SPHERES IN E^3

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1. Introduction. In this paper we shall be concerned with two dimensional spheres embedded in Euclidean three space E^3.

A 2-sphere is a set homeomorphic to the boundary of a ball. It might have an equation like $x^2+y^2+z^2=1$, but in general it is only homeomorphic with something having such an equation. Figure 1 shows some 2-spheres. The first picture shown in Figure 1 is that of a piece of lava. This particular piece of lava was not examined with enough care to determine whether or not its boundary is actually a 2-sphere. Such a 2-sphere (if indeed it is one) seems much more representative of a general 2-sphere, however, than the rather special ones shown in the other parts of Figure 1.

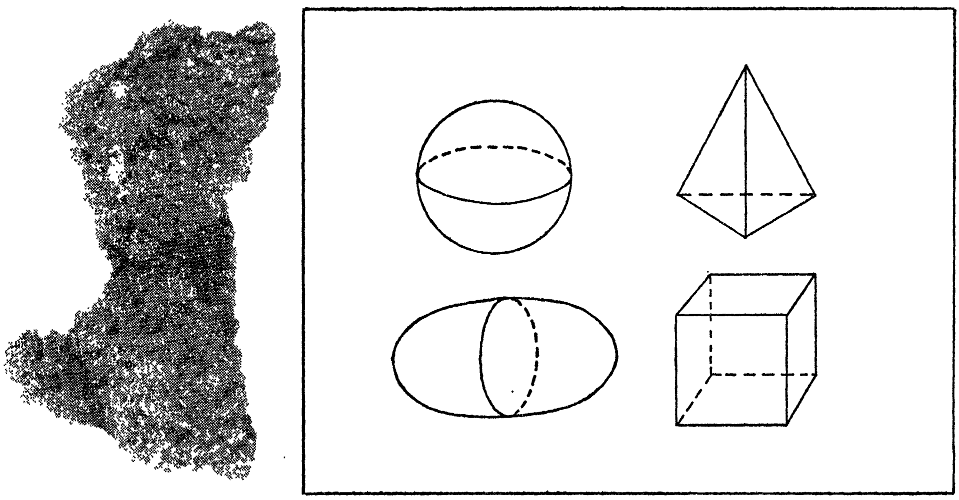

FIG. 1

There are certain properties possessed by all 2-spheres in E^3 irrespective of their shapes. For example, topologists have shown (see for example Theorem 8-38 of [17]) that the Jordan Curve Theorem for the plane E^2 extends to Euclidean spaces of higher dimensions. Hence, if S is a 2-sphere in E^3, S separates E^3 into precisely two pieces. These two pieces are called the interior and exterior of S as follows:

$$\operatorname{Int} S = \text{bounded component of } E^3 - S,$$
$$\operatorname{Ext} S = \text{unbounded component of } E^3 - S.$$

There are certain properties possessed by certain special 2-spheres that are not possessed by 2-spheres in general. The boundaries of the ball and the ellipsoid shown in Figure 1 are smooth and differentiable whereas the other 2-spheres shown are not. The boundaries of the tetrahedron and the cube are polyhedral while the others are not. The boundary of the rock stands a chance of having infinite area but the others do not. Some of the 2-spheres can be pierced by a straight line interval at each point but one cannot.

It may come as a surprise to some that the solid bounded by one 2-sphere may be topologically different from the solid bounded by another. Those who think the Schoenflies Theorem needs no proof in E^2 because it is intuitively obvious, might also think that the extension to E^3 is true—where indeed it is false. [The Schoenflies Theorem says that if J_1, J_2 are two 1-spheres in E^2 (topological circles, frequently called simple closed curves), then there is a homeomorphism of E^2 onto itself taking J_1 onto J_2. See Corollary 9.25 of [15]. The generalization to E^3 (which is false) would say that if S_1, S_2 are two 2-spheres in E^3, then there is a homeomorphism of E^3 onto itself taking S_1 onto S_2.] Once one realizes that 2-spheres can be so exotic, one may be surprised that they behave as well as they do. We mentioned in a preceding paragraph that the Jordan Curve Theorem extends. In Section 5 we discuss some results about retractions that apply to all 2-spheres in E^3. These and other results of this paper extend to Euclidean spaces of higher dimensions and even to abstract spaces of certain sorts but since this is an expository paper, we restrict ourselves to the simplest cases of interest.

