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Index Theorem and the Heat Equation

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The purpose of this paper is to review the recent developments in the heat equation proofs of the Atiyah-Singer Index Theorem for Dirac operators.

Let us briefly recall that if D_+ is half a Dirac operator and if D_- is its adjoint, the starting point of the method is the McKean-Singer formula for the index Ind D_+ [**MS**]:

$$|\mathrm{Ind}\, D_+ = \mathrm{Tr}[\exp(-tD_-D_+/2)] - \mathrm{Tr}[\exp(-tD_+D_-/2)], \qquad t > 0. \qquad (0.1)$$

As $t \downarrow\downarrow 0$, the right-hand side has an expansion starting with negative powers of t, and the problem is to show that in certain situations, when expressing the traces using kernels, even locally

- no negative powers of t arise;
- the constant term coincides with the local Atiyah-Singer polynomial.

After the pioneering papers of Patodi [**P1**, **P2**], Gilkey [**Gi1**] and Atiyah-Bott-Patodi [**ABP**] established that this is indeed the case for algebraic reasons. The method is indirect:

- an algebraic argument gives the general form of the local terms arising in the expansion, and then excludes negative powers of t;
- the zero order term is calculated using a similar classification argument, and also explicit computations on examples.

This approach has been extended by Patodi [**P3**] and Gilkey [**Gi2**] to include the fixed point formulas of Atiyah-Bott [**AB1**] and Atiyah-Singer [**AS**], and it is fully described in Gilkey's recent book [**Gi3**].

By using arguments based on supersymmetry considerations, physicists Witten [**W1**], Alvarez-Gaumé [**Al**], and Friedan-Windey [**FW**] strongly suggested that a direct proof of the local Index Theorem could be given, which would altogether prove the local cancellations and identify the local integrand by brute force.

That this is indeed possible has been proved by Getzler [**Ge1**, **Ge2**] for the Index Theorem, by Bismut [**B1**] and Berline-Vergne [**BV2**] for the Index Theorem and for the Lefschetz fixed point formulas. The proofs of Getzler are based on the asymptotic representation of heat kernels on supermanifolds [**Ge1**] and also on adequate rescaling in time, space, and Clifford variables [**Ge1**, **Ge2**]. Our proofs [**B1**] use a probabilistic asymptotic representation of the heat kernel [**B5**], together with certain stochastic area formulas of P. Lévy [**Le**]. The proofs of Berline-Vergne [**BV2**] are of group-theoretic nature.

On the other hand, Atiyah and Witten [**At**] have found a remarkable formal link between the Index Theorem for Dirac operators on the spin complex and localization formulas in equivariant cohomology of Duistermaat-Heckman [**DH**], Berline-Vergne [**BV1**]. In particular the $\hat{A}$ genus was interpreted in [**At**] as the inverse of an equivariant Euler form associated with an infinite-dimensional bundle. This suggested that an alternative approach to the Index Theorem, in relation with the equivariant cohomology of the loop space, was possible.

In [**B2**], we verified that the Atiyah-Witten formalism could be extended to the case of general Dirac operators, and also to fixed point theory. Also we showed in [**B4**] that the heat equation method is by itself such a reasonable proof of these formulas in infinite dimensions that it has a finite-dimensional counterpart, i.e., there is a proof of the formulas of [**BV1**] and [**DH**] which is at each step the finite-dimensional analogue of the probabilistic proof of the Index Theorem. This proof exhibits Patodi-like cancellations in finite dimensions. Conversely, it clearly demonstrates the purely geometric nature of these cancellations in Index Theory, the geometry to be considered being the geometry of the loop space.

Until recently, the heat equation formula (0.1) for the Index was considered a tool, which happened to work. The introduction of superconnections by Quillen [**Q1**] changed the situation dramatically. In [**Q1**], Quillen introduced a new class of objects, the superconnections on Z_2 graded finite-dimensional bundles, which makes (0.1) cry out to be considered as a formula for a Chern character. To briefly explain the analogy, let us just say that if E is a bundle with connection ∇, if ∇^2 is the curvature of E, then ch E is represented in cohomology by

$$\mathrm{ch}\, E = \mathrm{Tr}[\exp(-\nabla^2/2i\pi)]. \tag{0.2}$$

In [**B3**], we gave heat equation proofs of the Index Theorem of Atiyah-Singer for families of Dirac operators [**AS**], based on an infinite-dimensional analogue of Quillen's theory. To find the right choice of a superconnection, the finite-dimensional baby model of [**B4**] was of critical importance.

In relation with papers by Quillen [**Q2**] and Witten [**W3**] on determinant bundles and global anomalies, a transgressed form of Quillen's superconnection formalism has been introduced in Bismut-Freed [**BF**]. In particular, a remarkable argument of Witten [**W3**] relating the holonomy of determinant bundles to êta invariants has received a complete proof in [**BF**]. Another proof has recently been given by Cheeger [**Ch**].

