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LIV

New York Academy of Sciences

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This issue includes papers presented at

Graph Theory Day 54

held at

The Department of Mathematics and Computer Science

Manhattan College

Manhattan, New York

Saturday, October 27, 2007

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INTRODUCTORY REMARKS

Graph Theory Day 54 was hosted by Manhattan College, Riverdale, New York. The local organizers were Kathryn Weld and Richard Goldstone of Manhattan College. The meeting opened with welcoming remarks from Richard Goldstone. The invited speakers were Gary Gordon and Claude Tardif. Articles based on these talks can be found in this issue.

Although ***Graph Theory Notes of New York*** is not the proceedings of Graph Theory Days, we are always pleased when papers by the invited speakers and by presenters of contributed talks at Graph Theory Days are submitted for publication. In particular, we invite any readers of the ***Notes*** to submit articles for consideration as per Instructions for Contributors listed on the inside of the back cover.

With respect to Graph Theory Days, there is an ongoing need for hosts of these events. Hosting is both good for the host institution and for graph theory in general. Incidentally, taking students to Graph Theory Days is a great way to introduce them to the mathematical community outside of the class room.

Graph Theory Day 55 organized by Gary E. Stevens will take place at Hartwick College, Oneonta, New York on May 10, 2008. Also of interest to our readers is the complementary series of meetings, Discrete Mathematics Days in the Northeast, organized by the DMD-NE Steering Committee (Seth Chaiken, Albany, SUNY; Karen Collins, Wesleyan; Cristian Lenart, Albany, SUNY; Rosa Orellan, Dartmouth; and Lauren Rose, Bard). The next meeting in this series will take place on June 7, 2008 at the Stevens Institute of Technology, Hoboken, New Jersey. Organizer and contact person is Daniel Gross, Seton Hall University, <grossdan@shu.edu>. The web site for the meeting is <http://www.math.shu.edu/DMD08>.

Thanks to all of the supporters of ***Graph Theory Notes of New York*** and Graph Theory Days.

JWK/LVQ
New York
April 2008

GRAPH THEORY DAY 54

Organizing Committee

Katheryn Weld and Richard Goldstone
(Manhattan College)

John W. Kennedy (Queens College, CUNY)

Michael L. Gargano and Louis V. Quintas (Pace University)

Graph Theory Day 54, sponsored by the Mathematics Section of the New York Academy of Sciences, was organized and hosted by the Department of Mathematics and Computer Science, Manhattan College, New York on Saturday, October 27, 2007.

The featured presentations at Graph Theory Day 54 were:

Honey, I Shrunk the Hedetniemi Conjecture

Claude Tardif [See this issue page 46]
Royal Military College of Canada
PO Box 17000 Stn Forces
Kingston, Ontario, CANADA

From the Chromatic Polynomial of a Graph to the Tutte Polynomial of a Greedoid with several stops along the way

Gary Gordon [See this issue page 34]
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[GTN LIV:1] THE FIRE INDEX

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Dedicated to the memory of the 343 firefighters who lost their lives in New York City on the 11th of September 2001

Abstract

We introduce a new graph parameter $f^*(G)$, defined as the largest order among all fan digraphs associated with the edge-deleted colorings of the critical subgraphs of G. We show the fundamental importance of this parameter in edge coloring. In particular, we provide generalizations of Vizing's Theorem, Shannon's Theorem, and Vizing's Adjacency Lemma. We offer an extension to multigraphs of the simple graph version of Vizing's Theorem, obtained by proving that the chromatic index of an arbitrary multigraph must assume one of only two possible values. We call f^* the fire index.

1. Introduction

A *multigraph* G is, for the purposes of this paper, an ordered triple (V, E, ψ), where V and E are two disjoint finite sets called, respectively, the set of vertices and the set of edges of G, and $\psi: E \to V^{(2)}$ is a function, called the *incidence function*. Here and elsewhere in this paper, for a set X, the symbol $X^{(2)}$ is used to denote the set of unordered pairs of distinct elements of X. Thus, the incidence function associates to each edge e of G an unordered pair $\{u, v\}$ of distinct vertices of G, which e is said to *join*. The vertices u and v are also called the *endpoints* of e. Two edges are *adjacent* if they are distinct and have at least one common endpoint. The set of edges joining two vertices u and v is denoted by uv. The cardinality of uv is denoted by $\mu(uv)$ and is called (with a slight abuse of terminology) the *multiplicity* of the edge uv. The number

$$\mu(G) = \max_{\{u, v\} \in V^{(2)}} \{\mu(uv)\}$$

is called the *maximum multiplicity* of G.

If u is a vertex of G, then $\sum_{v \neq u} \mu(uv)$, the number of edges incident with u in G, is called the *degree* of u and is denoted by $\deg(u)$. The maximum degree of G, denoted by $\Delta(G)$, is defined to be the maximum among the degrees of the vertices of G.

We adopt the convention that, whenever appropriate, the name of the multigraph to which a certain symbol or quantity refers shall be attached to that symbol or quantity in any convenient manner, such as by means of a subscript or superscript. This is particularly useful when more than one graph is under discussion.

Let C be a set, whose elements we conventionally call *colors*. An *edge coloring* of a multigraph $G = (V, E, \psi)$ is a function $\varphi: E \to C$ such that $\varphi(e) \neq \varphi(f)$ for any pair $\{e, f\}$ of adjacent edges of G. The *chromatic index* of G, denoted by $\chi'(G)$, is defined by

$$\chi'(G) = \min\{|C|\},$$

where C ranges over the color sets in all edge colorings of G. An edge coloring of G is called *optimal* if its color set C satisfies the condition $|C| = \chi'(G)$.

It is easy to see that $\chi'(G) \geq \Delta(G)$ for any multigraph G. If $\chi'(G) = \Delta(G)$, we say that G is *Class 1*, otherwise we say that G is *Class 2*. Virtually nothing was known about the chromatic index of arbitrary multigraphs until 1949, when C.E. Shannon **[1]** proved the following theorem.

Theorem 1 (Shannon, 1949): For any multigraph G, $\chi'(G) \leq \lfloor 3\Delta(G)/2 \rfloor$. ■

Shannon showed that the upper bound in his theorem is attained by an infinite family of graphs. Several years later, V.G. Vizing **[2]** determined another formidable upper bound on the chromatic index of multigraphs, namely

Theorem 2 (Vizing, 1964): For any multigraph G, $\chi'(G) \leq \Delta(G) + \mu(G)$. ■

Vizing's result is particularly striking when $\mu(G) = 1$; that is, when G is a simple graph, because it restricts the range of the values of the chromatic index to only two possible (consecutive) integers. It should be noticed, however, that Theorem 1 is not an improvement of Theorem 2 nor is Theorem 2 an improvement of Theorem 1.

The main objective of this paper is to prove a result that generalizes both Theorem 1 and Theorem 2, and from which most theorems in edge coloring can be derived. Possibly this result could prove to be of use in an attack on the foremost unsolved problem on edge coloring of multigraphs; the conjecture of Goldberg **[3]** and Seymour **[4]**.

Conjecture 1 (Goldberg–Seymour Conjecture): Let G be a Class 2 multigraph such that $\chi'(G) > \Delta(G) + 1$. Then

$$\chi'(G) = \max\left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil,$$

where the maximum is over all submultigraphs H of G of order at least two. □

2. Edge Coloring Preliminaries

This paper is a natural continuation of **[5]**, the notation, terminology, and results from which will be assumed. However, for the convenience of the reader we reiterate a number of definitions given in **[5]**. An edge e of a multigraph G is called *critical* if $\chi'(G-e) < \chi'(G)$. A multigraph G is called *critical* if it is Class 2, has no isolated vertex, and all of its edges are critical. An *e-tense coloring* ϕ of G is a partial edge coloring of G that assigns no color to e and whose restriction to $E(G-e)$ is an optimal coloring of $G-e$. The color set of ϕ is defined to be the color set of its restriction to $G-e$. Given an e-tense coloring ϕ of G with color set C and a vertex $w \in V(G)$, we say that a color $\alpha \in C$ is *missing* at w (or that w is missing the color α) if there is no edge, having w as an endpoint, that is assigned the color α by ϕ. The set of colors missing at w is denoted by C_w and its cardinality is called the *color-deficiency* of w, denoted by $\mathrm{cdef}(w)$.

Let $u \in V(G)$. A *fan* at u with respect to ϕ is a sequence of edges of the form

$$F = [e_0, e_1, \ldots, e_{k-1}, e_k],$$

where $e_0 = e$, $e_i \in uv_i$, and where the vertex v_i is missing the color of edge e_{i+1}, for each $i = 0, 1, \ldots, k-1$. An edge f is called a *fan edge* at u if it appears in at least one fan at u. A vertex w is called a *fan vertex* at u if it is joined to u by at least one fan edge. The set of fan vertices is denoted by $V(\mathcal{F})$. A color $\alpha \in C$ is called a *fan color* if it is the color of a fan edge. The set of fan colors is denoted by $C_{\mathcal{F}}$. If w is a fan vertex at u, we denote by $\mu^*(uw)$ the number of fan edges joining u and w, and call $\mu^*(uw)$ the *fan multiplicity* of the edge uw. The main contribution of **[5]** was the introduction of a new concept in edge coloring, the *fan digraph*, which we now define.

Definition: Let G be a Class 2 multigraph and let $e \in uv$ be a critical edge. The *e-fan digraph* at u with respect to ϕ is the directed multigraph $\mathcal{F} = (V(\mathcal{F}), A(\mathcal{F}), \psi_{\mathcal{F}})$, where

1. $V(\mathcal{F}) = \{w : w \text{ is a fan vertex at } u\}$;
2. $A(\mathcal{F}) = C_{\mathcal{F}} = \{\alpha : \alpha \text{ is a fan color at } u\}$;
3. $\psi_{\mathcal{F}}\colon A(\mathcal{F}) \to V(\mathcal{F}) \times V(\mathcal{F}),\ \alpha \mapsto (w_\alpha, z_\alpha)$, where w_α is the unique fan vertex at u missing color α and z_α is the unique fan vertex at u joined to u by an edge colored α. ■

Note that the existence of the vertices w_α and z_α is far from being obvious, and follows from deep results of edge coloring (see Lemma 1 and Lemma 2 in **[5]**).

It follows immediately from definitions **[5]** that, under the above hypotheses, we have

(1) $|V(\mathcal{F})| \geq 2$

and, for each $w \in V(G)$,

$$\textbf{(2)} \qquad \operatorname{cdef}(w) = \begin{cases} \chi'(G) - \deg_G(w) & \text{if } w = u \text{ or } w = v \\ \chi'(G) - 1 - \deg_G(w) & \text{if } w \neq u, v. \end{cases}$$

The following theorem expresses a central property of tense colorings and was obtained in **[5]** by counting the number of arcs of the fan digraph in three different ways.

Theorem 3 (Fan Theorem): Let G be a Class 2 multigraph and let $e \in E(G)$ be a critical edge, where $e \in uv$. Let ϕ be an e-tense coloring of G and let $\mathcal{F}$ be the corresponding fan digraph. Then

$$\sum_{w \in V(\mathcal{F})} \operatorname{cdef}(w) = \sum_{w \in V(\mathcal{F})} \mu^*(uw) - 1 = |C_{\mathcal{F}}|. \qquad \blacksquare$$

3. A Formula for the Chromatic Index

We now deduce from Theorem 3 the formula for the *chromatic index* that was mentioned in the Introduction. We use the same assumptions and notation of Theorem 3. Substituting **(2)** in the first identity of Theorem 3, we obtain

$$\sum_{w \in V(\mathcal{F})} (\chi'(G) - 1 - \deg_G(w)) + 1 = \sum_{w \in V(\mathcal{F})} \mu^*(uw) - 1,$$

where we have used the fact that $v \in V(\mathcal{F})$ and $u \notin V(\mathcal{F})$.

Rearranging the terms, we obtain

$$|V(\mathcal{F})| \cdot \chi'(G) = \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + |V(\mathcal{F})| - 2,$$

from which

$$\textbf{(3)} \qquad \chi'(G) = \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) + \frac{|V(\mathcal{F})| - 2}{|V(\mathcal{F})|}.$$

Note that this is an exact expression for the chromatic index of a multigraph, and hence, by its own nature, is superior to all known upper and lower bounds on the chromatic index. However, a disadvantage of **(3)** is that it contains several quantities that may not be immediately computable. We comment briefly on this. First, note that the right-hand side of **(3)** can be evaluated only if we have a tense coloring ϕ of G (or, equivalently, a fan digraph), which can be obtained only if one knows the chromatic index of G (since $\chi'(G)$ is just the number of colors used by ϕ plus one). In other words, knowledge of the right-hand side of **(3)** presupposes knowledge of a tense coloring of G, which in turns implies *a priori* knowledge of the chromatic index of the multigraph. Despite this being true, in practical situations, we are most often faced with an entirely different problem; that is, we have only imprecise or partial information about the graph at hand and its chromatic properties, and we wish to test our information for accuracy. In this sense, Equation **(3)** may prove to be not only a powerful theoretical tool, but also one useful in applications. Suppose, for instance, that we have constructed an e-tense coloring of G, but we do not know if e is a critical edge; or suppose, conversely, that we know that e is a critical edge of G, but we are unable to tell if the coloring ϕ is an e-tense coloring. In both circumstances we may use **(3)** on the assumption that ϕ is an e-tense coloring and e is a critical edge, as a test that (at least in some cases) may allow us to decide if our assumption is false.

An expression that is even more compact may be obtained from **(3)** by taking **(1)** into account. Thus,

$$\textbf{(4)} \qquad \chi'(G) = \left\lceil \frac{1}{|V(\mathcal{F})|} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*(uw)) \right\rceil.$$

From **(4)** it is easily seen that **(3)** generalizes Vizing's Theorem since, obviously, $\deg_G(w) \leq \Delta(G)$ and $\mu^*(uw) \leq \mu(G)$ for any fan vertex w. In a similar way several other classical upper bounds on the chromatic index of multigraphs can be obtained (see **[6]**). The interesting feature of **(4)** is that it emphasizes the fact that the chromatic index of G can be expressed as the ceiling of the "average" of a certain quantity associated with each fan vertex w. This quantity depends, however, on the particular tense coloring (or fan digraph) chosen. From this point of view **(3)** is much more interesting, since it is obvious that the quantity at the left-hand side (and, consequently, at the right-hand side) of **(3)** is independent of the tense coloring chosen.

4. The Fire Index

For different reasons, some of which are indicated below, it is interesting to choose in **(3)** a tense coloring for which $|V(\mathcal{F})|$, the order of the corresponding fan digraph (or, equivalently, the number of fan vertices), is *maximum*. As a first justification for this choice, we now obtain a generalization of the well known *Vizing adjacency lemma* **[7]**. Suppose that u is a vertex of a critical simple graph G and v is a neighbor of u. Let $e \in uv$. Let $\mathcal{F}$ be an e-fan digraph at u of maximum order. Then, from Theorem 3,

$$\textbf{(5)} \qquad \sum_{w \in V(\mathcal{F})} \operatorname{cdef}(w) = |V(\mathcal{F})| - 1,$$

where we have used the fact that all fan multiplicities are equal to 1. Since G is simple and Class 2, we have $\chi'(G) = \Delta(G) + 1$, by Vizing's Theorem. Then from **(2)** we obtain

$$\textbf{(6)} \qquad \operatorname{cdef}(w) = \Delta(G) - \deg_G(w) \text{ if } w \in V(\mathcal{F})\backslash\{v\}$$

and

$$\textbf{(7)} \qquad \operatorname{cdef}(v) = \Delta(G) + 1 - \deg_G(w).$$

Define the *deficiency* of vertex x by

$$\operatorname{def}(x) = \Delta(G) - \deg_G(x).$$

Then, from **(5)**, **(6)**, and **(7)**, we obtain

$$\sum_{w \in V(\mathcal{F})} \operatorname{def}(w) = |V(\mathcal{F})| - 2.$$

Since $v \in V(\mathcal{F})$, we obtain

$$\operatorname{def}(v) + \sum_{w \in V(\mathcal{F})\backslash\{v\}} \operatorname{def}(w) = |V(\mathcal{F})| - 2,$$

and hence,

$$\textbf{(8)} \qquad \sum_{w \in V(\mathcal{F})\backslash\{v\}} \operatorname{def}(w) = |V(\mathcal{F})| - 2 - \operatorname{def}(v) = |V(\mathcal{F})| - 2 - \Delta(G) + \deg_G(v).$$

This statement is more informative than Vizing's adjacency lemma, which only gives information about the vertices of maximum degree adjacent to u. Indeed Vizing's adjacency lemma can be easily obtained from **(8)** by simply noticing that the number of vertices of positive deficiency in $V(\mathcal{F})\backslash\{v\}$ cannot exceed the right-hand side of **(8)**, and hence, the number of vertices of zero deficiency (that is, of maximum degree) in $V(\mathcal{F})\backslash\{v\}$ must be at least

$$(|V(\mathcal{F})| - 1) - (|V(\mathcal{F})| - 2 - \Delta(G) + \deg_G(v)) = \Delta(G) + 1 - \deg_G(v).$$

These are all vertices that are adjacent to u and distinct from v, whence Vizing's adjacency lemma follows. It should now be apparent that, the larger $V(\mathcal{F})$, the larger the number of neighbors of u for which we have some information. This example shows that it is of value to consider the largest possible fan digraph based at a given vertex. We now make one further step in this direction, and consider the largest possible fan digraph based at *any* vertex of G. In order to define this as a graph parameter, we initially start with an arbitrary Class 2 multigraph M. Let $\mathcal{G}$ be the class of submultigraphs of M that have the same chromatic index as M (and hence, are Class 2) and that have at least one critical edge. Note that G is non-empty, since there always exists a critical submultigraph of M with the same chromatic index as M **[7]**. Consider then, for each $G \in \mathcal{G}$, all the critical edges e of G and, for each of these, all the e-tense colorings ϕ of G. For each such tense coloring ϕ, consider the corresponding fan digraph $\mathcal{F}$. We define $f^*(M)$ to be the largest order among all the fan digraphs $\mathcal{F}$ described above. Thus,

(9) $f^*(M) = \max\{|V(\mathcal{F})|\}$,

where the maximum ranges over all the e-fan digraphs of G, G ranges over all the multigraphs in $\mathcal{G}$, and e ranges over all the critical edges of G. If M is a Class 1 multigraph, we simply define $f^*(M)$ to be ∞. We have now defined the parameter f^* for all multigraphs. We propose to call f^* the *fire index*.

We now offer additional evidence of the importance that this parameter has in edge coloring. A more comprehensive study will form the subject of other publications by the author.

First note that we are now able to provide an extension to multigraphs of the simple graph version of Vizing's theorem, as follows.

Theorem 4: Let M be a multigraph. Then either $\chi'(M) = \Delta(M)$ or there exists a submultigraph G of M that admits a fan digraph $\mathcal{F}$ such that

$$\chi'(M) = \frac{1}{f^*(M)} \cdot \sum_{w \in V(\mathcal{F})} (\deg_G(w) + \mu^*_G(uw)) + \frac{f^*(M) - 2}{f^*(M)}. \qquad \blacksquare$$

The importance, theoretical and practical, of the above statement cannot be underestimated, since it restricts the possible value of the chromatic index of an arbitrary multigraph to one of only two integers.

It is not too difficult to see (and it will be proved elsewhere) that, in order to evaluate the fire index, it suffices to evaluate it for only critical multigraphs. More precisely, we have the following.

Lemma 1: Let M be a Class 2 multigraph. Then

$$f^*(M) = \max\{f^*(H) : H \subseteq M, H \text{ critical}, \chi'(H) = \chi'(M)\}. \qquad \blacksquare$$

Note that, if H is a critical multigraph, $f^*(H)$ is simply the largest order of a fan digraph of H. Thus, Lemma 1 provides a significant conceptual simplification to the definition of the fire index.

Assume now that M is a Class 2 multigraph. Using **(1)** and the definition of the fire index, it is easily seen that

(10) $2 \le f^*(M) \le \Delta(M)$.

Hence, using Theorem 4 and the obvious inequalities

$$\deg_G(w) \le \Delta(G) \le \Delta(H)$$

and

$$\sum_{w \in V(\mathcal{F})} \mu^*_G(uw) \le \sum_{w \in V(\mathcal{F})} \mu_G(uw) \le \deg_G(u) \le \Delta(G) \le \Delta(H),$$

we obtain

(11) $$\chi'(M) \le \Delta(M) + \frac{\Delta(M) + f^*(M) - 2}{f^*(M)}.$$

Since the left-hand side of **(11)** is an integer, **(11)** implies

(12) $$\chi'(M) \le \Delta(M) + \left\lfloor \frac{\Delta(M) + f^*(M) - 2}{f^*(M)} \right\rfloor.$$

Because $f^*(M)$ is in the range **(10)**, this inequality provides a generalization of Shannon's theorem, which we state below. (Clearly, for Class 1 multigraphs, the right-hand side of **(12)** must be interpreted in the limit sense, that is, as $\Delta(M) + 1$).

Theorem 5: For any multigraph M,

$$\chi'(M) \le \Delta(M) + \left\lfloor \frac{\Delta(M) + f^*(M) - 2}{f^*(M)} \right\rfloor. \qquad \blacksquare$$

It is now clear from the monotonicity of the function $(\Delta + f^* - 2)/f^*$ with respect to f^*, that the larger the value of $f^*(M)$, the better is the bound on the chromatic index provided by Theorem 5. This provides a further motivation to the introduction of the fire index.