A 2-sphere S in E^3 is called *tame* if there is a homeomorphism of E^3 onto itself that takes S onto a polyhedral 2-sphere. Note that for each 2-sphere S' there is a homeomorphism of S' onto a polyhedral 2-sphere but in order that S' be tame we insist that the homeomorphism can be extended to E^3. It is known that, if a 2-sphere S is tame, then there are homeomorphisms of E^3 onto E^3 that take S onto the boundary of a round ball. Examples are known of 2-spheres in E^3 that are not tame. Such 2-spheres are called *wild*. We describe two wild 2-spheres in Sections 3 and 6 of this paper. Sketches of wild 2-spheres are found in [2], [6], [7], [10], [13], and [17] while descriptions of these and others are also found in [3] and [12].

Topologists are interested in the problem of determining conditions under which a 2-sphere in E^3 is tame. At Stockholm in 1962 one of the addresses [9] was devoted to reviewing known sets of conditions that imply that a 2-sphere in E^3 is tame. Some questions were raised, one of which is as follows: *Is a 2-sphere S in E^3 tame if S can be pierced by a straight line segment at each point?*

The question is not wholly topological since the notion of straightness is geometric rather than topological. However, the question seemed of interest even for topologists. A segment axb is said to pierce S at x if $S \cdot axb = \{x\}$ and a, b belong to different components of $E^3 - S$. M. K. Fort, Jr. found [12] a negative answer to the question. An example giving this answer is given in Section 3. The example relates to the game of pick-up-sticks which is discussed in the next section.

2. Pick-up-sticks. In pick-up-sticks one is confronted with a tangled pile of sticks and challenged to remove them one at a time in such a way that none of the remaining ones is moved. One finds that there are piles in which each stick has another resting on it so that it is impossible to proceed.

Suppose P_1, P_2 are two horizontal planes with P_2 above P_1. Let X be a collection of mutually exclusive straight line segments such that each has one end in P_1 and the other in P_2. See Figure 2. Is it possible to adjust the segments gradually so that each assumes a vertical position, each keeps its bottom end fixed, and during the movement the segments remain mutually exclusive?

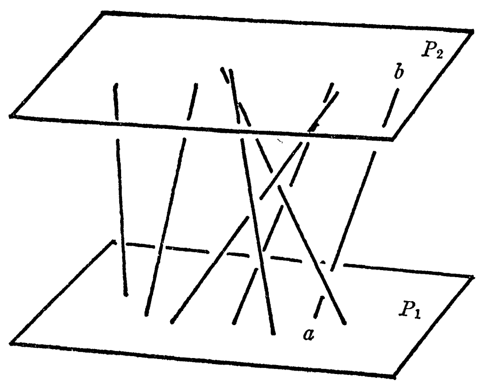

FIG. 2

An affirmative answer to the above question is provided by the following method suggested by R. H. Fox. Consider a segment ab of X with $\{a\}$ in P_1. The lower end of the segment ab is left fixed but at time t the modified segment intersects P_2 at the point which is the vertical projection of the point of ab which divides ab in the ratio $1-t$ to t. Note that at time $t=1$ the modified segment is vertical. In showing that the modified segments do not intersect each other we suppose that a and b have coordinates $(x_a, y_a, 0)$ and $(x_b, y_b, 1)$ respectively. The modified segment at time $t(0 \leqq t < 1)$ is the lower portion of the elongated segment from $(x_a, y_a, 0)$ to $(x_b, y_b, 1/(1-t))$. The elongated segments do not intersect each other since they are the same as the originals under a change of z coordinates. Hence the lower portions of these segments do not intersect each other either.

Let us consider a collection X of segments in a certain instructive position. Let V_1 be the sum of two segments each with ends in P_1, P_2 so that the segments intersect in their upper end points. The segments in V_1 are not elements of X but are merely used as a first approximation to the elements of X. Let N_1 be a tubular neighborhood on V_1. See Figure 3.

