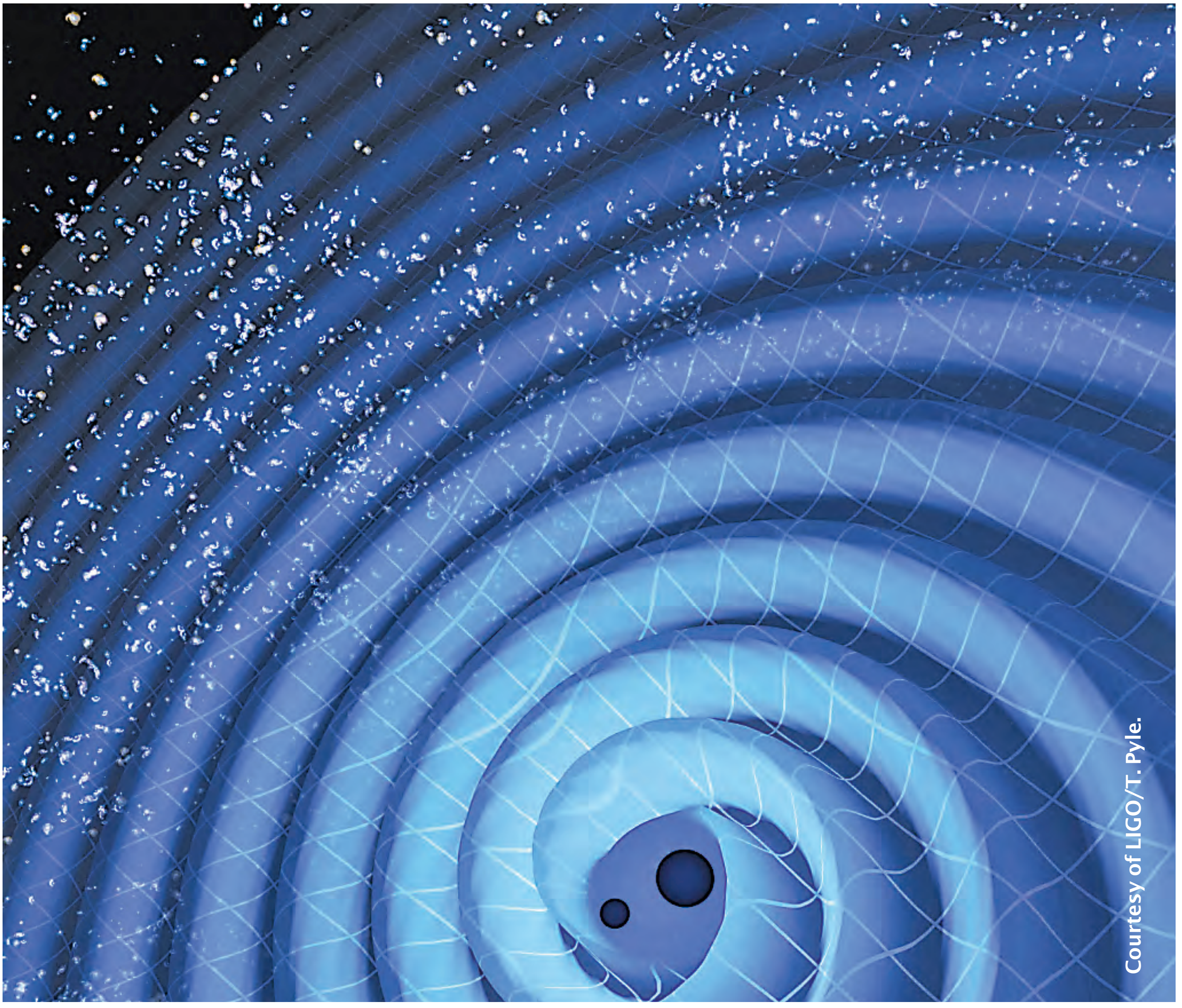
THE MATHEMATICS OF GRAVITATIONAL WAVES



This illustration shows the merger of two black holes and the gravitational waves that ripple outward as the black holes spiral toward each other. The black holes—which represent those detected by LIGO on December 26, 2015—were 14 and 8 times the mass of the sun, until they merged, forming a single black hole 21 times the mass of the sun. In reality, the area near the black holes would appear highly warped, and the gravitational waves would be difficult to see directly.

