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DYNAMICAL SYMMETRY BREAKING*,**

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The aim of these lectures is to present a description of dynamical symmetry breaking that closely parallels spontaneous symmetry breaking via scalar fields. The link is provided by the effective potential that can be defined whether or not elementary scalar fields are present in the theory. I wish to show that the effective potential is calculable in a non-trivial approximation to a gauge theory.

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1. *Introduction*

Spontaneous symmetry breaking (SSB) is an essential ingredient in our current understanding of elementary interactions. The now familiar phenomenon of SSB via scalar fields, as in σ models and Higgs potentials, is a subject of elegant simplicity. Finding the minimum of a potential function in order to determine the properties of the ground state is basic to our intuition about mechanical systems. However in current fundamental theories, elementary scalars either do not occur — as in QCD — or are introduced solely to generate SSB — as in electro-weak and grand unified theories. Scalar fields are not required for self consistency as for example the gauge fields are. The alternative is to look for the source for SSB in the dynamics of the gauge theories themselves i.e. dynamical symmetry breaking (DSB) [1, 2, 3]. Unfortunately DSB does not occur in a simple approximation in gauge theories. Infinite sets of graphs must be summed which can be conveniently stated as Schwinger-Dyson and Bethe-Salpeter equations. But generally these are complicated integral equations and little is known about their solutions. In spite of these technical complications similar principles govern these two phenomena. My plan for these lectures is to describe the latter in a way that closely parallels the former. The link is provided by the effective potential which can be defined whether or not elementary

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scalar fields are present in the theory. This brings our simple intuition about σ models to bear on the technical problems of DSB.

Specifically I plan to direct my lectures toward results of a recent paper by Perez--Mercader and myself [4] in which we show that the calculation of the effective potential is tractable in a widely studied model of DSB. The model is chiral invariant QED in four dimensions. The symmetry which is broken is chiral U(1) giving the fermion a mass. The approximation is a truncation of Feynman graphs which is rich enough to give DSB yet simple enough to allow an exact solution to the Schwinger-Dyson and Bethe-Salpeter equations and a closed form expression for the effective potential.

The basic mechanism of SSB of the Goldstone or Higgs type is the presence of an instability in the normal vacuum state, i.e. the vacuum state that is invariant under the symmetries of the Lagrangian. Suppose for a moment that scalar fields are present and that scalar sector of the Lagrangian is as follows:

$$\mathscr{L}_{\text{scalar}} = \tfrac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{1.1}$$

If μ^2 is taken to be negative then the classical field configuration $\phi = 0$ is not stable since the energy can be lowered by increasing ϕ. We refer to this as a tachyonic instability since imaginary mass excitations (tachyons) lead to exponentially growing field fluctuations. If λ is also negative then the theory has no ground state (at least classically). Alternatively a positive λ would lead to a ground state in which ϕ takes on the value $\pm(-6\mu^2/\lambda)^{1/2}$. The $\phi \to -\phi$ symmetry of the Lagrangian is broken by this ground state.

There need not be a scalar field present to generate a tachyonic instability since the tachyon could also be a bound state of elementary fermion fields for example. A bound state tachyon would mean that a certain $\bar{\psi}\psi$ field configuration is unstable. I wish to stress in these lectures that there is a calculable "first approximation" to DSB as there is for the scalar case, the latter being the tree approximation. These techniques I will discuss are applicable to problems in dynamical Higgs models, such as heavy color [5], and possibly to gauge hierarchy and the fermionic mass spectrum in grand unified theories [6]. However I will discuss here no symmetries beyond U(1) $\times$ U(1) and will stress only the "first approximation" to DSB.

Much of the work on the dynamics of DSB can be classified as studies of quartic interactions [1, 7], studies of non-linear Schwinger-Dyson equations for gauge and other trilinear interactions [8], studies of the stability of various phases close to a phase transition [9, 10] more recently the linking of chiral breaking to confinement [11] and of course the connection between the dynamical breaking of γ_5 invariance in QCD and PCAC [3, 12]. There have also been studies of the solutions of the Schwinger-Dyson equations [13]. It is on the latter that all the peculiarities of the non-linear problems enter.

My plan for these lectures is as follows: (2) I would like to give some background and then give the results of the paper by Mercader and myself [4] in order to clarify the direction of these lectures. (3) Then I will give a short review of the effective potential and Legendre transforms for the scalar field case. (4) Next I would like to show how DSB occurs without introducing any new formalism. This is a pedagogical digression to show how a negative mass-square bound state signals a phase transition. (5) I will present next the main formal developments. This consists of selecting appropriate classical source terms and defining the corresponding generating functionals. (6) There are two theorems on Legendre transforms that are needed in order to calculate the generalized effective potential and I will give them here. (7) We need solutions of Schwinger-Dyson (SD) and Bethe-Salpeter (BS) equations which fortunately are exactly soluble and are given here. (8) Finally I will give conclusions and discuss limitations, generalizations and speculations.

2. *Background*

Chiral symmetry breaking is the example I will discuss in these lectures and I would like to contrast here two early models of this: The Gell-Mann Levy σ model [14] which involves a scalar field and the Nambu Jona-Lasinio model [1] that does not. The σ model is very familiar to everybody but I would still like to review it to facilitate comparisons later.

(*i*) Gell-Mann Levy σ model

Consider the following Lagrangian involving fermions ψ, $\bar{\psi}$ interacting with a spin zero chiral doublet (π, σ)

$$\mathscr{L} = \bar{\psi}(i\gamma\cdot\partial - g(\sigma + i\gamma_5\pi))\psi + \tfrac{1}{2}((\partial_\mu\sigma)^2 + (\partial_\mu\pi)^2) - V(\sigma, \pi). \tag{2.1}$$

This is invariant under the chiral transformation:

$$\psi' = e^{i\frac{\alpha}{2}\gamma_5}\psi$$
$$\begin{pmatrix}\sigma' \\ \pi'\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}\sigma \\ \pi\end{pmatrix}, \tag{2.2}$$

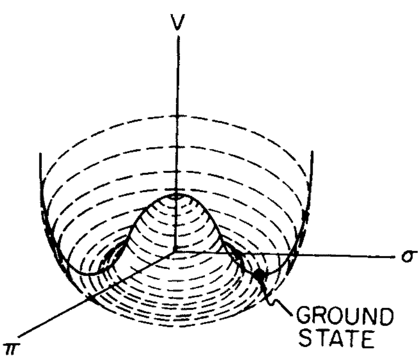

Fig. 1. An effective potential giving spontaneous symmetry breaking

where V is a function of the chiral invariant $\sigma^2 + \pi^2$. If we pick the simple form for $V = \frac{m^2}{2}(\sigma^2+\pi^2) + \frac{\lambda}{4!}(\sigma^2+\pi^2)^2$ the essential features of chiral breaking are very transparent. Ignoring the fermions for the moment and treating the meson fields classically we then wish to find the lowest energy configuration. The energy can be lowered by taking the field to be constant in space and time. The minimum energy configuration is determined by the potential $V(\sigma, \pi)$. If m^2 is positive then the minimum of V occurs at the origin and $\sigma = \pi = 0$ is the ground state. If one picks $m^2 < 0$ then the origin in the σ, π plane is an unstable stationary point and we have the situation depicted in Fig. 1. Any fluctuations of the field will grow exponentially and the system will pick a new ground state out of the continuum of minima on the circle $\sigma^2 + \pi^2 = \dfrac{-6m^2}{\lambda}$. We can make a chiral transformation if necessary to put the point of the σ axis.

A few simple results follow: There is clearly a zero frequency normal mode corresponding to displacements around the symmetry axis which is the massless pseudoscalar Goldstone mode. There is a "radial" mode corresponding to the massive scalar particle. Further note that the fermion picks up a mass given by $g\sigma$. This qualitative behavior of the spectrum is plotted in Fig. 2.

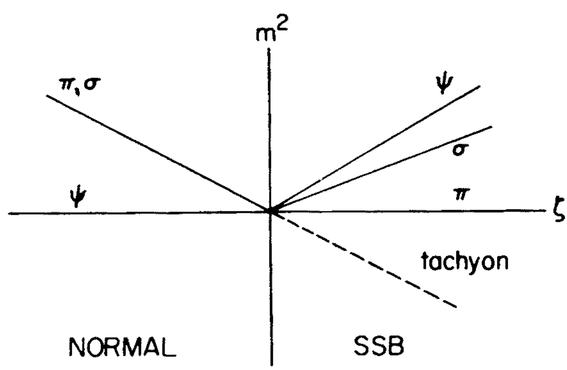

Fig. 2. Qualitative behaviour of the spectrum for spontaneous chiral breaking. For the σ model $\zeta = -\mu^2$. For the Nambu Jona-Lasinio model $\zeta = -1 + \dfrac{g\Lambda^2}{2\pi^2}$

(*ii*) Nambu Jona-Lasinio model

Consider now a Lagrangian that involves only fermion fields which have the same chiral transformation properties as above

$$\mathscr{L} = \bar{\psi} i\gamma \cdot \partial \psi + g\{(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\}. \tag{2.3}$$

This model has essentially the same features as the σ model but now the instability is dynamical in origin. The properties are less apparent and I will just give the results. In the $\bar{\psi}\ \psi$ channel the Bethe-Salpeter equation reveals a chiral doublet resonance with the quantum number of the σ and π as above. (Because the fermion constituents are massless prior to symmetry breaking the σ and π can decay and hence they are not bound states but rather are resonances.) As one increases the coupling constant, g, these states become tachyons and the qualitative features of the spectrum are identical to those in Fig. 2. The fermion mass arises from a Schwinger-Dyson equation shown in Fig. 3:

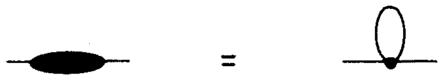

Fig. 3. Generation of a mass term (on left) from a four point interaction in the Nambu Jona-Lasinio model

$$M = 4g \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{M}{k^2+M^2} . \tag{2.4}$$

This equation has a trivial solution, $M = 0$, and a "self consistent" solution given implicitly by

$$1 = 4g \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{k^2+M^2} . \tag{2.5}$$

The lowest energy solution switches from one to the other as $(g\Lambda^2/2\pi^2)-1$ changes sign as shown in Fig. 2. The presence of a tachyon in the normal vacuum correctly signals that it is not the state of lowest energy just as it does in the σ model. Unlike the σ model this chiral breaking is dynamical in origin. It is a consequence of the strong attractive forces and not the result of putting an instability in by hand. The lowest order SD and BS equations in this model reduce to algebraic equations because the interaction is quartic. These become integral equations in the case of trilinear interactions and in general are not soluble in closed form.