Let V_2 be the sum of two inverted V's in N_1 such that the V's hook as shown in Figure 3. Let N_2 be a thin tubular neighborhood of V_2. Let V_3 be the sum of four inverted V's in N_2 so that the part of V_3 in a component of N_2 consists of two hooked V's linked like V_2 in N_1. We continue defining N_3, V_4, N_4, V_5, $\cdots$. Elements of the set X are the components of the intersection of $\overline{N}_1$, $\overline{N}_2$, $\overline{N}_3$, $\cdots$. Note that X is a Cantor set of segments.

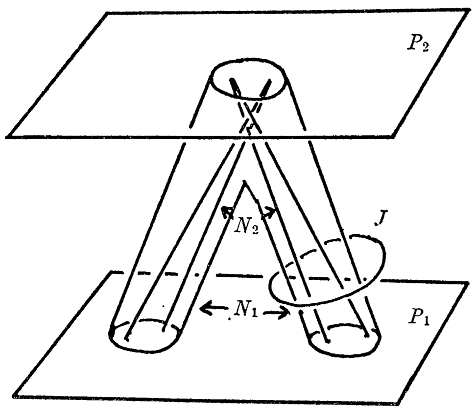

FIG. 3

To see that X is an unusual collection of segments, consider a simple closed curve J as shown in Figure 3 which circles one leg of N_1. In the halfplane above P_1, J cannot be lifted above P_2 without intersecting $\overline{N}_1$. Neither can it be so lifted without intersecting $\overline{N}_2$, nor $\overline{N}_3$ nor .... A rigorous proof that it cannot be lifted can be based on Theorem 9 of [7]. Hence it cannot be lifted without intersecting some element of X. If the elements of X were vertical, however, it could be so lifted. Is there a paradox here? Was the argument sound that the elements of X could be lifted into a vertical position?

It was not the verticalizing argument that was in error. The family of homeomorphisms making the segments vertical was defined on the segments only and not on other points of E^3. The family of homeomorphisms could not be extended in a continuous fashion to the rest of space.

3. A wild porcupine. We describe a wild 2-sphere S_0 which can be pierced at each point with a straight line segment. The example is a modification of that given by M. K. Fort, Jr. in [12].

Suppose that S' is the boundary of a cube in E^3 such that the upper face of the cube lies in P_1 of Section 2 and contains the lower portion of N_1 as shown in Figure 4.

Suppose two holes are cut in S' where N_1 intersects P_1 and the holes are replaced by cylinders which run halfway along the vertical sides on N_1 and capped with horizontal disks halfway between P_1 and P_2. Suppose two holes are cut in each cap and cylinders along the sides on N_2 are run up half the remaining distance to P_2. Caps are put on these cylinders but two holes are cut in each cap and cylinders along the sides of N_3 are run up half the remaining distance to P_2. The procedure is continued so as to get a 2-sphere S_0 as shown in Figure 4 so that all intervals of X lie, except for their upper ends, in Int S_0.

Each tame 2-sphere S has the property that each simple closed curve in E^3-S can be shrunk to a point in E^3-S. The simple closed curve J shown in Figure 3 cannot be shrunk to a point in E^3-S_0; hence S_0 is not tame. It can be shown that the embedding of S_0 is the same as the embedding of Alexander's Horned Sphere [2].

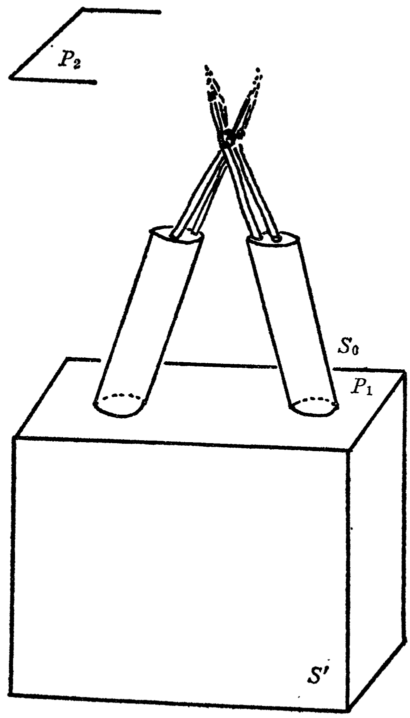

FIG. 4

At each point of S_0, S_0 can be pierced with a straight line segment. The only questionable points are the points of $P_2 \cdot S_0$ and the segments in X show that S_0 can be pierced there.

