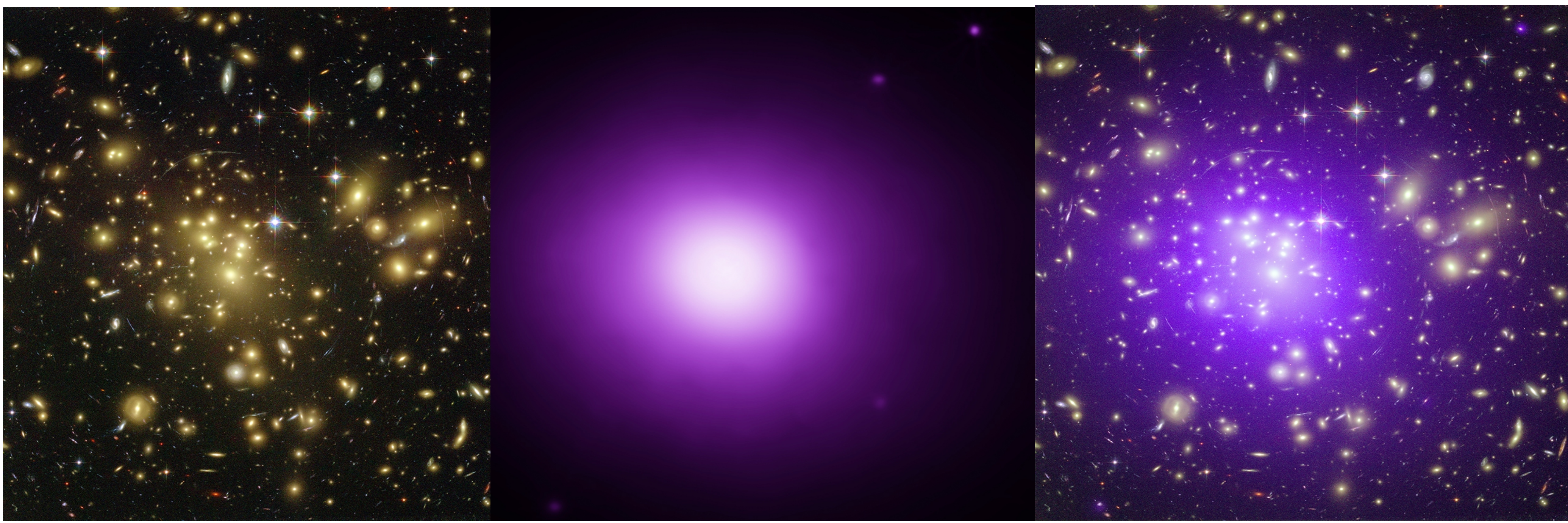
Measurement of acoustic oscillations imprinted in galaxy distribution



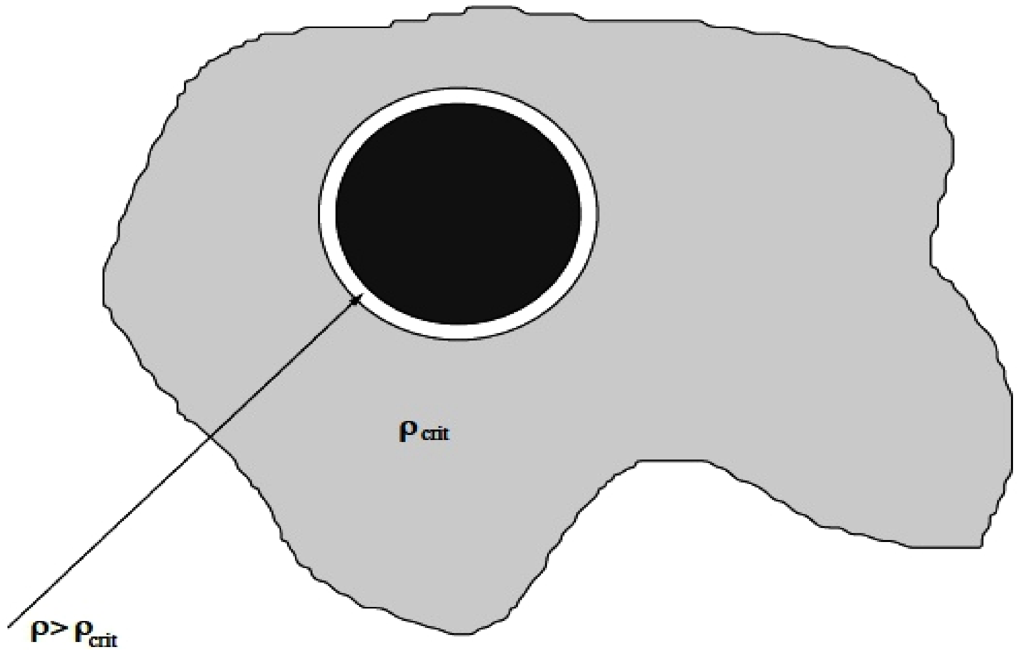
Evolution of density perturbations into the non-linear ($\delta\rho/\rho > 1$) regime

Two approaches: APPROXIMATE METHODS and NUMERICAL SIMULATIONS

1. SPHERICAL COLLAPSE



The galaxy cluster Abell 1989 seen in optical light and in X-rays



Consider the idealised case of
a spherical volume where the density is infinitesimally higher than the cosmic mean.