I would like to give a preview of the main result I will present in order to clarify where the discussion is leading. The model is massless QED in four dimensions which of course involves trilinear interactions. There are no scalar fields. This model also exhibits DSB. It has resonances which can become tachyonic and drive the instability. I will show that there is a quantity which I call the generalized effective potential, $V(\phi_1, \phi_2, \ldots)$ which serves the same role for this model as $V(\sigma, \pi)$ does for the σ model. The arguments $\phi_1, \phi_2, \ldots$ correspond to the bound states of $\bar{\psi}$ and ψ. This function is calculable in a "first approximation" and has a relatively simple form:

$$V(\phi_a) = -i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} \ln \left(\frac{\sum_b \phi_b F_b}{\mu^2} \right) + \frac{1}{\mu^2} \sum_b \left\{ i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} (S_0 F_b S_0 \gamma \cdot k) \phi_b - \frac{g^2 \phi_b^2}{2 g_b^2} \right\}, \tag{2.6}$$

where $S_0(k)$ is the fermion propagator and is the solution of a soluble SD equation, $F_n(k)$ and g_n are the normalized bound state eigenfunctions and eigenvalues of a soluble BS equation, g is the coupling constant, and μ^2 is a scale parameter which I will explain later. This function, Eq. (2.6), signals symmetry breaking in exactly the same way as $V(\sigma, \pi)$ does for scalar theories as in Fig. 1. In this way we are able to determine which of the candidate vacuum states that we may find correspond to the field configuration of lowest energy.

Before going on I would like to show that SSB in the σ model can be viewed as an effect arising from summing an infinite subset of Feynman graphs. Since this is a useful way to view the approximations which give DSB it is worthwhile to see how the effect works for this trivial case.

Consider the Lagrangian, Eq. (2.1) and add to it an external constant source term $J\sigma(x)$. (This extra term is just a useful device in the functional formalism and will be set to zero at the end.) Looking only at the σ sector:

$$V \to V + J\sigma = \frac{m^2}{2}\sigma^2 + \frac{\lambda}{4!}\sigma^4 + J\sigma. \tag{2.7}$$

With the external source on, the vacuum properties are determined by the stationary points of $V + J\sigma$:

$$\sigma = -\frac{1}{m^2}\left(J + \frac{\lambda}{6}\sigma^3\right). \tag{2.8}$$

This may be viewed as a truncated non-linear Schwinger-Dyson equation for the one-point function $\sigma = \langle \sigma_{\mathrm{op}} \rangle$. By truncated, I mean that it sums a subset of all one-point-function

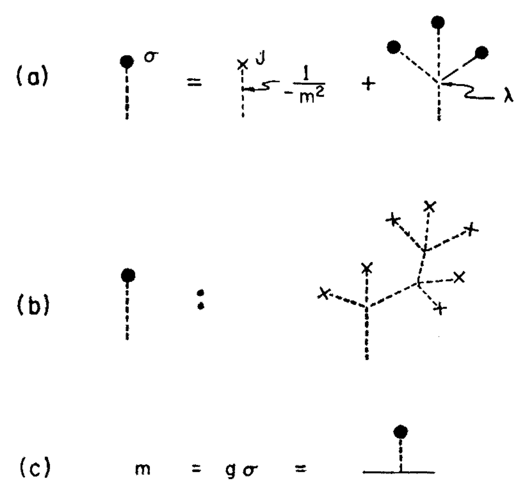

Fig. 4. (a) Graphical depiction of Eq. (2.8) for the σ model, (b) a representative graph, (c) fermion mass term

graphs — in fact tree graphs only, as shown in Fig. 4. The representative graph shown there can be obtained by iterating Eq. (2.8). Finally recall that the fermion mass is given by $m = g\sigma$ which is shown in Fig. 4c.

It is clear from Eqs. (2.7), (2.8) that when you turn off J, σ goes to either zero or $(-6m^2/\lambda)^{1/2}$ depending on which corresponds to the minimum of V. Looking at this result graphically, in Fig. 4 note that all the graphs are proportional to a power of J yet when J goes to zero the sum need not be zero. In our generalization to DSB this is precisely what we will look for: We will introduce a generalized source term which violates chiral invariance. As we turn it off we will look for solutions to SD equations that have a residual chiral breaking. The criterion to decide if such a solution is the one of lowest energy is to see if the corresponding stationary point of V is a minimum. The SD equation will sum subsets of graphs in the fermion mass term and the important property of that subset is that it contains *bound state poles* which yield the structure of *scalar field poles* shown in Fig. 4. In essences the bound states play the identical role of the scalar field in the σ model.

3. *The effective potential, scalar fields*

The classical potential function that determines the ground state symmetry, as in the σ model example in the last section, has a generalization to the full quantum theory and is called the effective potential. Here I will briefly review the concepts that are needed for later generalizations. For a more complete review, see for example Ref. [15].

In the functional formulation, it is useful to introduce external classical sources coupled to the fields. I will restrict these sources to be constant in space and time throughout these lectures. For the case of a single scalar field, we add to the lagrangian the following source term:

$$\mathscr{L} \to \mathscr{L}(\Phi, \partial_\mu \Phi) + J\Phi. \tag{3.1}$$

The vacuum to vacuum generating functional is then given by the functional integral:

$$\langle 0|0\rangle_J = e^{iW(J)\int d^4x} = \int [d\Phi] e^{i\int d^4x(\mathscr{L}+J\Phi)}. \tag{3.2}$$

$W(J)$ thus defined generates the n-point connected Green's function in momentum space at zero momentum.

$$W(J) = \sum_{n=1}^{\infty} \frac{J^n}{n!} G^{(n)}(0, 0, \ldots). \tag{3.3}$$

(I will not discuss the more general generating functional $W[J(x)]$ which replaces $W(J)\int d^4x$ in Eq. (3.2) and which gives Green's functions at finite momenta.) One can now define a classical field ϕ which is conjugate to the classical source J and the effective potential is just the Legendre transform of $W(J)$:

$$\phi = \frac{dW}{dJ} = \frac{\langle 0|\Phi|0\rangle}{\langle 0|0\rangle}, \tag{3.4}$$

$$V(\phi) = \phi J - W. \tag{3.5}$$

$V(\phi)$ has a simple graphical interpretation: $-V(\phi)$ is the generating function of one particle irreducible graphs at zero momentum

$$V(\phi) = -\sum_n \frac{\phi^n}{n!} \Gamma^{(n)}(0, 0, \ldots). \tag{3.6}$$

To see this it is worthwhile to expand out these functions in a Taylor series. The Taylor series for $V(\phi)$ is obtained by expanding the Legendre transform, Eq. (3.5).

$$\frac{dV}{d\phi} = J + \phi \frac{dJ}{d\phi} - \frac{dJ}{d\phi}\frac{dW}{dJ} = J, \tag{3.7a}$$

$$\frac{d^2V}{d\phi^2} = \frac{dJ}{d\phi} = \left\{\frac{d^2W}{dJ^2}\right\}^{-1} = D^{-1}, \tag{3.7b}$$

$$\frac{d^3V}{d\phi^3} = -D^{-2}\frac{dD}{dJ}\frac{dJ}{d\phi} = -D^{-3}\frac{d^3W}{dJ^3}, \tag{3.7c}$$

$$\frac{d^4V}{d\phi^4} = 3D^{-5}\left(\frac{d^3W}{dJ^3}\right)^2 - D^{-4}\frac{d^4W}{dJ^4}. \tag{3.7d}$$

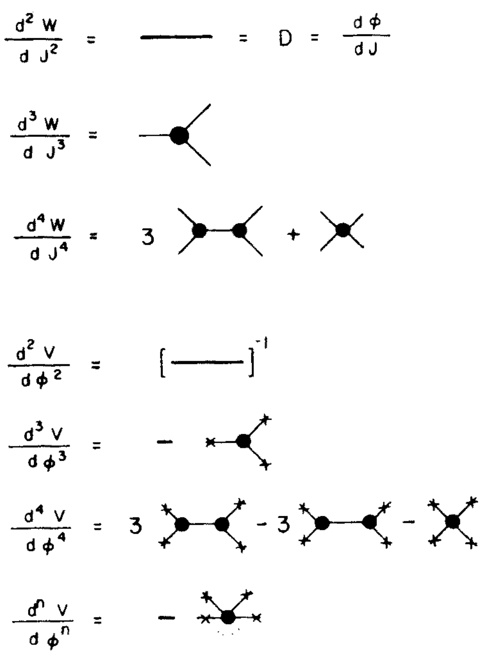

Fig. 5. Expansion of W and V for the scalar field case. Crosses mean to remove the propagator

The effect of the Legendre transform is shown in Fig. 5. The graphs in W have been classified as those with one-particle states and those without, see for example d^4W/dJ^4. The graphs in V have their external propagators removed and the one particle reducible graphs cancelled as in $d^4V/d\phi^4$ leaving just the one-particle-irreducible vertices as shown on the bottom line. As a check one can easily see that if one restricts the connected Green's functions to be just tree graphs, (classical approximation), then the effective potential is just the negative of the non-derivative terms in the Lagrangian which is just the classical potential. Let us note how the behavior of $V(\phi)$ is related to the question of stability of the vacuum. From Eq. (3.7a) it follows that

$$J = 0 \Rightarrow \frac{dV}{d\phi} = 0,$$

and thus V is stationary in ϕ at the point corresponding to $J = 0$. Next notice that from Eq. (3.7b) a positive curvature, $d^2V/d\phi^2 > 0$, corresponds to a positive value of the inverse propagator at zero momentum. In our convention, that corresponds to a positive mass--square pole in the propagator, i.e. a real particle as indicated in Fig. 6. Alternatively,

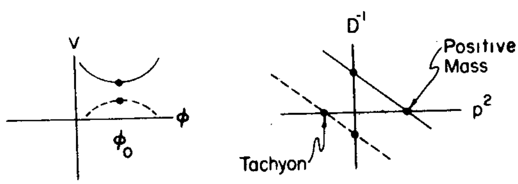

Fig. 6. A stable local minimum of V (solid line) corresponds to a positive intercept for D^{-1} and therefore a positive mass state. Likewise a local maximum of V (dashed line) corresponds to a negative intercept and therefore a tachyon

negative curvature of V corresponds to a negative mass-square pole which is a tachyon, indicating an instability as described in the introduction.