We can define the segments piercing S_0 so that the directions in which the segments run is continuous. If we do this, however, we will have to permit some of the segments that pierce near $P_2 \cdot S_0$ to be very short. Also, we could have defined the segments so that all of them reach the same distance into Int S_0 and Ext S_0, but under this definition we would not make the directions continuous. That we cannot do both at the same time follows from a result of John Hempel [16].

THEOREM 1. *Any 2-sphere in E^3 for which there is a continuous family of piercing segments is tame.*

The quills of S_0 are tangentially sharp since the cylinders get progressively smaller very fast. If one sharpened a pencil, it would not be possible to touch S_0 from the inside with the tip of the pencil.

Questions. The above considerations suggest the question as to whether or not a 2-sphere S is tame if at each point it can be wedged between two tangent round balls which lie, except for the point of contact, in different components of E^3-S. The left part of Figure 5 suggests the question. The answer is unknown. Similarly we do not know the answer to the corresponding question suggested by the right part of Figure 5 where we show an arbitrary point of S wedged between two cones. These questions go beyond the notions of Topology and involve the idea of straightness from Geometry.

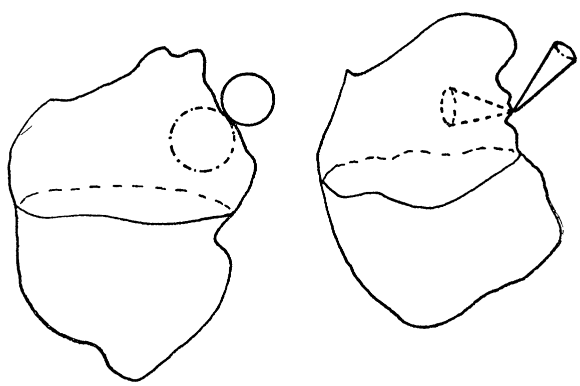

FIG. 5

4. Tietze Extension Theorem. Suppose that one is given an integrable function $y=f(x)$ defined on the line E^1. Even though $f(x)$ is not continuous, for each $\epsilon>0$ the function

$$g_\epsilon(x) = \int_{x-\epsilon}^{x+\epsilon} f(t)\,dt$$

is continuous. We get a corresponding result if we replace the constant ϵ by a continuous positive function $\epsilon(x)$.

We use the above notion in proving a version of the Tietze Extension Theorem which in turn will be used in the next section. We use calculus (something novel in point set topology) as an averaging tool to give the proof.

Although the Tietze Extension Theorem is normally proved for more abstract spaces (see for example pages 59–61 of [17]), we prove it here only for metric spaces since the proof is more elementary in this case and this is the context in which we use it. Furthermore, we get to exhibit an interesting application of calculus. Another elementary proof is found in [18].

THEOREM 2. (Tietze Extension Theorem for metric spaces.) *Each map f of a closed subset X of a metric space Y into the segment $[0, 1]$ can be extended to take Y onto $[0, 1]$.*

Proof. Our job is to consider a point p of $Y-X$ and decide what value of $[0, 1]$ to assign to it. If points of X near p are assigned a particular value of $[0, 1]$, we wish to assign p a value nearby. If D is the distance function for Y and

$$V(p, r) = \{q \mid q \in Y,\ D(p, q) \leqq r\},$$

it does no good to look on the interior of the ball $V(p, D(p, X))$ since there are no points of X there. However, there will be points of X in the hollow ball $V(p, 2D(p, X)) - \operatorname{Int} V(p, D(p, X))$, and we seek a certain average over the f values of X in this annulus or hollow ball. See Figure 6.