Finally we point out that, if we evaluate the right-hand side of the inequality of Theorem 5 by letting the fire index take successive even integer values 2, 4, …, 12, we obtain the quantities

$$\left\lfloor \frac{3\Delta}{2} \right\rfloor, \left\lfloor \frac{5\Delta+2}{4} \right\rfloor, \left\lfloor \frac{7\Delta+4}{6} \right\rfloor, \left\lfloor \frac{9\Delta+6}{8} \right\rfloor, \left\lfloor \frac{11\Delta+8}{10} \right\rfloor, \left\lfloor \frac{13\Delta+10}{12} \right\rfloor.$$

Curiously, these are precisely the quantities that appear in connection to the partial proofs of the Goldberg–Seymour conjecture obtained, respectively, by Vizing **[8]**, Goldberg **[9]**, Andersen **[10]**, Goldberg **[11]**, Nishizeki and Kashiwagi **[12]**, and Favrholdt *et al.* **[13]**. This clearly indicates a strong connection between the fire index and the most elusive unsolved question on the edge coloring of multigraphs. This we will try to clarify in the future.

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[GTN LIV:2] GRAPHS WHOSE VERTICES ARE FORESTS WITH BOUNDED DEGREE

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Abstract

Let $F(n, f)$ denote the graph whose vertex set is the set of all forests of order n with no vertex of degree greater than f. Vertices G and H of $F(n, f)$ are adjacent if and only if G and H differ (up to isomorphism) by exactly one edge. Results and open problems concerning the order, size, diameter, and traceability of $F(n, f)$ are presented.

1. Introduction

In **[1]** an extensive study of graphs with bounded vertex degree and the various contexts in which they appear is provided. Among the topics explored are graphs whose vertices are graphs with bounded degree. In turn, the study of these graphs suggests variations for consideration. For example, in **[2]** the distance properties of graphs whose vertices are forests with bounded degree are studied. In **[3]** other properties of the latter graphs are explored. In this paper, results and open problems concerning the order, size, diameter, and traceability of graphs whose vertices are forests with bounded degree are presented.

2. Graphs Whose Vertices are Forests

Let $F(n, f)$ denote the graph whose vertex set consists of all forests having order n and no vertex of degree greater than f. Vertices G and H of $F(n, f)$ are adjacent if and only if G and H differ (up to isomorphism) by exactly one edge. For the purpose of illustrating the various properties that are considered here it is useful to show all of the $F(n, f)$ with $4 \le n \le 7$ and $1 \le f \le 6$ (see Figures 1–18).

3. Order

Problem 1: What is the order of $F(n, f)$? □

Table 1 shows the order of $F(n, f)$ based on the order n of each *forest-vertex* (a vertex in $F(n, f)$) and f, the largest vertex degree in each forest-vertex.

Table 1: Order of $F(n,f)$ for $2 \le n \le 7$ and $1 \le f \le n - 1$.

n	$f = 1$	$f = 2$	$f = 3$	$f = 4$	$f = 5$	$f = 6$
2	2					
3	2	3				
4	3	5	6			
5	3	7	9	10		
6	4	11	17	19	20	
7	4	15	28	34	36	37

For any given number of vertices, there is a specific number of trees. The number of trees on i vertices is denoted by t_i. The values of t_i for $1 \le i \le 10$ are shown in Table 2. Values of t_i for $1 \le i \le 30$ can be found in Table A4 page 253 of **[1]**.

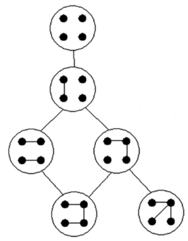

Figure 1: *F*(4, 3)

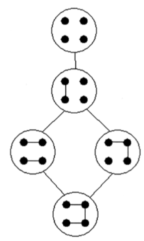

Figure 2: *F*(4, 2)

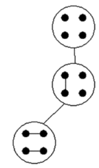

Figure 3: *F*(4, 1)

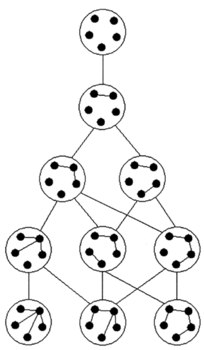

Figure 4: *F*(5, 4)

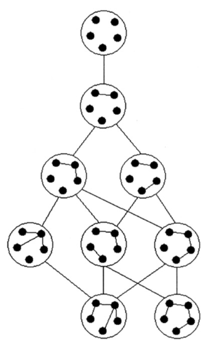

Figure 5: *F*(5, 3)

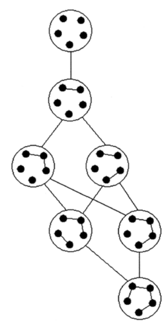

Figure 6: *F*(5, 2)

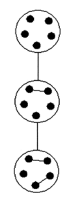

Figure 7: *F*(5, 1)

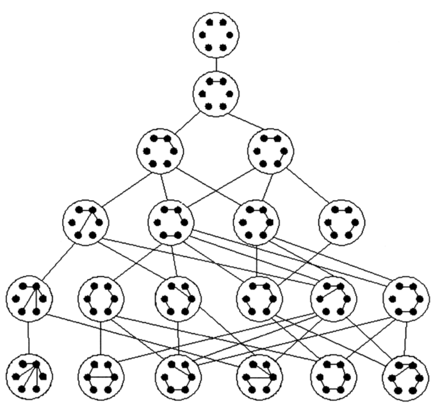

Figure 8: *F*(6, 5)

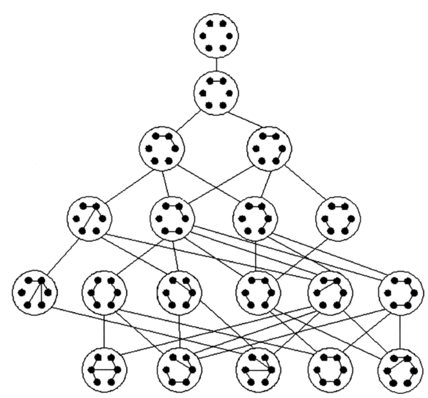

Figure 9: *F*(6, 4)

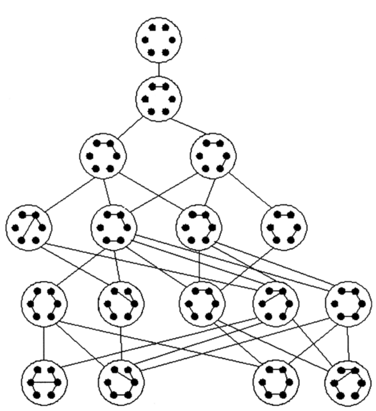

Figure 10: *F*(6, 3)

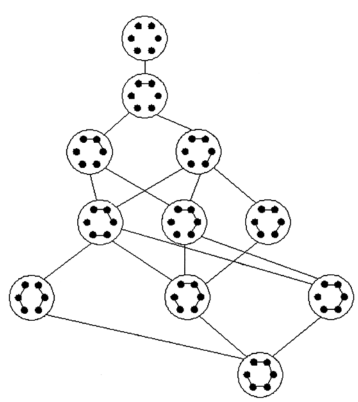

Figure 11: *F*(6, 2)

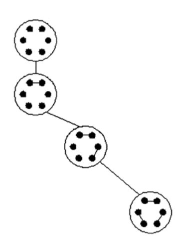

Figure 12: *F*(6, 1)

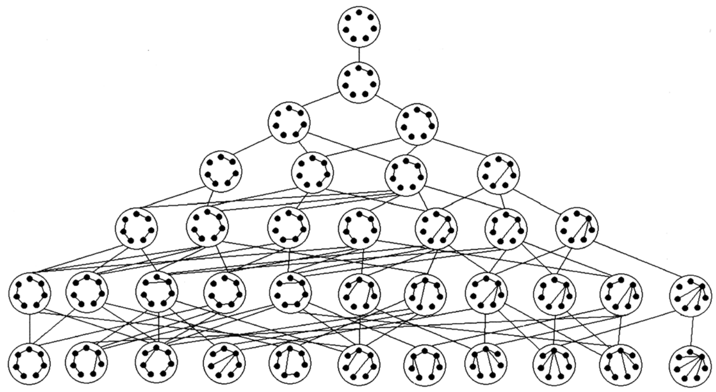

Figure 13: *F*(7, 6)

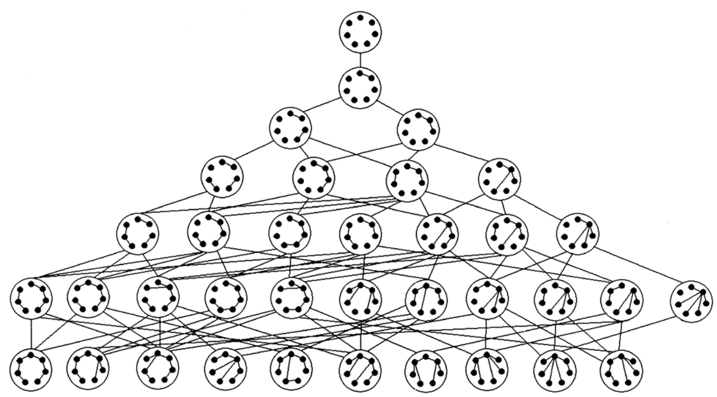

Figure 14: *F*(7, 5)

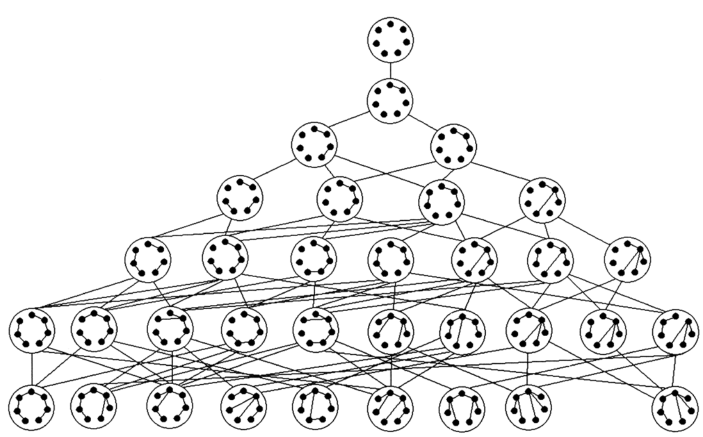

Figure 15: *F*(7, 4)

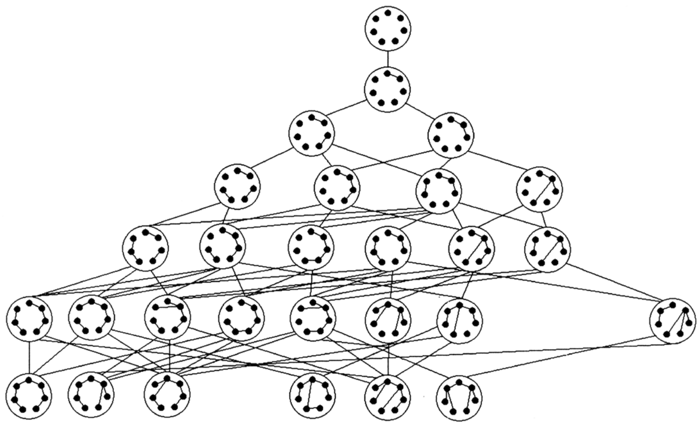

Figure 16: *F*(7, 3)

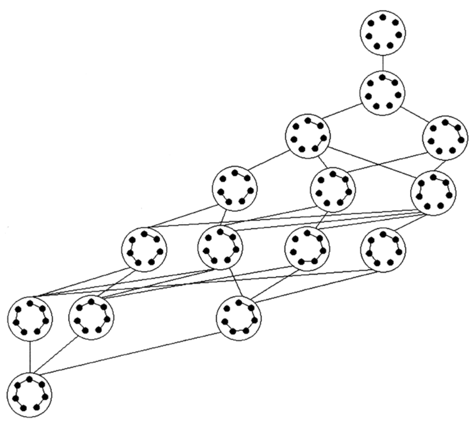

Figure 17: *F*(7, 2)

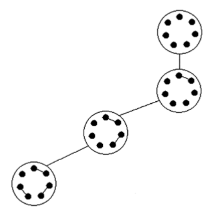

Figure 18: *F*(7, 1)

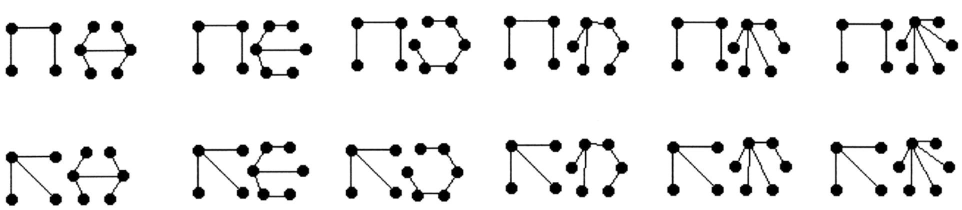

Figure 19: Forests on 10 vertices partitioned into 4 + 6.

Figure 20: Isomorphic forests on 10 vertices.

When looking at a diagram of $F(n, f)$, one can see that the vertices are partitioned into levels. These levels correspond to the size of the forest-vertices at each level and may be numbered accordingly. This results in the uppermost level being numbered 0 and the lowest level numbered $n-1$. Because the order n is constant throughout $F(n, f)$ and, on a given level, the size is constant, the number of components on that level is constant and equal to $n-e$. We can use this information to determine the number of vertices at a given level in $F(n, f)$ and by summing over all levels obtain the number of vertices in $F(n, f)$.

Table 2: Number of trees, t_i, on i vertices, $1 \le i \le 10$.

i	t_i
1	1
2	1
3	1
4	2
5	3
6	6
7	11
8	23
9	47
10	106

First, note the following. Consider 10 vertices partitioned into two parts. One possible partition is 4 + 6. On four vertices there are two trees and on six vertices there are six trees. Hence, there are 12 forests on 10 vertices partitioned into two parts as 4 + 6. These are shown in Figure 19.

Next, consider 10 vertices partitioned into two parts, each part having five vertices. Now $t_5 \times t_5 = 3 \times 3 = 9$. However, this includes graphs like *G* and *H*, shown in Figure 20, as two of the nine possible forests associated with the partition 5 + 5. However, *G* and *H* are isomorphic and should not be counted as distinct vertices. To avoid this problem we consider the number of combinations of *t* objects taken *m* at a time with repetition allowed; that is, $\binom{t+m-1}{m}$. Hence, for 10 partitioned into 5 + 5 we obtain $\binom{3+2-1}{2} = 6$. This can be seen in Figure 21 as the correct number of forests associated with this partition.

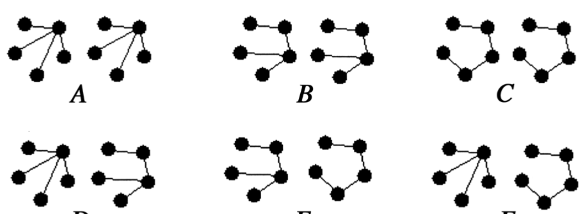

Figure 21: Forests on 10 vertices partitioned into 5 + 5.

We now show how the order of $F(6, 5)$ can be obtained. Consider level 4. There are two components since $6-4=2$. There are three ways to partition 6 into two parts. These are 1 + 5, 2 + 4, and 3 + 3. When the parts of the partitions are unequal, 1 + 5 and 2 + 4, the number of forests is easily seen to be $t_1 \times t_5 = 1 \times 3 = 3$ and $t_2 \times t_4 = 1 \times 2 = 2$, respectively. The values of t_i can be obtained from Table 2. For the partition 3 + 3 we must allow for isomorphic forest-vertices and thus the number of distinct forest-vertices with partition 3 + 3 is

$$\binom{t_1+2-1}{2} = \binom{2}{2} = 1.$$

Consequently, the number of vertices of $F(6, 5)$ on level 4 is $3+2+1 = 6$. To determine the order of $F(6, 5)$ we repeat this process for each level and sum the results. This is illustrated in Table 3.

Table 3: Order of $F(6, 5)$.

Level e	$c = 6 - e$	Partitions of 6 into c parts	Number of forests at level e
0	6	111111	$\binom{t_1+5}{6} = \binom{6}{6} = 1$
1	5	11112	$\binom{t_1+1}{4} \times t_2 = \binom{4}{4} \times 1 = 1$
2	4	1113, 1122	$\binom{t_1+2}{3} \times t_3 + \binom{t_1+1}{2} \times \binom{t_2+1}{2} = 2$
3	3	114, 123, 222	$\binom{t_1+1}{2} \times t_4 + t_1 \times t_2 \times t_3 + \binom{t_2+2}{3} = 4$
4	2	15, 24, 33	$t_1 \times t_5 + t_2 \times t_4 + \binom{t_3+1}{2} = 6$
5	1	6	$t_6 = 6$

On summing the right hand column we obtain 20, which is the order of $F(6, 5)$ (see Figure 8).

The above procedure generalizes to one for determining the order of any $F(n, n-1)$ and, in fact, can be extended to any $F(n, f)$ for which $f \neq 1$ by using t_i^f, *the number of trees having order i and no vertex with degree greater than f*, in place of t_i. This is summarized in the following theorem, where trees (or forests) having no vertex degree greater than *f* are called *f-trees* (or *f-forests*).

Theorem 1: Consider the graph $F(n, f)$, $2 \leq f \leq n-1$.
Let $a_1 m_1 + a_2 m_2 + \ldots + a_d m_d = n$ be a partition of *n* into *c* parts, where the a_j are nonzero and distinct and $\sum_{j=1}^{d} m_j = c$. Let t_i^f denote the number of *f*-trees of order *i*.
Then the number of vertices in $F(n, f)$ corresponding to *f*-forests of size *e* is

$$V_e = \sum \left(\prod_{i=1}^{d} \binom{t_{a_i}^f + m_i - 1}{m_i} \right)$$

where the sum is over all partitions of *n* into *c* parts, $1 \leq c \leq n$.
The order of $F(n, f)$ is

$$\sum_{e=0}^{n-1} V_e .$$

Proof: Consider the graph $F(n, f)$, $2 \leq f \leq n-1$. On level *e*, the *f*-forest-vertices have $c = n - e$ components, since *n* is constant over the entire graph and for each level *e* is fixed (see Corollary 4.5(b) in **[4]**).

Let $a_1 m_1 + a_2 m_2 + \ldots + a_d m_d = n$ be a partition of *n* into *c* parts, where the a_j are nonzero and distinct and let $\sum_{j=1}^{d} m_j = c$.

Let t_i^f denote the number of *f*-trees of order *i*. If the parts are distinct; that is, if there are *c* different parts, then the number of forests with that partition is

$$\prod_{i=1}^{c} t_i^f .$$

However, if any part is repeated; that is, if $m_j \geq 2$ for any *j*, then the number of forests on $m_j a_j$ vertices partitioned into m_j parts of size a_j is the number of combinations of *t* objects taken *m* at a time with repetition.

Table 4: t_i^f for $2 \le f \le 16$ and $1 \le i \le 17$

i	$f = 2$	$f = 3$	$f = 4$	$f = 5$	$f = 6$	$f = 7$	$f = 8$	$f = 9$	$f = 10$	$f = 11$	$f = 12$	$f = 13$	$f = 14$	$f = 15$	$f = 16$
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
4	1	2	2	2	2	2	2	2	2	2	2	2	2	2	2
5	1	2	3	3	3	3	3	3	3	3	3	3	3	3	3
6	1	4	5	6	6	6	6	6	6	6	6	6	6	6	6
7	1	6	9	10	11	11	11	11	11	11	11	11	11	11	11
8	1	11	18	21	22	23	23	23	23	23	23	23	23	23	23
9	1	18	35	42	45	46	47	47	47	47	47	47	47	47	47
10	1	37	75	94	101	104	105	106	106	106	106	106	106	106	106
11	1	66	159	204	223	230	233	234	235	235	235	235	235	235	235
12	1	135	355	473	520	539	546	549	550	551	551	551	551	551	551
13	1	265	802	1098	1223	1270	1289	1296	1299	1300	1301	1301	1301	1301	1301
14	1	552	1858	2633	2954	3081	3128	3147	3154	3157	3158	3159	3159	3159	3159
15	1	1132	4347	6353	7208	7536	7663	7710	7729	7736	7739	7740	7741	7741	7741
16	1	2410	10359	15641	17905	18785	19115	19242	19289	19308	19315	19318	19319	19320	19320
17	1	5098	24894	38789	44863	47207	48094	48424	48551	48598	48617	48624	48627	48628	48629

That is,

$$\binom{t_{a_j}^f + m_j - 1}{m_j}.$$

Therefore, the number of *f*-forests corresponding to the partition $\sum_{j=1}^{d} a_j m_j$ is

$$\prod_{i=1}^{d} \binom{t_{a_i}^f + m_i - 1}{m_i}.$$

Repeating this process for all possible partitions of n into c parts and summing results yields the number of vertices of $F(n, f)$ on level e. Thus,

$$V_e = \sum \left(\prod_{i=1}^{d} \binom{t_{a_i}^f + m_i - 1}{m_i} \right)$$

where the sum is over all partitions of n into c parts, $1 \le c \le n$.