Superconnections and the local form of the Index Theorem for families are currently used by Gillet and Soulé [**GS**] to construct direct images in Arakelov theory.

On the other hand, the results of Witten [**W2**] on the Morse inequalities, and the asymptotic Morse inequalities of Demailly for complex manifolds [**De**] have also been proved by us [**B6**, **B7**] using heat equation methods. Getzler [**Ge3**, **Ge4**] has given a degree-theoretic interpretation in infinite dimensions of certain Index problems. Current efforts are done to relate in a more direct way heat equation methods to the cyclic homology of Connes [**Co**].

This paper is organized in the following way. In §1, the current heat equation proofs of the Index Theorem for Dirac operators are briefly reviewed. The new proofs have been classified into

- proofs related to supersymmetry,
- probabilistic proofs,
- group-theoretic proofs.

The principle of the probabilistic proof is briefly described, to emphasize its relations with the localization formulas in equivariant cohomology of Duistermaat-Heckman [**DH**], Berline-Vergne [**BV1**]. These relations are made explicit in §2, along the lines of Atiyah [**At**] and ourselves [**B2**, **B4**].

In §3, we briefly describe Quillen's superconnections [**Q1**] and their applications to the heat equation proof of the Atiyah-Singer Index Theorem for families of Dirac operators [**B3**]. One application to anomalies is also briefly indicated [**BF**].

I. The heat equation proofs of the Index Theorem. In this section, we briefly review the heat equation proofs of the Index Theorem for Dirac operators.

In (a) and (b), we summarize the now-classical proofs which rely on algebraic arguments.

In (c), we indicate some of the ideas involved in the recent proofs in [**Ge1**, **Ge2**, **B1**, **BV2**].

(a) *The heat equation method.* Let M be a compact connected Riemannian manifold of even dimension $n = 2l$. Let $E = E_+ \oplus E_-$ be a Z_2 graded complex Hermitian bundle over M, such that E_+ and E_- are orthogonal. Let τ be the involution of E defining the grading, i.e., $\tau = \pm 1$ on $E_\pm$.

$\Gamma(E)$, $\Gamma(E_\pm)$ denote the sets of C^∞ sections of E, $E_\pm$. Clearly $\Gamma(E) = \Gamma(E_+) \oplus \Gamma(E_-)$ is also naturally Z_2 graded. We still denote by τ the involution defining the grading in $\Gamma(E)$.

Also $\Gamma(E)$ can be endowed with the L_2 Hermitian product

$$h, h' \to \int_M \langle h, h'\rangle(x)\, dx. \tag{1.1}$$

Let D_+ be a first-order elliptic differential operator mapping $\Gamma(E_+)$ into $\Gamma(E_-)$. Let D_- be the formal adjoint of D_+. Set

$$D = \begin{bmatrix} 0 & D_- \\ D_+ & 0 \end{bmatrix}. \tag{1.2}$$

End $\Gamma(E)$ is naturally Z_2 graded, the even (resp. odd) elements commuting (resp. anticommuting) with τ.

Clearly

$$D^2 = \begin{bmatrix} D_-D_+ & 0 \\ 0 & D_+D_- \end{bmatrix}. \tag{1.3}$$

For any $t > 0$, $\exp(-tD^2/2)$ is given by a C^∞ kernel $P_t(x,y)$, so that if $h \in \Gamma(E)$

$$\exp\left(-\frac{tD^2}{2}\right) h(x) = \int_M P_t(x,y)h(y)\,dy. \tag{1.4}$$

For any $x \in M$, $P_t(x,x)$ is even in $\operatorname{End}_x E$.

If A is a trace class operator acting on $\Gamma(E)$, we define its supertrace $\operatorname{Tr_s}[A]$ by

$$\operatorname{Tr_s}[A] = \operatorname{Tr}[\tau A].$$

Recall that the index $\operatorname{Ind} D_+$ of D_+ is given by

$$\operatorname{Ind} D_+ = \dim\ker D_+ - \dim\ker D_-. \tag{1.5}$$

The first step in the calculation of $\operatorname{Ind} D_+$ is the McKean-Singer formula [**MS**, **ABP**]:

$$\operatorname{Ind} D_+ = \operatorname{Tr_s}\left[\exp\left(-\frac{tD^2}{2}\right)\right] = \int_M \operatorname{Tr_s}[P_t(x,x)]\,dx. \tag{1.6}$$