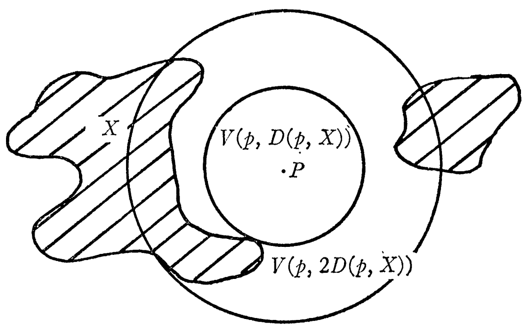

FIG. 6

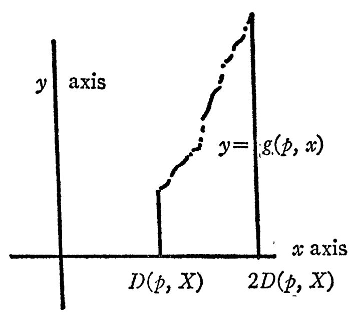

FIG. 7

Consider the function

$$g(p, r) = \text{least upper bound} f(x), \qquad x \in X \cdot V(p, r).$$

There is no reason to suppose that g is continuous in either p or r. It is not defined for $r < D(p, X)$, but for p fixed it is a bounded monotone nondecreasing function for $r > D(p, X)$ and hence integrable. The graph of $y = g(p, x)$ with p fixed and $D(p, X) < x \leqq 2D(p, X)$ might look somewhat like that shown in Figure 7. If we take the area under the curve and divide by the length of the base line, we find an average value of $g(p, x)$. It is immaterial as to whether or not g is defined on the left end of the base line.

For values of p not in X we define

$$f(p) = \frac{1}{D(p, X)} \int_{D(p,X)}^{2D(p,X)} g(p, t)\,dt.$$

It is an exercise in calculus to show that this function is continuous at each point of $Y-X$. It is a continuous extension of the given function f since for each point q of X and each $\epsilon > 0$, there is a positive number $\delta(q)$ such that if $q' \in X$ $V(q, \delta)$, $|f(q) - f(q')| < \epsilon$. Then for each $p \in V(q, \delta/3)$, $|f(q) - f(p)| \leqq \epsilon$.

It is sometimes convenient to generalize the Tietze Extension Theorem as follows.

THEOREM 3. *Each map of a closed subset of a metric space Y into a disk can be extended to all of Y.*

Proof. We regard the disk as a square in E^2 with opposite vertices at (0, 0) and (1, 1). For each point p on which f is defined let $(f_x(p), f_y(p))$ be the coordinates of $f(p)$. It follows from the preceding theorem that f_x and f_y can be extended to all of Y. If f_x, f_y are used to denote these extensions, the extended f may be defined so that $f(p)$ has coordinates $(f_x(p), f_y(p))$.

5. Retractions onto spheres. A *retraction* of a set Y onto a set X is a map f of Y onto X such that f is fixed on each point of X.

A round 2-sphere has the property that if p is a point of Int S, then there is a retraction of $E^3 - \{p\}$ onto S. As pointed out in Theorem 4, each wild 2-sphere has this property. For the round 2-sphere S, one can take the half lines emanating from p onto the points where these half lines pierce S. For arbitrary 2-spheres more ingenuity needs to be used in getting the retraction.

THEOREM 4. *If S is an arbitrary 2-sphere in E^3 and p is a point of* Int S, *then there is a retraction of $E^3 - \{p\}$ onto S.*

This theorem was proved in [8]. An effort was made to avoid the existence type of proof, but rather to exhibit a method for getting a retraction of $E^3 - \{p\}$ onto S. The proof given in [8] was somewhat constructive but failed to indicate what the retraction of a neighborhood of S into S would look like. The purpose of giving a proof of the Tietze Extension Theorem in the last section by defining the extension rather than by existence techniques was to enable us to show in Theorem 5 what the retraction of the neighborhood of S is.

A set X is an ANR (*absolute neighborhood retract*) if for each embedding of X into a metric space Y there is a neighborhood N of X in Y and a retraction of N onto X. It is known that each 2-sphere is an ANR. See for example Theorem 2-36 of [17]. Hence, no matter how wildly a 2-sphere S is embedded in E^3, there is a neighborhood N of S in E^3 and a retraction of N onto S.