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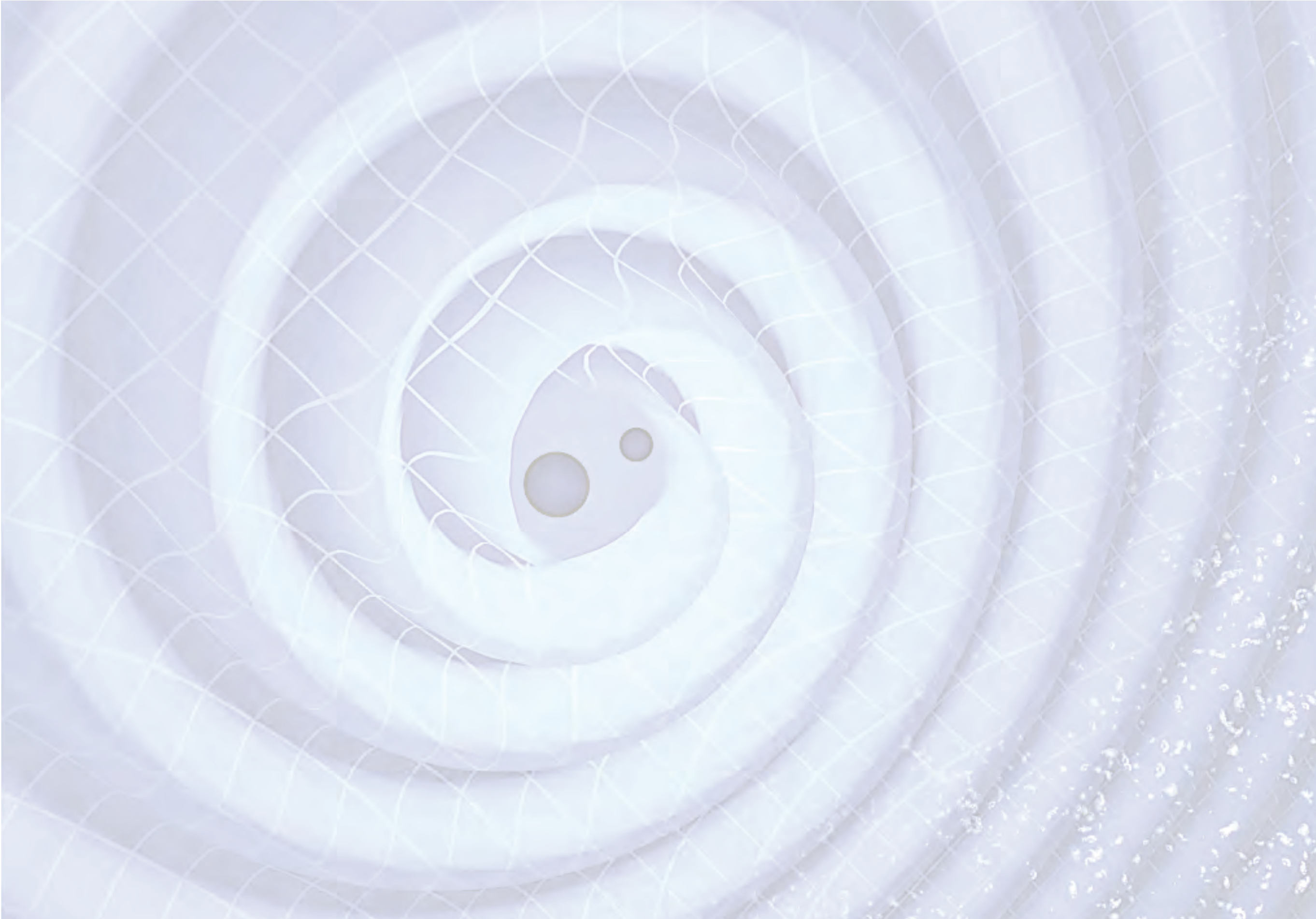

Introduction by Christina Sormani

The Mathematics of Gravitational Waves

A little over a hundred years ago, Albert Einstein predicted the existence of gravitational waves as a possible consequence of his theory of general relativity. Two years ago, these waves were first detected by LIGO. In this issue of *Notices* we focus on the mathematics behind this profound discovery.

Einstein's prediction of gravitational waves was based upon a linearization of his gravitational field equations, and he did not believe they existed as solutions to the original nonlinear system of equations. It was not until the 1950s that the mathematics behind Einstein's gravitational field equations was understood well enough even to define a wave solution. Robinson and Trautman produced the first family of explicit wave solutions to Einstein's nonlinear equations in 1962. Our first article, written by C. Denson Hill and Paweł Nurowski, describes this story of how the theoretical existence of gravitational waves was determined.

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Our second article, by Lydia Bieri, David Garfinkle, and Nicolás Yunes, describes the mathematics behind gravitational waves in more detail, beginning with a description of the geometry of spacetime. They discuss Choquet-Bruhat's famous 1952 proof of existence of solutions to the Einstein equations given Cauchy data. They then proceed to the groundbreaking work of Christodoulou-Klainerman and a description of the theory behind gravitational radiation: the radiation of energy in the form of gravitational waves.

Numerical methods are used to predict the gravitational waves emanating from specific cosmological events like the collision of black holes. Starting in Section 4 of their article, Bieri et al. describe these numerical methods beginning with linearized theory and the post-Newtonian approximation first developed by Einstein. They then describe the inward spiraling (as on the cover of this issue) of two black holes coming together and the resulting waves that occur as the black holes merge into one. They close with a description of the LIGO detector and how its measurements corroborated the predictions of the numerical teams. Ultimately the LIGO detection of gravitational waves not only validated Einstein's theory of general relativity, but also the work of the many mathematicians who contributed to an understanding of this theory.

Part one by C. Denson Hill and Paweł Nurowski

How the Green Light Was Given for Gravitational Wave Search

The recent detection of gravitational waves by the LIGO/Virgo team (B. P. Abbot et al. 2016) is an incredibly impressive achievement of experimental physics. It is also a tremendous success of the theory of general relativity. It confirms the existence of black holes, shows that binary black holes exist and that they may collide, and that during the merging process gravitational waves are produced. These are all predictions of general relativity theory in its fully nonlinear regime.