Let $P_t^\pm(x,x)$ be the restriction of $P_t(x,x)$ to $E_{\pm,x}$. Well-known results on zêta functions [**Se**] and heat kernels [**ABP**] show that as $t \downarrow\downarrow 0$, for any $k \in N$,

$$\operatorname{Tr}[P_t^\pm(x,x)] = \sum_{j=-n/2}^{k} a_j^\pm(x)t^j + o(t^k,x). \tag{1.7}$$

In (1.6), $(a_j^\pm(x))$ are C^∞ functions which only depend on the local symbol of D. For $j \geq -n/2$, set

$$a_j(x) = a_j^+(x) - a_j^-(x). \tag{1.8}$$

Clearly

$$\operatorname{Tr_s}[P_t(x,x)] = \sum_{j=-n/2}^{k} a_j(x)t^j + o(t^k,x). \tag{1.9}$$

From (1.5), (1.6), we find [**MS**, **ABP**],

$$\int_M a_j(x)\,dx = 0, \qquad j \neq 0. \tag{1.10}$$

$$\operatorname{Ind} D_+ = \int_M a_0(x)\,dx.$$

McKean and Singer [**MS**] conjectured that if $D = d + d^*$ acting on the de Rham complex, some extraordinary cancellations would show that for $j < 0$, $a_j = 0$, and that a_0 is exactly equal to the Chern-Gauss-Bonnet integrand for the Euler characteristic. In [**P1**], Patodi showed that this was indeed the case. In [**P2**], he extended his results to the Riemann-Roch theorem for Kähler manifolds.

(b) *Gilkey's theory of invariants.* In [**Gi1**], Gilkey established an algebraic theory of invariants. He showed that if $D = d + d^*$, the functions (a_j) belong to a certain class of local functions of the metric. After classifying such functions, Gilkey proved on a priori grounds that for $j < 0$, $a_j = 0$. The identification of a_0 was done in [**Gi1**] in an indirect way. Also Gilkey [**Gi1**] extended his approach to the Hirzebruch signature theorem.

In [**ABP**], Atiyah, Bott, and Patodi systematized the arguments of Gilkey to obtain the same type of result for twisted signature complexes, and derived the general Index Theorem. They developed Gilkey's theory in the realm of Riemannian geometry.

This point of view is systematically described in Gilkey's recent book [**Gi3**]. The theory of invariants has been also successfully applied [**Gi2, 3**] to prove the Lefschetz fixed point formulas of Atiyah-Bott [**AB1**] and Atiyah-Singer [**AS**].

(c) *Direct proofs of the cancellations and identification of the local integrand.* We now assume that M is orientable and spin. $F = F_+ \oplus F_-$ denotes the Z_2 graded Hermitian bundle of spinors over M. The Levi-Civita connection ∇^L of TM lifts into a unitary connection on F.

Let ξ be a complex Hermitian bundle over M, endowed with a unitary connection ∇^ξ.

Set $E = F \otimes \xi$, $E_\pm = F_\pm \otimes \xi$. $E_\pm$ are Hermitian bundles, naturally endowed with the connection $\nabla^L \otimes 1 + 1 \otimes \nabla^\xi$, which we denote by ∇.

Recall that if $e \in TM$, e acts on F by Clifford multiplication. $E = E_+ \oplus E_-$ is then a TM Clifford module.

We now define the Dirac operator. Let $e_1, \ldots, e_n$ be an orthonormal base of TM.

DEFINITION 1.1. D denotes the operator acting on $\Gamma(E)$,

$$D = \sum_1^n e_i \nabla_{e_i}. \tag{1.11}$$

$D_\pm$ is the restriction of D to $\Gamma(E_\pm)$. $D_\pm$ maps $\Gamma(E_\pm)$ into $\Gamma(E_\mp)$.

Let R be the curvature of TM, K the scalar curvature of M, L the curvature of ξ. Let Δ^H be the horizontal Laplacian on $\Gamma(E)$. Lichnerowicz's formula [**Li**] asserts that

$$D^2 = -\Delta^H + \frac{K}{4} + \frac{1}{2} \sum e_i e_j \otimes L(e_i, e_j). \tag{1.12}$$

1. *The supersymmetric proofs.* We first briefly review the arguments of Witten [**W**], Alvarez-Gaumé [**Al**], Friedan-Windey [**FW**], and Zumino [**Z**] leading to a supersymmetric derivation of the Index Theorem for D_+.