To summarize, $V(\phi)$ is a function of the *c-number* variable ϕ, (not to be confused with the *q-number* Φ which is integrated out in Eq. (3.2)). In the classical approximation it agrees with the classical potential energy function. As the external source J is turned off, ϕ goes to a stationary local minimum of $V(\phi)$. In fact the point must be the global minimum in order for the vacuum to be stable — but that is a longer story.

4. *Phase transitions*

The beauty of the effective potential is that it can display phase transitions in the full quantum theory with the same simplicity as for example, the classical σ model case as explained above. It is not as apparent, but nevertheless true, that if a bound state becomes tachyonic, $M_B^2 < 0$, where M_B is the bound state mass, due to strong binding in some channel, the minimum of $V(\phi)$ will also shift but in this case onto another branch of the function. I would like to demonstrate in this section what I mean by this and how this happens. This helps motivate the generalizations of the effective potential I will introduce later and at the same time shows that the new formalism is a convenience, not a necessity, for studying dynamical symmetry breaking.

Let us consider a massive ϕ^4 theory. For weak coupling, the effective potential reduces to the classical or tree approximation:

$$V = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4. \tag{4.1}$$

If λ and μ^2 are positive this potential has a minimum at $\phi = 0$ as indicated in Fig. 7a. Let us assume that the coupling is increased and a deep bound state of two ϕ's is formed. Even if we can not calculate $V(\phi)$ for strong coupling, one can argue that the bound state

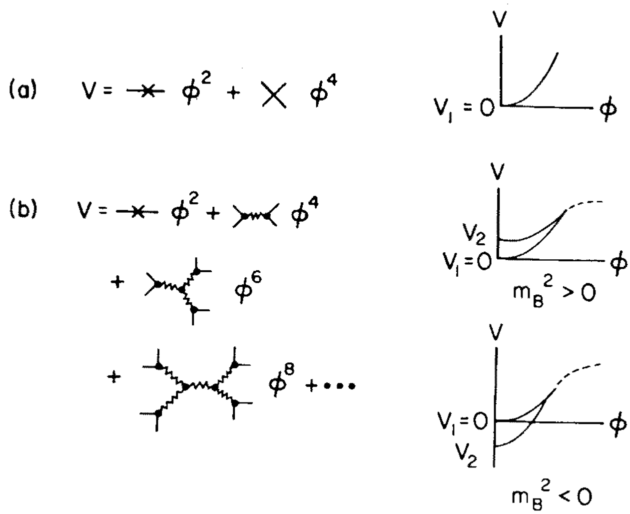

Fig. 7. The effective potential in ϕ^4 theory (a) for weak coupling, (b) for coupling such that there is a deep bound state of mass M_B. The graphs indicate the dominant bound state pole contribution to the one--particle-irreducible vertices. The resulting branch structure of V is indicated

contribution dominates the behavior of $V(\phi)$ for M_B^2 close to zero. To see this, note the following: $V(\phi)$ is the generating function for one-ϕ-particle-irreducible m point functions at zero momentum $\Gamma^{(m)}(0, 0, \ldots)$. Recall that the $\Gamma^{(m)}(p_i)$ do not have ϕ poles since these are removed by the Legendre transform. But if there are bound states the corresponding poles will occur in $\Gamma^{(m)}(p_i)$ because these are two-particle-reducible and the bound states occur in such channels. A typical pole term would be:

$$\Gamma^{(m)} = \tilde{\Gamma}^{(m-k)} \frac{1}{p_i^2 - M_B^2} \tilde{\Gamma}^{(k+2)}, \tag{4.2}$$

as shown, for example, for $\Gamma^{(8)}$ in Fig. 8. This dominates over background for $p_i^2 \approx M_B^2$. Since we are interested in this function at zero momentum, this pole term dominates over background for $M_B^2 \approx 0$. If we isolate the pole structure in Γ with the maximum number of poles, then at zero momentum it will have the maximum power of $1/M_B^2$ and will give the leading bound state contribution to $V(\phi)$.

To isolate the dominant pole structures, choose a particular pole in $\Gamma^{(m)}$ and pick up the residue as in Eq. (4.2). Then pole-dominate the residue factors and continue until all possible pole terms are isolated as indicated in the example in Fig. 8. This gives the

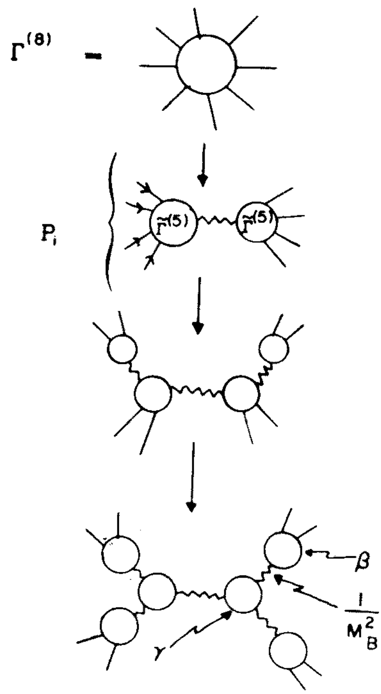

Fig. 8. Exhibiting bound state poles in the manner described in Sec. 4

dominant contribution and it clearly is just the tree structures shown in Fig. 7b. The first few terms in V are:

$$V(\phi) = \frac{\mu^2}{2}\phi^2 - \frac{3\beta^2}{M_B^2}\frac{\phi^4}{4!} - \frac{15\gamma\beta^3}{M_B^6}\frac{\phi^6}{6!} - \frac{315\gamma^2\beta^4}{M_B^{10}}\frac{\phi^8}{8!} + \ldots \tag{4.3}$$

where β is the $\phi\phi$ B coupling, γ is the BBB coupling and the combinatoric factors count the number of ways of inserting pole terms. This series can be summed to give:

$$V(\phi) = \frac{\mu^2}{2}\phi^2 - \frac{M_B^2}{3\gamma^2}\{(1-\xi)^{3/2} - 1 + \tfrac{3}{2}\xi\}, \tag{4.4}$$

where

$$\xi = \frac{\beta\gamma}{M_B^4}\phi^2. \tag{4.5}$$

The important properties of this function are shown in Fig. 7b. It has a branch point at $\xi = 1$ and is complex for ξ larger that this value. The interesting features of this result are that (*i*) there is a second branch to $V(\phi)$, (*ii*) the second branch has a minimum at $\phi = 0$, denoted by V_2 in Fig. 7b and (*iii*) the minimum V_2 drops below the old minimum V_1 as M_B^2 goes negative. Hence as the theory becomes unstable due to a composite tachyon a new vacuum state at lower energy appears. This leading bound state contribution to $V(\phi)$ is sufficient to show the effect.

Let us contrast this with the situation in which the small mass particle is elementary with a corresponding field χ. Then the effective potential would be a function of the two fields ϕ and χ of the form:

$$V(\phi, \chi) = \frac{\mu^2}{2}\phi^2 + \frac{M_B^2}{2}\chi^2 + \frac{\beta}{2}\phi^2\chi + \frac{\gamma}{3!}\chi^3, \tag{4.6}$$

where β and γ are the same as before. For $M_B^2 > 0$ a local minimum of V is $\phi = \chi = 0$. For $M_B^2 < 0$ the minimum shifts in the usual way to $\phi = 0$, $\chi \neq 0$ defining a new vacuum with positive mass excitations.

Any difference between the bound state and elementary-field cases in this discussion is illusory. The two expressions for V — Eqs (4.4) and (4.6) — are related simply by the constraint:

$$V(\phi) = V(\phi, \chi(\phi)), \tag{4.7}$$

where $\chi(\phi)$ is given by the equation

$$\frac{\partial V(\phi, \chi)}{\partial \chi} = 0; \quad M_B^2\chi + \frac{\beta}{2}\phi^2 + \frac{\gamma}{2}\chi^3 = 0. \tag{4.8}$$

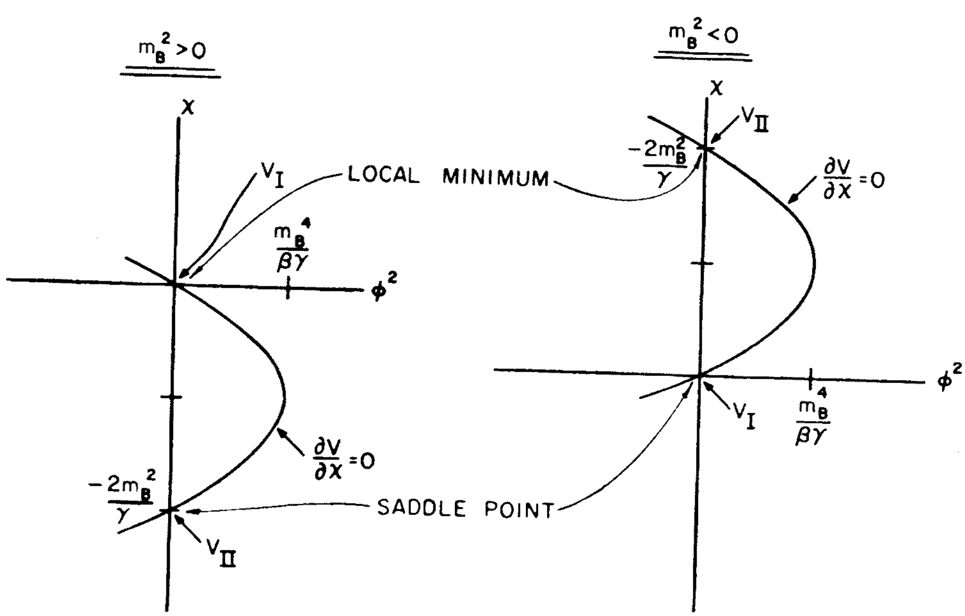

Fig. 9. The constraint curve given by Eq. (4.8) indicating how the local minimum of V shifts as M_B^2 changes sign

In Fig. 9 the parabolic constraint is shown. The function developed in Eq. (4.4) is just the value of the polynomial, Eq. (4.6), on this constraint. Because the constraint is a stationary condition in one of the variables all stationary points of $V(\phi, \chi)$ must lie on it. Hence all the stationary points of $V(\phi, \chi)$ are given by the solution to the equation $\partial V/\partial\phi = 0$. In other words no stationary points are missed by imposing the constraint and the two formulations lead to identical conclusions about the vacuum.