Summing over all levels in $F(n, f)$ results in the order of $F(n, f)$. That is, the order of $F(n, f)$ is

$$\sum_{e=0}^{n-1} V_e. \qquad \blacksquare$$

Note that if $m_j = 1$, then

$$\binom{t_{a_j}^f + m_j - 1}{m_j} = \binom{t_{a_j}^f}{1} = t_{a_j}^f.$$

Drawings of all trees with order $1 \le i \le 10$ are shown in **[4]** (pp. 233–234). From these drawings t_i^f can be obtained for $1 \le i \le 10$ and $1 \le f \le 9$. This data is given in Table 4. The number of trees $t_i = t_i^{i-1}$ can be generated through the counting series for trees given in Theorem 15.11 in **[4]**. Values for t_i for $i \le 30$ are given in Table A.4 of **[1]**.

The number of trees t_i^f for $f = 1, 2, 3, 4$ and for $i \le 25$ can be found by using Tables F and N in **[5]**; see also Table 1 in **[6]** for $i = 25$. Using the McKay packed Nauty geng algorithm, values of t_i^f can be generated **[7]**. Values of t_i^f for $2 \le f \le 16$ and $1 \le i \le 17$ are shown in Table 4.

4. Size

Problem 2: What is the size of $F(n, f)$? □

For $2 \le n \le 7$ and $1 \le f \le n - 1$, the size of $F(n, f)$ can be obtained from Figures 1–18, as shown in Table 5. A general solution to Problem 2 is not known.

Table 5: Size of $F(n,f)$ for $2 \le n \le 7$ and $1 \le f \le n - 1$.

n	$f = 1$	$f = 2$	$f = 3$	$f = 4$	$f = 5$	$f = 6$
2	1					
3	1	2				
4	2	5	6			
5	2	9	13	14		
6	3	18	32	36	37	
7	3	28	67	82	87	88

5. Distance Problems

Problem 3: What is the diameter of $F(n, f)$? □

In **[2]** various distance properties of $F(n, f)$ are investigated. In particular, the following theorem concerning the *diameter* of $F(n, f)$, $\mathrm{diam}(F(n, f))$ is obtained.

Theorem 2 ([2]): If $f = 2, 3, n-2, n-1$, then

$$\mathrm{diam}(F(n, f)) = \max\{n-1, n+f-5\} = \begin{cases} n-1 & \text{if } f = 2, 3 \\ n+f-5 & \text{if } f = n-2, n-1. \end{cases}$$

Values for $\mathrm{diam}(F(n, f))$ can be obtained from Figures 1–18. Note that $\mathrm{diam}(F(7, 4)) = 6$ and that $\mathrm{diam}(F(n, 1)) = \lfloor n/2 \rfloor$.

In a detailed study **[1]** of graphs with bounded degree the concepts of height and width of $U(n, f)$, a graph whose vertices are graphs with bounded degree, are defined. Analogously, define the *height* of $F(n, f)$ to be the length of a shortest path from level 0 to the lowest level in $F(n, f)$. Thus, the height of $F(n, f)$ is $n-1$ for all *f* except $f = 1$, where the height is $\lfloor n/2 \rfloor$. That is, for $f \geq 2$ the height is the distance from the empty graph of order *n* to an *f*-tree of order *n*. On the other hand, the diameter of $F(n, f)$ can be thought of as the *width* of $F(n, f)$. In $U(n, f)$, for small values of *n*, the width is equal to the height; however, for large values of *n*, the width exceeds the height. Let $w(f)$ denote the least integer such that the width is greater than the height. When considering $F(n, f)$, $w(f)$ is the least integer such that $\mathrm{diam}(F(w(f), f)) > n-1$.

Problem 4: What is the least order $w(f)$ for which the width exceeds the height of $F(n, f)$? □

There are two known values for $w(f)$. From $F(6, 5)$, $\mathrm{diam}(F(6, 5)) = 6 > 6-1$, so that $w(5) = 6$. Similarly, $w(6) = 7$ since $\mathrm{diam}(F(7, 6)) = 7 > 7-1$.

The value of $w(f)$ in general is an open problem for both $F(n, f)$ and $U(n, f)$. For more information with respect to this problem for $U(n, f)$ see Theorem 8,23 in **[1]**.

6. Hamiltonian Paths

A graph is call *traceable*, if it contains a Hamilton path. The following theorem follows from Lemma 2.2 in **[8]** adapted to apply to *f*-forests. This theorem enables us to eliminate the possibility for some $F(n, f)$ to have a Hamilton path. On the other hand, only by finding a Hamilton path are we able to state that a given $F(n, f)$ is traceable. Let $e_f(n)$ and $o_f(n)$ denote the number of unlabeled *f*-forests of order *n* having even size and odd size, respectively.

Theorem 3: If $F(n, f)$ contains a Hamilton path $K_n^c \to G$, then

(1) The order $N(n, f)$ of $F(n, f)$ and the size of *G* have opposite parity.

(2) $e_f(n) - o_f(n) = 0$ when $N(n, f)$ is even; $e_f(n) - o_f(n) = 1$ when $N(n, f)$ is even.

(3) $F(n, f)$ has no more than two pendant edges. ■

Problem 5: For what *n* and *f* does $F(n, f)$ contain a Hamilton path? □

From Figures 4, 8–10, 13–16 and Theorem 3(2), it follows that $F(n, f)$ does not have a Hamilton path when $n = 7$, $f = 3, 4, 5, 6$; $n = 6$, $f = 3, 4, 5$; and $n = 5$, $f = 4$. Using Theorem 1, the number of vertices of even and odd size can be determined. When this is calculated for $n = 8$, $f = 2, 3, 4, 5, 6, 7$ and $n = 9$, $f = 2, 3, 4, 5, 6, 7, 8$, it can be determined from Theorem 3(2) that there is no Hamilton path. By inspection, it is easy to find a Hamilton path when $n = 4$, $f = 1, 2, 3$; $n = 5$, $f = 1, 2, 3$; $n = 6$, $f = 1, 2$; and $n = 7$, $f = 1$. Figure 22 shows a Hamilton path for $F(7, 2)$.

Determining traceability in general is an open problem for both $F(n, f)$ and $U(n, f)$, for elaboration on the latter see Chapter 9 in **[1]**.

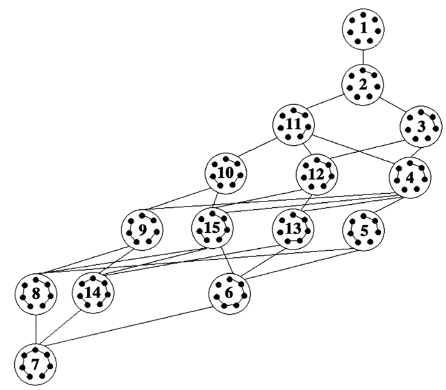

Figure 22: A Hamilton path on $F(7, 2)$.

Acknowledgement

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[GTN LIV:3] TOTAL NUMBER OF DOMATIC PARTITIONS FOR SPECIAL GRAPHS

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Abstract

A partition of $V(G)$, all of whose classes are dominating sets in G, is called a domatic partition of G. The maximum number of classes in a domatic partition of G is called the domatic number of G. If W is a domatic partition of G with cardinality k, then k is at most the domatic number of G. In this paper, we first define recursively the number of all domatic partitions of a path and a cycle on n vertices. Next, we give the solutions of recurrences using generating functions for Fibonacci or Lucas numbers.

1. Introduction

Let G be a finite, undirected graph with no loop or multiple edge, $V(G)$ the set of vertices of G, and $E(G)$ the set of edges of G. A set $D \subseteq V(G)$ is called a *dominating set* in G if every vertex not in D is adjacent to at least one vertex in D. A *domatic partition* of a graph G is a partition of $V(G)$ into pairwise disjoint dominating sets. The *domatic number* of G is the maximum cardinality among all domatic partitions of G and is denoted by $d(G)$. The domatic number was introduced by E.J. Cockayne and S.T. Hedetniemi in **[1]**.

We note the simple observation.

Observation 1: If $W = \{D_1, \ldots, D_k\}$ is a domatic partition of G, then $k \le d(G)$. Additionally, if $k = 1$, then $W = \{D_1\}$, where $D_1 = V(G)$ is a unique domatic partition of G. ■

We now define the special classes of graphs that will be considered in this paper.

Let P_n and C_n, respectively, denote the path and cycle graphs of order n, with $n \ge 1$ for the path graphs and $n \ge 3$ for the cycle graphs. It is immediately seen that:

Observation 2:

(1) $$d(P_n) = \begin{cases} 1 & \text{for } n = 1 \\ 2 & \text{otherwise.} \end{cases}$$

(2) $$d(C_n) = \begin{cases} 2 & \text{if } n \not\equiv 0 \pmod 3 \\ 3 & \text{otherwise.} \end{cases}$$ ■

For concepts not defined in this paper see **[2]**.

Because we express recurrences using Fibonacci numbers, F_n, or Lucas numbers, L_n, we recall their recurrence forms:

$$F_0 = 1,\ F_1 = 2, \text{ and for } n \ge 1\ \ F_{n+1} = F_n + F_{n-1},$$

$$L_0 = 2,\ L_1 = 1, \text{ and for } n \ge 1\ \ F_{n+1} = F_n + F_{n-1}.$$

The following notation is used in our investigation: Let S denote any assertion that may be true or false, then

$$[S] = \begin{cases} 1 & \text{if } S \text{ is true} \\ 0 & \text{otherwise.} \end{cases}$$

We make use of the following elementary generating functions:

Observation 3 [3]:

(1) $\sum_{n \geq 0} [n = m] z^n = z^m$, for $m \in \mathbb{N} \cup \{0\}$,

(2) $\sum_{n \geq 0} c^n z^n = \dfrac{1}{1 - cz}$,

(3) $\sum_{n \geq 0} [m/n] z^n = \dfrac{1}{1 - z^m}$. ■

Our aim is to determine the number of domatic partitions for the path graph P_n and the cycle graph C_n.

2. The Number of Domatic Partitions of the Path Graph, P_n

Initially, we establish the recurrence definition for the *number of domatic partitions* of the path P_n, $n \geq 1$. It is clear that $W = \{\{x_1\}\}$ is the unique domatic partition for P_1. Therefore, assume $n \geq 2$. Let $W = \{D_1, \ldots, D_k\}$ be a domatic partition of P_n. Recall that $d(P_n) = 2$ and, therefore, $k \leq 2$. Taking account of Observation 1, we consider only the case $k = 2$. Let

$I(P_n) = \{W : W$ is a domatic partition of P_n and $|W| = 2\}$, and

$i(P_n) = |I(P_n)|$.

The number $i(P_n)$ of domatic partitions of the path P_n into two sets $\{D_1, D_2\}$ trivially satisfies the following Fibonacci recurrence, as can be seen by simply considering whether x_{n-1} is in the same set as x_{n+1} or in the set containing x_n.

Lemma 1: For $n \geq 3$, $i(P_{n+1}) = i(P_n) + i(P_{n-1})$, $i(P_2) = 1$, and $i(P_3) = 1$. ■

Denote by $i_c(P_n)$ the cardinality of all possible domatic partitions of the graph P_n ($n \geq 2$). From the previous lemma we obtain the following result.

Theorem 1: For $n \geq 2$, $i_c(P_n) = i(P_n) + 1$. ■

We now specify explicitly the sequence of numbers $i(P_n)$, $n = 3, 4, \ldots$, described by the recurrence relation from Lemma 1. Our aim is to solve this recurrence and to express its explicit formula in terms of Fibonacci numbers. We begin with the relationship between the number $i(P_n)$ and the Fibonacci number F_n.

Lemma 2: For $n \geq 3$, $i(P_n) = F_{n-3}$.

This is simply a corollary of Lemma 1 and the definition of the Fibonacci number F_n. ■

Using Theorem 1 and Lemma 2 we obtain

Theorem 2: For $n \geq 3$, $i_c(P_n) = F_{n-3} + 1$. ■

Next, we give a more explicit formula for the sequence $i(P_n)$, $n = 2, 3, \ldots$, described by recurrence of Lemma 1. For this purpose we use the method of generating functions, see **[3]**.

Theorem 3: For $n \geq 2$, $i(P_n) = B\phi^n + (1 - B)(1 - \phi)^n$,
where $B = (5 - \sqrt{5})/10$ and $\phi = (1 + \sqrt{5})/2$.

Proof: We write the recurrence from Lemma 1 using one equality having the form

$$i(P_n) = i(P_{n-1}) + i(P_{n-2}) + [n = 2],$$

which is satisfied for all integers n, whence we define $i(P_{-n}) = 0$ for $n \geq -1$.

For brevity, instead of $i(P_n)$ we write i_n. Let the term i_n be associated with the generating function

$$I(z) = \sum_n i_n z^n.$$

Indeed, one has

$$I(z) = \sum_n i_{n-1}z^n + \sum_n i_{n-2}z^n + \sum_n [n = 2]z^n = \sum_n i_n z^{n+1} + \sum_n i_n z^{n+2} + z^2$$
$$= zI(z) + z^2 I(z) + z^2.$$

Thus, the generating function for i_n has the form

$$I(z) = -1 + \frac{-z+1}{(1-\phi z)(1-\hat{\phi} z)},$$

where ϕ is the golden ratio (that is, $\phi = (1+\sqrt{5})/2$ and $\hat{\phi} = 1 - \phi$. Decomposing this rational function we obtain

$$I(z) = -1 + \frac{B}{1-\phi z} + \frac{1-B}{1-\hat{\phi} z}, \text{ where } B = \frac{5-\sqrt{5}}{10}.$$

Consequently, the generating function has the form

$$I(z) = \sum_{n \geq 0} \{-[n = 0] + B\phi^n + (1-B)\hat{\phi}^n\} z^n,$$

by (1) and (2) from Observation 3.

This means that if $n \geq 1$, then $i(P_n) = B\phi^n + (1-B)\hat{\phi}^n$, where $B = (5-\sqrt{5})/10$ and $\phi = (1+\sqrt{5})/2$. ■

3. The Number of Domatic Partitions of the Cycle Graph, C_n

In this section we define a recurrence for the *number of domatic partitions of the cycle* C_n, $n \geq 3$. Let $W = \{D_1, \ldots, D_k\}$ be a domatic partition for C_n. By Observation 1 and Observation 2 (2), if $n \not\equiv 0$ (mod 3), then $|W| \leq 2$; if $n \equiv 0$ (mod 3), then $|W| \leq 3$. Recall that if $k = 1$, then $W = \{D_1\}$, where $D_1 = V(C_n)$. It is not difficult to see that if $n \equiv 0$ (mod 3), then there exists a unique domatic partition W of C_n such that $W = \{D_1, D_2, D_3\}$, where $D_1 = \{x_1, x_4, \ldots, x_{n-2}\}$, $D_2 = \{x_2, x_5, \ldots, x_{n-1}\}$, and $D_3 = \{x_3, x_6, \ldots, x_n\}$. For this reason, it remains to consider domatic partitions W with two classes; that is, $k = 2$. For clarity, let $W = \{D_1, D_2\}$ and let

$$I(C_n) = \{W : W \text{ is a domatic partition of } C_n \text{ and } |W| = 2\}.$$

This implies that the following subsets of $I(C_n)$ be considered:

$$I_1(C_n) = \{W : W = \{D_1, D_2\}, x_1, x_3, x_n \in D_1, x_2 \in D_2\},$$
$$I_2(C_n) = \{W : W = \{D_1, D_2\}, x_2, x_3, x_n \in D_1, x_1 \in D_2\},$$
$$I_3(C_n) = \{W : W = \{D_1, D_2\}, x_1, x_n \in D_1, x_2, x_3 \in D_2\},$$
$$I_4(C_n) = \{W : W = \{D_1, D_2\}, x_3, x_n \in D_1, x_1, x_2 \in D_2\},$$
$$I_5(C_n) = \{W : W = \{D_1, D_2\}, x_2, x_n \in D_1, x_1, x_3 \in D_2\}.$$

It is not difficult to see that there is no domatic partition of C_4 that belongs to $I_1(C_4)$ or $I_2(C_4)$, and that there is no domatic partition of C_5 that belongs to $I_3(C_5)$. Thus, assume that $I_1(C_4) = I_2(C_4) = I_3(C_5) = \varnothing$. In other cases, all of the determining sets listed above are well defined.

The cardinalities of these sets are as follows:

$$i(C_n) = |I(C_n)|,$$
$$i_s(C_n) = |I_s(C_n)|, \quad \text{for } s = 1, \ldots, 5 \text{ and } n \geq 4.$$

Since there exists no domatic partition $W = \{D_1, D_2\}$ where $x_1, x_2, x_n \in D_1$ and $x_3 \in D_2$, we conclude that

(1) $$i(C_n) = \sum_{s=1}^{5} i_s(C_n) \text{ for } n \geq 4.$$

Let $i_c(C_n)$ denote the cardinality of the set of all possible domatic partitions of the cycle C_n, for $n \geq 3$. Then we deduce the following:

Observation 4: For $n \geq 3$,

$$i_c(C_n) = \begin{cases} i(C_n) + 2 & \text{if } n \equiv 0 \pmod 3 \\ i(C_n) + 1 & \text{otherwise.} \end{cases} \qquad \blacksquare$$

In the next theorem, we give a recurrence for the number of domatic partitions of C_n with cardinality 2. To do this we first prove the necessary lemmas.

Lemma 3: Let $n \geq 5$. Then $i_1(C_{n+1}) = i_5(C_n)$.

Proof: Suppose that W is a domatic partition of C_{n+1} for $n \geq 5$, such that $W = \{D_1, D_2\}$. Moreover, let $x_1, x_3, x_{n+1} \in D_1$ and $x_2 \in D_2$. Since for C_{n+1}, $x_1, x_{n+1} \in D_1$, then $x_n \in D_2$. Thus, $\{D_1 \backslash \{x_1\}, D_2\}$ of the set $A = V(C_{n+1}) \backslash \{x_1\}$ is a domatic partition of the graph $H_1 \cong C_n$, such that $V(H_1) = A$ and $E(H_1) = (E(C_{n+1}) \backslash \{x_1x_2, x_1x_{n+1}\}) \cup \{x_2x_{n+1}\}$. Furthermore, $\{D_1 \backslash \{x_1\}, D_2\} \in I_5(H_1)$. Thus,

$$i_1(C_{n+1}) = i_5(H_1) = i_5(C_n). \qquad \blacksquare$$

The next two lemmas can be proved analogously.

Lemma 4: Let $n \geq 5$. Then $i_2(C_{n+1}) = i_5(C_n)$. $\blacksquare$

Lemma 5: Let $n \geq 5$. Then $i_3(C_{n+1}) = i_5(C_{n-1})$. $\blacksquare$

Lemma 6: Let $n \geq 5$. Then $i_4(C_{n+1}) = i_5(C_n) + i_5(C_{n-1})$.

Proof: Assume $W = \{D_1, D_2\}$ is a domatic partition of C_{n+1} for $n \geq 5$. Let $x_3, x_{n+1} \in D_1$ and $x_1, x_2 \in D_2$. Consider two cases.

<u>Case 1</u>: Let $x_4 \in D_1$. Since $x_3 \in D_1$ it follows that $x_5 \in D_2$. Hence, $\{D_1 \backslash \{x_3\}, D_2 \backslash \{x_2\}\}$ of the set $B = V(C_{n+1}) \backslash \{x_2, x_3\}$ is a domatic partition of the graph $H_1 \cong C_{n-1}$, where $V(H_1) = B$ and

$$E(H_1) = (E(C_{n+1}) \backslash \{x_1x_2, x_2x_3, x_3x_4\}) \cup \{x_1x_4\}.$$

Thus, $\{D_1 \backslash \{x_3\}, D_2 \backslash \{x_2\}\} \in I_5(H_1)$.

<u>Case 2</u>: Let $x_4 \notin D_1$. Then $x_4 \in D_2$. It is clear that $\{D_1, D_2 \backslash \{x_2\}\}$ of the set $C = V(C_{n+1}) \backslash \{x_2\}$ is a domatic partition of the graph $H_2 \cong C_n$, such that $V(H_2) = C$ and

$$E(H_2) = (E(C_{n+1}) \backslash \{x_1x_2, x_2x_3\}) \cup \{x_1x_3\}.$$

Thus, $\{D_1, D_2 \backslash \{x_2\}\} \in I_5(H_2)$.

From these two cases, we obtain $i_4(C_{n+1}) = i_5(H_1) + i_4(H_2) = i_5(C_{n-1}) + i_5(C_n)$, as required. $\blacksquare$

A similar proof can be used to obtain the next lemma.

Lemma 7: Let $n \geq 5$. Then $i_5(C_{n+1}) = i_4(C_n) + i_4(C_{n-1})$. $\blacksquare$

Theorem 4: For $n \geq 4$,

$$i(C_{n+1}) = \begin{cases} i(C_n) + i(C_{n-1}) + 2 & \text{if } n \equiv 2 \pmod 3 \\ i(C_n) + i(C_{n-1}) - 1 & \text{otherwise} \end{cases}$$

and $i(C_3) = i(C_4) = 3$.

Proof: It is not difficult to see that the initial conditions hold. Thus, we prove the recurrence relation for $n \geq 4$. First, let $n = 4$. Construct the domatic partitions that belong to each of the sets $I_i(C_5)$ for $i = 1, 2, 4, 5$ (we already know that $I_3(C_5) = \varnothing$). Evidently, these sets have the form:

$$I_1(C_5) = \{\{\{x_1, x_3, x_5\}, \{x_2, x_4\}\}\}$$
$$I_2(C_5) = \{\{\{x_2, x_3, x_5\}, \{x_1, x_4\}\}\}$$
$$I_4(C_5) = \{\{\{x_3, x_5\}, \{x_1, x_2, x_4\}\}\}$$
$$I_5(C_5) = \{\{\{x_2, x_4, x_5\}, \{x_1, x_3\}\}, \{\{x_2, x_5\}, \{x_1, x_3, x_4\}\}\}.$$

This means that $i_1(C_5) = i_2(C_5) = i_4(C_5) = 1$, $i_3(C_5) = 0$, and $i_5(C_5) = 2$. In consequence, by Equation **(1)**, $i(C_5) = 5 = i(C_4) + i(C_3) - 1$, using the initial conditions, and for $n = 4$ this recurrence holds.