Our density perturbation will then evolve like a closed universe with $\Omega_m = 1 + \delta$. As we saw in lecture 2, the scale factor a(t) of such a universe reaches a maximum value a_{max} and then decreases again—in other words, our perturbation will grow to a maximum size $r=r_{max}$ at time $t=t_{max}$ and then collapse.

The Friedmann equation for a closed Universe:

$$\frac{1}{a}\frac{da}{dt} = H_0\left(\Omega_{\mathrm{m},0}a^{-3} + (1-\Omega_{\mathrm{m},0})a^{-2}\right)^{1/2}$$

has a parametric solution in terms of a development angle θ:

$$\theta = H_0\,\eta\,(\Omega_{\mathrm{m},0} - 1)^{1/2}$$

so that $$r(\theta) = A(1-\cos\theta)$$

and $$t(\theta) = B(\theta - \sin\theta)$$

with $$A = r_0\frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0}-1)}; \quad B = \frac{1}{H_0}\frac{\Omega_{\mathrm{m},0}}{2(\Omega_{\mathrm{m},0}-1)^{3/2}}$$

the development angle θ is a scaled form of the ‘conformal time’ (the time travelled by a photon since the Big Bang)

The maximum size which the perturbation will grow is given by

$$\frac{dr}{d\theta} = A\sin\theta = 0$$

Which is satisfied at $\theta = 0$, π and 2π. $\theta = \pi$ corresponds to the time of turn-around when the over-density reaches its maximum size before collapsing. At this time $t = t_{max}$, we have

$$r_{\mathrm{max}} = 2A = r_0 \frac{\Omega_{\mathrm{m},0}}{\Omega_{\mathrm{m},0} - 1}$$

and, more generally

$$\frac{r}{r_{\mathrm{max}}} = \frac{1}{2}(1 - \cos\theta)$$

$$t_{\mathrm{max}} = t(\pi) = \pi B; \quad H_0 t_{\mathrm{max}} = \frac{\pi}{2}\frac{\Omega_{\mathrm{m},0}}{(\Omega_{\mathrm{m},0} - 1)^{3/2}}$$

and

$$\frac{t}{t_{\mathrm{max}}} = \frac{1}{\pi}(\theta - \sin\theta)$$

The constants A and B are related through the enclosed mass

$$M = \frac{4\pi}{3} r_0^3 \Omega_{\mathrm{m},0} \rho_{\mathrm{crit}} = \frac{4\pi}{3} r_0^3 \Omega_{\mathrm{m},0} \frac{3H_0^2}{8\pi G}$$

by the simple relation $A^3 = GMB^2$

In the linear regime, we can follow the growth of the perturbation by using the Maclaurin expansions for $\cos\theta$ and $\sin\theta$, to yield

$$\lim_{\theta \to 0} r(\theta) = A \left(\frac{1}{2}\theta^2 - \frac{1}{24}\theta^4 \right)$$

$$\lim_{\theta \to 0} t(\theta) = B \left(\frac{1}{6}\theta^3 - \frac{1}{120}\theta^5 \right)$$

The leading order, $r = A\theta^2/2$ and $t = B\theta^3/6$, just gives the expansion of the background (i.e. outside the volume including the overdensity) universe where

$$r = a = \frac{A}{2} \left(\frac{6t}{B} \right)^{2/3}$$

Our overdensity grows according to the equations:

$$\frac{r}{r_{\max}} \simeq \frac{\theta^2}{4} - \frac{\theta^4}{48}, \qquad \frac{t}{t_{\max}} \simeq \frac{1}{\pi}\left(\frac{\theta^3}{6} - \frac{\theta^5}{120}\right)$$

which can be combined to give the linearised scale factor of our closed Universe:

$$\frac{a_{\rm lin}}{a_{\max}} \simeq \frac{1}{4}\left(6\pi\frac{t}{t_{\max}}\right)^{2/3}\left[1 - \frac{1}{20}\left(6\pi\frac{t}{t_{\max}}\right)^{2/3}\right]$$

Again, the first term is just the expansion of the background in a flat matter dominated universe. Including both terms in the square brackets gives the linear theory expression for the growth of a perturbation.