In this discussion so far nothing has been said about the symmetry. The generalization is clear: The effective potential, Eq. (4.6), would be an invariant function of ϕ_i and χ_a and the constraints, Eq. (4.8), would read:

$$\frac{\partial V(\phi_i, \chi_a)}{\partial \chi_b} = 0. \tag{4.9}$$

The vacuum properties are defined by the stationary points of $V(\phi_i, \chi_a)$ and the Goldstone phenomenon follows exactly as in the elementary field case.

In summary, the point of this discussion was to show that the effective potential, as defined in Sec. 3, shows the signal of a phase transition due to a bound state tachyon and further that the one-variable form, Eq. (4.4) and the two-variables form, Eq. (4.6) give identical conclusion about the vacuum state. Since Eq. (4.6) is the form of a tree order effective potential for elementary ϕ and χ fields, what then distinguishes it from the case where ϕ is elementary and χ is a bound state? At the level of this discussion, the *only* difference is that M_B^2, β, and γ are either free or calculable parameters respectively. It is gratifying to see that there is a close correspondence in spite of the many dynamical complications of the bound state problem.

As a final remark, I should say that one can include higher powers of fields in Eq. (4.6). One can easily see that this would correspond to including non-leading terms in $1/M_B^2$ in Eq. (4.3). Since the phase transition is second order it is governed by the properties of V near the origin in ϕ and χ space. The higher powers will not alter any of these conclusions as long as β and γ are non-zero. This statement is just the two dimensional generalization of the obvious statement that the position of the local minimum of a function near the origin is governed by the first few terms in a power series expansion.

5. Sources and generating functions for DSB

In Section 3 I reviewed the effective potential formalism for scalar fields. In Section 4 I showed that this potential can exhibit the phase transition caused by a tachyonic bound state. What we need now is a systematic formalism that implements these ideas. The effective potential introduced thus far, Sec. 3, is not the optimal starting point for DSB studies. The discussion of Sec. 4 suggests that we would like the arguments of $V(\phi_1, \phi_2 \ldots)$ to be classical fields representing bound states not constituents since it is the shift in the minimum of V in these variables that signal the phase transition. This can be achieved as we will see by introducing sources coupled bilinearly to the constituent fields, rather than linearly as in Eq. (3.1). Further we need to discuss dynamics, i.e. to outline a calculational scheme which is rich enough to give DSB yet simple enough to be tractible.

The model I will discuss is massless QED but first consider a massless free Dirac field. Consider the following action:

$$A = \int d^4x\bar{\psi}i\gamma \cdot \partial\psi - \int d^4xd^4y\bar{\psi}(x)J(x-y)\psi(y). \tag{5.1}$$

I have included an external source term coupled bilinearly to the fermion fields. This external sources is translation invariant and hence in momentum space it attaches a zero momentum J line to a fermion propagator $S(p)$, giving $S(p)$ $J(p)$ $S(p)$. $J(p)$ is an arbitrary 4×4 matrix function of the four-vector p. It is convenient to expand it in a complete set of Dirac matrix functions of p:

$$J(p) = \sum_n J_nF_n(p). \tag{5.2}$$

I will specify the particular set of functions later. As we will see, dynamical considerations will dictate a particular choice in an interesting way. The vacuum to vacuum generating functional can be defined (analogously to Eq. (3.2)):

$$e^{iW\int d^4x} = \int [d\psi d\bar{\psi}]e^{iA}. \tag{5.3}$$

W is the generating functional for connected graphs with external "J lines" at zero momentum. Because of the expansion, Eq. (5.2), it is in fact an ordinary function of the set of variables $\{J_n\}$. Evaluating the gaussian integral gives:

$$W(J_1, J_2, \ldots) = -i\int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} \ln\left(\gamma\cdot k - \sum_n J_nF_n(k)\right). \tag{5.4}$$

The N'th derivative of W,

$$\frac{\partial^N W}{\partial J_d \ldots \partial J_b \partial J_a} \tag{5.5}$$

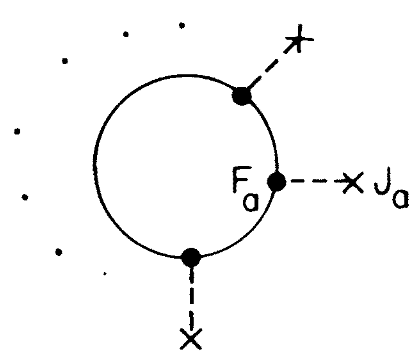

Fig. 10. Graphs generated by a free Dirac field coupled bilinearly to an external field in Eqs (5.4, 5)

is the N-point function with all external momentum set to zero shown in Fig. 10.

This free field coupled to a classical source serves to introduce the needed sources and generating functionals for DSB and provides a closed form expression for the limit in which the interactions are turned off. I will show in the next section that for a judicious choice of expansion functions, $F_a(p)$, the Legendre transform can be evaluated to give the generalized effective potential. Of course we must turn on interactions in order to get DSB effects. The reasons for introducing the above quantities should then become more apparent because then poles will develop in $\bar{\psi}\, F_a\psi$ channels and the N-point functions generated by $W(J_a)$ will have the pole structure discussed in the previous section. The next step is to introduce a tractible approximation for the interacting case.

Consider the complete fermion propagator in massless QED:

$$S(p)^{-1} = A(p^2)\gamma\cdot p - B(p^2). \tag{5.6}$$

The manifestation of spontaneous chiral symmetry breaking is a non-zero $B(p^2)$. In perturbation theory $B = 0$ to any finite order. One should instead look at the infinite hierarchy of Schwinger Dyson (SD) equations. The fermion propagator equation is:

$$S(p)^{-1} = \gamma\cdot p - J(p) + ig^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu S(k)\Gamma_\nu(k, p)D^{\mu\nu}(k-p), \tag{5.7a}$$

where $J(p)$ is the external source Eq. (5.1, 2), Γ_ν is the ψ-$\bar{\psi}$-photon vertex and $D_{\mu\nu}$ is the photon propagator. As a first approximation one can truncate the hierarchy by taking the vertex function and photon propagator to lowest order.

$$D^{\mu\nu}(k) = \frac{1}{k^2}\left(g^{\mu\nu} + (\alpha-1)\frac{k^\mu k^\nu}{k^2}\right), \tag{5.7b}$$

$$\Gamma_\mu = \gamma_\mu. \tag{5.7c}$$

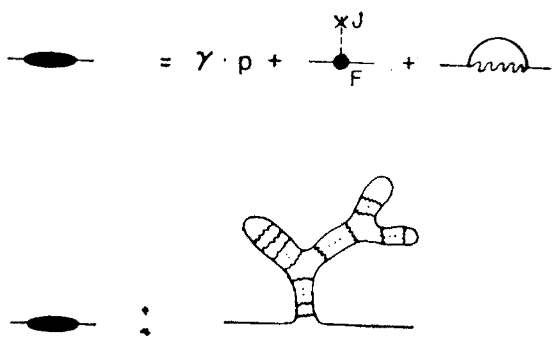

Fig. 11. The SD equation for the fermion propagator, Eq. (5.7a) with approximations given by Eq. (5.7b, c). The typical graph is shown. It is a branching of ladder graphs

This gives a non-linear integral equation for S which sums an infinite set of graphs, Fig. 11, and as will see, can exhibit the chiral phase transition. The signal is a non-zero $B(p^2)$ in the limit in which $J(p)$ goes to zero. There are no corrections to the photon propagator or the fermion-photon vertex in this approximation.

Equations (5.7a, b, c) define a starting point for studies of DSB by specifying a subset of graphs in the propagator and this is the approximation in which I will work in these lectures. It coincides with the first term in the $1/N_{\rm color}$ expansion of an SU($N_{\rm color}$) gauge theory but for this case $N_{\rm color} = 1$. As with most bound state approximations this is not gauge invariant but it is hoped that the exact solutions presented here will make this a viable first approximation upon which corrections can be made and for which a "best gauge" can be determined. Finally, this aproximation is the first term in a systematic expansion of the SD equations but I will not develop that here. The goal of this section is to show how to calculate the generating function $W(J_a)$ in an approximation that is consistent with the dynamics specified by Eqs. (5.7a, b, c). In the next section I will give the generalized effective potential which enables one to choose which of the solutions to the propagator equation corresponds to lowest energy.

In Fig. 11 I showed a representative graph in the inverse propagator. It is a combination of ladder graphs connected with the topology shown. To understand why this approximation is a judicious starting point one should compare this with Fig. 4b, c. In that case the mass term contains branching tree graphs. The crucial point in what follows is that the infinite sum of ladder graphs will contain poles corresponding to bound states and the bound state pole terms will form branching trees in exactly the same way as we will see.