Now let $n \geq 5$. Combining **(1)** with Lemmas 3–7, the number $i(C_{n+1})$ has the form

$$i(C_{n+1}) = 3i_5(C_n) + 2i_5(C_{n-1}) + i_4(C_n) + i_4(C_{n-1}).$$

Additionally, put $i_4(C_3) = i_5(C_2) = 1$ and $i_5(C_3) = 0$ (then the following statements for $n = 5$ and $n = 6$ are true). By substituting for $i_5(C_n)$, $i_5(C_{n-1})$, and again $i_5(C_{n-1})$ using Lemma 7, Lemma 4, and Lemma 3, respectively, we obtain

$$i(C_{n+1}) = 2i_5(C_n) + i_4(C_n) + 2i_4(C_{n-1}) + i_4(C_{n-2}) + i_2(C_n) + i_1(C_n).$$

Replacing $i_4(C_{n-1})$ by the equation from Lemma 6 and applying Lemma 5 to $i_5(C_{n-2})$ using **(1)** we obtain

$$i(C_{n+1}) = i(C_n) + i_5(C_n) + i_5(C_{n-3}) + i_4(C_{n-1}) + i_4(C_{n-2}).$$

Now using Lemma 5 with $i_5(C_{n-3})$, then adding $i_1(C_{n-1})$, $i_2(C_{n-1})$, $i_5(C_{n-1})$, subtracting ones, and further using **(1)** (replacing n by $n - 1$), we conclude that

$$i(C_{n+1}) = i(C_n) + i(C_{n-1}) + i_5(C_n) - i_5(C_{n-1}) + i_4(C_{n-2}) - i_4(C_{n-1}) + i_1(C_{n-1}).$$

Replacing $i_1(C_{n-1})$ and $i_2(C_{n-1})$ by the equations from Lemma 3 and Lemma 4, respectively, and using Lemma 6, then

$$i(C_{n+1}) = i(C_n) + i(C_{n-1}) + i_5(C_n) - i_5(C_{n-2}) + i_4(C_{n-2}) - i_4(C_n).$$

Next, adding terms $i_4(C_{n-1})$, $i_5(C_{n-1})$, subtracting ones, and applying Lemma 6 and Lemma 7 twice, we obtain

(2) $$i(C_{n+1}) = i(C_n) + i(C_{n-1}) + i_5(C_n) - i_5(C_{n+1}) + i_4(C_{n+1}) - i_4(C_n).$$

Finally, applying induction on n, $n \geq 4$, we arrive at

(3) $$i_5(C_n) - i_5(C_{n+1}) + i_4(C_{n+1}) - i_4(C_n) = \begin{cases} 2 & \text{if } n \equiv 2 \pmod 3 \\ -1 & \text{otherwise.} \end{cases}$$

Finally, it is sufficient to apply **(3)** to **(2)** to conclude the proof of the theorem. ■

We now explicitly specify the sequence of numbers $i(C_n)$, $n = 5, 6, \ldots$, described by the recurrence relation in Theorem 4. Our aim is to present an explicit form for this sequence expressing it in terms of Lucas numbers.

Proposition 1: For $n \geq 5$ and $n \equiv 2 \pmod 3$,

$$i(C_{n+1}) = 3L_{n-3} + 2L_0 - \sum_{i \in A_{n-4}} L_i + 2\sum_{i \in B_{n-4}} L_i,$$

where $A_{n-4} = \{i \in \mathbb{N} \cup \{0\} : (i \equiv 0 \pmod 3) \vee i \equiv 1 \pmod 3)) \wedge i < n-4\}$

and $B_{n-4} = \{i \in \mathbb{N} \cup \{0\} : i \equiv 2 \pmod 3 \wedge i < n-4\}$.

Proof: Using $k+1$ times the recurrence from Theorem 4 for $n + 1 \equiv 0 \pmod 3$ and expressing the numbers $i(C_{n+1})$, for $n \geq 5$ in terms of Lucas numbers we obtain

(4) $$i(C_{n+1}) = L_k \cdot i(C_{n-k}) + L_{k-1} \cdot i(C_{n-k-1}) + 2L_0 - \sum_{i \in A_k} L_i + 2\sum_{i \in B_k} L_i,$$

where

$$A_k = \{i \in \mathbb{N} \cup \{0\} : (i \equiv 0 \pmod 3 \vee i \equiv 1 \pmod 3) \wedge i < n-4\} \text{ and}$$
$$B_k = \{i \in \mathbb{N} \cup \{0\} : i \equiv 2 \pmod 3 \wedge i < n-4\},$$

and $k = 1, \ldots, n-4$ (this equality may be easily proved by applying induction on k). Letting $k = n-4$ the equality **(4)** takes the form

$$i(C_{n+1}) = L_{n-4} \cdot i(C_4) + L_{n-5} \cdot i(C_3) + 2L_0 - \sum_{i \in A_{n-4}} L_i + 2 \sum_{i \in B_{n-4}} L_i.$$

Applying the initial conditions from Theorem 4 and applying the definition of the Lucas number L_n to the terms L_{n-4} and L_{n-5} we complete the proof. ■

Analogously, the next two results can be derived.

Proposition 2: For $n \geq 5$ and $n \equiv 0 \pmod 3$,

$$i(C_{n+1}) = 3L_{n-3} - L_0 + 2\sum_{i \in X_{n-4}} L_i - \sum_{i \in Y_{n-4}} L_i, \text{ where}$$

$$X_{n-4} = \{i \in \mathbb{N} \cup \{0\} : i \equiv 0 \pmod 3 \wedge i < n-4\}, \text{ and}$$
$$Y_{n-4} = \{i \in \mathbb{N} \cup \{0\} : (i \equiv 1 \pmod 3 \vee i \equiv 2 \pmod 3) \wedge i < n-4\}.$$ ■

Proposition 3: For $n \geq 5$ and $n \equiv 1 \pmod 3$,

$$i(C_{n+1}) = 3L_{n-3} - L_0 - \sum_{i \in S_{n-4}} L_i + 2\sum_{i \in T_{n-4}} L_i, \text{ where}$$

$$S_{n-4} = \{i \in \mathbb{N} \cup \{0\} : (i \equiv 0 \pmod 3 \vee i \equiv 2 \pmod 3) \wedge i < n-4\}, \text{ and}$$
$$T_{n-4} = \{i \in \mathbb{N} \cup \{0\} : i \equiv 1 \pmod 3 \wedge i < n-4\}.$$ ■

In the last part of this paper we establish an explicit formula for the sequence $i(C_n)$, $n = 3, 4, \ldots$, using the method of generating functions.

Theorem 5: For $n \geq 2$, $i(C_{n+1}) = (\alpha^{n+1} + \hat{\alpha}^{n+1} + \phi^{n+1} + \hat{\phi}^{n+1})/2$, where $\alpha = (-1-\sqrt{3}i)/2$, $\phi = (1+\sqrt{5})/2$, $\hat{\alpha} = -1-\alpha$, and $\hat{\phi} = 1-\phi$.

Proof: We can write the recurrence in Theorem 4 using one equality with the form

(5) $$i(C_n) = i(C_{n-1}) + i(C_{n-2}) - [n > 4] + 3[3\backslash n, n > 0],$$

which is satisfied for all integers n (we define $i(C_n) = 0$ for $n \leq 2$). For convenience, instead of $i(C_n)$ we write i_n. Recall the term i_n is associated with the generating function $I(z) = \sum_n i_n z^n$.

Subsequently, applying **(5)** to the above we have

$$\begin{aligned} I(z) &= \sum_n i_{n-1} z^n + \sum_n i_{n-2} z^n - \sum_{n>4} z^n + 3\sum_{n>0} [3\backslash n] z^n \\ &= \sum_n i_n z^{n+1} + \sum_n z^{n+2} - \sum_{n \geq 0} z^n + \sum_{n=0}^{4} z^n + 3\sum_{n \geq 0} [3\backslash n] z^n - 3\sum_{n=0} [3\backslash n] z^n \\ &= zI(z) + z^2 I(z) - \frac{1}{1-z} + \frac{3}{1-z^3} + z^4 + z^3 + z^2 + z - 2, \end{aligned}$$

by Observation 3(2 and 3). This means that the generating function of i_n takes the form

$$I(z) = -z^2 - 2 + \frac{z^4 - z^2 - 2z + 2}{(1-z)(1-\alpha z)(1-\hat{\alpha} z)(1-\phi z)(1-\hat{\phi} z)},$$

where $\alpha = (-1-\sqrt{3}i)/2$, $\phi = (1+\sqrt{5})/2$, $\hat{\alpha} = -1-\alpha$, and $\hat{\phi} = 1-\phi$.

Decomposing this rational function we obtain

$$I(z) = -z^2 - 2 + \frac{1}{2}\left(\frac{1}{1-\alpha z} + \frac{1}{1-\hat{\alpha} z} + \frac{1}{1-\phi z} + \frac{1}{1-\hat{\phi} z}\right).$$

According to Observation 3(1 and 2), the generating function has the form

$$I(z) = \sum_{n \geq 0} \left\{ -[n = 2] - 2[n = 0] + \frac{1}{2}(\alpha^n + \hat{\alpha}^n + \phi^n + \hat{\phi}^n) \right\} z^n .$$

This completes the proof. ■

Theorem 5 and the equality $\alpha^{n+1} = \hat{\alpha}^{n+1} = 1$ if $n \equiv 2 \pmod 3$, imply the following corollary.

Corollary 1: If $n \equiv 2 \pmod 3$, then $i(C_{n+1}) = 1 + (\phi^{n+1} + \hat{\phi}^{n+1})/2$. ■

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[GTN LIV:4] THE EXPONENT OF THE PRIMITIVE PARTIAL JOIN OF GRAPHS

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Abstract

A digraph D is primitive if there exists an integer $k > 0$ such that for all ordered pairs of vertices $u, v \in V(D)$ (not necessarily distinct), there is a directed walk (vertices and arcs may be repeated) from u to v of length k. The smallest such k is called the exponent of the digraph D and is denoted by $\exp(D)$. A symmetric digraph is primitive, if and only if it is connected and contains at least one odd cycle. Replacing two opposite arcs in the symmetric digraph D by an edge, we obtain a graph G, that is called primitive if the digraph D is primitive. In this sense, we consider the primitivity of graphs. We define the partial join $G_0(G) + H$ of graphs G and H with respect to an induced subgraph G_0 of the G (for short, $G_0 < G$) as follows: $G_0(G) + H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup E_1$, where $E_1 = \{uv : u \in V(G_0) \text{ and } v \in V(H)\}$. We estimate the exponent of the primitive partial join using parameters related to the graph G.

1. Introduction

Let D be a directed graph (digraph) on n vertices, $n \geq 2$. Loops are permitted but not multiple arcs. A sequence W: $v_0, a_1, v_1, \ldots, a_k, v_k$ (where $a_i = v_{i-1}v_i$ for $i = 1, \ldots, k$) of vertices and arcs is called a *walk* from v_0 to v_k (a $v_0 \to v_k$ walk) in D. The *length* of a walk is its number of arcs. Note that vertices can be repeated in a walk. In a *closed walk* $v_0 = v_k$. If all vertices are distinct, then a closed walk is called a *simple cycle*. Thus, the length of a simple cycle is no greater than n. The set of the lengths of all simple cycles in D is denoted by $l(D)$. Loops are considered to be simple cycles of length 1.

A digraph D is called *primitive* if there is a positive integer t such that there exists a walk from u to v of length t in D, for every ordered pair $u, v \in V(D)$ (not necessarily distinct). It is not difficult to observe, that if the above condition is satisfied for t, then it is also satisfied for $t + 1$. The smallest such t is denoted by $\exp(D)$ and it is called the *exponent* of the primitive digraph D. If D is primitive, then we also define the exponent of a pair of vertices $u, v \in V(D)$. Denote by $\exp_D(u, v)$ the minimum positive integer t, such that there for each integer $p \geq t$, there is a $u \to v$ walk in D of length p. Obviously,

$$\exp(D) = \max_{u, v \in V(D)} \{\exp_D(u, v)\}.$$

The following necessary and sufficient condition for a digraph to be primitive is well known. In the statement, $\gcd(p_1, \ldots, p_t)$ denotes the greatest common divisor of the integers $p_1, \ldots, p_t$.

Theorem 1 [1]: Let $l(D) = \{p_1, \ldots, p_t\}$, $t \geq 2$. Then D is primitive if and only if D is strongly connected and $\gcd(p_1, \ldots, p_t) = 1$. ■

We discuss *primitivity* in the class of symmetric digraphs. If a digraph D is symmetric, then $2 \in l(D)$. Moreover, a symmetric digraph is primitive if and only if it is strongly connected and contains an odd simple cycle. Let G_D be the graph obtained from a symmetric digraph D by replacing two opposite arcs by an edge. In other words, every simple cycle of length 2 in the symmetric digraph D is an edge in the graph G_D. Thus, the primitivity of a symmetric digraph can be discussed in terms of the resulting graph. If D is primitive, then the graph G_D is also called *primitive*. The existence of a $u \to v$ walk is equivalent to the existence of a $v \to u$ walk, thus we use the notation u—v walk in G_D instead of a $u \to v$ walk in D.

It is easy to see that $\exp(D) = \exp(G_D)$.

Corollary 2: G is primitive if and only if G is connected and contains at least one cycle of odd length. ■

The *exponent set problem* and *generalized exponents problem* were considered in **[2]** and **[3]**.

The exponent of a primitive graph has an interpretation in modeling a *memoryless communication network* associated with a graph G **[3]**. Consider a system of n points in which there is one bit of information at each point. Every point passes its information to each of its neighbors and then forgets its information. A point also receives information from another point. The system continues in this way, until each point knows all n bits of information. The shortest time problem for this to occur is the exponent problem.

The preceding application can be described using an operation called the *partial join* of two graphs. Let G_0 be an induced subgraph of G and $|V(G_0)| = n_0$. Let H be an arbitrary graph. Define the *partial join* to be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup E_1$, where

$$E_1 = \{uv : u \in V(G_0) \text{ and } v \in V(H)\}.$$

Denote the resulting graph by $G_0(G) + H$. Note that if $G_0 = G$, then we obtain the *join* $G + H$ of two graphs G and H.

Suppose a new vertex with a new bit of information is added to the system of points and the new vertex is joined to some of the original vertices. It seems natural to ask about the time needed for this process in the new system of vertices. This question was resolved in **[3]** for a special case.

Theorem 3 [3]: Let G be primitive graph and $v \in V(G)$. If $V(G_0) = N_G(v)$, then $\exp(G_0(G) + K_1) = \exp(G)$. ■

Beasley and Kirkland obtained an upper bound for primitive graphs that contain a primitive subgraph. **[4]**

Theorem 4 [4]: Let D be a strongly connected digraph on $n + k$ vertices that contains a primitive directed subdigraph D_1 of order n. Then D is primitive and $\exp(D) \le \exp(D_1) + 2k$. ■

The graph G is, in particular, a subgraph of $G_0(G) + H$. Since $N_G(h) = V(G_0)$, for $h \in V(H)$, Theorem 3 implies that $\exp(G_0(G) + H)$ does not depend on the number of isolated vertices in H. In general, the exponent of the supergraph of the primitive graph $G_0(G) + H$, which has the same set of vertices, is no greater than $\exp(G_0(G) + H)$. Therefore, it is most interesting to study edgeless graphs H. If the graph H is edgeless, then, by Theorem 3, we can put $k = 1$ in Theorem 4 and an upper bound on the exponent of $G_0(G) + H$ is obtained.

Corollary 5: If G is a primitive graph and $G_0 < G$, then $\exp(G_0(G) + H) \le \exp(G) + 2$. ■

Therefore, we can put $H = K_1 = \{h\}$ and write $G_0(G) + h$ instead of $G_0(G) + K_1$. We restrict our attention to such graphs. Our purpose is to study the primitivity of the partial join for the class of simple graphs (graphs with no loop).

The following results will be useful in expressing upper bounds on the exponent of $G_0(G) + h$ using various parameters of G.

Theorem 6 [3]: If G is primitive and r is the length of an odd simple cycle in G, then $\exp(G) \le 2n - r - 1$. ■

Lemma 7 [3]: Let G be primitive and $u, v \in V(G)$. Let a and b be the lengths of the shortest odd and the shortest even walk, respectively. Then $\exp_G(u, v) = \max\{a, b\} - 1$. ■

In particular, $\exp_G(u, v)$ is even.

2. Main Results

Proposition 8: Let G be primitive and $v \in V(G)$. $G_0 < G$ such that $V(G_0) \subseteq N_G(v)$, then $\exp(G_0(G) + h) \ge \exp(G)$.

Proof: Choose $x, y \in V(G)$ satisfying $\exp_G(x, y) = \exp(G)$. Then there is no x—y walk of length $\exp(G) - 1$ in G. Now suppose that $\exp(G_0(G) + h) \leq \exp(G) - 1$. There must be an x—y walk of length $\exp(G) - 1$ containing the vertex h. Obviously, if there is an x—y walk of length k in $G_0(G) + h$ that contains the vertex h, then there is an x—y walk of length k in G that contains the vertex v. Thus, there is an x—y walk of length $\exp(G) - 1$ in G, contrary to the fact that $\exp_G(x, y) = \exp(G)$. ■

This means that if $|V(G_0)| \leq \Delta(G)$, then there is a partial join $G_0(G) + h$ satisfying $\exp(G_0(G) + h) \geq \exp(G)$. However, in general, the number $\exp(G_0(G) + h)$ does not depend on the number $\Delta(G)$.

Example 9: Let G be the graph shown in the Figure. Clearly, G is primitive. It is easy to observe that $\Delta(G) = 4$, $|V(G)| = 2^k + 1$, and $\exp(G) = 2k$. If $V(G_0) \subseteq \{v_{2^{k-1}+1}, \ldots, v_{2^k}\}$, $k \geq 1$, then

$$\exp(G_0(G) + h) = \exp(G) + 2 = 2k + 2.$$

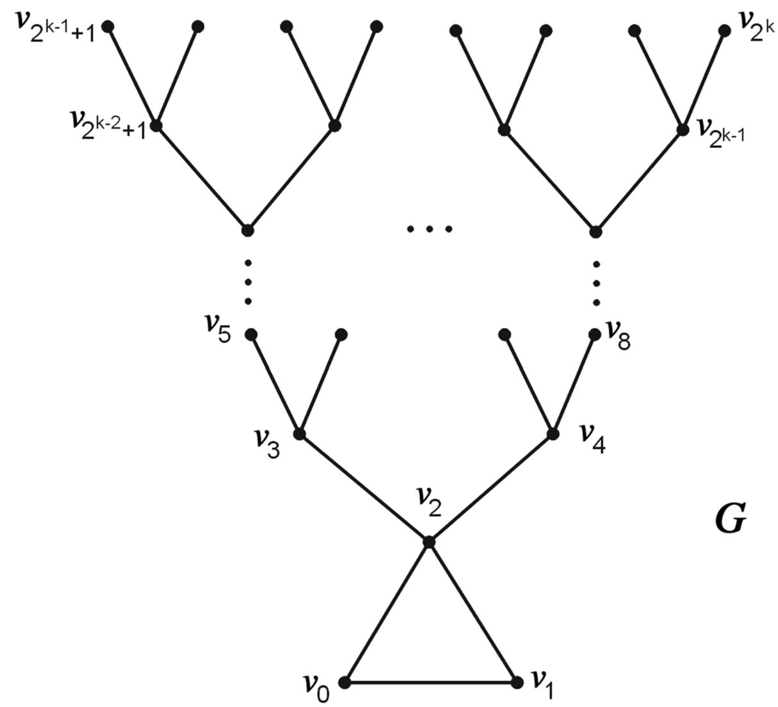

Figure.

Proposition 10 [4]: Let D be a strongly connected digraph on $n + k$ vertices that contains a primitive subdigraph D_1 on n vertices. If $\exp(D) = \exp(D_1) + 2k$, then there are pairs of (not necessarily distinct) vertices of D_1, $(u_1, v_1), (u_2, v_2), \ldots, (u_p, v_p)$, such that $\exp_{D_1}(u_i, v_i) = \exp(D_1)$, for $1 \leq i \leq p$. ■

The following property provides a complete characterization of primitive graphs for which the upper bound is attained.

Proposition 11: Let G be a primitive graph and $G_0 < G$. Then $\exp(G_0(G) + h) = \exp(G) + 2$ if and only if $\exp_G(u, v) = \exp(G)$, for every $u, v \in V(G_0)$.

Proof: Assume that $\exp_G(u, v) = \exp(G)$, for all $u, v \in V(G_0)$. Thus, there is no u—v walk of length $\exp(G) - 1$ in G for any $u, v \in V(G_0)$. This implies, that there is no h—h walk of length $\exp(G) + 1$ in $G_0(G) + h$ and, by Corollary 5, $\exp(G_0(G) + h) = \exp(G) + 2$.