Let LM be the loopspace of M. The idea is to rewrite (1.6) in the form

$$\operatorname{Ind} D_+ = \int_{LM} \exp\{\mathcal{L}^t(x)\}\, dD(x), \tag{1.13}$$

where $\mathcal{L}^t(x)$ is a supersymmetric Lagrangian, and $dD(x)$ is the "volume element" of LM. Let us just say that $\mathcal{L}^t(x)$ is a Lagrangian involving anticommuting variables ψ, $\overline{\psi}$. Supersymmetry here means that $\mathcal{L}^t$ is invariant under transformations which involve x and the anticommuting variables ψ, $\overline{\psi}$. By making $t \downarrow\downarrow 0$, and using arguments in particular from spectral theory, [**A1**, **FW**, **Z**] derive the local formula for the index

$$\operatorname{Ind} D_+ = \int_M \hat{A}\left(\frac{R}{2\pi}\right) \operatorname{Tr}\left[\exp\, -\frac{L}{2i\pi}\right]. \tag{1.14}$$

In (1.14) $\hat{A}$ is the Hirzebruch polynomial on antisymmetric matrices:

$$\hat{A}(C) = \prod_1^l \frac{x_i/2}{\sinh(x_i/2)}. \tag{1.15}$$

In [**Ge1**], Getzler gave a rigorous formulation to the previous arguments. He used the supermanifold T^*M to give an asymptotic representation of the supertrace $\operatorname{Tr_s}[P_t(x,x)]$ in terms of the graded symbol of $\exp(-tD^2/2)$. The local formula for the index is finally obtained by using a quadratic Gaussian approximation.

Recently, Getzler [**Ge2**] has given a new proof of the local convergence closely related to [**Ge1**] and also to [**B1**]. In [**Ge2**], Getzler adequately rescales the time, space, and Clifford variables to show that if D^ε is adequately rescaled with the factor ε, $(D^\varepsilon)^2$ converges to the partial differential operator on $T_{x_0}M$

$$\mathcal{L} = -\sum_1^n \left(\nabla_{e_i} + \frac{1}{4} R_{x_0}(e_i, x_j e_j)\right)^2 + L.$$

The explicit computations of the fundamental solution of $\partial/\partial t + \mathcal{L}$ leads again to the formula (1.14).

2. *The probabilistic proof.* We now briefly summarize our proof of the Index Theorem [**B1**]. To simplify, we assume that ξ is here the trivial bundle C.

Let $p_t(x,y)$ be the scalar heat kernel on M. Let $E^t_{x_0,x_0}$ be the law on $C([0,1];M)$ of the Brownian bridge x^t_s starting at x_0 at time 0 and ending at x_0 at time 1 associated with the scaled metric g_M/t [**B5**, §2].

Let $\tau_0^{1,t}$ be the parallel transport operator from F_{x_0} into F_{x_0} along the loop $x^t_\cdot$. An easy application of Itô's formula [**B1**] shows that

$$\operatorname{Tr_s}[P_t(x_0,x_0)] = p_t(x_0,x_0) E^t_{x_0,x_0}\left[\exp\left\{-\frac{t\int_0^1 K(x^t_s)ds}{8}\right\} \operatorname{Tr_s}[\tau_0^{1,t}]\right]. \tag{1.16}$$

As $t \downarrow\downarrow 0$,

$$p_t(x_0,x_0) \simeq 1/(\sqrt{2\pi t})^n. \tag{1.17}$$

Also using the techniques of [**B5**], we describe $E^t_{x_0,x_0}$ by means of a Brownian bridge $w^1_\cdot$ in $T_{x_0}M$ with $w^1_0 = w^1_1 = 0$, so that approximately

$$x^t_s \sim \exp_{x_0}\left(\sqrt{t} w^1_s\right). \tag{1.18}$$

Then $\tau_0^{1,t}$ acting on $T_{x_0}M$ has the expansion

$$\tau_0^{1,t} = I - \frac{t}{2}\int_0^1 R_{x_0}(dw^1, w^1) + o(t). \tag{1.19}$$

An argument from representation theory shows that

$$\frac{\mathrm{Tr_s}\tau_0^{1,t}}{t^{n/2}} \to (-i)^l \mathrm{Pf}\left[\frac{\int_0^1 R_{x_0}(dw^1, w^1)}{2}\right]. \tag{1.20}$$

We thus find that

$$\lim_{t\downarrow\downarrow 0} \mathrm{Tr_s}[P_t(x_0,x_0)] = \int \mathrm{Pf}\left[\frac{-i}{4\pi}\int_0^1 R_{x_0}(dw^1,w^1)\right] dP_1(w^1). \tag{1.21}$$

If η is the Riemannian orientation form of TM, we get

$$\lim_{t\downarrow\downarrow 0} \mathrm{Tr_s}[P_t(x_0,x_0)]\eta(x_0) = \int \exp^\wedge\left\{\frac{-i}{4\pi}\int_0^1 R_{x_0}(dw^1,w^1)\right\} dP_1(w^1), \tag{1.22}$$

where $\exp^\wedge\{\dots\}$ is the exponential in $\Lambda(T^*M)$ of the corresponding 2 form. Using well-known symmetries of R, we find that

$$\lim_{t\downarrow\downarrow 0} \mathrm{Tr_s}[P_t(x_0,x_0)]\eta(x_0) = \int \exp^\wedge\left\{\frac{-i}{4\pi}\int_0^1 \langle R_{x_0}(\cdot,\cdot)w^1, dw^1\rangle\right\} dP_1(w^1). \tag{1.23}$$

A formula of P. Lévy [**Le**], known as the stochastic area formula, shows that the r.h.s. of (1.23) is equal to $\hat{A}(R/2\pi)$.