The existence of gravitational waves was predicted by Albert Einstein in 1916 within the framework of linearized Einstein theory. Contrary to common belief, even the very *definition* of a gravitational wave in the fully nonlinear Einstein theory was provided only after Einstein's death. Actually, Einstein advanced erroneous arguments against the existence of nonlinear gravitational waves, which stopped the development of the subject until the mid 1950s. This is what we refer to as the *red light* for gravitational wave research.

In this note we explain how the obstacles concerning gravitational wave existence were successfully overcome at the beginning of the 1960s, giving the *green light* for experimentalists to start designing detectors, which eventually produced the recent LIGO/Virgo discovery.

Gravitational Waves in Einstein's Linearized Theory

The idea of a gravitational wave comes directly from Albert Einstein. Immediately after formulating General Relativity Theory, still in 1916 Einstein [3] linearized his field equations

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} \tag{0.1}$$

by assuming that the metric $g_{\mu\nu}$ representing the gravitational field has the form of a slightly perturbed Minkowski metric $\eta_{\mu\nu}$,

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon h_{\mu\nu}.$$

Here $0 < \epsilon \ll 1$, and his linearization simply means that he developed the left hand side of (0.1) in powers of ϵ and neglected all terms involving ϵ^k with $k > 1$. As a result of this linearization Einstein found the field equations of *linearized* general relativity, which can conveniently be written for an unknown

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h_{\alpha\beta}\eta^{\alpha\beta}$$

as

$$\Box\bar{h}_{\mu\nu} = 2\kappa T_{\mu\nu}, \qquad \Box = \eta_{\mu\nu}\partial^\mu\partial^\nu.$$

These equations, outside the sources where

$$T_{\mu\nu} = 0,$$

constitute a system of decoupled relativistic wave equations

$$\Box h_{\mu\nu} = 0 \tag{0.2}$$

for each component of $h_{\mu\nu}$. This enabled Einstein to conclude that *linearized* general relativity theory admits solutions in which the perturbations of Minkowski spacetime $h_{\mu\nu}$ are plane waves traveling with the speed of light. Because of the *linearity*, by superposing plane wave solutions with different propagation vectors k_μ, one can get waves having any desirable wave front. Einstein named these *gravitational waves.* He also showed that within the linearized theory these waves carry energy, and he found a formula for the energy loss in terms of the third time derivative of the quadrupole moment of the sources.

Since far from the sources the gravitational field is very weak, solutions from the linearized theory should coincide with solutions from the full theory. Actually the wave detected by the LIGO/Virgo team was so weak that it was treated as if it were a gravitational plane wave from the linearized theory. We also mention that essentially all visualizations of gravitational waves presented during popular lectures or in the news are obtained using linearized theory only.

The Red Light

We focus here on the fundamental problem posed by Einstein in 1916, which bothered him to the end of his life. The problem is: Do the fully nonlinear Einstein equations admit solutions that can be interpreted as gravitational waves?

If "yes," then far from the sources, it is entirely reasonable to use linearized theory. If "no," then it makes no sense to expend time, effort, and money to try to detect such waves: solutions from the linearized theory are not physical; they are artifacts of the linearization.

If the answer is "no" we refer to it as a "red light" for gravitational wave search. This red light can be switched to "green" only if the following subproblems are solved:

(1) What is a definition of a *plane* gravitational wave in the full theory?
(2) Does the so defined plane wave exist as a solution to the full Einstein system?
(3) Do such waves carry energy?
(4) What is a definition of a gravitational wave with *nonplanar front* in the full theory?
(5) What is the energy of such waves?
(6) Do there exist solutions to the full Einstein system satisfying this definition?

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(7) Does the full theory admit solutions corresponding to the gravitational waves emitted by bounded sources?

To give a green light here, one needs a satisfactory answer to all these subproblems. Let us explain: Suppose that only the questions (1)–(3) had been settled in a satisfactory manner. Could we have a green light? The answer is no, because, contrary to the linear theory, unless we are very lucky, there is no way of superposing plane waves to obtain waves with arbitrary fronts. Thus the existence of a plane wave does not mean the existence of waves that can be produced by bounded sources, such as for example binary black hole systems.

Search for Plane Waves in the Full Theory

Naive Approach

A naive answer to our question (1) could be: a gravitational plane wave is a spacetime described by a metric, which in some coordinates (t, x, y, z), with t being timelike, has metric functions depending on $u = t - x$ only; preferably these functions should be sin or cos. This is not a good approach as is seen in the following example:

Consider the metric

$$
\begin{aligned}
g =& (\eta_{\mu\nu} + h_{\mu\nu})\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2 \\
&+ \cos(t-x)\Big(2 + \cos(t-x)\Big)\mathrm{d}t^2 \\
&- 2\cos(t-x)\Big(1 + \cos(t-x)\Big)\mathrm{d}t\mathrm{d}x \\
&+ \cos^2(t-x)\mathrm{d}x^2.
\end{aligned}
$$

We see here that the terms after the first row give the perturbation $h_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$ of the Minkowski metric $\eta = \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$. They are *oscillatory*, and one sees that the *ripples of the perturbation move with the speed of light*, $c = 1$, along the x-axis. A closer look shows also that the coefficients $h_{\mu\nu}$ of the perturbation satisfy the wave equation (0.2) (since they depend on a single null coordinate u only), and more importantly, that the full metric g has Ricci curvature 0 (is "*Ricci flat*").