Now let $\exp(G_0(G) + h) = \exp(G) + 2$. By Proposition 10, there are pairs of vertices of G, $(u_1, v_1), (u_2, v_2), \ldots, (u_p, v_p)$, satisfying $\exp_G(u_i, v_i) = \exp(G)$ for $1 \leq i \leq p$. It suffices to observe that every pair of vertices $x, y \in V(G_0)$ is one of these pairs. Proof of this fact is analogous to the proof of Proposition 10 and is omitted. ■

Note that if $\exp(G)$ is odd, then $\exp_G(u, u) < \exp(G)$ for all $u \in V(G)$, consequently, $\exp(G_0(G) + h) \le \exp(G) + 1$.

In general, primitivity of a connected graph G is not necessary for a partial join to be primitive. The existence of an odd u—v walk in G for some $u, v \in V(G_0)$ implies, by Corollary 2, that there is an odd simple cycle in $G_0(G) + h$. Hence, $G_0(G) + h$ is primitive. If there is an odd u—v walk in G for at least one pair of vertices $u, v \in V(G_0)$, then primitivity of G is not required. We can bound the exponent of the partial join using the length of such a walk.

Proposition 12: Let G be a connected graph, $G_0 < G$, and $u, v \in V(G_0)$. Suppose that there is an odd simple u—v walk in $G_0(G) + h$ and let p be the length of this walk. Then $\exp(G_0(G) + h) \le 2n - p - 1$.

Proof: There are simple u—v walks of lengths 2 and of length p in $G_0(G) + h$, so there is a subgraph of $G_0(G) + h$ that is isomorphic with C_{p+2}. Hence, by Theorem 4,

$$\exp(G_0(G) + h) \le 2(n + 1) - p - 3 = 2n - p - 1. \qquad \blacksquare$$

In particular, if $u, v \in V(G_0)$ and $d_G(u, v) \ge 3$ is odd, then $\exp_{G_0(G)+h}(u, v) \le d_G(u, v) - 1$. Since $d_{G_0(G)+h}(u, v) = 2$ and $d_G(u, v)$ is the length of a shortest odd u—v walk in $G_0(G) + h$, then, by Lemma 5,

$$\exp_{G_0(G)+h}(u, v) = d_{G_0(G)+h}(u, v) - 1 \le d_G(u, v) - 1.$$

Let $X \subset V(G)$ and $v \in V(G)$. The distance between the vertex v and the set X is defined by

$$d_G(v, X) = \min_{u \in X} d_G(v, u).$$

Theorem 13: Let G be connected, $G_0 < G$ have no isolated vertex, and $|V(G_0)| = n_0$. Then $\exp(G_0(G) + h) \le 2n - 2n_0 + 2$. Moreover, equality holds if and only if the vertices can be relabeled so that:

(1) $G - V(G_0) \cong P_{n-n_0}$, where $V(G - V(G_0)) = \{v_{n_0+1}, v_{n_0+2}, \ldots, v_n\}$;

(2) $N_G(V(G_0)) = \{v_{n_0+1}\}$; and

(3) the induced subgraph $G[N_{G_0}(v_{n_0+1})]$ is edgeless.

Proof: The induced subgraph $G_0 + h$ of the graph $G_0(G) + h$ is primitive. Since G_0 has no isolated vertex, then $\exp(G_0 + h) = 2$. Thus, the graph $G_0(G) + h$ is a supergraph of $G_0 + h$. We can put $k = n - n_0$ and, by Theorem 4, obtain $\exp(G_0(G) + h) \le 2n - 2n_0 + 2$. This proves the first part of the theorem.

Now let G be a graph satisfying (1), (2), and (3). Let $V(G_0) = \{v_1, \ldots, v_{n_0}\}$. It is easy to see that there is no v_n—v_n walk of length $2(n - n_0) + 1$ in $G_0(G) + h$, consequently, $\exp(G_0(G) + h) \ge 2n - 2n_0 + 2$. Hence, by the first part of the proof, $\exp(G_0(G) + h) = 2n - 2n_0 + 2$.

Now let $G_0 < G$ and $\exp(G_0(G) + h) = 2n - 2n_0 + 2$. Then there exist $v, v' \in V(G_0(G) + h)$ such that $\exp_{G_0(G)+h}(v, v') = 2n - 2n_0 + 2$. Note that

$$\exp_{G_0+h}(v, v') \le 2 + d_{G_0(G)+h}(v, V(G_0)) + d_{G_0(G)+h}(v', V(G_0)).$$

Since $d_{G_0(G)+h}(v, V(G_0)) \le n - n_0$ and $d_{G_0(G)+h}(v', V(G_0)) \le n - n_0$, the equality

$$d_{G_0(G)+h}(v, V(G_0)) + d_{G_0(G)+h}(v', V(G_0)) = 2n - 2n_0$$

implies that $d_{G_0(G)+h}(v, V(G_0)) = n - n_0$ and $d_{G_0(G)+h}(v', V(G_0)) = n - n_0$, thus $v = v'$. This proves (1) and (2). Thus, each vertex of $G - V(G_0)$ can be labeled using its distance from G_0. We choose to order the vertices of $G - V(G_0)$ in such a way that $d_{G_0(G)+h}(v_{n_0+i}, V(G_0)) = i$, for $i = 1, 2, \ldots, n - n_0$. Suppose that there are $x, y \in N_{G_0}(v_{n_0+1})$ that are adjacent. Then v_{n_0+1} belongs to a cycle of length 3, and hence,

$$\exp_{G_0(G)+h}(v_n, v_n) \le 2 + 2(n - n_0 - 1) = 2n - 2n_0,$$

a contradiction. $\blacksquare$

The exponent of the partial join $G_0(G) + h$ depends on the choice of G_0. If there is no odd simple u—v walk in G, for any pair $u, v \in V(G_0)$, then the primitivity of G is necessary for the partial join to be primitive. If

G is primitive then, by Corollary 2, every connected supergraph is primitive. The exponent of a pair of vertices in $G_0(G)+h$ can be estimated using the exponent in G. Clearly, the following results can be also applied to primitive graphs with the property that there is an odd u—v walk for at least one pair $u, v \in V(G_0)$.

Remark 14: Let G be primitive, $G_0 < G$, and $u, v \in V(G)$. Then

(1) $\exp_{G_0(G)+h}(h, h) \leq \min\limits_{x, y \in V(G_0)} \{\exp_G(x, y)\} + 2$

(2) $\exp_{G_0(G)+h}(h, v) \leq \min\limits_{w \in V(G_0)} \{\exp_G(w, v)\} + 1$

(3) $\exp_{G_0(G)+h}(u, v) \leq \exp_G(u, v)$.

These bounds are useful if there is no simple u—v walk in G.

The next proposition provides a characterization of the class of partial joins that satisfy $\exp(G_0(G)+h) = \exp(G)+1$.

Proposition 15: Let G be a primitive graph and $G_0 < G$. Then, $\exp(G_0(G)+h) = \exp(G_0(G)+h) = \exp(G)+1$ if and only if $\exp_G(x, y) = \exp(G)-1$ for all $x, y \in V(G_0)$ or there exists $u' \in V(G)\backslash V(G_0)$ such that $\exp_G(u, u') = \exp(G)$ for all $u \in V(G_0)$.

Proof: We prove only the necessary condition; proof of the sufficient condition is the same as in the proof of Theorem 13 and is omitted.

If $\exp(G_0(G)+h) = \exp(G)+1$, then by **(3)** there are two cases.

(1) If $\exp(G_0(G)+h) = \exp_{G_0(G)+h}(h, h)$, then $\exp(G)$ is odd. There is an h—h walk of length $m+2$ if and only if there is a pair of vertices $x, y \in V(G_0)$ such that there is an x—y walk of length m in G. Thus, there is no x—y walk of length $\exp(G)-2$ for any $x, y \in V(G_0)$ and, hence, there is no simple x—y walk of odd length in G. On the other hand, by **(1)**,

$$\min_{u, v \in V(G_0)} \{\exp_G(u, v)\} \geq \exp(G) - 1,$$

therefore, $\exp_G(u, v) \in \{\exp(G)-1, \exp(G)\}$ for all $u, v \in V(G_0)$.

The case $\exp_G(u, v) = \exp(G)$ means, by Lemma 7, that $d_G(u, v)$ is odd, a contradiction. Hence $\exp_G(u, v) = \exp(G)-1$ for all $u, v \in V(G_0)$.

(2) If $\exp_{G_0(G)+h}(h, h) \leq \exp(G)$, then there is a $v \in V(G)$ such that $\exp(G_0(G)+h) = \exp_{G_0(G)+h}(h, v)$. Thus, by **(2)**,

$$\min_{w \in V(G_0)} \{\exp_G(w, v)\} \geq \exp(G),$$

so that $\exp_G(w, v) = \exp(G)$ for every $w \in V(G_0)$. ■

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[GTN LIV:5] CHROMATIC AND TUTTE POLYNOMIALS FOR GRAPHS, ROOTED GRAPHS, AND TREES

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Abstract

The chromatic polynomial of a graph is a one-variable polynomial that counts the number of ways the vertices of a graph can be properly colored. It was invented in 1912 by G.D. Birkhoff in his unsuccessful attempt to solve the four-color problem. In the 1940s, Tutte generalized Birkhoff's polynomial by adding another variable and analyzing its combinatorial properties. The Tutte polynomial itself has been generalized to other combinatorial objects, with connections to knot theory, state changes in physics, probability, and other areas. We concentrate on an extension to rooted trees where the polynomial is a complete invariant; that is, two rooted trees T_1 and T_2 are isomorphic iff their Tutte polynomials are equal.

1. Historical Introduction

How many colors are needed to color a map so that regions sharing a border receive different colors? This problem traces its origin to October 23, 1852, when Francis Guthrie asked his professor, Augustus De Morgan, whether he knew of a solution. De Morgan, in turn, asked his friend, Sir William Rowan Hamilton, who was also stumped. This innocent question raised by a student inspired an enormous amount of research in graph theory, with a final resolution of the question occurring more than 100 years later.

It is easy to see that at least four colors are necessary. For instance, in South America (see Figure 1) note that the four countries Argentina, Brazil, Bolivia, and Paraguay must all receive different colors since each pair of these countries shares a border. To prove that four colors also suffice for maps in the plane (or, equivalently, on a sphere) occupied some of the best mathematicians of the Nineteenth and Twentieth Centuries.

Instead of coloring the regions of maps, most mathematicians prefer to color the vertices of graphs. A proper *vertex coloring* of a graph G is an assignment of colors to the vertices of G so that adjacent vertices receive different colors. Converting a map to a graph is straightforward: Each region of the map becomes a vertex of the graph, and two vertices of the graph are joined by an edge precisely when the corresponding regions share a boundary.

Guthrie's problem can now be stated in graph-theoretic terms:

> **Conjecture 1.1 (Four Color Conjecture):** Every planar graph can be properly colored using at most four colors. □

A very quick history of the progress on the conjecture is given below, culminating in the famous proof of Appel and Haken in 1976 that made extensive use of computers. This proved to be a source of controversy at the time, and many mathematicians are still not completely satisfied with the nature of this proof. (A shorter proof that still requires some computer checking can be found in **[1]**.) See **[2]** for a fairly extensive recounting of the colorful history of this problem.

- 1852 — Problem posed by Guthrie.
- 1871 — De Morgan dies.
- 1878 — Arthur Cayley revives interest in the problem.
- 1879 — Alfred Kempe "solves" the problem.
- 1889 — Percy Heawood finds a flaw in Kempe's proof.
- 1912 — G.D. Birkhoff introduces the chromatic polynomial.
- 1976 — Appel and Haken publish "correct" proof.

Figure 1: A famous continent.

This paper is organized as follows. In Section 2, we define the chromatic polynomial and develop its basic properties. Section 3 does the same for the Tutte polynomial, including the connections between the Tutte polynomial and three one-variable polynomials: the chromatic, flow, and reliability polynomials.

In Section 4, we extend the definition of the Tutte polynomial to rooted graphs; that is, graphs with a distinguished vertex, concentrating especially on rooted trees. Section 5 explores the difference between the ordinary Tutte polynomial and the rooted version. In sharp contrast to the situation for ordinary trees and the (ordinary) Tutte polynomial, the rooted polynomial characterizes rooted trees (see Theorem 5.1).

If T is an ordinary tree, then it is possible to define two different Tutte polynomials, one based on cycle-rank (the ordinary Tutte polynomial) and one based on a greedoid rank function. The greedoid version is much sharper at distinguishing among different trees, but there is no characterization result analogous to that for rooted trees.

It is possible to give combinatorial interpretations to the two greedoid-based Tutte polynomials defined in Sections 4 and 5. In each case we can formulate the Tutte polynomial in purely graph theoretic terms, allowing us to obtain tree reconstruction results.

We conclude with open problems in Section 6.

2. The Chromatic Polynomial of a Graph

In 1912, George Birkhoff introduced a polynomial, the *chromatic polynomial* to count the number of proper colorings of a graph. More precisely, let $\chi(G;\lambda)$ be the number of proper colorings of a graph G using λ (or fewer) colors.

Example 2.1: Let G be the graph obtained by removing an edge from K_4, as shown in Figure 2.

In this case, we can determine the chromatic polynomial "greedily". There are λ colors available for vertex 1, $\lambda - 1$ colors left for vertex 2, and $\lambda - 2$ colors for vertex 3. Now there are also $\lambda - 2$ colors available for vertex 4, so we obtain

$$\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2.$$

Note that $\chi(G;3) = 6 > 0$, so it is possible to properly color the vertices of G using three colors. Birkhoff hoped that studying the roots of $\chi(G;\lambda)$ could lead to a proof of the four-color theorem.

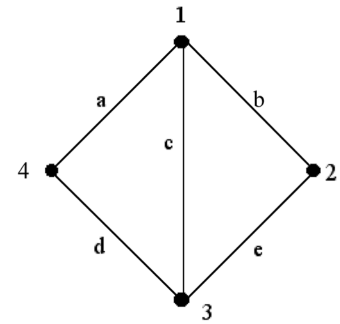

Figure 2: $\chi(G;\lambda) = \lambda(\lambda-1)(\lambda-2)^2$.

Theorem 2.2 (Four Color Theorem): If G is a planar graph, then $\chi(G;4) > 0$. ■

Computing $\chi(G;\lambda)$ for an arbitrary graph is difficult. In fact, determining if the chromatic number of a graph is equal to k (for $k \geq 3$) is an NP-complete problem. This was one of the first problems to be shown to be NP-complete, in 1972—see **[3]**. More on the complexity of computing chromatic invariants can be found in **[4]**.

$\chi(G;\lambda)$ satisfies an important recursive formula that allows for inductive proofs. Recall that if G is a graph and e is an edge in G, then the *deletion* $G - e$ is formed by simply removing the edge e from G. The *contraction* G/e of the (non-loop) edge e is obtained from G by identifying the two endpoints of e and then removing e. Thus, if G has n edges, then $G - e$ and G/e each have $n - 1$ edges.

Suppose u and v are the two endpoints of the edge e and partition the proper colorings of $G - e$ as follows: Colorings in which u and v receive different colors, and colorings in which they receive the same color. In the former case, we obtain a proper coloring of G; in the latter case, we obtain a proper coloring of G/e. Furthermore, all proper colorings of G and G/e arise in this way. This proves the next result.

Theorem 2.3. (Deletion–Contraction): Let G be a graph and e be a non-loop edge. Then

$$\chi(G;\lambda) = \chi(G-e;\lambda) - \chi(G/e;\lambda).$$ ■

It follows from Theorem 2.3 and induction that $\chi(G;\lambda)$ is actually a *polynomial*. Note that this is not immediate from Birkhoff's definition.

Corollary 2.4: $\chi(G;\lambda)$ is a polynomial in λ. ■

Example 2.5: As an example of the deletion–contraction method, we again compute $\chi(G;\lambda)$ for the graph G shown in Figure 2. In Figure 3, we show the result of deleting and contracting the edge c. Note that $G - c$ is isomorphic with C_4, the 4-cycle, and G/c is isomorphic with a join of two 2-cycles. One can check $\chi(G/c;\lambda) = \lambda(\lambda-1)^2$, whereas $\chi(G-c;\lambda) = \lambda(\lambda-1)(\lambda^2-3\lambda+3)$. This agrees with our previous calculation:

One can use the deletion-contraction formula to establish other results by induction. For instance, the coefficients of $\chi(G;\lambda)$ alternate in sign. More information about the chromatic polynomial and its elementary properties can be found in **[5]** or **[6]**, for instance.

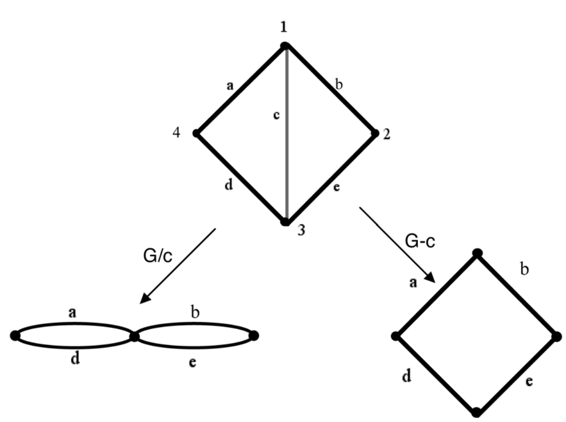

Figure 3: Deletion–contraction.

We need the next result, which is a standard exercise.

Proposition 2.6: Let T be a tree with n edges. Then $\chi(T;\lambda) = \lambda(\lambda - 1)^{n-1}$. ■

Study of the roots of $\chi(G;\lambda)$ is still a very active area of research. For instance, a somewhat surprising application to statistical mechanics is explained in **[7]**, where the chromatic polynomial models state changes.

3. The Tutte Polynomial of a Graph

William Tutte was one of the giants of graph theory and combinatorics in the Twentieth Century. His work at Bletchley Park as a code breaker has been called “one of the greatest intellectual feats of world war II.”

While working on a recreational problem involving the partition of a square into squares of distinct sizes (the *squared square problem*), Tutte noticed that the number of spanning trees $n(G)$ in a connected graph obeys a deletion–contraction recurrence: $n(G) = n(G-e) + n(G/e)$ (provided e is neither an *isthmus* (also called a *bridge*) nor a loop). Tutte investigated other invariants that satisfied similar recursive formulæ, leading to the following definition.

Definition 3.1: Let G be a graph on the edge set E. The *Tutte polynomial* $f(G;x, y)$ is defined as follows:

(1) If e is neither an isthmus (bridge) nor a loop, then $f(G;x, y) = f(G/e;x, y) + f(G-e;x, y)$.

(2) If e is an isthmus, then $f(G;x, y) = x \cdot f(G/e;x, y)$.

(3) If e is a loop, then $f(G;x, y) = y \cdot f(G-e;x, y)$.

To ensure $f(G;x, y)$ is well-defined, we must make sure the polynomial obtained from repeated deletions and contractions does not depend on the order of operating on the edges. The following theorem establishes this.

Theorem 3.2: Let G be a graph with edge set E. For $A \subseteq E$, let $r(A)$ be the size of the largest cycle-free subset of A. Then

$$f(G;x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

A proof of Theorem 3.2, using induction, is straightforward. ■

The expression $r(A)$ is usually called the *cycle rank* of A, and it allows us to extend the definition of the Tutte polynomial to any objects that possess a rank function. In particular, this invariant has been extended to *matroids* and *greedoids*; see **[8]**. Borrowing terminology from linear algebra, the expression $r(E) - r(A)$ is called the *corank* of A and $|A| - r(A)$ is the *nullity* of A.

The reader is encouraged to use either Definition 3.1 or Theorem 3.2 to establish the following.

Proposition 3.3:

(1) Let T be a tree with n edges. Then $f(T;x, y) = x^n$.

(2) Let C_n be a cycle with n edges. Then $f(C_n;x, y) = x^{n-1} + x^{n-2} + \ldots + x + y$. ■

Example 3.4: As an example, we compute the Tutte polynomial $f(G;x, y)$ for the graph G shown in Figure 2. Note that the deletion–contraction definition allows us to use the decomposition illustrated in Figure 3. Thus, $f(G) = f(G-c) + f(G/c) = f^2(C_2) + f(C_4)$. This uses the fact that $f(G_1 \oplus G_2) = f(G_1)f(G_2)$, where $G_1 \oplus G_2$ is the graph obtained from G_1 and G_2 by identifying a vertex of G_1 with a vertex of G_2. Now we can use Proposition 3.3(2) to obtain $f(G;x, y) = x^3 + 2x^2 + 2xy + x + y + y^2$.

The Tutte polynomial has a certain universal property: Any invariant satisfying a deletion–contraction recurrence is (essentially) an evaluation of the Tutte polynomial. This is made precise by a theorem of Brylwaski—see 6.2.2 of **[8]**. For now, we list several important evaluations that can be obtained from $f(G;x, y)$.

Theorem 3.5: Let G be a graph.

(1) Subsets: The number of subsets of the edges of G is $f(G;2, 2)$.

(2) Spanning trees: The number of spanning trees of G is $f(G;1, 1)$.

(3) Spanning sets: The number of subsets of edges of G that contain a spanning tree is $f(G;1, 2)$.

(4) Acyclic subsets of edges: The number of subsets of the edges of G that are contained in a spanning tree is $f(G;2, 1)$.

(5) Acyclic orientations: The number of acyclic orientations of the edges of G is $f(G;2, 0)$.

(6) Acyclic orientations with a unique specified source: The number of acyclic orientations of the edges of G in which a specified vertex v is the unique source is $f(G;1, 0)$ (and, in particular, does not depend on the vertex v chosen as the source).