The Lefschetz fixed point formulas of Atiyah-Bott [**AB1**] and Atiyah-Singer [**AS**] were also proved in [**B1**] using the same sort of arguments and formulas of P. Lévy [**Le**].

3. *The group-theoretic proof.* In [**BV**], Berline and Vergne have given a proof of the Index formula and of the Lefschetz formulas by considering the scalar heat kernel on the bundle of orthonormal frames of TM. This idea is of course motivated by the $G \to G/H$ situation in group theory. The $\hat{A}$ polynomial appears naturally in [**BV2**], being related to the jacobian of the exponential mapping in $\mathrm{SO}(n)$.

II. Index Theorem and equivariant cohomology of the loop space. In this section, we discuss the relations of the Index Theorem for Dirac operators to the equivariant cohomology of the loop space.

In (a), we summarize the observations of Atiyah and Witten [**At**]. In (b), we describe the baby model of [**B4**], where a proof of the localization formulas of [**BV1,DH**] is given, which is strictly parallel to the proof of [**B1**]. Patodi's cancellations in finite dimensions are exhibited.

(a) *The remark of Atiyah and Witten.* We now summarize the observation in [**At**].

Namely, the space LM of smooth loops $s \in R/Z \to x_s \in M$ is an infinite-dimensional manifold with the Riemannian metric

$$Y \in T_xLM \to \int_0^1 |Y_s|^2\, ds.$$

S_1 acts naturally on LM by $x. \to k_t x_0 = x_{.+t}$, and the k_t are isometries. Let X be the Killing vector field generating $k.$ so that

$$X(x)_s = dx/ds. \tag{2.1}$$

Let X' be the 1 form on LM;

$$Y \in TLM \to X'(Y) = \langle X, Y \rangle. \tag{2.2}$$

One easily verifies that if $Y \in TLM$

$$dX'(Y,Z) = 2 \int_0^1 \left\langle \frac{DY}{Ds}, Z \right\rangle ds, \tag{2.3}$$

where DY/Ds is the covariant derivative of Y along x for the Levi-Civita connection.

The parallel transport operator τ_0^1 along the loop x acts like an element of $\mathrm{SO}(n)$ on $T_{x_0}M$. Let $\pm\theta_j$ be the angles of τ_0^1. One verifies easily that the eigenvalues of D/Ds acting on $T_x LM$ are given by

$$\pm 2i\pi m \pm i\theta_j, \qquad m \in N. \tag{2.4}$$

The Pfaffian $\mathrm{Pf}(-dX'/2)$ is given formally by

$$\mathrm{Pf}\left(-\frac{dX'}{2}\right) = \prod_{j=1}^{l} \theta_j \prod_1^{+\infty} [4\pi^2 m^2 - \theta_j]^2. \tag{2.5}$$

Dividing (2.5) formally by the infinite $(\prod_1^l 4\pi^2 m^2)^l$, we get

$$\frac{\mathrm{Pf}(-dX'/2)}{\prod_1^{+\infty}(4\pi^2 m^2)^l} = \prod_1^l 2\sin\left(\frac{\theta_j}{2}\right). \tag{2.6}$$

On the other hand, a formula from representation theory shows that if $\mathrm{Tr_s}[\tau_0^1]$ is the supertrace of τ_0^1 acting on $F_{\pm, x_0}$,

$$\mathrm{Tr_s}[\tau_0^1] = \pm(i)^l \prod_1^l 2\sin\frac{\theta_j}{2}. \tag{2.7}$$

Using (1.6), (2.6), and (2.7), Atiyah gives the following formal formula

$$\mathrm{Ind}\, D_+ = \frac{(\prod_1^{+\infty} m^2)^l}{(2\pi)^l} i^l \int_{LM} \exp\left\{-\frac{(d+i_X)X'}{2t}\right\}. \tag{2.8}$$

Also $L_X X' = 0$, and so

$$(d+i_X)[(d+i_X)X'] = 0. \tag{2.9}$$

The differential form μ_t appearing in the r.h.s. of (2.9) is such that $(d+i_X)\mu_t = 0$.

Now observe that $M = \{X = 0\}$.