Thus the above metric is not only an example of a "gravitational wave" in the linearized Einstein theory, but also it provides an example of a solution of the vacuum Einstein equations $R_{\mu\nu} = 0$ in the *fully nonlinear* Einstein theory. With all this information in mind, in particular having in mind the sinusoidal change of the metric with the speed of light in the x direction, we ask: is this an example of a plane gravitational wave?

The answer is *no*, as we created the metric g from the flat Minkowski metric $\eta = \mathrm{d}\bar{t}^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ by a *change of the time coordinate*: $\bar{t} = t + \sin(t - x)$. In view of this, the metric g is just the flat Minkowski metric, written in nonstandard coordinates. As such it does not correspond to any gravitational wave!

The moral from this example is that attaching the name of a "gravitational wave" to a spacetime that just satisfies an intuitive condition in some coordinate system is a wrong approach. As we see in this example we can always introduce a sinusoidal behaviour of the metric coefficients and their 'movement' with speed of light, by an appropriate change of coordinates.

Figure 1. Herman Bondi (left) here pictured with Peter G. Bergmann at the Jabłonna Relativity Conference, 1962, was one of the first to establish the possibility of planar gravitational waves.

We need a mathematically precise definition of even a plane wave.

Red Light Switched on: Einstein and Rosen

The first ever attempt to define a plane gravitational wave in the full theory is due to Albert Einstein and Nathan Rosen [4]. It happened in 1937, twenty years after the formulation of the concept of a plane wave in the linearized theory. They thought that they had found a solution of the vacuum Einstein equations representing a plane polarized gravitational wave. They observed that their solution had certain singularities and as such must be considered as *unphysical*. Their opinion is explicitly expressed in the subsequent paper of Rosen [7], which has the following abstract:

The system of equations is set up for the gravitational and electromagnetic fields in the general theory of relativity, corresponding to plane polarized waves. It is found that all nontrivial solutions of these equations contain singularities, so that one must conclude that strictly plane polarized waves of finite amplitude, in contrast to cylindrical waves, cannot exist in the general theory of relativity.

The Einstein-Rosen paper [4] was refereed by Howard P. Robertson, who recognized that the singularities encountered by Einstein and Rosen are merely due to the wrong choice of coordinates and that, if one uses correct coordinate patches, the solution may be interpreted as a cylindrical wave, which is nonsingular everywhere except on the symmetry axis corresponding to an infinite line source. This is echoed in Rosen's abstract quoted above in his phrase "in contrast to cylindrical waves," and is also mentioned in the abstract of the earlier Einstein-Rosen paper [4], whose first sentence is: *The rigorous solution for cylindrical gravitational waves is given.* Nevertheless, despite the clue given to them by Robertson, starting from 1937, neither Einstein nor Rosen believed that physically acceptable plane gravitational waves were admitted by the full Einstein theory. This belief of Einstein affected the views of his collaborators, such as Leopold Infeld, and more generally many other relativists. If a plane gravitational wave is not admitted by the theory, and if this statement comes from, and is fully supported by, the authority of Einstein, it was hard to believe at any fundamental level that the predictions of the linearized theory were valid.

Towards the Green Light: Bondi, Pirani, and Robinson

It is now fashionable to say that a new era of research on gravitational waves started at the International Conference on Gravitation held at Chapel Hill on 18–23 January 1957. To show that not everybody was sure about the existence of gravitational waves during this conference we quote Herman Bondi [1], one of the founding fathers of gravitational wave theory:

Polarized plane gravitational waves were first discovered by N. Rosen, who, however, came to the conclusion that such waves could not exist because the metric would have to contain certain physical singularities. More recent work by Taub and McVittie showed that there were no unpolarized plane waves, and this result has tended to confirm the view that true plane gravitational waves do not exist in empty space in general relativity. Partly owing to this, Scheidegger and I have both expressed the opinion that there might be no energy-carrying gravitational waves at all in the theory.

The last sentence in the quote refers to Bondi's opinion expressed during the Chapel Hill Conference. Interestingly, the quote is from Bondi's *Nature* paper announcing the discovery of a singularity-free solution of a plane gravitational wave that carries energy, received by the journal on March 24, 1957. A dramatic change of opinion between January and March of the same year!

Bondi in the *Nature* paper invokes the solution of Einstein's equations found in the context of gravitational waves by Ivor Robinson. This paper, and the subsequent paper written by Bondi, Felix Pirani, and Robinson [2], answers in positive our problems (1), (2), and (3).

In particular (1) is answered with the following definition of a *plane wave in the full theory*: The gravitational plane wave is a spacetime that (a) satisfies vacuum Einstein's equations $R_{\mu\nu} = 0$ and (b) has a 5-dimensional group of isometries. The motivation for this definition is the fact that a plane electromagnetic wave has a 5-dimensional group of symmetries. Bondi, Pirani, and Robinson do *not* assume that the 5-dimensional group of isometries is isomorphic to the symmetry of a plane electromagnetic wave. They inspect all Ricci flat metrics with symmetries of dimension greater than or equal to 4 given by A. Z. Petrov (1957), and find exactly one class of solutions with the same 5-dimensional group of isometries, which by a miracle *is* isomorphic to the symmetry group of the electromagnetic field.