(7) Totally cyclic orientations: The number of orientations of G in which every edge is in some cycle is $f(G;0, 2)$.

(8) Distinct score vectors: The number of distinct score vectors that arise from orientations of the edges of G is $f(G;2, 1)$. ■

Note that the evaluation $f(G;2, 1)$ counts two distinct invariants; both the number of acyclic sets and the number of score vectors. Property (5) is due to Stanley **[9]**, and Properties (6) and (7) are proved in **[10]**. We also draw the attention of the reader to Property (2), the prototypical evaluation Tutte noticed during the 1940s. Tutte was able to use *basis activities* to set up a bijection between spanning trees and individual terms of the polynomial. His work is valid in the more general context of *matroids*; see 6.6.A of **[8]** for more.

In addition to these numerical invariants, the Tutte polynomial also encodes three important one-variable graph polynomials. We now define the *flow polynomial* of a graph. A λ-flow is obtained by choosing an orientation of the edges of G and assigning an element of the additive group $\mathbb{Z}_\lambda$ to each edge so that *Kirchoff's law* is satisfied at each vertex v (i.e., the sum of the weights of the edges directed toward v is equal to the sum of the weights of the edges directed away from v). A λ-flow is *nowhere zero* if no edge is assigned 0.

Definition 3.6: The *flow polynomial*, $\chi^*(G;\lambda)$ is the number of nowhere zero λ-flows of the graph G. ■

It is worth pointing out two features of this polynomial:

- $\chi^*(G;\lambda)$ does not depend on the initial orientation chosen for the edges of G: reversing the direction of an edge of weight x is equivalent to replacing its weight with $-x$.
- $\chi^*(G;\lambda)$ does not depend on the Abelian group of order λ used in the definition. We chose $\mathbb{Z}_\lambda$, but any Abelian group of order λ would work equally well.

Our last one-variable invariant is the *reliability polynomial*. This polynomial is treated in depth in **[4]**.

Definition 3.7: Let G be a graph and suppose each edge is independently operational with probability p. The *reliability polynomial*, $R(G;p)$ is the probability that the number of components of G does not increase. ■

The chromatic, flow, and reliability polynomials can all be found from evaluations of the Tutte polynomial.

Theorem 3.8: Let G be a graph with m vertices, n edges, and c components.

(1) $\chi(G;\lambda) = \lambda^c(-1)^{m-c}f(G;1-\lambda, 0)$.

(2) $\chi^*(G;\lambda) = (-1)^{n-m+c}f(G;0, 1-\lambda)$.

(3) $R(G;p) = (1-p)^{n-m+c}p^{m-c}f(G;0, \frac{1}{1-p})$. ■

Proofs of these results follow from applying deletion and contraction and using induction; see Propositions 6.3.1, 6.3.4, and Example 6.2.7 of **[8]**. The close connection between the evaluations giving the chromatic and flow polynomials has an interpretation via duality.

We conclude this section with a fundamental result concerning duality that extends to matroids.

Theorem 3.9: Let G be a planar graph with planar dual G^*. Then $f(G^*;x, y) = f(G;y, x)$. ■

Thus, the flow polynomial carries information about coloring the dual graph when G is planar. When G is a planar graph, one can then view the Tutte polynomial as simultaneously carrying information about coloring a graph and its dual. In fact, Tutte called (a version of) $f(G;x, y)$ the *dichromate* for this reason.

4. The Tutte Polynomial of a Rooted Graph

Rooted graphs are simply graphs with a distinguished vertex. Such graphs are important in many applications, especially in communication theory in which one vertex plays a special role (a server, for instance). Thus, it is worthwhile to search for an extension of the Tutte polynomial to rooted graphs.

We now define the *Tutte polynomial of a rooted graph*; this definition follows the lines of Theorem 3.2.

> **Definition 4.1:** Let G_v be a rooted graph with edge set E and root vertex v. For $A \subseteq E$, define the rank $r(A)$ via subtrees rooted at v:
>
> $$r(A) = \max_{F \subseteq A}\{|F| : F \text{ is a rooted subtree of } T\}.$$
>
> Then the Tutte polynomial $f(G_v;x, y)$ is defined by
>
> $$f(G_v;x, y) = \sum_{A \subseteq E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}. \qquad \blacksquare$$

In this context, $r(A)$ is the *branching rank* of the *greedoid* associated with the edge set of the rooted graph. Greedoids generalize matroids, which can be thought of as a simultaneous generalization of graphs and finite subsets of a vector space. Greedoids were introduced by Korte and Lovász in a series of papers during the 1980s; see **[11]** for general background in the subject. For our purposes, we will not need this level of generality.

Given the variety of evaluations (Theorem 3.5) and the very important deletion–contraction property (Definition 3.1) the (ordinary) Tutte polynomial satisfies, it is reasonable to try to extend as many of these results as possible to the rooted case.

We write $r(G_v)$ for $r(E)$ and $r(G_v - e)$ for $r(E - e)$ when computed in the rooted graph $G_v - e$, and so on.

> **Theorem 4.2:** Let G_v be a rooted graph on n edges and let e be an edge adjacent to v. Then
>
> $$f(G_v;x, y) = f(G_v/e;x, y) + (x-1)^{r(G_v)-r(G_v-e)}f(G_v-e;x, y).$$
>
> If there is no edge adjacent to the root, then $f(G_v;x, y) = y^n$. $\blacksquare$

Theorem 4.2 is proved in **[12]**]. The term $(x-1)^{r(G_v)-r(G_v-e)}$ does not appear in the recurrence for ordinary graphs (or matroids, for that matter) because $r(G) = r(G-e)$ provided e is not an isthmus (bridge). (An alternative approach is taken in **[11]**, where a one-variable greedoid Tutte polynomial is defined. That polynomial can be obtained from the two-variable polynomial defined here by setting $x = 1$. This allows the deletion–contraction recursion to match that given in Definition 3.1, but the one-variable polynomial carries much less information than the two-variable version.)

Example 4.3: We compute $f(G_1;x, y)$ for the graph G shown in Figure 2, rooted at vertex 1. We need the following formula for rooted cycles.

> **Proposition 4.4:** Let C_n be a cycle on n edges, where the root can be placed at any vertex. Then
>
> $$f(C_n;x, y) = n - 1 + y + \sum_{k=1}^{n-1}(n-k)(x-1)^k y^{k-1}.$$

The proof of Proposition 4.4 follows from Theorem 4.2 and induction. $\blacksquare$

To determine $f(G_1;x, y)$, we use deletion–contraction from Theorem 4.2. As before, we obtain $f(G_1) = f(G_1 - c) + f(G_1/c) = f^2(C_2) + f(C_4)$. This time, using the formula from Proposition 4.4, we obtain

$$f(G_1;x, y) = 3x + x^2 + 3y - 2xy + 2x^2y + 3xy^2 - 3x^2y^2 + x^3y^2.$$

Unlike the ordinary Tutte polynomial, $f(G_v;x, y)$ can have negative coefficients. Although it is still true that $f(G_1;1, 1)$ counts the number of spanning trees of G_v, it is not possible to set up a direct bijection between the spanning trees and the terms of the polynomial. Despite the problem with negative coefficients, the rooted Tutte polynomial shares many of the same evaluations the ordinary polynomial possesses.

Proposition 4.5: Let G_v be a graph rooted at vertex v.

(1) Subsets: The number of subsets of edges of G_v is $f(G_v;2, 2)$.

(2) Spanning trees: The number of spanning trees of G_v is $f(G_v;1, 1)$.

(3) Spanning sets: The number of subsets of edges of G_v that contain a spanning tree is $f(G_v;1, 2)$.

(4) Rooted subtrees: The number of rooted subtrees of G_v is $f(G_v;2, 1)$.

(5) Acyclic orientations with unique source v: The number of acyclic orientations of the edges of G_v in which v is the unique source is $f(G_v;1, 0)$. ■

Properties (1)–(4) can be proved in much the same way as before. Property (5) is also straightforward using induction **[13]**. Connections between the different expansions of the ordinary Tutte polynomial are developed in **[14]**. For rooted graphs, it is still possible to develop a theory of activities; see **[15]**.

5. Chromatic and Tutte Uniqueness

A class of graphs C is said to be *Tutte unique* if any two graphs in C have different Tutte polynomials. A simple example of a Tutte-unique class of graphs is the class consisting of all cycles.

In the opposite direction, it is easy to find two graphs with the same Tutte polynomial: Any tree on n edges has $f(T;x, y) = x^n$ (Proposition 3.3(1)). It follows from Theorem 3.8(1) (or Proposition 2.6) that the chromatic polynomial also fails to distinguish two trees with the same number of edges. Even if we ignore trees (which are not 2-connected), it is hopeless to expect the Tutte polynomial to distinguish all graphs; there are simply more graphs than there are potential Tutte polynomials. It follows from the pigeonhole principle that for any positive integer N, there are N non-isomorphic 2-connected graphs all having the same Tutte polynomial. See Exercise 6.9 in **[8]** for the matroid version of this argument.

5.1. Rooted Trees

The situation for rooted trees and the Tutte polynomial is much different. In this case, using the *branching rank function*, we obtain the following.

Theorem 5.1: Let T_1 and T_2 be rooted trees. Then $f(T_1;x, y) = f(T_2;x, y)$ *iff* T_1 and T_2 are isomorphic.

This is the main result of **[12]**. We give a sketch of the proof, which follows from two lemmas, both of which are proved in **[12]**.

Lemma 5.2: Let G_1 and G_2 be two disjoint graphs, rooted at vertices v_1 and v_2, respectively. Let $G_1 \oplus G_2$ be the rooted graph formed by gluing G_1 and G_2 together, with new root vertex created by identifying the two roots v_1 and v_2. Then

$f(G_1 \oplus G_2) = f(G_1)f(G_2)$. ■

Lemma 5.3: Let T_v be a rooted tree and suppose the root vertex v has degree 1. Then $f(T_v;x, y)$ is an irreducible polynomial in the polynomial ring $\mathbb{Z}[x, y]$. ■

Sketch of Proof for Theorem 5.1: We show how to reconstruct the rooted tree T_v from the polynomial $f(T_v;x, y)$. The proof uses the two lemmas and mathematical induction.

<u>Case 1</u>: The polynomial $f(T_v;x, y)$ factors in a non-trivial way. Then each irreducible factor of $f(T_v)$ must correspond to a rooted subtree in which the degree of the root is 1 (this follows from Lemmas 5.2 and 5.3). Now reconstruct each factor by induction, then glue them all together by identifying all of the roots, as illustrated in Figure 4.

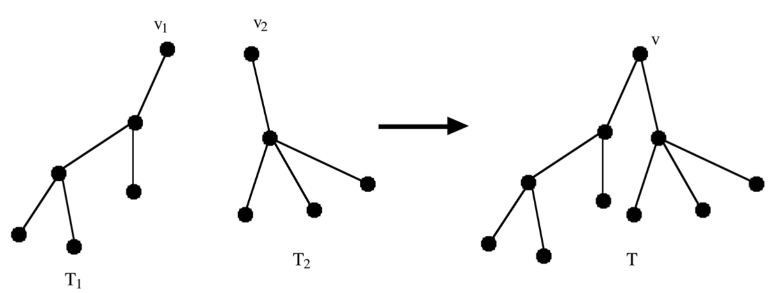

Figure 4: Tree identification.

Case 2: $f(T_v;x, y)$ does not factor. Then the root vertex v must have degree 1 (again from Lemmas 5.2 and 5.3). Let e be the edge incident to v and suppose the highest power of x appearing in $f(T_v;x, y)$ is n. Then $f(T_v/e;x, y) = f(T_v;x, y) - (x-1)^n y^{n-1}$. By induction, we can now reconstruct T_v/e, so we can reconstruct T_v by adding the edge e. ■

5.2. Unrooted Trees

When T is an ordinary (unrooted) tree, the ordinary Tutte polynomial is well-defined. Unfortunately, $f(T) = x^n$ gives no information about the structure of T. We now give a finer invariant for ordinary trees. We first need to define the *rank of a subset of edges*.

Definition 5.4: Let T be an unrooted tree with edge set E, and let $|E| = n$. For $A \subset E$, define the rank of A to be the size of the largest subtree complement contained in A:

$$r(A) = \max_{F \subseteq A}\{|F| : E - F \text{ is a subtree}\}.$$ ■

It is probably easier to think of $r(A)$ algorithmically. Given a subset A, we repeatedly prune the *leaves* of A: More precisely, given $A \subseteq E$, remove all leaves in A (a leaf is an edge incident on a vertex of degree 1), then remove all edges of A that became leaves after the first batch of leaves is removed, and so on. Let F be the collection of all edges of A that were removed at some stage in this process. Then $r(A) = |F|$. F is called a *feasible set*.

This rank function is usually called the *pruning rank* and it gives T an *antimatroid* structure. As usual, we will not need that level of generality. Antimatroids are an important class of greedoids with closed sets that satisfy a certain anti-exchange condition. See **[11]** for more on antimatroids and their relationship to greedoids.

This definition of rank enables us to define a Tutte polynomial $g(T;x, y)$ exactly as before:

$$g(T;x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

In light of Theorem 5.1, it is reasonable to make the following conjecture:

Conjecture 5.5: Let T_1 and T_2 be unrooted trees. Then $g(T_1;x, y) = g(T_2;x, y)$ *iff* T_1 and T_2 are isomorphic. □

However, this conjecture is false. We give a counterexample from **[16]**.

Example 5.6: Let T_1 and T_2 be the two trees shown in Figure 5. Note that each tree has 10 edges, so the definition of $g(T)$ requires us to compute the rank of all 2^{10} subsets of the edges of each tree. We postpone the computation of $g(T_i)$ until we give a combinatorial interpretation and we then have a better way to express $g(T)$. We promise to return to this.

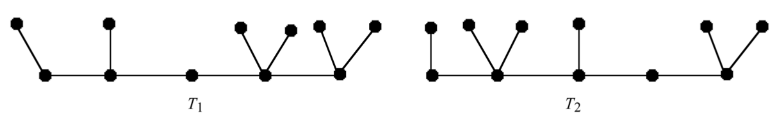

Figure 5: $g(T_1) = g(T_2)$.

T_1 and T_2 are examples of *caterpillars*; that is, trees in which every leaf is incident on a single path. The central path is called the *spine* of the caterpillar, even though caterpillars are invertebrates.

5.3. Combinatorial interpretations

The Tutte polynomial of a rooted or unrooted tree can be expressed in purely graph-theoretic terms. The basic idea is to group the terms in the subset expansions of the polynomials so that we can sum over subtrees instead of subsets.

We need the following notation. For a rooted tree T_v having n edges, let $\mathcal{S}$ be the collection of all subtrees of T rooted at v. If $S \in \mathcal{S}$, then let $m(S)$ be the number of edges x that can be added to S so that $S \cup \{x\}$ is a rooted subtree. If T is unrooted, let $\mathcal{S}$ be the collection of all subtrees of T, and if $S \in \mathcal{S}$, then let $i(S)$ be the number of *internal edges* of S; that is, the edges of S that are not leaves.

The following Theorem is proved in **[17]**.

Theorem 5.7:

(1) Let T_v be a rooted tree. Then $f(T_v;x,y) = \sum_{S \in \mathcal{S}} (x-1)^{n-|S|} y^{n-|S|-m(S)}$.

(2) For an unrooted tree T, we have $g(T;x,y) = \sum_{S \in \mathcal{S}} (x-1)^{|S|} y^{i(S)}$.

Sketch of Proof: We give the idea for part (2). Let S be a subtree and consider all subsets $A \subseteq E$ with $F = E - S \subseteq A$ and $|F|$ maximum. For each such A, F is the *unique* subtree complement contained in A of maximum size. This follows because the union of two subtree complements is also a subtree complement (equivalently, the intersection of any two subtrees is a subtree). Then $r(A) = |F| = n - |S|$ for all these subsets A, so they all have $r(E) - r(A) = |S|$.

What can A look like? Using our idea of repeated pruning to determine $r(A)$, we note that A must contain $F = E - S$. What additional edges can be in A? Precisely those edges that cannot be pruned after F has been pruned; that is, the internal edges of S. The binomial theorem allows us to group these terms together:

$$\sum (x-1)^{|S|}(y-1)^{|A|-|F|} = (x-1)^{|S|} y^{i(S)}. \qquad \blacksquare$$

Example 5.6, continued: We now apply Theorem 5.7 to complete the computation of $g(T)$ from Example 5.6 (see Figure 5)—thus, keeping our promise. Note that T_1 and T_2 have the same degree sequence. Since $i(T_1) = i(T_2) = 4$, we also have $i(S) \le 4$ for any subtree of T_1 or T_2. In the Table below we list the sizes of all the subtrees, together with the number of internal edges each subtree has. We omit entries in the table with no subtrees. For example, there are no subtrees with 8 edges and 2 internal edges. We also point out that the counts are facilitated by noticing that the only edges that can be internal in any subtree are the four edges along the spine of T_1 or T_2.

Table: Subtree data for T_1 and T_2 of Figure 5.

$\|E(S)\|$	$i(S)$	Count	Term
0	0	1	1
1	0	10	$10(x-1)$
2	0	14	$14(x-1)^2$
3	0	6	$6(x-1)^3$
	1	13	$13(x-1)^3 y$
4	0	1	$(x-1)^4$
	1	14	$14(x-1)^4 y$
	2	9	$9(x-1)^4 y^2$
5	1	6	$6(x-1)^5 y$
	2	15	$15(x-1)^5 y^2$
	3	7	$7(x-1)^5 y^3$
6	1	1	$(x-1)^6 y$
	2	9	$9(x-1)^6 y^2$
	3	18	$18(x-1)^6 y^3$
	4	2	$2(x-1)^6 y^4$

Table: Subtree data for T_1 and T_2 of Figure 5.

$\|E(S)\|$	$i(S)$	Count	Term
7	2	2	$2(x-1)^7y^2$
	3	17	$17(x-1)^7y^3$
	4	7	$7(x-1)^7y^4$
8	3	7	$7(x-1)^8y^3$
	4	9	$9(x-1)^8y^4$
9	3	1	$(x-1)^9y^3$
	4	5	$5(x-1)^9y^4$
10	4	1	$(x-1)^{10}y^4$

Summing the final column gives the shared Tutte polynomial. For convenience, we set $t = x-1$,

$$\begin{aligned} g(T_i;t+1,y) = {} & y^4t^{10} + 5y^4t^9 + y^3t^9 + 9y^4t^8 + 7y^3t^8 + 7y^4t^7 + 17y^3t^7 \\ & + 2y^2t^7 + 2y^4t^6 + 18y^3t^6 + 9y^2t^6 + yt^6 + 7y^3t^5 + 15y^2t^5 \\ & + 6yt^5 + 9y^2t^4 + 14yt^4 + t^4 + 13yt^3 + 6t^3 + 14t^2 + 10t + 1. \end{aligned}$$

We now restate Theorem 5.1 in combinatorial terms. We also include a related result obtained by using a pruning rank function for rooted trees **[17]**. For a rooted tree T_v, let $c_{i,l}$ be the number of subtrees S rooted at v with precisely i internal edges and l external edges. Let $d_{s,m}$ be the number of rooted subtrees S on s edges having exactly m edges e of $T_v - S$ with $S \cup \{e\}$ a rooted subtree.

Corollary 5.8: Let T_v be a rooted tree.
(1) T_v can be uniquely reconstructed from the sequence $\{c_{i,l}\}$.
(2) T_v can be uniquely reconstructed from the sequence $\{d_{s,m}\}$. ■

More colloquially, we can reconstruct a rooted tree from the knowledge of the number of subtrees with i internal and l external edges for all i and l.

The counterexample of Example 5.6 shows that the same is not true for unrooted trees. In purely combinatorial terms, we have:

Unrooted Trees:
It is not possible in general to reconstruct a tree from the sequence $\{c_{i,l}\}$. ■

6. Open Problems

(1) In the spirit of the study of chromatic- and Tutte-uniqueness for ordinary graphs, it would be interesting to extend Theorem 5.1 to other classes of rooted graphs. For instance, is it true that any two rooted graphs on n edges and n vertices have distinct rooted Tutte polynomials? More generally, we propose the following:

Problem 6.1: Find a non-trivial class of rooted graphs C so that, for $G_1, G_2 \in C$, we have $f(G_1) = f(G_2)$ *iff* $G_1 \cong G_2$. □

(2) There are several interesting evaluations of the Tutte polynomial—see Theorem 3.5. In Proposition 4.5, we list a few evaluations of the rooted version of the Tutte polynomial. It should be possible to extend or interpret other evaluations of the ordinary Tutte polynomial in the rooted case.

Problem 6.2: Extend all parts of Theorem 3.5 to the rooted case. □

(3) The chromatic polynomial is an evaluation of the ordinary Tutte polynomial (Theorem 3.8):

$$\chi(G;\lambda) = \lambda^c(-1)^{m-c}f(G;1-\lambda,0).$$

This suggests that it should be possible to extend the definition of the chromatic polynomial to rooted graphs by applying the same evaluation to the rooted Tutte polynomial:

$$\chi(G_v;\lambda) = \lambda^c(-1)^{m-c}f(G_v;1-\lambda, 0)\,.$$

It should be worthwhile to study this polynomial and the combinatorial interpretations of its evaluations at positive integers. A similar comment applies to the flow polynomial.