Assume temporarily that LM is instead a finite-dimensional compact manifold. A formula of Duistermaat-Heckman **[DH]** and Berline-Vergne **[BV1]** (also see [**AB2**]) asserts that if e is the equivariant Euler class of the normal bundle to M in LM, then

$$\int_{LM} \mu_t = \int_M \frac{\mu_t}{e}. \tag{2.10}$$

Now by calculating e formally in terms of the Levi-Civita connections on M and LM, one shows easily that $\hat{A}(R/2\pi)$ represents in cohomology

$$\frac{\prod_1^{+\infty}(m^2)^l}{(2\pi)^l} i^l \frac{1}{e}.$$

Also $\mu_t = 1$ on M_0. We thus find that, rather surprisingly, a formal application of the formula of [**DH**, **BV1**] on LM "proves" the Index Theorem for the Dirac operators on the spin complex.

In [**B2**], we have shown that the observations of [**At**] extend to the case of Dirac operators acting on twisted spin complexes.

(b) *From infinite to finite dimensions: Patodi's cancellations in finite dimensions.* We noticed in [**B4**] that formula (2.8) could lead to a proof of the localization formulas of [**BV1**, **DH**] which would be strictly parallel to the heat equation proof of the Index Theorem.

In fact let N be a compact Riemannian orientable manifold. X is a Killing vector field, X' the corresponding 1 from. N^X is the submanifold $N^X = (X = 0)$. μ is a smooth section of $\Lambda(T^*N)$ such that $(d + i_X)\mu = 0$. We claim that for any $s \geq 0$

$$\int_N \mu = \int_N \exp\{-s(d + i_X)X'\}\mu. \tag{2.11}$$

In fact the derivative of the r.h.s. of (2.1) is given by

$$-\int_M (d + i_X)[X' \wedge \exp\{-s(d + i_X)X'\}\mu] = 0, \tag{2.12}$$

and so for any $t > 0$

$$\int_N \mu = \int_N \exp\left\{-\frac{(d + i_X)X'}{2t}\right\}\mu. \tag{2.13}$$

As $t \downarrow\downarrow 0$, the integral in the r.h.s. localizes on N^X. To make the analogy with §1c, we now assume that $\mu = 1$. Then

$$\int_N \exp - \frac{(d + i_X)X'}{2t} = \int_N \exp\left\{-\frac{|X|^2}{2t}\right\} \frac{\mathrm{Pf}(-dX'/2)}{t^{\dim N/2}}\, dx. \tag{2.14}$$

If B is the normal bundle of N^X in N, let J^X be the infinitesimal action of X in B. By taking geodesic coordinates in the normal bundle and doing the change of variables $y = \sqrt{t}y'$ in B, we find that as $t \downarrow\downarrow 0$, (2.14) is close to

$$\int_{N^X} dx \int_B \exp\left\{-\frac{|X|^2(x, y\sqrt{t})}{2t}\right\} \frac{\mathrm{Pf}_{TN^X}[-dX'(x, y\sqrt{t})/2]}{t^{\dim N^X/2}} [\mathrm{Pf}_B J_X]\, dy. \tag{2.15}$$

Let R be the curvature of TN. B is stable under $R(Y, Z)$, for $Y, Z \in TN^X$. Since X is Killing, $\nabla_Y(\nabla \cdot X) + R(X, Y) = 0$, and so

$$\frac{\mathrm{Pf}_{TN^X}[-dX'(x, \sqrt{t}y)/2]}{t^{\dim N^X/2}} \to \mathrm{Pf}_{TN^X}\left[-\frac{R}{2}(J_X y, y)\right]. \tag{2.16}$$

So as $t \downarrow\downarrow 0$, (2.14) converges—while staying constant—to

$$\int_{N^X} \int_B \exp^\wedge \left\{ -\frac{|J_X y|^2}{2} - \frac{R}{2}(J_X y, y) \right\} \mathrm{Pf}_B[J_X]\, dy. \tag{2.17}$$

At this stage the similarity of (2.17) with (1.20) and (1.22) should be obvious. (2.16) is a version of Patodi's cancellations in finite dimensions. It also gives a geometric origin to such cancellations.

III. Superconnections and the families Index Theorem. We now describe Quillen's superconnections [**Q1**] and their applications to the Index Theorem for families [**B3**, **BF**].

In (a), we describe the results of Quillen [**Q1**]. In (b), we summarize our heat equation proof of the Atiyah-Singer Index Theorem for families of Dirac operators [**B3**]. In (c), we summarize the results of [**BF**], in relation with [**Q1**, **W3**].