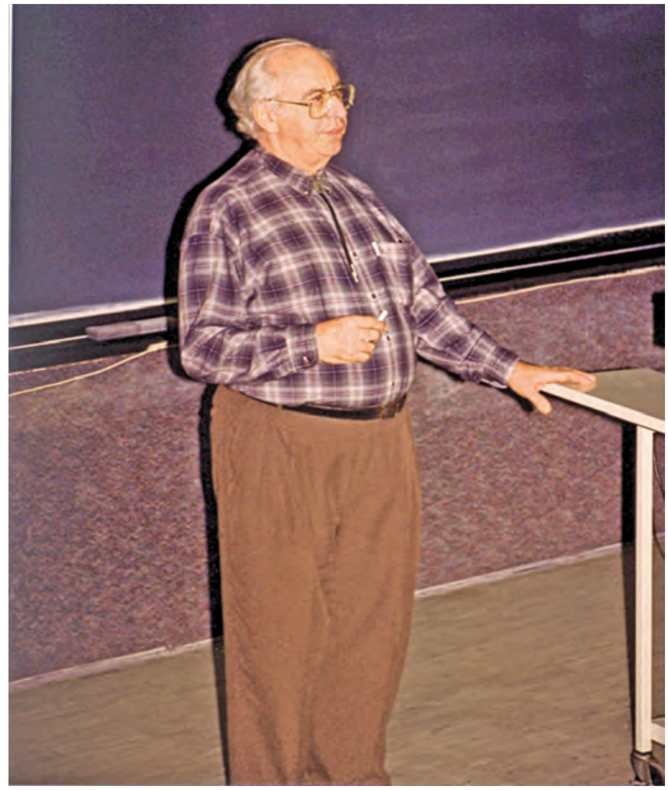


Figure 2. Ivor Robinson, shown here during Journées Relativistes in Dublin, 2001, was an independent discoverer of an exact solution describing planar gravitational waves.

It follows that the class of metrics obeying the Bondi-Pirani-Robinson definition of a plane gravitational wave depends on two *free functions of one variable* that can be interpreted as the wave amplitude and the direction of polarization. Using these free functions Bondi, Pirani, and Robinson obtained a *sandwich wave*, i.e. a gravitational wave that differs from the Minkowski spacetime only in a 4-dimensional strip moving in a given direction with the speed of light. They used this sandwich wave and analyzed what happens when it hits a system of test particles. It follows that the wave *affects* their motion, which leads to the conclusion that *gravitational plane waves in the full theory carry energy*.

In this way, the *Nature* paper of Bondi [1], together with the later paper of Bondi, Pirani, and Robinson [2], *solves our problems* (1), (2) *and* (3): the plane wave in the full theory is defined, it is realized as a class of solutions of Einstein field equations $R_{\mu\nu} = 0$, and it carries energy, since passing through the spacetime in a form of a sandwich it affects test particles.

As a last comment in this section we mention that the Bondi-Pirani-Robinson gravitational plane waves, sought

Figure 3. Felix Pirani, shown here in 1937, when Einstein and Rosen were writing their controversial paper, and in May 2015, a few months before his death, collaborated with Bondi and Robinson and gave an algebraic local criterion for gravitational waves.

with great effort by physicists for forty years, were actually discovered already in 1925 by a *mathematician*, H. W. Brinkmann. He discovered what are known as *pp-waves*, a class of Ricci flat metrics having radiative properties, which include Bondi-Pirani-Robinson plane waves as a special case. His discovery was published in English in *Mathematische Annalen* **94** (1925), 119–145. If only there had been better communication between mathematicians and physicists.

General Gravitational Waves

Closer to the Green: Pirani

The development of the theory of gravitational waves at the turn of the 1950s and 1960s was very rapid. The story, as we are presenting it here now, is more topical than chronological, so, breaking the chronology, we will now discuss an important paper of Felix Pirani [5], which appeared before Bondi's *Nature* announcement of the existence of a plane wave in Einstein's theory. It is also worthwhile to note that Pirani's paper [5] was submitted a few months *before* the Chapel Hill conference. For us, this paper is of fundamental importance, since, among other things, it gives the first attempt at a purely geometric *definition of a gravitational wave spacetime*.

Pirani argues that gravitational radiation should be detectable by analysis of the Riemann tensor. He suggests that a spacetime containing gravitational radiation should be *algebraically special*. This suggestion uses the so-called *Petrov classification* of gravitational fields. At every point it consists in the enumeration of the distinct *eigendirections* of the Weyl tensor (the traceless part of the Riemann tensor). These eigendirections are called *principal null directions* (PNDs). If at a point all four PNDs are distinct, the spacetime at this point is called *algebraically general*. If at least two of the PNDs coincide, the spacetime at this point is called *algebraically special*. At each point various coincidences of PNDs may occur, resulting in the stratification of the algebraically special spacetime points into four *Petrov types*: type II (two PNDs coincide, the other two are distinct), type III (three PNDs coincide), type N (four PNDs coincide), and type D (four PNDs are grouped in two different pairs of coinciding PNDs). Pirani's suggestion that spacetimes containing radiation should be algebraically special *everywhere* was not very precise, as all the Petrov types (II, III, D, N) had not yet been correctly spelled out (the fully correct Petrov classification was given later by Roger Penrose in 1960).