Although it is not possible to give a closed form expression for $W(J_a)$ for the interacting case, it can be conveniently expressed as a variational principle. Consider the following:

$$\mathscr{W}[S, J_n] = -i\int\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\ln S^{-1}(k)+\left(\gamma\cdot k-\sum_n J_nF_n(k)\right)S(k)\right\}$$
$$-\frac{i}{2}g^2\int\frac{d^4k}{(2\pi)^4}\,i\int\frac{d^4q}{(2\pi)^4}\,\mathrm{Tr}\,\{S(k)\gamma_\mu S(q)\gamma_\nu\}D^{\mu\nu}(k-q). \tag{5.8}$$

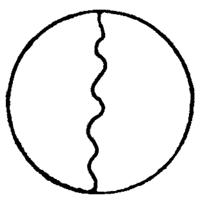

Fig. 12. A vacuum graph which contributes to $\mathscr{W}[S, J_a]$, Eq. (5.8)

$\mathscr{W}$ is functional of the fermion propagator $S(p)$ (for the moment an independent variable) and an ordinary function of the set of variables $\{J_a\}$. The g^2 term in Eq. (5.8) is the vacuum graph shown in Fig. 12. Calculate now the variational derivative $\delta\mathscr{W}/\delta S(p)$. Setting it zero gives

$$S^{-1}(p) = \gamma\cdot p - J(p) + ig^2\int\frac{d^4k}{(2\pi)^4}\gamma_\mu S(k)\gamma_\nu D^{\mu\nu}(k-p), \tag{5.9}$$

which is just the SD equation Eq. (5.7a, b, c). Hence $\mathscr{W}$ is stationary under first order variations of $S(p)$ about the solution, Eq. (5.9), by construction. As I will show in the remainder of this section, $\mathscr{W}$ evaluated at the stationary point is the generating function $W(J_a)$:

$$W(J_a) = \mathscr{W}[S, J_a]|_{\frac{\partial\mathscr{W}}{\partial S}=0} = \mathscr{W}[S(p; J_a), J_a], \tag{5.10}$$

where $S(p;\, J_a)$ is determined by the stationary condition, Eq. (5.9). If we could solve Eq. (5.9) we would have a closed form expression for $W(J_a)$. We can not but we nevertheless have a useful expression for deriving the generalized effective potential.

In order to show that Eq. (5.10) gives $W(J_a)$, I will take derivatives and see what n-point functions are generated. Consider the first derivative:

$$\frac{\partial W}{\partial J_a} = \left.\frac{\partial \mathcal{W}}{\partial J_a}\right|_{\text{explicit}} + \int d^4p\, \frac{\delta \mathcal{W}}{\delta S(p)} \frac{\partial S(p)}{\partial J_a} \tag{5.11}$$

$$= i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} S(k) F_a(k). \tag{5.12}$$

The second term in Eq. (5.11) is zero because $\mathcal{W}$ is stationary. All the higher derivatives are easily calculated from the simple expression, Eq. (5.12). The second derivative is:

$$\frac{\partial^2 W}{\partial J_b \partial J_a} = -i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr}\left(F_a S \frac{\partial S^{-1}}{\delta J_b} S\right). \tag{5.13}$$

To evaluate this we need to know $\partial S^{-1}/\partial J_b$. Taking the derivative of Eq. (5.9) gives a Bethe--Salpeter equation:

$$\frac{\partial S(p)^{-1}}{\partial J_b} = -F_b(p) - i \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) \frac{\partial S^{-1}(k)}{\partial J_b} S(k) \gamma_\nu D^{\mu\nu}(k-p). \tag{5.14}$$

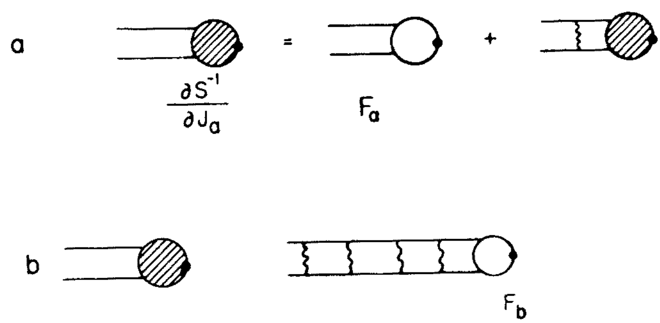

Fig. 13. (a) Graphical form of the BS equation, Eq. (5.14); (b) Typical graphs summed by this equation

This equation generates ladder graphs as shown in Fig. 13. From Eq. (5.14), we see that the fermion propagators on the sides of the ladder are not the free propagators but are the solutions to the SD equation Eq. (5.9). Higher derivatives of W will require higher derivatives of S^{-1}. The second derivative gives:

$$\frac{\partial^2 S^{-1}}{\partial J_c \partial J_b} = i \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S \frac{\partial S^{-1}}{\partial J_c} S \frac{\partial S^{-1}}{\partial J_b} S \gamma_\nu D^{\mu\nu} + (b \to c)$$

$$-i \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S \frac{\partial^2 S^{-1}}{\partial J_c \partial J_b} S \gamma_\nu D^{\mu\nu}. \tag{5.15}$$

This is also a Bethe-Salpeter equation as shown in Fig. 14 and in fact has the same kernel as the first one: $\gamma_\mu S \ldots S\gamma_\nu D^{\mu\nu}$. The inhomogeneous term in Eq. (5.15) is presumed known from the solution of Eq. (5.14). A little thought will convince the reader that in this hier-

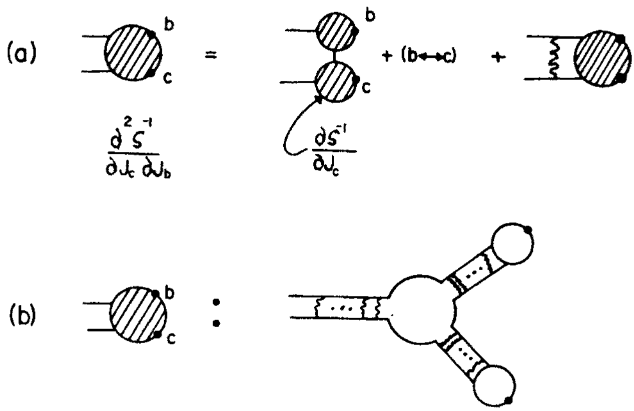

Fig. 14. (a) Graphical form of the BS equation, Eq. (5.15); (b) Typical graphs summed by this equation

archy of Bethe-Salpeter equations all have the same kernel and are soluble when taken in turn if the first one, Eq. (5.14), is soluble. We will see in Section 7 that it is soluble for the case that is needed.

Calculation of the N-th derivative of W evaluated at $J_a = 0$ is straightforward: in the Bethe-Salpeter equations and the formulas for $\partial^N W$, S is replaced everywhere by S_0, where S_0 is the solution of the SD equation with the source off, $J_a = 0$,

$$S_0(p)^{-1} = \gamma \cdot p + ig^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S_0(k)\gamma_\nu D^{\mu\nu}(k-p). \tag{5.16}$$

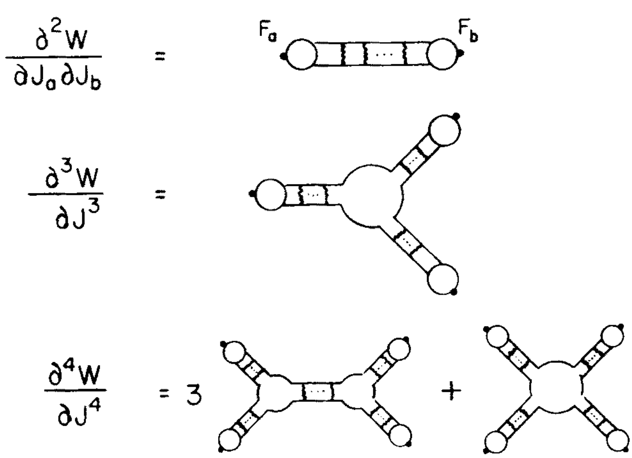

Fig. 15. Typical graphs in the N-point functions $\partial^N W/\partial J^N$ generated from Eq. (5.10)

The first few derivatives are shown in Fig. 15. The graphs can be characterized as follows: The N-point function has N factors of F_a on the single fermion loop. Photon exchange ladder graphs occur in the manner shown in Fig. 15. Poles corresponding to bound states in those channels will occur in the sum of ladder graphs as we will see and this provides the mechanism for the phase transition as discussed in Section 4.

Thus far the expansion functions, F_a in Eq. (5.2), are unspecified. If we choose them to be eigenfunctions of the Bethe-Salpeter kernel encountered in Eq. (5.14) and (5.15) great simplification occurs. Hence one should choose the $F_a(p)$'s to be the complete set of solutions to the following homogeneous BS equation.

$$F_a(p) = -ig_a^2 \int \frac{d^4k}{(2\pi)^4} \gamma_\mu S(k) F_a(k) S(k) \gamma_\nu D^{\mu\nu}(k-p). \tag{5.17}$$

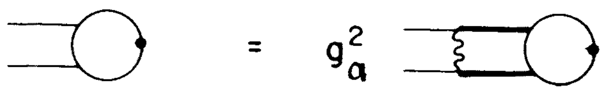

Fig. 16. BS eigenvalue equation, Eq. (5.17)

This is an eigenvalue problem in g^2 with eigenvalue g_a^2 shown in Fig. 16. The orthogonality relation for this eigenvalue problem is

$$i \int \frac{d^4k}{(2\pi)^4} \operatorname{Tr} F_a(k) S_0(k) F_b(k) S_0(k) = \mu^2 \delta_{ab}. \tag{5.18}$$

The completeness relation is

$$\sum_a [F_a(k)]_{\alpha\beta} [S_0(p) F_a(p) S_0(p)]_{\gamma\delta} = -i(2\pi)^4 \mu^2 \delta_{\beta\gamma} \delta_{\alpha\delta} \delta^4(k-p). \tag{5.19}$$

Taking the eigenfunctions to be dimensionless we note that it is necessary to introduce a normalization constant μ^2 with dimensions of [mass^2]. I will break the scale invariance of this model with cut-offs and this parameter μ^2 is calculable in terms of these dimensionful cutoffs but it requires us to go the bound state poles at finite momentum in order to normalize the bound state wave functions to one particle. This is straight-forward but will not be given here.