(a) *Quillen's superconnections.* Let N be a connected manifold. $E = E_+ \oplus E_-$ is a Z_2 graded vector bundle on N. End $E \hat{\otimes} \Lambda(T^*N)$ is a Z_2 graded algebra. The supertrace $\mathrm{Tr_s}$ defined on End E extends to End $E \hat{\otimes} \Lambda(T^*N)$ and takes its values in $\Lambda(T^*N)$. Let ∇ be a connection on E preserving the grading. ∇ defines a first-order differential operator acting on smooth sections of $\Lambda(T^*N) \hat{\otimes} E$.

Let u be an odd smooth section of End $E \hat{\otimes} \Lambda(T^*N)$. $\nabla + u$ is a superconnection in the sense of Quillen [**Q1**]. $(\nabla + u)^2$ is an even section of $\Lambda(T^*N) \hat{\otimes}$ End E and is the curvature of $\nabla + u$.

We now have the result of Quillen [**Q1**].

THEOREM 3.1. $\mathrm{Tr_s}\exp\{-(\nabla + u)^2/2\}$ *is a closed form on N which is a representative of the scaled Chern character of $E_+ - E_-$.*

In particular if D is an odd section of End E, $\nabla + D$ is a superconnection.

In [**Q1**], Quillen used superconnections to study differential forms and K-theory with support conditions and was also motivated by the Index Theorem for families. Mathai and Quillen [**MQ**] have used superconnections to study various problems related to localization and Thom forms.

(b) *The heat equation proof of the Index Theorem for families of Dirac operators.* Formula (1.6) for Ind D_+ is now crying out to be considered as a formula for a Chern character in the special case of one single operator.

In fact let $M \xrightarrow[\pi]{} B$ be a fibering of compact manifolds, with compact connected fibers Z of even dimension $n = 2l$. We assume that TZ is spin. Let g_Z be a smooth metric on TZ.

Let $F = F_+ \oplus F_-$ be the bundle of spinors of TZ. Let ξ be a Hermitian bundle on M, endowed with a unitary connection ∇^ξ.

For each $y \in B$, there is a well-defined Dirac operator

$$D_y = \begin{bmatrix} 0 & D_{-,y} \\ D_{+,y} & 0 \end{bmatrix}$$

on Z_y.

The Atiyah-Singer Index Theorem for families [**AS**] calculates $\ker D_+ - \ker D_- \in K(B)$.

In [**B3**], we have adapted Quillen's formalism in an infinite-dimensional situation. For $y \in B$, let $H_y^\infty = H_{+,y}^\infty \oplus H_{-,y}^\infty$ be the Z_2 graded bundle of C^∞ sections of $F \otimes \xi$ over Z_y. D is odd in End H^∞.

Let $T^H M$ be a subbundle of TM such that $TM = T^H M \oplus TZ$. $T^H M$ identifies with $\pi^* TB$. Any metric g_B on TB lifts to $T^H M$. Let ∇^L be the Levi-Civita connection on TM endowed with the metric $g_B \oplus g_Z$. If P_Z is the projection operator from TM on TZ, let ∇^Z be the Euclidean connection on TZ,

$$\nabla^Z = P_Z \nabla^L. \tag{3.1}$$

We proved in [**B3**] that ∇^Z does not depend on g_B, and is canonically defined by $T^H M$ and g_Z. ∇^Z and ∇^ξ define a unitary connection ∇ on $F \otimes \xi$.

For $Y \in TB$, let Y^H be the lift of Y in $T^H M$. If $h \in H^\infty$, set

$$\tilde{\nabla}_Y h = \nabla_{Y^H} h. \tag{3.2}$$

$\tilde{\nabla}$ is a connection on H^∞. For any $t > 0$, $\tilde{\nabla} + \sqrt{t} D$ is a superconnection on H^∞. The curvature $(\tilde{\nabla} + \sqrt{t} D)^2$ is a second-order elliptic operator acting fiberwise.

The following result is proved in [**B3**].

THEOREM 3.2. *For any $t > 0$, $\mathrm{Tr_s}[\exp\{-(\tilde{\nabla} + \sqrt{t} D)^2/2\}]$ is a C^∞ closed form on B, which represents the scaled Chern character of* $\ker D_+ - \ker D_-$.

As $t \downarrow\downarrow 0$, $\mathrm{Tr_s}[\exp\{-(\tilde{\nabla} + \sqrt{t} D)^2/2\}]$ does not converge in general.