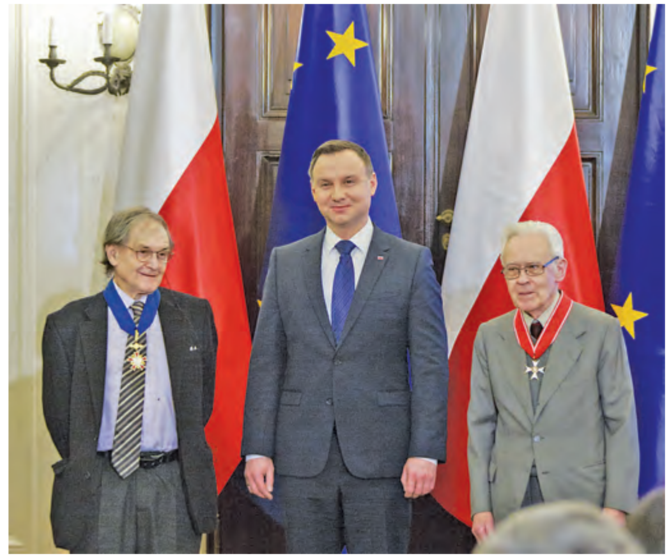


Figure 4. Roger Penrose (left), President of the Republic of Poland Andrzej Duda (center), and Andrzej Trautman at the ceremony at which Penrose got the highest Polish medal of merit for a foreigner and Trautman for a Pole, Warsaw 2016. Penrose and Trautman developed a nonlocal theory of radiation.

Pirani's intuition about the importance of algebraic speciality in the theory of gravitational waves was brilliant. However, he was wrong in insisting on algebraical speciality of radiative spacetimes everywhere. We know now ([9], p. 411, eq. (21)) that the Weyl tensor of a radiative spacetime must be of type N very *far from the sources*, or better said, *asymptotically*.

Switching on Green: Radiation is Nonlocal

Pirani's algebraic speciality condition for a gravitational wave spacetime refers to pointwise defined objects—the PNDs. As the Weyl tensor can change its algebraic type from point to point, the criterion is local. On the other hand, even in Maxwell theory, radiation is a nonlocal phenomenon. To illustrate this we recall a well-known conundrum:

Q: Does a unit charge hanging on a thread attached to the ceiling of Einstein's lift radiate or not?
A: Well… viewed by an observer in the lift—NO!, as it is at rest; but, on the other hand, viewed by an observer on the Earth—YES!, as it falls down with constant acceleration $\vec{g}$.

Here, the confusion in the answers is of course due to the fact that one tries to apply a *purely local* physical law—the equivalence principle[1]—to the very nonlocal phenomenon, which is radiation in electromagnetic theory.

This gives a hint as to how to define what radiation is in general relativity. One can not expect that in this nonlinear theory radiation can be defined in terms of local notions. This point is raised and consequently developed by Andrzej Trautman, in two papers [8, 9] submitted to *Bulletin de l'Academie Polonaise des Sciences*, behind the Iron Curtain, in April 1958. This led him to finally solve our problems (4)–(5), [9], and (6)–(7), [6], thereby switching the red light to green.

It is worthwhile to mention that although Trautman's two papers [8, 9] were published behind the Iron Curtain, their results were exposed to the Western audience. In the next two months after their submission to the Polish *Bulletin* (May–June, 1958) Trautman, on the invitation of Felix Pirani, gave a series of lectures at King's College London presenting their theses. The audience of his lectures included H. Bondi and F. Pirani, and the lectures were mimeographed and spread among Western relativists.

Another interesting thing is that Trautman's two papers were an abbreviated version of his PhD thesis. It had two supervisors: the official one—Leopold Infeld, the closest collaborator of Albert Einstein, who following Einstein did not believe in gravitational waves, and the unofficial one—Jerzy Plebański, for whom the existence of gravitational waves was obvious. It was Plebański who proposed gravitational waves as a subject of Trautman's PhD. Despite Infeld's disbelief in gravitational waves, Trautman obtained his PhD under Infeld.