Having defined the F_a as eigenfunctions, it is very informative to do a Hilbert-Schmidt expansion of the BS equations, i.e. to expand the $\partial^N W/\partial J^N$ in terms of eigenfunctions. Consider the following expansion

$$\frac{\partial S^{-1}}{\partial J_a} = \sum_b c_{ab} F_b(p). \tag{5.20}$$

Insert this into the BS equation, Eq. (5.14), and we obtain

$$\frac{\partial S^{-1}}{\partial J_a} = -\frac{g_a^2}{g_a^2 - g^2} F_a(p). \tag{5.21}$$

Putting this result in Eq. (5.13) and using orthogonality, Eq. (5.18), gives

$$\frac{\partial^2 W}{\partial J_b \partial J_a} = \mu^2 h_a \delta_{ab}, \tag{5.22}$$

where

$$h_a \equiv \frac{g_a^2}{g_a^2 - g^2}. \tag{5.23}$$

Similarly the three point function is:

$$\frac{\partial^3 W}{\partial J_c \partial J_b \partial J_a} = -h_a h_b h_c i \int \frac{d^4k}{(2\pi)^4} (\mathrm{Tr}\, F_a S_0 F_b S_0 F_c S_0 + (b \leftrightarrow c)). \tag{5.24}$$

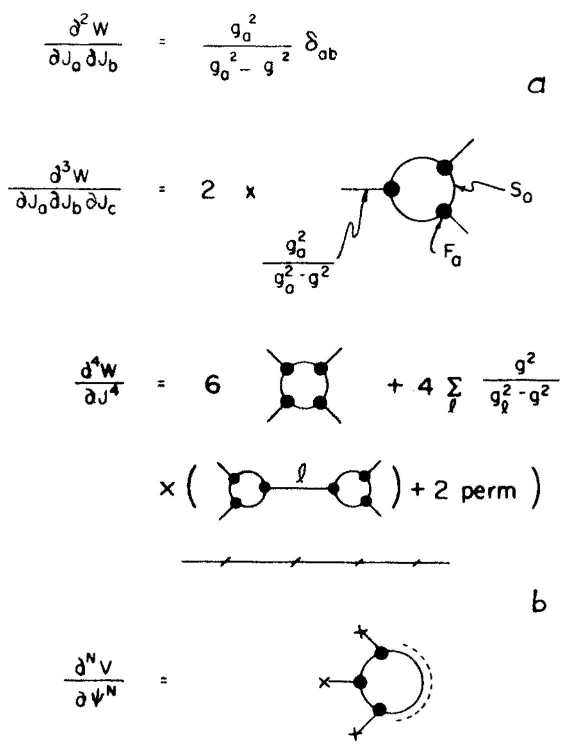

Fig. 17. (a) Graphical form of the Hilbert Schmidt expansion of the graphs in W, Eq. (5.10) and Fig. 15. (b) Graphs in the generalized effective potential, Eq. (6.2). Crosses indicate the poles are removed

These graphs are shown in Fig. 17a. The fourth and higher derivatives involve a sum over the eigenfunctions. We can now characterize these N-point functions as follows: They are *tree graph structures of bound state poles* coupled at vertices made up of a closed fermion loop. The n'th bound state couples to the loop with a wave function $F_n(p)$. This approximation has led to a bound-state-analog of $W(J)$ for scalar fields that was described in Sec.

3. The one particle decomposition of the N-point functions in Fig. 17 has the same structure shown in Fig. 5.

In summary, the important steps are: (i) We chose a complete set of sources F_a, Eq. (5.2.). (ii) We wrote a variational expression for $W(J_a)$ which defines a subset of Feynman graphs, Eqs. (5.8, 9, 10). (iii) We further choose the F_a to be eigenfunctions of the Bethe--Salpeter kernel, Eq. (5.17). And (iv) A Hilbert-Schmidt expansion led to the results summarized in Fig. 17a. In the next section I will show that the Legendre transform gives the generating function of one-particle-irreducible bound state structures analogous to the scalar field case shown in Fig. 17b.

6. *Generalized effective potential*

The effective potential for this model is defined in the same way as for the scalar field case described in Section 3.

$$\phi_a = \frac{\partial W}{\partial J_a}, \tag{6.1a}$$

$$V(\phi_a) = \sum_b \phi_b J_b - W(J_a). \tag{6.1b}$$

There are now an infinite number of variables J_a and ϕ_a corresponding to the bound states of ψ and $\bar{\psi}$. The effective potential thus defined can be evaluated in closed form to give:

$$V(\phi_a) = -i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\ln\left(\frac{\sum_b \phi_b F_b}{\mu^2}\right) + \frac{1}{\mu^2}\sum_b \left\{ i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\,(S_0 F_b S_0 \gamma\cdot k)\phi_b - \frac{g^2\phi_b^2}{2g_b^2}\right\}. \tag{6.2}$$

This is the result given in Section 2. The function $F_a(p)$ and g_a^2 are the eigenfunctions and eigenvalues given by Eq. (5.17), $S_0(p)$ is given by the solution to Eq. (5.16), and μ^2 is a scale parameter arising in the Bethe-Salpeter normalization, Eq. (5.18). The N'th derivative $\partial^N V/\partial\phi^N$ generates a fermion loop graphs shown in Fig. 17b. The simple result, Eq. (6.2), is not surprising. $V(\phi_a)$ generates one bound state irreducible graphs and these are just the N bound state vertices that go to make up the connected graphs in say Fig. 17a. Fig. 17 is the bound state version of Fig. 5. I would like to derive Eq. (6.2) by showing that it follows from an interesting property of Legendre transforms and complete orthonormal functions. I will first consider the free field result obtained by setting $g^2 = 0$ in Eq. (5.8, 9, 10) and Eq. (6.2):

$$W = -i\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\ln\left(\gamma\cdot k - \sum_a J_a F_a\right), \tag{6.3a}$$

$$V = -i\frac{d^4k}{(2\pi)^4}\,\mathrm{Tr}\left\{\ln\left(\frac{\sum_b \phi_b F_b}{\mu^2}\right) - \left(S_0\frac{\sum_b \phi_b F_b}{\mu^2}S_0\gamma\cdot k\right)\right\}. \tag{6.3b}$$

To derive this let us consider the following:

Theorem: Let $\{G_n(x)\}$ be a complete orthonormal set of functions on the interval [0, 1] with an arbitrary, positive weight function $S_0(x)^2$:

$$\int_0^1 G_m G_n S_0^2 dx = \delta_{mn} \tag{6.4a}$$

$$\sum_n G_n(x) G_m(x) = \frac{\delta(x-y)}{S_0^2(x)}. \tag{6.4b}$$

Consider the following generating function:

$$W(J_a) = -\int_0^1 dx \ln(1 - S_0 \sum_n J_n G_n) - \sum_n J_n \int_0^1 dx G_n S_0. \tag{6.5}$$

With the conjugate variables and Legendre transform defined by Eq. (6.1) then

$$V(\phi_n) = -\int_0^1 dx \ln(1 + S_0 \sum_n \phi_n G_n) + \sum_n \phi_n \int_0^1 dx G_n S_0. \tag{6.6}$$

Proof: Evaluate ϕ_n using Eq. (6.1a). Use completeness to obtain:

$$1 - S_0(x) \sum_n J_n G_n(x) = \{1 + S_0(x) \sum_n \phi_n G_n(x)\}^{-1}. \tag{6.7}$$

Using orthogonality one can obtain J_m as a function of ϕ_n. Substituting this into Eq. (6.1b) gives the result Eq. (6.6).

It is instructive to expand out W and V in order to see the role of completeness in the Legendre transform. Expanding W about $J_a = 0$ gives:

$$\frac{\partial W}{\partial J_a} = 0 \quad \text{(by construction)}, \tag{6.8a}$$

$$\frac{\partial^2 W}{\partial J_b \partial J_a} = D_{ab} = \int_0^1 G_a G_b S_0^2 dx = \delta_{ab}, \tag{6.8b}$$

$$\frac{\partial^N W}{\partial J_{a_N} \dots \partial J_{a_1}} = (N-1)! \int_0^1 G_{a_1} \dots G_{a_N} S_0^N dx. \tag{6.8c}$$

Expanding V about $\phi_a = 0$ gives:

$$\frac{\partial V}{\partial \phi_a} = 0, \tag{6.9a}$$

$$\frac{\partial^2 V}{\partial \phi_b \partial \phi_a} = D_{ab}^{-1} = \delta_{ab} \tag{6.9b}$$

$$\frac{\partial^3 V}{\partial\phi_c\partial\phi_b\partial\phi_a} = -\frac{\partial^3 W}{\partial J_c\partial J_b\partial J_a} = -2\int_0^1 dx G_a G_b G_c S_0^3 \tag{6.9c}$$

$$\frac{\partial^4 V}{\partial\phi_d\partial\phi_c\partial\phi_b\partial\phi_a} = \left\{\sum_l \frac{\partial^3 W}{\partial J_a\partial J_b\partial J_l}\frac{\partial^3 W}{\partial J_c\partial J_d\partial J_l} + 2 \text{ perm.}\right\} - \frac{\partial^4 W}{\partial J_d \ldots \partial J_a}$$
$$= +\frac{\partial^4 W}{\partial J_d\partial J_c\partial J_b\partial J_a}. \tag{6.9d}$$

The things to note are the following: The N'th derivative of V generates quantities involving all the derivatives of W up to the N'th. However if the set of functions G_a are complete, then the sums, as in the fourth derivative, are completeness sums and the derivatives reduce to simple expressions. Without completeness I suspect that one could not evaluate V in closed form.

The generalization to our case Eq. (6.3 b) where the set of functions also have matrix indices is straightforward.