Let S be defined by

$$\nabla^L - \nabla^B \oplus \nabla^Z = S.$$

Let $e_1, \dots, e_n$ be an orthonormal base of TZ. $f_1, \dots, f_m$ is a base of TB which lifts into a base of $T^H M$; $dy^1, \dots, dy^M$ is the corresponding dual base. In [**B3**, §3], we introduce the Levi-Civita superconnection

$$\begin{aligned}\tilde{\nabla}^{L,t} + \sqrt{t} D = \sum_{i,j,\alpha,\beta} \Bigg[e_i \Bigg(\sqrt{t}\, \nabla_{e_i} &+ \frac{1}{2} \langle S(e_i) e_j, f_\alpha \rangle e_j \, dy^\alpha \\ &+ \frac{1}{4\sqrt{t}} \langle S(e_i) f_\alpha, f_\beta \rangle \, dy^\alpha \, dy^\beta \Bigg) \\ + dy^\alpha \Bigg(\nabla_{f_\alpha} &+ \frac{1}{2\sqrt{t}} \langle S(f_\alpha) e_i, f_\beta \rangle e_i \, dy^\beta \Bigg) \Bigg].\end{aligned} \tag{3.3}$$

Let K be the scalar curvature of the fiber Z and L the curvature of ξ.

The following formula is proved in [**B3**, §3].

THEOREM 3.3. *The curvature of the Levi-Civita superconnection is given by*

$$\begin{aligned}(\tilde{\nabla}^{L,t} + \sqrt{t}D)^2 = & -t\left(\nabla_{e_i} + \frac{1}{2t}\langle S(e_i)e_j, f_\alpha\rangle\sqrt{t}e_j\,dy^\alpha \right. \\ & \left. \qquad + \frac{1}{4t}\langle S(e_i)f_\alpha, f_\beta\rangle dy^\alpha\,dy^\beta\right)^2 \\ & + \frac{tK}{4} + \frac{1}{2}te_ie_j \otimes L(e_i,e_j) + \frac{1}{2}dy^\alpha\,dy^\beta \otimes L(f_\alpha, f_\beta) \\ & + \sqrt{t}e_i dy^\alpha \otimes L(e_i, f_\alpha).\end{aligned} \tag{3.4}$$

We prove in [**B3**, §4] that as $t \downarrow\downarrow 0$, $\mathrm{Tr_s}[\exp\{-(\tilde{\nabla}^{L,t} + \sqrt{t}D)^2/2\}]$ converges. More precisely, we obtain a local version of this convergence. After adequately scaling the limit, we find that if R^Z is the curvature of TZ, the rescaled limit is

$$\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \mathrm{Tr}\ \exp\left[-\frac{L}{2i\pi}\right] \tag{3.5}$$

We thus find that (3.5) represents $\mathrm{ch}(\ker D_+ - \ker D_-)$. Recently, Berline and Vergne [**BV3**] have given a different proof of the convergence, using group-theoretic ideas.

(c) *Determinant bundles and the holonomy theorem.* In [**Q2**] Quillen has constructed a metric and a holomorphic connection on the determinant bundle of a family of $\overline{\partial}$ operators on Riemann surfaces. This construction has been extended in Bismut-Freed [**BF**] to the case of the family of Dirac operators considered in §3(b).

Namely, set

$$\lambda = (\det\ker D_+)^* \otimes \det\ker D_-. \tag{3.6}$$

λ is a well-defined C^∞ line bundle on B, even if B is noncompact [**Q2**, **BF**].

The first result of [**BF**] is that if the bundle λ is endowed with the Quillen metric, there is a unitary connection ${}^1\nabla$ on λ, whose curvature is given by

$$2i\pi\left[\int_Z \hat{A}\left(\frac{R^Z}{2\pi}\right) \mathrm{Tr}\ \exp\ -\frac{L}{2i\pi}\right]^{(2)}. \tag{3.7}$$

In [**W3**] Witten has given an argument showing that in certain situations, the holonomy of a loop c in B could be calculated using the êta invariant of a Dirac operator on the cylinder $\pi^{-1}(c)$.

This result has been fully proved in [**BF**] for a family of Dirac operators. Recall that the êta function of a selfadjoint elliptic operator has been defined in Atiyah-Patodi-Singer [**APS**].

THEOREM 3.4. *Let c be a smooth loop in B. For $\varepsilon > 0$, let D'^ε be the Dirac operator on $\pi^{-1}(c)$ associated with the metric $g_B/\varepsilon \oplus g_Z$. Let $\eta^\varepsilon(s)$ be the êta function of D'^ε. Set*

$$\overline{\eta}^\varepsilon = (\eta^\varepsilon(0) + \dim\ker D'^\varepsilon)/2.$$

Then as $\varepsilon \downarrow\downarrow 0$, $[\overline{\eta}^\varepsilon]$ has a limit $[\overline{\eta}]$ in R/Z. Also if τ is the holonomy of λ over c for the connection ${}^1\nabla$, then

$$\tau = (-1)^{\mathrm{Ind}\,D_+}\exp(-2i\pi[\overline{\eta}]). \tag{3.8}$$

The result of Theorem 3.4 is strongly connected with Atiyah-Donelly-Singer [**ADS**]. A new proof of Theorem 3.4 has been recently given by Cheeger [**Ch**].

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