The Cauchy equation

Definition: A function $f : \mathbb{R} \to \mathbb{R}$ is *additive* if it satisfies the *Cauchy equation* (pron. Ko-'shee):

$$f(x+y) = f(x) + f(y) \quad \text{for all } x, y \in R.$$

In this note, *linear function* will mean a function of the form $f(x) = cx$ (zero intercept).

Clearly, *linear functions are additive.* Are there any other additive functions? After experimenting for a while, you'll be convinced that there are none. And in a sense there are none (namely among well-behaving functions), but in a sense there are (the existence of some erratically behaving non-linear additive functions follows from the Axiom of Choice).

Our first theorem is about "tame" additive functions:

Theorem 1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function, and let $I = [a, b]$ be an arbitrary interval (with $a < b$).*

If f is monotone on I then f is linear (that is, $(\exists c)(\forall x) f(x) = cx$).

If f is continuous on I then f is linear.

If f is bounded on I then f is linear.

The proof of Theorem 1 is based on the following lemma which says that *all* additive functions are linear on $\mathbb{Q}$ (the set of rational numbers) as well as on all "copies" $a\mathbb{Q}$ of $\mathbb{Q}$ (but the slope involved might be varying from a to a).

Lemma 2. *Let f be an arbitrary additive function. Then, $(\forall x \in \mathbb{Q}) f(x) = f(1)x$. In general, for any $a \in \mathbb{R}$ we have $(\forall x \in \mathbb{Q}) f(ax) = f(a)x$.*

Proof steps for Lemma 2:

Fix $a \in \mathbb{R}$. Use induction to show $(\forall n \in \mathbb{N}) f(an) = nf(a)$.
Use induction to show $(\forall n \in \mathbb{N}) f(a/n) = f(a)/n$.
And finally, use induction to show $(\forall m, n \in \mathbb{N}) f(a(m/n)) = f(a)(m/n)$.

And now the"wild" functions:

Theorem 3 (assuming the Axiom of Choice). *There are additive functions that are not linear.*

Proof. We describe *all* additive functions at once ("most" are easily seen not be linear):

Recall that since both $\mathbb{Q}$ and $\mathbb{R}$ are fields and $\mathbb{Q}$ is a subfield of $\mathbb{R}$, so $\mathbb{R}$ can be considered as a vector space over $\mathbb{Q}$; let B be a basis in this vector space (a so-called Hamel basis). [The AC guarantees that *every* vector space has a basis!]

Define f arbitrarily on B, and extend it to $\mathbb{R}$ in the obvious way: if $x = \sum q_i b_i$ with some $q_i \in \mathbb{Q}$ and $b_i \in B$ then let $f(x) := \sum q_i f(b_i)$. Since such a representation of x is unique (definition of linear basis!), f is well-defined, and it is easy to see that f is additive. $\square$

The following homework (6.6 in the LBB) shows that the adjective "wild" above is well-deserved: *Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function. Show that if f is not linear, then the graph of f is everywhere dense on the plane.* (That means that every rectangle in the plane - however small - contains at least one point of the graph of f.)

Subadditive sequences

The pathological behavior of certain additive functions resulted from the richness of the set of real numbers. For sequences (that is, functions with domain $\mathbb{N}$), no such erratic behavior is possible as the following trivial fact shows (use induction on n):

Fact 4. *Let (x_n) be a sequence of real numbers satisfying the* additivity condition

$$x_{m+n} = x_m + x_n \quad \textit{for all } m, n \in \mathbb{N}.$$

Then x_n is linear; indeed, $x_n = nx_1$ for all $n \in \mathbb{N}$.

The next theorem says that if we relax the condition of additivity to subadditivity, then the sequence will still asymptotically behave as linear, in that $\lim_n x_n/n$ exists (possibly $-\infty$).

Theorem 5 (Subadditivity Lemma - Fekete 1923). *If a sequence of real numbers (x_n) satisfies the* subadditivity condition

$$x_{m+n} \leq x_m + x_n \quad \textit{for all } m, n \in \mathbb{N},$$

then

$$\lim_{n\to\infty} \frac{x_n}{n} = \inf_{m\geq 1} \frac{x_m}{m}.$$

Sketchy proof (for those who are familiar with lim and lim sup):
(1) Induction on k shows that $(\forall m \in \mathbb{N})(\forall k \in \mathbb{N})x_{km} \leq kx_m$.
(2) Writing $C_m = \max\{x_r : 1 \leq r < m\}$, we get for all $r \in [1, r-1]$, all $k \in \mathbb{N}$, and $n = km + r$: $\quad x_n = x_{km+r} \leq x_{km} + x_r \leq x_{km} + C_m \leq kx_m + C_m$. Hence,

$$\frac{x_n}{n} \leq \frac{km}{n} \cdot \frac{x_m}{m} + \frac{C_m}{n}.$$

(3) Letting $k \to \infty$, we get

$$\limsup_{n\to\infty} \frac{x_n}{n} \leq \frac{x_m}{m} \quad \text{for all } m \in \mathbb{N}, \quad \text{whence} \quad \limsup_{n\to\infty} \frac{x_n}{n} \leq \inf_{m\in\mathbb{N}} \frac{x_m}{m}.$$

(4) But since $x_n/n \geq \inf_{m\in\mathbb{N}} x_m/m$ for all $n \in \mathbb{N}$, so $\lim_{n\to\infty} x_n/n = \inf_{m\in\mathbb{N}} x_m/m$. $\square$

Example (hereditary properties): Let S_n be the set of all strings of English letters of length n which do not contain the substring **hello**. Then S_n is of exponential size, in that $|S_n|^{1/n}$ exists. Indeed, since the required property is hereditary (to segments of a string), so $S_{m+n} \subset S_m S_n$, where $S_m S_n := \{xy : x \in S_m,\ y \in S_n\}$ (concatenated strings). Hence $|S_{m+n}| \leq |S_m||S_n|$, and the sequence $x_n := \log |S_n|$ is subadditive. The claim easily follows.