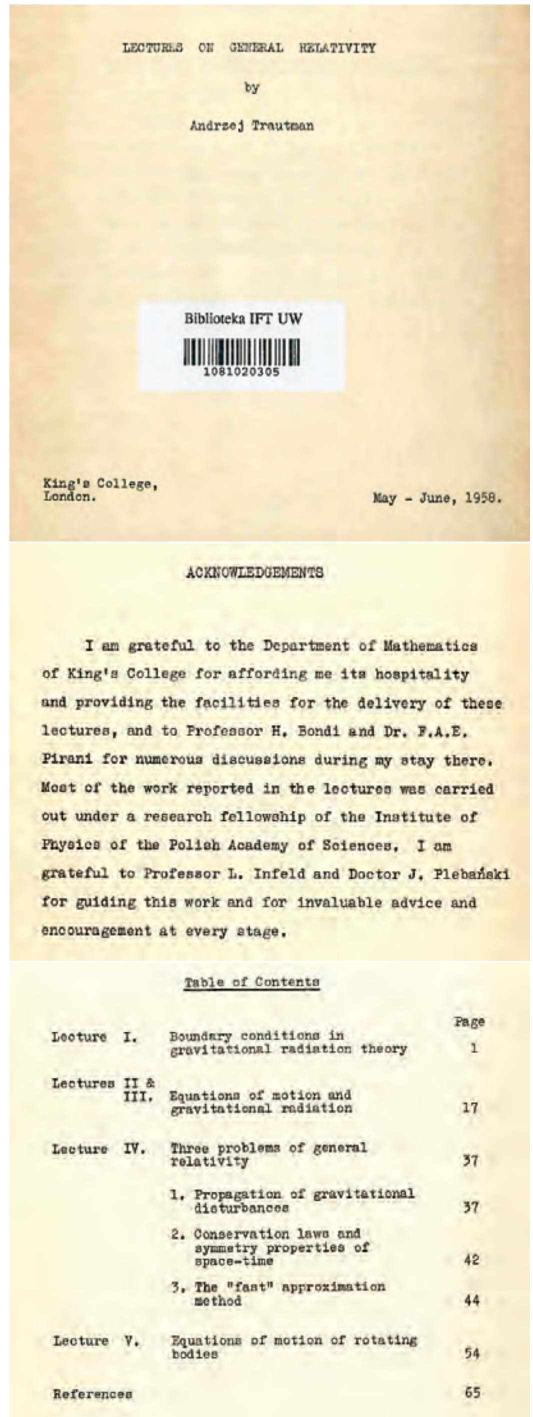
Green Light: Trautman

Trautman's general idea in defining what a gravitational wave is in the full Einstein theory was to say that it should satisfy certain boundary conditions at infinity. More precisely, from all spacetimes, i.e. solutions of Einstein's equations in the full theory, he proposed to select only those that satisfied boundary conditions at infinity, which were his *generalizations* of Sommerfeld's radiation conditions. These are known in the linear theory of a scalar field, and Trautman [8, 9] generalizes them to a number of *physical theories*. He reformulates Sommerfeld's radiation boundary conditions for the scalar inhomogeneous wave equation into a form that is then generalized to other field theories. As an example he shows how this generalization works in Maxwell's theory and that it indeed selects the outgoing radiative Maxwell fields from all solutions of Maxwell's equations.

In the next paper [9] Trautman does the same for Einstein's general relativity. Trautman defines the *boundary conditions to be imposed on gravitational fields due to isolated systems of matter*. This is the first step in solving our problems (4) and (5).

He then passes to the treatment of our problem (5). He uses the *Freud superpotential* 2-form $\mathcal{F}$ to split the Einstein tensor E into $E = \mathrm{d}\mathcal{F} - \kappa\mathfrak{T}$ so that the Einstein equations $E = \kappa T$ take the form

$$\mathrm{d}\mathcal{F} = \kappa(T + \mathfrak{T}).$$


LECTURES ON GENERAL RELATIVITY

by

Andrzej Trautman

King's College,
London.

May - June, 1958.

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I am grateful to the Department of Mathematics of King's College for affording me its hospitality and providing the facilities for the delivery of these lectures, and to Professor H. Bondi and Dr. F.A.E. Pirani for numerous discussions during my stay there. Most of the work reported in the lectures was carried out under a research fellowship of the Institute of Physics of the Polish Academy of Sciences. I am grateful to Professor L. Infeld and Doctor J. Plebański for guiding this work and for invaluable advice and encouragement at every stage.

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Figure 5. The first three pages of Trautman's mimeographed King's College Lectures, which carried Trautman's work behind the Iron Curtain to Western relativists.

[1] *The inability to locally distinguish between gravitational and inertial forces.*

Figure 6. A blackboard discussion between Trautman's two advisors, Jerzy Plebański (left) and Leopold Infeld, at the Institute of Theoretical Physics of University of Warsaw. Ironically, Trautman got his PhD on gravitational waves as recommended by Plebański under Infeld, who didn't believe in them.

Here T is the energy-momentum 3-form, and κ is a constant related to the gravitational constant G and the speed of light c via $\kappa = \frac{8\pi G}{c^4}$ (in the following we work with physical units in which $c = 1$).

Since $\mathfrak{T}$ is a 3-form totally determined by the geometry, it is interpreted as the energy-momentum 3-form of *pure gravity*. The closed 3-form $T + \mathfrak{T}$ is then used to define the 4-momentum $P^\mu(\sigma)$ of a *gravitational field attributed to every space-like hypersurface* σ of a spacetime satisfying his radiative boundary conditions. He shows that $P^\mu(\sigma)$ is *finite* and *well defined*, i.e. that it does not depend on the coordinate systems adapted to the chosen boundary conditions. Using his boundary conditions he then calculates how much of the gravitational energy $p^\mu = P^\mu(\sigma_1) - P^\mu(\sigma_2)$ contained between the spacelike hypersurfaces σ_1 (initial one) and σ_2 (final one) *escapes to infinity*.

Finally, he shows that p^0 is *nonnegative*, saying that radiation is present when $p^0 > 0$.

Taken together, everything we have said so far about Trautman's results, *solves our problems* (4) and (5): What in popular terms is called a *gravitational wave in the full GR theory* is a *spacetime satisfying Trautman's boundary conditions with* $p^0 > 0$; the *energy of a gravitational wave* contained between hypersurfaces σ_1 and σ_2 is given by p^0.