THEOREM 5. *If S is a 2-sphere in E^3, then there is a neighborhood N of S and a retraction of N onto S.*

Proof. Let D_1, D_2, D_3 be three disks in S with $D_1 \subset \text{Int } D_2 \subset D_2 \subset \text{Int } D_3$. Let f be the identity map of D_3 onto itself and use Theorem 3 to extend f so as to obtain a retraction of E^3 onto D^3. Let N be a neighborhood of Bd D_2 such that $\overline{N} \cdot S + f(\overline{N}) \subset D_3 - D_1$. Let N_1, N_2 be neighborhoods of D_2 and $S - \text{Int } D_2$ respectively such that $\overline{N}_1 \cdot \overline{N}_2 \subset N$. Let g be a map of $(S - \text{Int } D_1) + \overline{N}$ onto $S - \text{Int } D_1$ that is the identity on $S - \text{Int } D_1$ and f on $\overline{N}$. It follows from Theorem 3 that the map g can be extended to a retraction of $(S - \text{Int } D_1) + \overline{N} + \overline{N}_2$ onto $S - \text{Int } D_1$. Then the retraction of $N_1 + N + N_2$ given by f on $N_1 + N$ and by the extended g on $N_2 + N$ satisfies the requirements of Theorem 5.

Fixed point properties of cubes imply that if p is not removed in Theorem 4, there is no retraction. Again, this theorem holds as well for arbitrary 2-spheres in E^3 as for nice ones.

THEOREM 6. *There is no 2-sphere in E^3 which is a retract of E^3.*

Proof. Assume there is a retraction r of E^3 onto a particular 2-sphere S. Consider a homeomorphism h of S onto itself that moves every point of S. That here is such a homeomorphism follows from the fact that the antipodal homeomorphism of a round 2-sphere onto itself throws each point into its diametrically opposite point and hence moves every point. The map hr would be a fixed point free map of E^3 onto a compact subset of E^3. Theorems like Theorem 6-39 of [17] say that this is impossible.

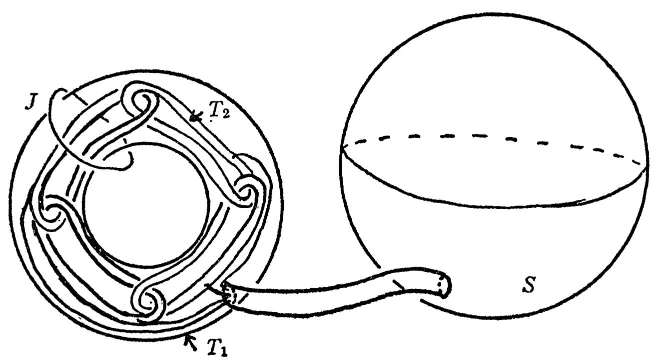

FIG. 8

6. Antoine's necklace. An Antoine's necklace is illustrated in the left portion of Figure 8. It is described in [3, 4, 5, 6, 10, 11, 17] and is the intersection of a solid torus T_1, a set T_2 which is the sum of several small solid tori which form a chain circling T_1, a set T_3 which is the sum of some very small solid tori which intersect each component of T_2 in the same fashion that T_2 lies in T_1, $\cdots$. There are countably many layers of T's, and the set of points belonging to their intersection is Antoine's necklace. We denote it by A. It may be shown that A is topologically equivalent to an ordinary Cantor set. Its complement, however, is topologically different from the ordinary Cantor set, because the simple closed curve J shown in Figure 8 cannot be shrunk to a point without hitting A. See [11] for proof of this.

Consider a dendron running down to a Cantor set as shown in Figure 9. It looks something like a form used to record results of an elimination tournament in which only winners remain at each stage to engage other winners. There is no starting round, however, since at each stage there was a preceding one. We use a combination of the notion of a dendron of this type and of Antoine's necklace to describe what was perhaps the first known wild 2-sphere [3].

Let S be a 2-sphere missing T_1 as shown in Figure 8. Alter S by removing a small disk and replacing the hole with a cylinder leading over to T_1 with a cap on Bd T_1. Holes are cut in this cap and tubes are run in T_1 over to the components of T_2 where the tubes are capped. Holes are cut in these caps and tubes are run in T_2 over to the components of T_3. The process is continued. Note that the tubes get shorter since they lie in the T's which at the later stages get arbitrarily small.