In matter-dominated universes, the mass-energy density varies as a^{-3}. Throughout the evolution of the perturbation, the following relation holds:

$$1 + \delta_{\mathrm{lin}} = \left(\frac{a_{\mathrm{back}}}{a_{\mathrm{lin}}}\right)^3$$

Substituting this into the previous equation where a_{back} is given by the leading order term, and with the substitution $(1 + \delta)^{-1/3} \sim 1 - 1/3\,\delta$ valid for $\delta \ll 1$, we have:

$$\delta_{\mathrm{lin}} = \frac{3}{20}\left(6\pi \frac{t}{t_{\mathrm{max}}}\right)^{2/3}$$

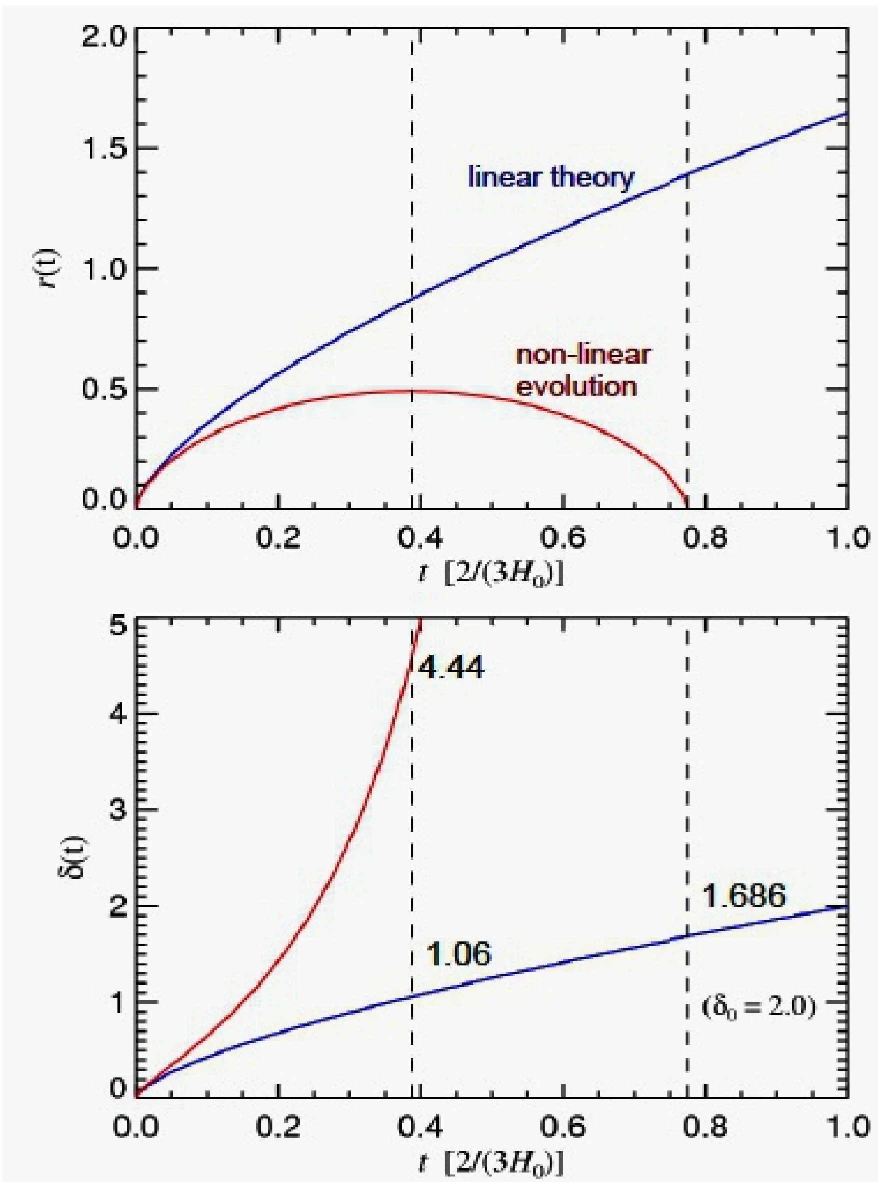
So at $t = t_{max}$: $$\delta_{\mathrm{lin}}^{\mathrm{turn}} = \frac{3}{20}(6\pi)^{2/3} = 1.06$$

Of course, turnaround also represents the breakdown of linear theory. The actual nonlinear density contrast at turnaround is

$$1 + \delta_{\mathrm{nonlin}}^{\mathrm{turn}} = \left(\frac{a_{\mathrm{back}}}{a_{\mathrm{max}}}\right)^3 = \left[\frac{1}{4}\left(6\pi \frac{t}{t_{\mathrm{max}}}\right)^{2/3}\right]^3 = \frac{(6\pi)^2}{4^3} = 5.55$$

This again just considers leading order terms.

After turnaround, the evolution of the overdensity mirrors the expansion phase until the object collapses at $t = 2t_{max}$. At this time the linear density contrast has become



$$\delta_{\rm lin}^{\rm coll} = \delta_{\rm c} = \frac{3}{20}(12\pi)^{2/3} = 1.686$$

Thus, a linear density contrast δ_C ~1.7 corresponds to the epoch of complete gravitational collapse of a spherically symmetric perturbation.
This value of δc~ 1.7 is used in analytical treatments of the growth of structure in the universe, such as the Press-Schechter formalism.