Let us now alter this example so that it corresponds to the interacting case. Consider the following BS equation:

$$T(x, y) = g^2 D(x, y) + g^2 \int_0^1 dz D(x, z) S_0^2(z) T(z, y), \tag{6.10}$$

and the corresponding eigenvalue problem:

$$G_n(x) = g_n^2 \int_0^1 dz D(x, z) S_0^2(z) G_n(z). \tag{6.11}$$

Now make a Hilbert-Schmidt expansion of these quantities in eigenfunctions.

$$T(x, y) = \sum_n G_n(x) \frac{g^2}{g_n^2 - g^2} G_n(y), \tag{6.12a}$$

$$D(x, y) = \sum_n G_n(x) \frac{g^2}{g_n^2} G_n(y). \tag{6.12b}$$

Consider now the following generating functions:

$$W(J_n) = \mathscr{W}[S, J_n]_{\text{stationary point}}, \tag{6.13}$$

where

$$\mathscr{W} = -\int_0^1 dx\left(-\ln\frac{S}{S_0} + (S - S_0)\left(\tilde{S}_0^{-1} - \sum_n J_n G_n\right)\right)$$
$$+\tfrac{1}{2} g^2 \int_0^1 dx \int_0^1 dy \{S(x) D(x, y) S(y) - S_0(x) D(x, y) S_0(y)\}. \tag{6.14}$$

The stationary requirement, $\delta\mathscr{W}/\delta S(x) = 0$, gives a constraint between S and J_n.

$$S^{-1}(x) = \tilde{S}_0^{-1}(x) - \sum_n G_n(x)\left\{J_n + \frac{g^2}{g_n^2}\int_0^1 dy G_n(y) S(y)\right\}. \tag{6.15}$$

$\tilde{S}_0$ is the value of S with $J_n = 0$ and $g^2 = 0$. This is the analog of the SD equation, Eq. (5.9), except that we have used the eigenfunction expansion of D, Eq. (6.12a). $S_0(x)$ is defined by Eq. (6.15) with $J_n = 0$.

Theorem: The Legendre transform, Eq. (6.1), of the interacting $W(J_a)$, Eq. (6.13) and Eq. (6.14) is

$$V(\phi_n) = V(\phi_n)|_{\text{free field}} - \tfrac{1}{2}\sum_n \frac{g^2}{g_n^2}\phi_n^2, \tag{6.16}$$

where the first term is given by Eq. (6.6).

Proof: Because $\mathscr{W}$ is stationary in S, (cf. Eq. (5.11,12)) the derivative of W with respect to J_n gets a contribution only from its explicit J_n dependence:

$$\phi_n = \frac{\partial W}{\partial J_n} = \int_0^1 dx G_n(x)\,(S(x) - S_0(x)). \tag{6.17}$$

Using Eq. (6.15) and Eq. (6.17) and completeness, Eq. (6.4b), we obtain

$$1 - S_0(x)\sum_n G_n(x)\left(J_n + \frac{g^2}{g_n^2}\phi_n\right) = \left\{1 + S_0(x)\sum_n \phi_n G_n(x)\right\}^{-1}. \tag{6.18}$$

Using orthogonality, Eq. (6.4a), one can obtain J_n as a function of $\{\phi_n\}$. This allows a straightforward evaluation of V, giving Eq. (6.16).

Again I will expand out W and V in order to see the role of completeness. Expanding W about $J = 0$ gives:

$$\frac{\partial W}{\partial J_a} = 0 \text{ (by construction)}, \tag{6.19a}$$

$$\frac{\partial^2 W}{\partial J_b \partial J_a} = h_a \delta_{ab}; \qquad h_a \equiv \frac{g_a^2}{g_a^2 - g^2}, \tag{6.19b}$$

$$\frac{\partial^3 W}{\partial J_c \partial J_b \partial J_a} = 2h_a h_b h_c (abc), \tag{6.19c}$$

$$\frac{\partial^4 W}{\partial J_d \partial J_c \partial J_b \partial J_a} = h_a h_b h_c h_d \left\{6(abcd)\right.$$

$$\left. + 4\sum_l \left[(abl)h_l \frac{g^2}{g_l^2}(lcd) + 2 \text{ perm.}\right]\right\}, \tag{6.19d}$$

where

$$(a_1 a_2 \dots a_N) = \int_0^1 dx G_{a_1} G_{a_2} \dots G_{a_N} S_0^N. \tag{6.20}$$

Expanding out V about $\phi_a = 0$ gives:

$$\frac{\partial V}{\partial \phi_a} = 0, \tag{6.21a}$$

$$\frac{\partial^2 V}{\partial \phi_b \partial \phi_a} = D_{ab}^{-1} = \frac{\delta_{ab}}{h_a}. \tag{6.21b}$$

The second derivative differs from the free case in that the propagator D_{ab} has a pole in g^2. However in all the higher derivatives all the g^2 dependence drops out. The N'th derivative gives:

$$\frac{\partial^N V}{\partial \phi_{a_N} \dots \partial \phi_{a_2} \partial \phi_{a_1}} = (N-1)!\,(a_1 a_2 \dots a_N) \tag{6.22}$$

which is the same as for the free case, Eq. (6.9). All the intermediate state sums in W, as for example in Eq. (6.19d), become completeness sums in V. Hence without completeness, none of this simplification would occur.

The generalization of this interacting case to the effective potential result Eq. (6.2) is straightforward. The result is indicated in Fig. 17b. Note that in Eq. (6.2) the effect of interactions is to add the g^2 dependent quadratic term. There is also a g^2 dependence coming from the fact that S_0 is the solution of a SD equation which depends on g^2. Further $F_a(p)$ are eigenfunctions of a Bethe-Salpeter kernel which involves S_0. For a particular case in which we are interested, that in which the vacuum is chiral invariant, and for the Landau gauge, S_0^{-1} is just $\gamma \cdot p$ *then the only g^2 dependence is in the quadratic term and it occurs in the trivial form.*

7. *Solutions to the SD and BS equations*

I wish to show here that these equations can be solved in the chiral invariant vacuum. This allows the evaluation of the effective potential as an expansion in powers of ϕ_a. I will restrict the external source $J(p)$ here to be of the form

$$J(p) = I \sum_n J_n f_n^{(s)}(p^2), \tag{7.1}$$

where I is the 4×4 Dirac identity and $f_n(p^2)$ is an invariant function of p. The solutions given here can be easily generalized for arbitrary $J(p)$.

Using the A, B decomposition of the propagator, Eq. (5.6), in the SD equation, Eq.

(5.9), gives the following coupled equations after Wick rotating and integrating over angles.

$$A(x) = 1+\frac{\alpha g^2}{16\pi^2}\left\{\frac{1}{x^2}\int_{\lambda^2}^{x}\frac{dy\,y^2A}{A^2y+B^2}+\int_{x}^{\Lambda^2}\frac{dy\,A}{A^2y+B^2}\right\}, \tag{7.2a}$$

$$B(x) = \sum_n J_n f_n^{(s)}(x)+\frac{(3+\alpha)g^2}{16\pi^2}\left\{\frac{1}{x}\int_{\lambda^2}^{x}\frac{dy\,yB}{A^2y+B^2}+\int_{x}^{\Lambda^2}\frac{dy\,B}{A^2y+B^2}\right\}. \tag{7.2b}$$

The variables x, y are the euclidean variables, p^2 and k^2, α is the gauge parameter in the photon propagator, Eq. (5.7b). I have introduced cut-offs at both ends. These will be discussed further in the next section. Let us turn off the source, $J(p) = 0$, and look for solutions with vanishing $B(x)$:

$$A(x) = 1+\frac{\alpha g^2}{16\pi^2}\left\{\frac{1}{x^2}\int_{\lambda^2}^{x}\frac{y\,dy}{A(y)}+\int_{x}^{\Lambda^2}\frac{dy}{yA(y)}\right\}. \tag{7.3}$$

In the Landau gauge, ($\alpha = 0$), $A = 1$. In other gauges this is a non-linear equation which can be solved. First note that this can be converted to a differential equation by applying the operator $\frac{d}{dx}x^3\frac{d}{dx}$. By the change of variables $x = \tilde{x}\exp(t)$, this becomes:

$$\frac{d^2A}{dt^2}+2\frac{dA}{dt} = -\frac{\alpha g^2}{8\pi^2}\frac{1}{A}. \tag{7.4}$$

The solution to this equation is given in the paper by Mercader and myself [4]. We have found the general solution of this equation in parametric form $A(\xi)$ and $t(\xi)$. For the two cases $\alpha > 0$ and $\alpha < 0$ we have respectively A_+ and A_-:

$$A_\pm(\xi) = -\tfrac{1}{2}\sqrt{\frac{\pm\gamma}{2}}\,e^{\mp\xi^2}/H_\pm(\xi), \tag{7.5a}$$

$$t_\pm(\xi) = \tfrac{1}{2}\int_{\xi_0}^{\xi} d\eta\,\frac{e^{\mp\eta^2}}{H_\pm(\eta)}, \tag{7.5b}$$

$$H_\pm(\xi) = \int_{\eta_0}^{\xi} e^{\mp\eta^2}d\eta, \tag{7.5c}$$

where $\gamma = \alpha g^2/8\pi^2$ and η_0 and ξ_0 are integration constants.

I will not go into a detailed discussion of this solution here, my main point is that this non-linear equation is soluble. $A(p^2)$ is quite a complicated function of p^2. It has an infinite number of branch points with two of them occurring in the real euclidean region, making $A(p^2)$ unexpectedly complex there. These pathologies have been noted before

[13] and are presumably a consequence of the approximation, Eq. (5.7b, c). There are delicate questions as to how to impose cut-offs and how to renormalize. The cutoffs we introduced in fact automatically select a region where A is real between them. That is, one can see without solving the integral equations that the solutions are real for $\lambda^2 \leqslant x \leqslant \Lambda^2$. There are hints that as the cut-offs are removed α must go to zero and hence the Landau gauge is forced on us. There is much to clarify here which will be published elsewhere.