(4) Although the Tutte polynomial of an unrooted tree (Section 5) does not uniquely determine the tree, there may be interesting classes of trees for which this data is a complete invariant. Information in this direction appears in **[18]**. More precisely, we propose the following:

> **Problem 6.3:** Find a non-trivial class of trees $\mathcal{T}$ so that, for $T_1, T_2 \in \mathcal{T}$, we have $g(T_1) = g(T_2)$ *iff* $T_1 \cong T_2$. □

(5) It is possible to extend the Tutte polynomial for rooted trees to posets. This is the focus of **[19]** and **[20]**. In this context, it is not difficult to find combinatorial interpretations for the polynomial. It is also possible to find two posets with the same Tutte polynomial: $f(N) = f(3 \oplus 1)$. We conjecture that the Tutte polynomial is a complete invariant for series–parallel posets, however.

> **Conjecture 6.4:** If P_1 and P_2 are series–parallel posets with $f(P_1) = f(P_2)$, then $P_1 \cong P_2$. □

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[GTN LIV:6] HEDETNIEMI'S CONJECTURE, 40 YEARS LATER

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Abstract

Hedetniemi's conjecture states that the chromatic number of a categorical product of graphs is equal to the minimum of the chromatic numbers of the factors. We survey the many partial results surrounding this conjecture, to review the evidence and the counter evidence.

1. Introduction

The *categorical product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G) \times V(H)$, where $(u, v)(u', v') \in E(G \times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. It is easy to derive a proper vertex coloring of $G \times H$ from a proper vertex coloring of G or of H. More than forty years ago, Hedetniemi conjectured that this is essentially the best way to color the categorical product of two graphs:

Conjecture 1.1 [1]: $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$,
where $\chi(K)$ is the *chromatic number* of the graph K. □

The year was 1966 and Hedetniemi was a graduate student at the University of Michigan. To put matters in perspective, the four color problem was still the biggest open problem in graph coloring, only probabilistic constructions were known for graphs with large girth and large chromatic number, and the concept of NP-completeness had not yet been formulated. Arguably, the supporting evidence for Hedetniemi's conjecture is scanty by today's standards. Yet the conjecture survived and the research that grew out of it over the years revealed the depth and richness of the subject of product colorings.

There have been two previous surveys **[2][3]** of Hedetniemi's conjecture. In the present survey, we classify partial results surrounding it as supporting evidence or counter evidence, and discuss stronger and weaker conjectures and related problems. For the sake of fluidity, we omit proofs and limit the presentation of some useful auxiliary concepts such as homomorphisms, topological bounds, fractional graph theory; the interested reader should consult the references provided on these subjects. Our goal is to give a clear picture of the modesty of our state of knowledge concerning colorings of products of general graphs, and indicate that the study of Hedetniemi's conjecture for specific classes of factors is perhaps worthy of more attention than it has received so far.

2. Reformulations

The inequality

$$\chi(G \times H) \leq \min\{\chi(G), \chi(H)\}$$

follows easily from the fact that for any coloring c of a factor (say G), we can define a coloring c' of $G \times H$ by $c'(u, v) = c(u)$. The difficulty lies in deriving a coloring of a factor from a coloring of the product, to prove the second inequality,

$$\chi(G \times H) \geq \min\{\chi(G), \chi(H)\}.$$

It is convenient to consider each chromatic number separately, by introducing the statements

$\boldsymbol{H}(n)$: If $G \times H$ is n-colorable, then G or H is n-colorable.

Thus, Hedetniemi's conjecture holds if and only if $\boldsymbol{H}(n)$ holds for all n. This section of the paper consists of various reformulations of the conjecture in the language of graph homomorphisms.

2.1. Multiplicative Graphs

Following **[4]**, a *homomorphism* from G to K is a map $\phi\colon V(G) \to V(K)$ such that, if $uv \in E(G)$, then $\phi(u)\phi(v) \in E(K)$. The statement: “there exists a homomorphism from G to K” is denoted by $G \to K$ and its negation by $G \nrightarrow K$. We write $G \leftrightarrow H$ when $G \to H$ and $H \to G$; G and H are then called *homomorphically equivalent*.

A graph K is called *multiplicative* if, whenever we have $G \times H \to K$, then $G \to K$ or $H \to K$. Let K_n denote the complete graph on n vertices, then $G \to K_n$ if and only if $\chi(G) \le n$. This allows the reformulation:

> **Proposition 2.1 [5]:** $\boldsymbol{H}(n)$ is equivalent to the statement: K_n is multiplicative. ■

The adjective *multiplicative* has become standardized although the property was previously introduced under the name *productivity* **[6]**. Hedetniemi’s conjecture falls within the more general problem of characterizing multiplicative graphs, digraphs, and relational structures (see also **[2][7]–[9]**). It is difficult to prove the multiplicativity of a single graph, since this property depends on the structure of the whole category of graphs. In fact, the question as to whether a graph is multiplicative is not even known to be decidable. *Exponential graphs*, introduced next, have been the most successful tools in the study of multiplicative graphs.

2.2. Exponential Graphs

Given two graphs G and K, the *exponential graph* K^G has for vertices the set of all functions $\varphi\colon V(G) \to V(K)$ (not just homomorphisms), and edges the pairs (φ_1, φ_2) such that for every $uv \in E(G)$, $\varphi_1(u)\varphi_2(u) \in E(K)$. Exponentiation and exponential structures have many applications in algebra and elsewhere. It follows from the definition of K^G that the *evaluation map* $\varepsilon\colon G \times K^G \to K$ defined by $\varepsilon(u, \varphi) = \varphi(u)$ is a homomorphism. Moreover, we have $G \times H \to K$ if and only if $H \to K^G$. Thus, for an integer n and a graph G, there exists a graph H such that $\chi(G \times H) \le n$ if and only if $H = K_n^G$ has this property. This implies:

> **Proposition 2.2 [10]:** $\boldsymbol{H}(n)$ is equivalent to the statement:
> If $\chi(G) > n$, then $\chi(K_n^G) = n$. ■

In **[5][10][11]** this version is exploited, not by considering K_n^G itself, but suitable subgraphs of K_n^C, where $C \to G$. Indeed, if $C \to G$, then $K_n^G \to K_n^C$ and thus it is sufficient to color a subgraph of K_n^C containing a homomorphic image of K_n^G. It is not clear that this method can be generalized to prove more cases of Hedetniemi’s conjecture. However, a second enlightening use of exponentiation is to apply it to an entire category of graphs rather than to a single graph.

Let $K_n^{\mathcal{G}}$ be the set of all graphs of the form K_n^G. The relation $\to$ induces a preorder on $K_n^{\mathcal{G}}$. The quotient structure $K_n^{\mathcal{G}}/{\leftrightarrow}$ is well known to be a Boolean lattice (see **[8]**). Thus, in a sense, the structure $K_n^{\mathcal{G}}/{\leftrightarrow}$ is much better understood than that of the single elements in $K_n^{\mathcal{G}}$. $\boldsymbol{H}(n)$ is equivalent to the statement that $K_n^{\mathcal{G}}/{\leftrightarrow}$ is a two-element lattice. The maximal element of $K_n^{\mathcal{G}}/{\leftrightarrow}$ consists of all graphs K_n^G containing loops; that is, all graphs K_n^G such that $G \to K_n$. (The homomorphisms $\varphi\colon G \to K_n$ are the loops of K_n^G.) The minimal element of $K_n^{\mathcal{G}}/{\leftrightarrow}$ consists of all graphs K_n^G such that $K_n^G \to K_n$; conjecturally, this coincides with the class of all graphs K_n^G such that $G \nrightarrow K_n$.

2.3. Retracts and Products

A graph K is called a *retract* of a graph G if there exist homomorphisms $\rho\colon G \to K$ and $\gamma\colon K \to G$ such that $\rho \circ \gamma$ is the identity on K. In particular, if the complete graph K_n is a retract of G, then $\chi(G) = \omega(G) = n$, where $\omega(G)$ is the *clique number* of G. For general graphs, the clique number is often smaller than the chromatic number. Thus, to express the chromatic number of a graph G in terms of retracts we need to add a disjoint copy of K_n: $\chi(G) \le n$ if and only if K_n is a retract of the disjoint union of G and K_n. Doing this to both factors of a product, we obtain the following.

> **Proposition 2.3 [12]:** $\boldsymbol{H}(n)$ is equivalent to the statement: Whenever K_n is a retract of a product of graphs, it is a retract of one of the factors. ■

A graph is called a *core* if it has no proper retracts. Every (finite) graph has a core that is unique up to isomorphism, and Proposition 2.3 generalizes to a characterization of the multiplicative cores as the cores that cannot be expressed as a retract of a product without being a retract of a factor. In many structure theories, one objective is to similarly characterize “irreducible” elements that cannot be built up as retracts of products in a nontrivial way.

The statement can also be presented from an order-theoretic point of view: The relation $\to$ induces a preorder on the category $\mathcal{G}$ of all graphs. It is well known (see **[4]**) that the natural quotient $\mathcal{G}/\leftrightarrow$ is a distributive lattice, with the meet operation induced by the categorical product and the join operation induced by the disjoint union. The multiplicative graphs turn out to correspond to the meet-irreducible elements of $\mathcal{G}/\leftrightarrow$. Thus Hedetniemi's conjecture states that the complete graphs are meet-irreducible.

3. Supporting Evidence

The statements $\boldsymbol{H}(1)$ and $\boldsymbol{H}(2)$ are relatively trivial. $\boldsymbol{H}(3)$ is nontrivial and was proved by El-Zahar and Sauer.

> **Theorem 3.1 [10]:** The chromatic number of the product of two 4-chromatic graphs is 4. ■

Note that the formulation of Theorem 3.1 (which is the title of **[10]**) is contrapositive. The exposition in **[10]** is closer to a direct proof of $\boldsymbol{H}(3)$, through a 3-coloring of some connected components of K_3^C, where C is an odd cycle contained in G or H. In **[13]**, it is shown that this gives a polynomial procedure to derive a 3-colouring of G or of H from a 3-coloring of $G \times H$.

None of the other statements $\boldsymbol{H}(n)$ are proved. However, many partial results for large chromatic numbers have been obtained by restricting the class of factors considered.

> **Theorem 3.2 [14]:** Let G be a graph such that every vertex of G is in an n-clique. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

> **Theorem 3.3 [15]:** Let G be a graph such that $\chi(G) \geq n$ and for every pair e_1, e_2 of edges of G, there is an edge e_3 incident to both of them. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

The *Hajos sum* of two graphs G and H with respect to the edges $[uv \in E(G)$ and $u'v' \in E(H)$ is the graph obtained from the disjoint union of G and H by removing uv and $u'v'$, identifying u and u' to a single vertex, and adding the edge vv'.

> **Theorem 3.4 [16]:** Let G be a Hajos sum of two graphs A and B, where $\chi(K_n^A) = n$ and B is obtained from copies of K_{n+1} by means of adding vertices and edges, taking Hajos sums, and at most one identification of nonadjacent vertices. For every graph H, if $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

In the preceeding three results, the factor G is heavily constrained (to find an n-coloring of K_n^G, but the factor H is free. The next results show that it is possible to impose weaker conditions, but on both factors at the same time.

> **Theorem 3.5 [17][18]:** Let G and H be connected graphs containing n-cliques. If $\chi(G \times H) = n$, then $\min\{\chi(G), \chi(H)\} = n$. ■

> **Theorem 3.6 [11]:** Let G and H be connected graphs containing odd wheels. If $\chi(G \times H) = 4$, then $\min\{\chi(G), \chi(H)\} = 4$. ■

It is interesting to compare Theorem 3.5 with the products and retracts version of Hedetniemi's conjecture. Theorem 3.5 can be shown to be equivalent to the statement that whenever the complete graph K_n is a retract of a product of two connected graphs, it is a retract of a factor. By Proposition 2.3, if we drop the requirement that the factors be connected, we get a statement that is equivalent to Hedetniemi's conjecture. Theorem 3.1 can be seen as a stronger version of Theorem 3.5, when $n = 3$, replacing the condition that the factors contain triangles by the (trivial) condition that the factors contain odd cycles. This point of view inspired Theorem 3.6 from Theorem 3.5 when $n = 4$, replacing the condition that the factors contain 4-cliques by the condition that the factors contain odd wheels. However, this approach has serious limitations (see Theorem 4.3).

Other results of this type involve well known lower bounds for the chromatic number. In **[19]**, the value of two plus "the connectivity of the geometric realization of the neighborhood complex of a graph" is introduced as a lower bound on the chromatic number of a graph. Here we call it the *topological bound*, although there are many similarly defined "topological bounds" (see **[20]**).

Theorem 3.7 [21]: Let G and H be graphs for which the topological bound on the chromatic number is tight. Then $\chi(G \times H) = \min\{\chi(G), \chi(H)\}$. ■

Another such bound is the *fractional chromatic number* $\chi_f(G)$ of a graph G; that is, the common linear relaxation of its clique number and its chromatic number (under suitable integer programming formulations, see **[22]** and Section 7). In particular, for every graph G, we have $\chi_f(G) \leq \chi(G)$.

Theorem 3.8 [23]: $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/2$. ■

The partial results of this section are susceptible of refinements and variations. It is unlikely that any of them will yield a complete proof of Hedetniemi's conjecture, but even partial results are interesting in their own right. For instance, suppose $\chi(G) = \chi(H) = n$, the topological bound is tight on G and $\chi_f(H) = n$, then Hedetniemi's conjecture states that $\chi(G \times H) = n$ but a common refinement of Theorems 3.7 and 3.8 would be needed to confirm this. This suggests the following.

Problem 3.9: Is there a natural lower bound on the chromatic number of a graph that is a common refinement of the topological bound and the fractional chromatic number? □

In the same vein, consider the following two results.

Theorem 3.10 [24]: Let G be a *Cayley graph* on $\mathbb{Z}_2^n$. If G contains an odd cycle, then $\chi(G) \geq 4$. ■

Theorem 3.11 [25]: Let G be a *vertex-transitive* graph such that G contains a triangle and $|V(G)|$ is not a multiple of 3. Then $\chi(G) \geq 4$. ■

If G and H satisfy the hypotheses of Theorems 3.10 and 3.11, respectively, then by Theorem 3.1, $\chi(G \times H) \geq 4$. It would be interesting to prove this result directly in the context of vertex-transitive graphs.

Problem 3.12: Is there a natural class of non 3-colorable vertex-transitive graphs that generalizes both the hypotheses of Theorem 3.10 and Theorem 3.11? □

4. Stronger Conjectures

4.1. Fiber Products

El-Zahar and Sauer actually proved a stronger result than Theorem 3.1.

Theorem 4.1 [10]: Let G and H be connected 4-chromatic graphs, and let C and C' be odd cycles contained in G and H, respectively. Then the subgraph of $G \times H$ induced by $(C \times H) \cup (G \times C')$ is 4-chromatic. ■

They conjectured that a similar phenomenon holds for higher chromatic numbers.

Conjecture 4.2 [10]: Let G and H be connected $(n+1)$-chromatic graphs, and let G' and H' be n-chromatic subgraphs of G and H, respectively. Then the subgraph of $G \times H$ induced by $(G' \times H) \cup (G \times H')$ is $(n+1)$-chromatic. □

This, however, turns out to be false.

Theorem 4.3 [11]: There exists a 4-chromatic graph K such that for each $n \geq 5$ there exists an n-chromatic graph G_n containing K as a subgraph, such that the subgraph of $G_n \times G_n$ induced by $(K \times G_n) \cup (G_n \times K)$ is 4-chromatic. ■

The fiber product yields another hypothesis on the structure of n-chromatic subgraphs of products of n-chromatic graphs. Given two n-chromatic graphs with n-colorings $c_G\colon G \to K_n$ and $c_H\colon H \to K_n$, their *fiber product* over c_G and c_H is the subgraph $(G, c_G) \times (H, c_H)$ of $G \times H$ induced by

$$V((G, c_G) \times (H, c_H)) = \{(u, v) \in V(G \times H) : c_G(u) = c_H(v)\}.$$

Conjecture 4.4 [26]: The fiber product of two n-chromatic graphs over n-colorings is n-chromatic. □

In **[26]**, the conjecture is proved for $n = 3$; it is still open for $n = 4$. Note that both Conjectures 4.2 and 4.4 suggest that the critical subgraphs of $G \times H$ are smaller that $G \times H$ itself.

4.2. Uniquely Colorable Graphs

In **[27]** it is proved that if G is connected and $\chi(G) > n$, then $G \times K_n$ is uniquely n-colorable. This motivates the following refinements of $\boldsymbol{H}(n)$ in terms of *uniquely colorable graphs*.

> $\boldsymbol{A}(n)$: If G and H are uniquely n-colorable, then each n-coloring of $G \times H$ is induced by a coloring of G or H.
>
> $\boldsymbol{B}(n)$: If G is uniquely n-colorable, H is connected, and $\chi(H) > n$, then $G \times H$ is uniquely n-colorable.

In **[17]** it is proved that $\boldsymbol{A}(n)$ implies $\boldsymbol{B}(n)$ and $\boldsymbol{B}(n)$ implies $\boldsymbol{H}(n)$. It is also conjectured that $\boldsymbol{A}(n)$ holds for all n.

4.3. Circular Colorings

Following **[28]**, for relatively prime integers r, s such that $2r \le s$ the *circular complete graph* $K_{s/r}$ has the elements of $\mathbb{Z}_s$ for vertices and for edges the pairs (i, j) such that $j - i \in \{r, r+1, \ldots, s-r\}$. The *circular chromatic number* $\chi_c(G)$ of a graph G is the (well-defined) smallest rational q such that $G \to K_q$. In particular, $\chi(G) = \lceil \chi_c(G) \rceil$; hence, χ_c is a refinement of χ. Zhu proposed the following strengthening of Hedetniemi's conjecture:

> **Conjecture 4.5 [28]:** $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$. □

The result of El-Zahar and Sauer has been adapted to the circular case, as well as the result of Duffus–Sands–Woodrow and Welzl.

> **Theorem 4.6 [9]:** If $\min\{\chi_c(G), \chi_c(H)\} \le 4$, then $\chi_c(G \times H) = \min\{\chi_c(G), \chi_c(H)\}$. ■
>
> **Theorem 4.7 [29]:** If G and H are connected graphs containing $K_{s/r}$ and $\chi_c(G \times H) = s/r$, then $\min\{\chi_c(G), \chi_c(H)\} = s/r$. ■

To date, the only graphs known to be multiplicative (up to homomorphic equivalence) are the circular complete graphs K_1 and K_q, $2 \le q < 4$.

5. Counter Evidence

There is no known counterexample to Hedetniemi's conjecture. However, there are natural extensions of this conjecture that are known to be false, most notably for directed graphs and infinite chromatic numbers.

5.1. Infinite Graphs

The chromatic number of an infinite graph is a (finite or infinite) cardinal. The product of two infinite chromatic graphs is still infinite chromatic, but nonetheless Hajnal notes that Hedetniemi's conjecture fails for infinite chromatic numbers.

> **Theorem 5.1 [30]:** For every infinite cardinal κ, there exist graphs G and H such that $\chi(G) = \chi(H) = \kappa^+$ and $\chi(G \times H) = \kappa$. ■

The examples provide an interesting application of the theory of stationary sets, although they cannot be directly adapted to the finite case. Extensions of Theorem 5.1 involve models of set theory; Soukup **[31]** proved that the following statement:

> There are two graphs G and H both of cardinality $\aleph_2$ such that $\chi(G) = \chi(H) = \aleph_2$ and $\chi(G \times H) = \aleph_0$.

is consistent with the axioms of Zermelo–Fraenkel set theory, with the axiom of choice, and the generalized continuum hypothesis, although Hajnal notes that it cannot be proved in this axiom system.

In the finite case, Hedetniemi's conjecture implies that the formula

$$\chi\left(\prod_{i=1}^{n} G_i\right) = \min\{\chi(G_1), \ldots, \chi(G_n)\}$$

is valid for any finite n. Miller **[32]** notes that it cannot be extended to an infinite number of factors. Indeed, the product of any family of graphs with unbounded odd girth is bipartite.

5.2. Directed Graphs

A coloring c of the vertices of a directed graph $\vec{G}$ is called *proper* if for every arc (u, v), $c(u) \neq c(v)$. Thus the constraints are not affected by the orientation, and the minimum number $\chi(\vec{G})$ of colors needed to properly color $\vec{G}$ is just the chromatic number of the underlying undirected graph obtained from $\vec{G}$. The *categorical product* $\vec{G} \times \vec{H}$ of two directed graphs $\vec{G}$ and $\vec{H}$ has vertex set $V(\vec{G}) \times V(\vec{H})$, and its arcs are the couples $((u, u'), (v, v'))$ such that (u, v) is an arc of $\vec{G}$ and (u', v') is an arc of $\vec{H}$. Categorical products of directed graphs have fewer edges than their undirected counterparts and tend to be easier to color. Poljak and Rödl **[33]** noted that Hedetniemi's conjecture fails for directed graphs, even with relatively small examples. To date, the strongest results in this direction are the following.