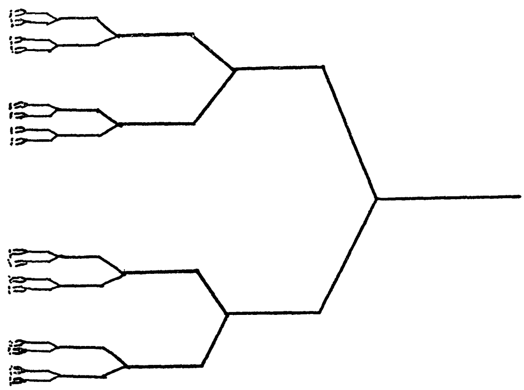

FIG. 9

The resulting 2-sphere is wild because the simple closed curve J shown cannot be shrunk to a point in E^3-A and hence not in the complement of the resulting 2-sphere.

7. The grooved ball. There are many criteria known for determining whether or not a 2-sphere is tame. One of the earliest of these [1] shows that a 2-sphere is tame if it is polyhedral. More understandable proofs of this are found in [**14**] and [**19**].

A condition that is somewhat in doubt is involved in the following question: *Is a 2-sphere S in E^3 tame if each horizontal cross section of S is either a point or a simple closed curve?*

An affirmative answer is suggested in the literature but the proof is very sketchy, saying merely that the result follows by "well-known theorems on logarithmic potential in 2-dimensions."

We give an example to show that even if the above condition does imply that a 2-sphere is tame, it is not enough to ensure that there is a homeomorphism of E^3 onto itself which is invariant on horizontal planes and which takes S onto a round 2-sphere.

Suppose B is a croquet ball to be left out in the rain. One wishes to groove the ball so that as water runs down the ball, one section of the equator does not have water run over it. See Figure 10. The grooves are very shallow so that each horizontal cross section of the ball is a disk. The cross section through the

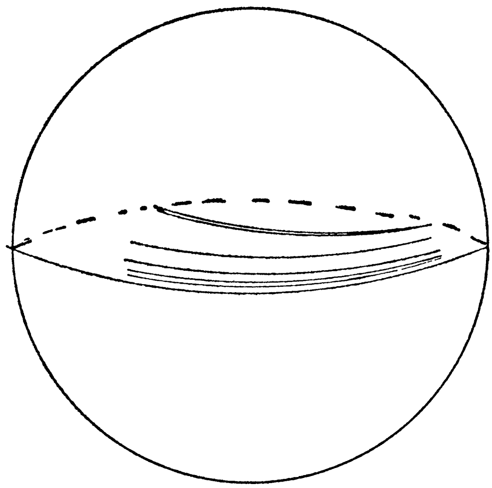

FIG. 10

equator is round but some close to it are disks with long feelers. These disks converge, but not homeomorphically, to the disk containing the equator. If we chose to make the groves spiral we could get even longer feelers.

We have learned much about E^3 but there is much we do not know. For the person with ingenuity and ability this is a fruitful area in which to work.

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THE MATHEMATICS USED IN MATHEMATICAL PSYCHOLOGY

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Introduction. The main issue in applying mathematics to psychological problems today, and most likely for some time to come, is the formulation of these problems in mathematical terms. The solution of difficult but well-formulated mathematical problems and the analysis of complex applied problems in terms of precise and well-confirmed theories are more secondary efforts. We do not yet have the basic concepts and variables staked out in a way that makes the introduction of mathematics the relatively straightforward business that it has become in much of physical science. We are in a situation somewhat analogous to sixteenth, or hopefully seventeenth, century physics, but the analogy is far from complete. We resemble the early physicists in our effort, often fumbling and always slow, to isolate and purify the fundamental variables from the myriad, vague, commonsense psychological ideas and concepts. We differ in the range of techniques available to us. The modern electronics technology, including high speed computers, provides us with a control over experimental conditions and a computational capacity for data analysis incomparably more extensive and subtle than those with which the early physicists had to contend. In addition, most of the mathematics and statistics we now use was quite unknown three centuries ago.

Applications to what psychology? The current applications of mathematics to parts of psychology are precisely that—applications to portions of the total field. To the great satisfaction of many who do not necessarily view with favor the increasing mathematical complication of the psychological literature, huge portions of both academic and applied psychology are essentially free from mathematical inroads. The main areas that have been affected are those usually grouped together as "experimental" psychology, which is a misnomer because