Next let us look at the BS eigenvalue equation for this model. A complete set of Dirac matrix-O(4) spherical harmonics will block-diagonalize Eq. (5.17) to a radial problem. Here I will only look for solutions of the form Eq. (7.1). Wick rotating and using the same variables as for the SD equations, Eq. (5.17) becomes after integrating over angles:

$$f_n(x) = (3+\alpha)\frac{g^2}{16\pi^2}\left\{\frac{1}{x}\int_{\lambda^2}^{x}\frac{dyf_n(y)}{A^2(y)} + \int_{x}^{\Lambda^2}\frac{dyf_n(y)}{yA^2(y)}\right\}. \tag{7.6}$$

This involves the $A(x)$ given implicitly in the above solution. However in the Landau gauge $A = 1$ and this equation is homogeneous in x and is trivially soluble and will be given below. In other gauges this can be solved by a change of variables close to the one for the SD equations. Convert Eq. (7.6) to a differential equation using the operator

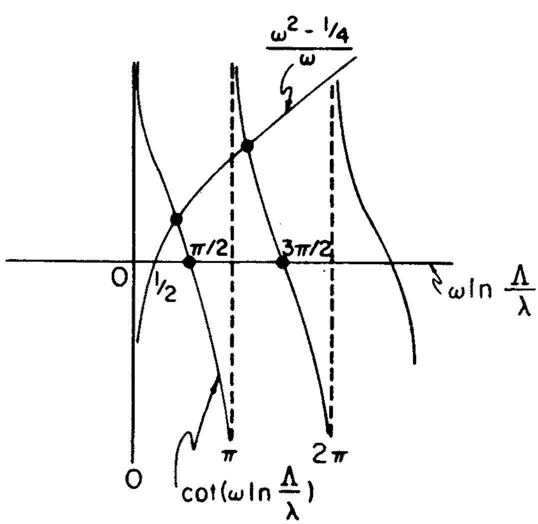

Fig. 18. Graphical solution to the eigenvalue condition, Eq. (7.8)

$\frac{d}{dx}x^2\frac{d}{dx}$. Now make a different change of variables $x = \tilde{x}\exp(-t)$. Now change to the variable ξ using Eq. (7.5b) where the $\pm$ sign refers to the sign of γ. This gives the following differential equation for f_n:

$$\frac{d^2f_n}{d\xi^2} - 2\xi\frac{df_n}{d\xi} + 4\frac{\alpha+3}{\alpha}f_n = 0. \tag{7.7}$$

This is the confluent hypergeometric equation and reduces to Hermite's equation for special gauges. Therefore this BS differential equation is soluble for arbitrary values of the gauge parameter. Similar changes of variables will cover the more general cases of more general Dirac matrices and more general angular dependence. The branch point problems mentioned for the SD equations must be addressed here also.

Returning to Eq. (7.4) in the Landau gauge, note that the equation is homogeneous in $x(A = 1)$. The two solutions are $x^{-\frac{1}{2}\pm i\omega}$ where $\omega^2 = \frac{3g^2}{16\pi^2} - \frac{1}{4}$. The boundary conditions can be satisfied only if $\omega^2 > 0$ which means that $\frac{g^2}{4\pi^2} > \frac{\pi}{3}$. The eigenvalue condition is

$$\frac{1}{\omega}(\omega^2 - \tfrac{1}{4}) = \cot\left(\omega \ln \frac{\Lambda}{\lambda}\right). \tag{7.8}$$

The graphical solution of this is shown in Fig. 18.

8. *Conclusions*

It was my aim in these lectures to focus attention on the close correspondence between the problems of dynamical symmetry breaking and the simple intuitive ideas of symmetry breaking in σ models. I have given an example of a system that exhibits DSB and shown that the stability of the vacuum state is governed by an effective potential function that can be evaluated in closed form. As input to this function one needs to solve a linear eigenvalue problem. Further the kernel for the eigenvalue problem involves the solution of a non-linear Schwinger-Dyson equation. For the example at hand these equations can be solved.

Let us look at the result, Eq. (2.6) or (6.2), in a little more detail by expanding it out in a power series in ϕ_n:

$$V = \sum_n \left(1 - \frac{g^2}{g_n^2}\right)\phi_n^2 + O(\phi^3). \tag{8.1}$$

As pointed out in the previous section, in the Landau gauge, the only g^2 dependence is that explicitly given in the quadratic term. Notice that for g^2 less than the smallest eigenvalue, the origin is a local minimum. As g^2 is increased and passes the first eigenvalue, the origin is no longer a stable minimum but becomes a saddle point. Hence the only effect of interactions on V in the Landau gauge is to change the sign of each quadratic term in turn as the coupling constant is increased. The whole question of stability is governed by the interplay between the g^2 dependent quadratic term and the effective potential of a free field theory (which has been perversely expanded in terms of these BS eigenfunctions).

There are a number of interesting questions that I would like to discuss but either I do not have time or I do not know the answers. Juan Perez-Mercader, Piotr Rembiesa, and I are presently pursuing further some of the questions raised by this approach. In closing I would like to mention a few:

(*i*) Surveying V for minima: We would like to find the trajectory of the minimum as a function of g^2 and look for possible bifurcation of the trajectory. This determines the pattern of symmetry breaking. Recall that the source $J(p)$ is expanded in a complete set of Dirac matrix functions of p, Eq. (5.2). In Sec. 7 I discussed the special case in which $J(p)$ is proportional to the Dirac identity matrix Eq. (7.1). But this expansion also includes terms proportional to the γ_ν matrices, e.g.

$$J(p) = \gamma_\mu \sum_n J_n^\mu f_n^{(V)}(p^2). \tag{8.2}$$

Corresponding to each J_n^μ there is a Legendre transform variable ϕ_n^μ. If the minimum of V occurs for non-zero ϕ_n^μ then Lorentz invariance will be spontaneously broken. This seems certain to be the case if g^2 is larger than the smallest eigenvalue g_n^2 corresponding to a vector source, but it may also occur for smaller values of g^2. If it does occur, well then back to the drawing boards but it may be connected with the next point.

(*ii*) There are problems with unphysical singularities in the particular model I discussed here. The propagator $S(p)$ should be analytic in the complex p^2 plane except for poles and normal thresholds for $p^2 \geqslant 0$. However the solutions to the propagator equation, Eq. (7.5), can have singularities in the space-like region, $p^2 < 0$. Without looking at the details of the solution one can see an example of this by noting that the differential equation for A, Eq. (7.4), has a classical mechanical analog. Rewrite it in the form:

$$\ddot{r}+2\dot{r} = -\left(\frac{\alpha g^2}{8\pi^2}\right)\frac{1}{r}, \tag{8.3}$$

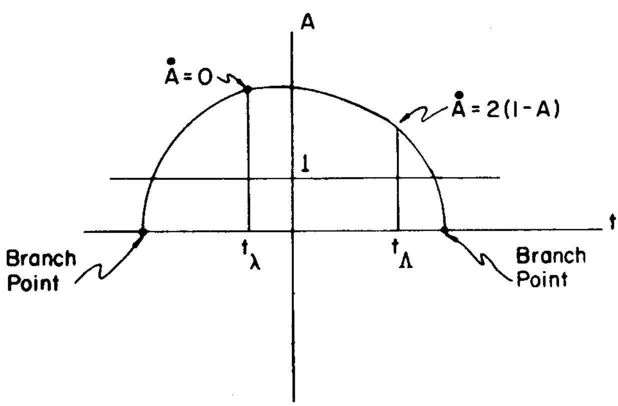

Fig. 19. Qualitative behavior of $A(t)$ Eq. (8.3)

where I substituted r for A. For positive α this describes the motion of a particle subject to dissipation and to an attractive $1/r$ force. Starting with the initial conditions of a finite r and finite velocity $\dot{r}$ such a particle will fall into the origin ($r = 0$) in a finite time dissipating all its energy (no bounce). There are no real solutions for larger times (unless the initial conditions are changed) but there are complex solutions. That is, $r(t)$ has a branch point on the real t axis and translating into the old variables, $A(p^2)$ has a branch point in the p^2 plane in the space-like region. The complex interval extends from the branch point to $-\infty$. A recurring theme in these lectures is that unphysical singularities e.g. tachyons, are associated with instabilities. Could it be that these singularities are also signals of an instability?

(*iii*) Removal of the cut-offs: The cut-offs were imposed on the coupled propagator equation, Eq. (7.2) to make them finite. Beside the divergence problem there are other complications in letting $\Lambda/\lambda \to \infty$. It is clear from Eq. (7.8) and Fig. 18 that the eigenvalues become dense in this limit. The spectrum is continuous because we are looking at the states of two massless constituents at total energy zero which is the normal threshold. Another complication in removing the cut-offs is connected with the singularity problem just mentioned. Let us write the integral equation for A, Eq. (7.3) in terms of the t variable $x = \tilde{x}e^t$:

$$A(t) = 1 + \frac{\alpha g^2}{16\pi^2}\left\{e^{-2t}\int_{t_\lambda}^{t}\frac{e^{2s}ds}{A(s)} + \int_{t}^{t_\Lambda}\frac{ds}{A(s)}\right\}. \tag{8.4}$$

The qualitative behavior of the solution is shown in Fig. 19. In the language of the mechanical analog, the particle starts at the left branch point with an infinite positive energy. It rises out of the well and falls back in. The boundary conditions implied by Eq. (8.3) are indicated on the curve. It is clear that if $t_\Lambda \to \infty$ the branch point is pushed in front of it, and similarly for $t_\lambda \to -\infty$. We have a hunch that this limit forces us into the Landau gauge ($A = 1$).

(*iv*) Gauge invariance: The set of graphs I have summed in these lectures is not a gauge invariant subset of all graphs. Problems with gauge invariance are ever-present with bound state problems. As with other bound state problems we hope that the scheme itself will reveal to us the "best" gauge or possibly the gauge will be chosen in the manner just described above. Since we now have an exact solution for any value of the gauge parameter we are in a position to look for these effects. We clearly also want to see if these techniques could possibly be applied to find a gauge invariant approximation.

(*v*) Non-Abelian Gauge Theories: Nothing in the presentation here restricted us to abelian gauge theories. However the set of graphs that are summed here clearly do not include gauge particle self interactions. One could partly correct for this defect by putting in a modified propagator (e.g. a confining propagator in QCD). The techniques developed here will unfortunately not sum the leading term in the $1/N$ expansion of an SU(N) gauge theory since the gauge particle self interactions contribute to this.

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