> **Theorem 5.2 [34]:** For every $\varepsilon > 0$ and $N > 0$, there exists an integer $n > N$ and directed graphs $\vec{G}_n$ and $\vec{H}_n$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = n$ and $\chi(\vec{G}_n \times \vec{H}_n) \leq \frac{2}{3}n + \varepsilon$. ■

> **Theorem 5.3 [35][36]:** For each $n \geq 4$, there exist directed graphs $\vec{G}$ and $\vec{H}$ such that $\chi(\vec{G}_n) = n$, $\chi(\vec{H}_n) = 4$, and $\chi(\vec{G}_n \times \vec{H}_n) = 3$. ■

> **Theorem 5.4 [35]:** There exist directed graphs $\vec{G}$ and $\vec{H}$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = 5$ and $\chi(\vec{G}_n \times \vec{H}_n) = 3$. ■

The *Poljak–Rödl function* $f\colon \mathbb{N} \to \mathbb{N}$ is defined by

$$f(n) = \min\{\chi(\vec{G} \times \vec{H}) : \chi(\vec{G}) = \chi(\vec{H}) = n\}.$$

Its undirected counterpart is the function $g\colon \mathbb{N} \to \mathbb{N}$ defined by

$$g(n) = \min\{\chi(G \times H) : \chi(G) = \chi(H) = n\}.$$

Thus, Hedetniemi's conjecture states that $g(n) = n$ for all n, whereas Theorem 5.2 shows that asymptotically $f(n) \leq 2n/3$. The two functions are obviously nondecreasing, and $f(n) \leq g(n)$ for all n. The values $f(1) = g(1) = 1$, $f(2) = g(2) = 2$, $f(3) = g(3) = 3 = f(4) = f(5)$, and $g(4) = 4$ are the only known values of f and g. In fact, the functions are not even known to be unbounded. Building on Poljak and Rödl's work with the *arc-graph construction*, Poljak, Schmerl,[1] and Zhu independently proved the following intriguing statements.

> **Proposition 5.5 [3][37]:**
> (1) Either f is unbounded or $f(n) \leq 3$ for all n.
> (2) Either g is unbounded or $g(n) \leq 9$ for all n. ■

Theorem 5.3 suggests that f may be bounded. However, this would contradict Hedetniemi's conjecture.

> **Proposition 5.6 [38]:** f is bounded if and only if g is bounded. ■

Perhaps the strongest argument against Hedetniemi's conjecture is the difficulty in proving that g is unbounded. Many interesting developments center around this question (see Section 6).

5.3. Hypergraphs

Following **[39]**, the *box product* of two graphs G and H is the hypergraph $G \blacksquare H$ with vertex set $V(G) \times V(H)$ and hyperedges $\{uu', uv', vu', vv'\}$ for every $uv \in E(G)$ and $u'v' \in E(H)$. A coloring of the vertices of a hypergraph is called *proper* if it has no monochromatic hyperedge, and the chromatic number of a hypergraph is the minimum number of colors needed to properly color the hypergraph.

Mubayi and Rödl **[39]** propose a general conjecture that admits the following particular case.■

> **Conjecture 5.7:** There exists a bound c such that for every n, there exist graphs G_n and H_n such that $\chi(G_n) = \chi(H_n) = n$ and $\chi(G_n \blacksquare H_n) \leq c$. □

Note that a proper coloring of $G \blacksquare H$ can be derived from a proper coloring of any product $\vec{G}_n \times \vec{H}_n$ of an orientation of G and orientation of H. Thus, the *fallacy* of Conjecture 5.7 would imply that the Poljak-Rödl function f is unbounded. Nonetheless Mubayi and Rödl base their conjecture on the fact that the statement is true for products of hypergraphs, when the size of the hyperedges of the factors is not bounded.

[1] J. Schmerl's unpublished result was obtained at the 1984 Banff conference *Graphs and Order*, after hearing of Poljak and Rödl's result. His contribution is mistakenly attributed to Schelp in **[3]**.

> **Theorem 5.8 [39]:** For every integer n there exists a hypergraph H_n such that $\chi(H_n) \geq n$ and $\chi(H_n \blacksquare H_n) = 2$. ■

This refutes a conjecture of **[40]**, although **[41]** shows that variants of Theorems 3.2 and 3.5 can be adapted to products of hypergraphs. In a sense, this weakens the support that Theorems 3.2 and 3.5 offer to Hedetniemi's conjecture.

5.4. Multicolorings

For an integer r, an r-coloring of a graph G is an assignment of an r-set $c(u)$ to each vertex u of G, such that if $uv \in E(G)$, then $c(u) \cap c(v) = \varnothing$. The *r-chromatic number* $\chi_r(G)$ of G is the least integer s such that G admits an r-coloring c with $\left|\bigcup_{u \in V(G)} c(u)\right| = s$. In particular, a 1-coloring is an ordinary coloring and $\chi_1(G) = \chi(G)$.

An r-coloring of $G \times H$ can be derived from an r-coloring of G or H, but also from the disjoint union of an r_1-coloring of G with an r_2-coloring of H, where $r_1 + r_2 = r$. This is the basis of the following result:

> **Theorem 5.9 [42]:** There are graphs G and H and integers r such that $\chi_r(G \times H) < \min\{\chi_r(G), \chi_r(H)\}$. ■

Multichromatic numbers can also be defined in terms of homomorphisms to Kneser graphs. For integers r and s, the *Kneser graph* $K(r, s)$ has the r-subsets of $\{1, 2, \ldots, s\}$ as vertices and two vertices are adjacent if they correspond to disjoint subsets. Thus, for a graph G, $\chi_r(G)$ is the least integer s such that G admits a homomorphism in $K(r, s)$. Hence, Theorem 5.9 admits the following reformulation.

> **Theorem 5.10 [42]:** There are nonmultiplicative Kneser graphs. ■

Nonetheless, the adaptation of Theorem 3.5 is valid for multichromatic numbers.

> **Theorem 5.11 [12]:** Whenever a Kneser graph is a retract of a product of connected graphs, it is a retract of a factor. ■

Theorems 5.10 and 5.11 weaken the support that Theorem 3.5 gives to Hedetniemi's conjecture.

6. Weaker Conjectures

6.1. Ramsey Theory

Burr, Erdős, and Lovász **[14]** rediscovered Hedetniemi's conjecture independently, and applied it to a problem in Ramsey Theory. For any graph G, there exists at least one graph F such that whenever the edges of F are colored in red and blue, then F contains a copy of G with all its edges having the same color. The smallest possible chromatic number for such a graph F is called the *chromatic Ramsey number* $r_c(G)$ of G. These authors proved the inequality $r_c(G) \geq (\chi(G) - 1)^2 + 1$, and proposed the following conjecture.

> **Conjecture 6.1 [14]:** For every integer $r \geq 1$, there exists a graph G_r that $\chi(G_r) = r$ and $r_c(G_r) = (r-1)^2 + 1$. □

There is a natural candidate G_r constructed as follows. For every coloring p_i of the edges of $K_{(r-1)^2+1}$ in red and blue, there exists a monochromatic subgraph H_i of $K_{(r-1)^2+1}$ such that $\chi(H_i) \geq r$. Let G_r be the categorical product of all the graphs H_i. Assuming Hedetniemi's conjecture, then $\chi(G_r) = r$, and using this hypothesis, Burr, Erdős and Lovász **[14]** conclude that G_r verifies Conjecture 6.1. Theorem 3.2 is used to prove Conjecture 6.1 for $r \leq 4$. In **[43]**, this is extended to the case $r = 5$.

6.2. The Weak Hedetniemi Conjecture

The main open problem surrounding Hedetniemi's conjecture is the asymptotic behavior of the Poljak–Rödl functions f and g.

> **Conjecture 6.2 (The Weak Hedetniemi Conjecture):** For every integer n, there exists an integer m_n such that if $\chi(G) = \chi(H) = m_n$, then $\chi(G \times H) \geq n$. □

By Theorem 5.5, it suffices to prove the result for $n = 10$, or a directed version of the result for $n = 4$. Thus, a strengthening of Theorem 3.1 to prove $\boldsymbol{H}(9)$ would be sufficient to prove Conjecture 6.2. Duffus and Sauer **[44]** observe that a common strengthening of Theorems 3.2 and 3.5 would also be sufficient.

Proposition 6.3 [44]: Consider the following hypothesis:
For any integer n and for any graphs G and H such that G is connected, G contains K_n, and $\min\{\chi(G), \chi(H)\} > n$, then $\chi(G \times H) > n$.
If this hypothesis is true, then the weak Hedetniemi conjecture is also true. ■

Let $\mathcal{G}$ be the class of finite undirected graphs. The lattice $K_9^{\mathcal{G}}/\leftrightarrow$ is well known to be Boolean (see **[8]**). If the lattice is finite or even if it has an atom $G/\leftrightarrow$ then for $n > \chi(K_9^{\mathcal{G}})$, $g(n) \geq 10$, whence, by Proposition 5.5, g is unbounded and the weak Hedetniemi conjecture is true. If, however, $K_9^{\mathcal{G}}/\leftrightarrow$ has no atoms, then it is dense, and up to isomorphism there is only one countable dense Boolean lattice. Therefore, at least one of the following statements holds.

(1) The weak Hedetniemi conjecture is true.

(2) The lattice $K_9^{\mathcal{G}}/\leftrightarrow$ is the unique dense countable Boolean lattice.

Both eventualities suggest a rich, and as yet undiscovered, structure in the category of graphs. Now, let $\mathcal{D}$ be the class of finite directed graphs. By Theorem 5.3, the lattice $K_3^{\mathcal{D}}/\leftrightarrow$ is at least known to be infinite. Perhaps it is possible to determine its exact structure.

Problem 6.4: Is $K_3^{\mathcal{D}}/\leftrightarrow$ isomorphic with the unique dense countable Boolean lattice? □

The weak Hedetniemi conjecture has interesting potential within the field of graph coloring (see Problem 3.9), but also in other fields of mathematics. In particular, Schmerl **[45]** linked the weak Hedetniemi conjecture to models of *Peano arithmetic*.

Proposition 6.5 [45]: Let $\mathcal{M}$ be an arithmetically saturated model of Peano arithmetic that is not a model of true arithmetic. Then $\mathcal{M}$ has a generic automorphism if and only if the weak Hedetniemi conjecture holds. ■

The weak Hedetniemi conjecture is reasonable, but so is the hypothesis that the Poljak–Rödl function g is bounded by 4. Indeed, El-Zahar and Sauer's proof of $\boldsymbol{H}(3)$ relies on odd cycles, hence on the fact that the topological bound is (essentially) tight for 3-chromatic graphs, whereas it is not always tight for larger chromatic numbers. However, it is harder to imagine the true bound on the Poljak–Rödl function being a number larger than 4. The case of directed graphs does not present such unacceptable alternatives: Either the function f is unbounded, or $f(n) = 3$ for all $n \geq 3$. Thus, it would be interesting to refine Proposition 5.5, and eventually prove that g is either unbounded or bounded by 4. The circular versions of the Poljak–Rödl functions considered next provide insight in this direction.

6.3. The Circular Weak Hedetniemi Conjecture

The circular versions of the Poljak–Rödl functions are the functions $f_c, g_c\colon [2, \infty) \to [2, \infty)$ defined by

$$f_c(x) = \inf\{\chi_c(\vec{G} \times \vec{H}) : \chi_c(\vec{G}), \chi_c(\vec{H}) \geq x\},$$

$$g_c(x) = \inf\{\chi_c(G \times H) : \chi_c(G), \chi_c(H) \geq x\}.$$

Obviously, f_c and g_c are bounded if and only if f and g are. By Theorem 4.6, we have $g(x) = x$ for $x \in [2, 4]$. Thus, if g_c is bounded, the bound is somewhere between 4 and 9. If f_c is bounded, the bound is at most 3, but to date there is not even a proof that f_c is not identically equal to 2. We can formulate the conjecture that f_c is not identically equal to 2 in terms of homomorphisms to odd cycles as follows.

Conjecture 6.6 (The Circular Weak Hedetniemi Conjecture): There exists an odd cycle C_{2k+1} and an integer n such that if $\chi_c(\vec{G}), \chi_c(\vec{H}) \geq n$, then there is no homomorphism from $\vec{G} \times \vec{H}$ to C_{2k+1}. □

Hedetniemi's conjecture trivially implies the weak Hedetniemi conjecture, which implies that f and f_c are unbounded (by Proposition 5.6), which implies Conjecture 6.6. However, even the latter seems to be a hard problem.

Tighter bounds on f_c would also improve the bounds on g, as shown by the following result.

Proposition 6.7: If f_c is bounded by 5/2, then g is bounded by 8, and if f_c is bounded by 7/3, then g is bounded by 7.

Proof: We include a sketch of proof here, since this result has not appeared elsewhere. Suppose that f_c is bounded by 5/2. Then, for every $n \geq 3$, there exist directed graphs $\vec{G}_n$ and $\vec{H}_n$ such that $\chi(\vec{G}_n) = \chi(\vec{H}_n) = n$ and there exists a homomorphism ϕ_n: $\vec{G}_n \times \vec{H}_n \to C_5$. Put

$$m_n = \max\{\chi(K_{n-1}^{\vec{G}_n}), \chi(K_{n-1}^{\vec{H}_n})\} + 1 .$$

The undirected graphs A_n and B_n are defined as follows: A_n is obtained from $\vec{G}_n \times \vec{G}_{m_n}$ by ignoring its orientation, and B_n is obtained by ignoring the orientation in $\vec{H}_n \times \vec{R}$, where $\vec{R}$ is the digraph obtained from $\vec{H}_{m_n}$ by reversing its orientation. By definition of m_n, we have $\chi(A_n) = \chi(B_n) = n$. Define a function ψ_n: $V(A_n \times B_n) \to V(C_5 \times C_5)$ by

$$\psi_n((u, v), (w, x)) = (\phi_n(u, w), \phi_{m_n}(v, x)) .$$

Let $((u, v), (w, x)), ((u', v'), (w', x'))$ be an edge of $A \times B$. Without loss of generality we can assume that $((u, v), (u', v'))$ is an arc of $\vec{G}_n \times \vec{G}_{m_n}$. If $((w, x), (w', x'))$ is also an arc of $\vec{H}_n \times \vec{R}$, then $((u, w), (u', w'))$ is an arc of $\vec{G}_n \times \vec{H}_n$, whence $\phi_n(u, w)\phi_n(u', w')$ is an edge of C_5. Otherwise, $((w', x'), (w, x))$ is an arc of $\vec{H}_n \times \vec{R}$, whence $((v, x), (v', x'))$ is an arc of $\vec{G}_{m_n} \times \vec{H}_{m_n}$, and $\phi_{m_n}(v, x)\phi_{m_n}(v', x')$ is an edge of C_5. Therefore, n is a homomorphism from $A_n \times B_n$ to the *very strong product* $C_5 \star C_5$ defined by

$$E(C_5 \star C_5) = \{((i, j), (i', j')) : ii' \in E(C_5) \text{ or } jj' \in E(C_5)\}.$$

Therefore, $\chi(A_n \times B_n) \leq \chi(C_5 \star C_5)$. It is known (see **[46]**) that $\chi(C_5 \star C_5) = 8$ and this proves the first statement. The second statement is proved in a similar way, using the fact that $\chi(C_7 \star C_7) = 7$. ■

For every k we have $\chi(C_{2k+1} \star C_{2k+1}) \geq 7$. Therefore, the proof method of Proposition 6.7 cannot exhibit further links between the potential upper bounds of f_c and g.

7. Related Problems

7.1. Multiplicative Graphs

Many variations and strengthenings of Hedetniemi's conjecture, such as Conjecture 4.5, fall within the larger framework of characterizing multiplicative graphs in general. The problem is indeed interesting and deep, in view of the scarcity of known examples. Delhommé and Sauer **[7]** have shown that the class of square-free graphs is a promising candidate, where it might be possible to adapt the methods of **[10]**. In particular, they prove that if G and H are connected graphs each containing a triangle and K is square-free, then $G \times H \to K$ implies $G \to K$ or $H \to K$. This suggests the following question.

Problem 7.1: Are all square-free graphs multiplicative? □

Of course, the multiplicativity of square-free graphs does not impact directly on Hedetniemi's conjecture, but clearly a better understanding of the class of multiplicative graphs would be an asset, and the work in **[9]** shows that it is sometimes possible to prove the multiplicativity of some graphs by using the known multiplicativity of other graphs.

Other variations on Hedetniemi's conjecture are independent of multiplicativity. We present two of them next.

7.2. The Fractional Chromatic Number

The fractional chromatic number of a graph can also be defined in terms of multicolorings or homomorphisms into Kneser graphs:

$$\chi_f(G) = \min_{r \in \mathbb{N}} \frac{\chi_r(G)}{r} = \min\left\{\frac{s}{r} : G \to K(s, r)\right\}.$$

The fractional version of Hedetniemi's conjecture is the following.

Problem 7.2: Does the identity $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ always hold? □

Note that the identity would follow directly from the multiplicativity of all Kneser graphs; however, by Theorem 5.10, there are nonmultiplicative Kneser graphs. Nonetheless, in **[47]**, it is shown that the identity $\chi_f(G \times H) = \min\{\chi_f(G), \chi_f(H)\}$ holds whenever one factor belongs to a well behaved class of graphs, and in **[36]**, it is shown that the inequality $\chi_f(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/4$ always holds. Thus, there are some grounds for believing that the identity is always true.

When G is vertex-transitive, then $\chi_f(G) = |V(G)|/\alpha(G)$, thus Problem 7.2 admits as a subquestion the vertex-transitive case of the problem of determining of the independence number in categorical products of graphs. This is a very active research area **[29][48]–[50]**, where not only the cardinality of large independent sets but also their structure is analyzed.

The examples in Theorem 5.2 are tournaments, which implies that the directed version of the fractional Hedetniemi conjecture fails. The fractional version of the Poljak–Rödl functions

$$f_f(x) = \inf\{\chi_f(\vec{G} \times \vec{H}) : \chi_f(\vec{G}), \chi_f(\vec{H}) \geq x\},$$

$$g_f(x) = \inf\{\chi_f(G \times H) : \chi_f(G), \chi_f(H) \geq x\},$$

are essentially linear. Using lexicographic products with complete graphs, it can be shown (see **[36]**) that $g_f(x) \simeq c \cdot x$, with $c \in [1/4, 1]$, and $f_f(x) \simeq c \cdot x$ with $c \in [1/4, 2/3]$.

Either Hedetniemi's conjecture or its fractional version would imply the inequality $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}$. The inequality $\chi(G \times H) \geq \min\{\chi_f(G), \chi_f(H)\}/2$ of Theorem 3.8 is known to hold both for directed and undirected graphs. The interesting phenomenon arising in these fractional versions of Hedetniemi's conjecture is that the lower bounds obtained so far for undirected graphs also hold for directed graphs. Distinguishing the undirected case from the directed case could perhaps help to understand the structure of independent sets in exponential graphs, and possibly impact on the non-fractional Hedetniemi conjecture.

7.3. The Local Chromatic Number

The *local chromatic number* of a graph G is the value

$$\chi_l(G) = \min\{\max_{u \in V(G)} |\{c(v) : uv \in E(G)\}| + 1 : c \text{ is a proper coloring of } G\},$$

that is, the maximum number of colors used "locally" (around a vertex) in a proper coloring of G. It can also be presented in terms of homomorphisms, by defining the graph $L(r, s)$ of "local r-colorings with s colors" as follows. The vertices of $L(r, s)$ are the couples (i, A) where A is an r-subset of $\{1, \ldots, s\}$ and $i \in A$, and the edges of $L(r, s)$ are the pairs $((i, A), (j, B))$ such that $i \in B$ and $j \in A$. Thus,

$$\chi_l(G) = \min\{r : \text{there exists } s \text{ such that } G \to L(r, s)\}.$$

At the 2007 *Canadian Discrete and Algorithmic Mathematics* conference in Banff, G. Simonyi asked about the validity of the local version of Hedetniemi's conjecture:

Problem 7.3: Does the identity $\chi_l(G \times H) = \min\{\chi_l(G), \chi_l(H)\}$ always hold? □

As in the fractional version, the identity in Problem 7.3 would follow from the multiplicativity of the graphs $L(r, s)$, although it does not depend on it. In **[51]** the inequalities $\chi_f(G) \leq \chi_l(G) \leq \chi(G)$ are proved, and in **[52]** another lower bound is obtained; namely, one half of (one version of) the topological bound on the chromatic number. Therefore, $\chi_l(G \times H)$ could differ significantly from $\min\{\chi_l(G), \chi_l(H)\}$ only both the fractional chromatic number and the topological bound are poor estimates for the chromatic number of a factor. The same criteria are also necessary for $\chi(G \times H)$ to differ significantly from $\min\{\chi(G), \chi(H)\}$.

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(1) Short contributions, surveys, and research articles in graph theory, related fields, and applications.

(2) Graph theory problems (clean or dirty), partial solutions to problems.

(3) Information about availability of graph theory information such as, preprints and technical reports.

(4) Developments on articles in previous *Graph Theory Notes of New York.*

Graph Theory Notes of New York is intended to provide for informal exchange of ideas within the graph theory community. Thus, a good selection of references should be included in each submission.

Please submit contributions to the Editors following the format of a recent issue.

All manuscripts submitted to *Graph Theory Notes of New York* are refereed. Acceptance is the responsibility of the Editors. Accepted articles will not necessarily be published in the order they are received but with the objective of balancing the content of each issue with respect to diversity of topics and authors.

Preferred articles should be concise, self-contained with respect to definitions, provide adequate background to the subject matter. In general, proofs should be in outline form rather than detailed. Papers containing open problems or partial solutions are especially welcome.

Excessive use of figures is discouraged. If used, art work must be submitted in camera ready form (good quality ink drawings or 1200 dpi laser printer output are acceptable). Any text used as items contained in the art work should be in Times Roman and any variables used should be italicized. Please note that figures may be subjected to considerable size reduction and this should be allowed for when preparing the originals.

Electronic submissions are welcome but must be accompanied by a printed copy of the manuscript.

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