Trautman proves only that $p^0 \geq 0$. If the inequality were sharp, $p^0 > 0$, this would give a proof of the statement that spacetimes satisfying Trautman's boundary conditions, or better said, the gravitational waves associated with them, *carry energy*. Trautman does not have such a proof. To handle this problem, one can try to find an example of an *exact solution* to the Einstein equations satisfying Trautman's boundary conditions, and to show that in this example p^0 is *strictly* greater than zero. This approach is taken by I. Robinson and Trautman [6], and we will comment on this later.

Figure 7. Andrzej Trautman established gravitational waves in the full Einstein theory.

As regards Trautman's paper [9], it is worthwhile to mention that Trautman shows there two other interesting things implied by his boundary conditions. The first of them is the fact that in the presence of electromagnetic radiation a spacetime satisfying his boundary conditions has far from the sources Ricci tensor in the form of a *null dust* $R_{\mu\nu} = \rho k_\mu k_\nu$, with k a *null vector*. This in particular means that the electromagnetic/gravitational radiation in his spacetimes travels with the speed of light. The second interesting feature he shows is that *far from the sources the Riemann tensor of a spacetime satisfying his radiative boundary conditions is of Petrov type* N. Since far from the sources $Riemann = Weyl$, this verifies the *intuition* of Pirani [5]: spacetimes satisfying radiative boundary conditions satisfy the algebraic speciality criterion, and from all the possibilities of algebraic speciality they choose a type N Weyl tensor as the leading term at infinity. This was later developed into the celebrated *peeling-off theorem* attributed to Ray Sachs.

The last two of our problems (6)–(7) were addressed by I. Robinson and Trautman [6]. There they *found a large class of exact solutions* of the full system of Einstein equations satisfying Trautman's boundary conditions.

Figure 8. Paul A. M. Dirac with Trautman and Infeld during the 1962 Jabłonna Conference.

The solutions describe waves with *closed fronts* so *they can be interpreted as coming from bounded sources.*

These solutions solve our last two problems (6) *and* (7). For some of them $p^0 > 0$, so they correspond to gravitational waves that *do carry energy*.

To conclude, we say that the Bondi-Pirani-Robinson papers [1, 2] and the Trautman-Robinson papers [9, 6] solve all our problems (1)–(7), giving the green light to further research on gravitational radiation. We will not comment on these further developments since they are well documented; see e.g. D. Kennefick's recent book *Traveling at the Speed of Thought: Einstein and the Quest for Gravitational Waves.*

Figure 9. Ivor Robinson and Andrzej Trautman, shown here in Trieste in the late 1980s, found a large class of exact solutions for gravitational waves with closed fronts in the full Einstein theory.

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Part two by Lydia Bieri, David Garfinkle, and Nicolás Yunes

Gravitational Waves and Their Mathematics

Introduction

In 2015 gravitational waves were detected for the first time by the LIGO team [1]. This triumph happened 100 years after Albert Einstein's formulation of the theory of general relativity and 99 years after his prediction of gravitational waves [4]. This article focuses on the mathematics of Einstein's gravitational waves, from the properties of the Einstein vacuum equations and the initial value problem (Cauchy problem), to the various approximations used to obtain quantitative predictions from these equations, and eventually an experimental detection.

General relativity is studied as a branch of astronomy, physics, and mathematics. At its core are the Einstein equations, which link the physical content of our universe to geometry. By solving these equations, we construct the spacetime itself, a continuum that relates space, time, geometry, and matter (including energy). The dynamics of the gravitational field are studied in the Cauchy problem for the Einstein equations, relying on the theory of nonlinear partial differential equations (pde) and geometric analysis. The connections between astronomy, physics, and mathematics are richly illustrated by the story of gravitational radiation.

Gravitational waves are vibrations in spacetime propagating at the speed of light.

In general relativity, the universe is described as a spacetime manifold with a curved metric whose curvature encodes the properties of the gravitational field. While sometimes one wants to use general relativity to describe the whole universe, often we just want to know how a single object or small collection of objects behaves. To address that kind of problem, we use the idealization of the isolated system: a spacetime consisting of just the objects we want to study and nothing else. We might consider the solar system as an isolated object, or a pair of black holes spiraling into one another until they collide. We ask how those objects look to a distant, far away observer in a region where presumably the curvature of spacetime is very small. Gravitational waves are vibrations in spacetime that propagate at the speed of light away from their source. They may be produced, for example, when black holes merge. This is what was detected by Advanced LIGO (aLIGO) and this is the focus of this article.

Figure 1. Albert Einstein predicted gravitational waves in 1916.

First we describe the basic differential geometry used to define the universe as a geometric object. Next we describe the mathematical properties of the Einstein vacuum equations, including a discussion of the Cauchy problem and gravitational radiation. Then we turn to the various approximation schemes used to obtain quantitative predictions from these equations. We conclude with the experimental detection of gravitational waves and the astrophysical implications of this detection. This detection is not only a spectacular confirmation of Einstein's theory, but also the beginning of the era of gravitational wave astronomy, the use of gravitational waves to investi-

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gate aspects of our universe that have been inaccessible to telescopes.[2]

The Universe as a Geometric Object

A spacetime manifold is defined to be a 4-dimensional, oriented, differentiable manifold M with a Lorentzian metric tensor, g, which is a nondegenerate quadratic form of index one,

$$g = \sum_{\mu,\nu=0}^{3} g_{\mu\nu} dx^{\mu} \otimes dx^{\nu},$$

defined in T_qM for every q in M varying smoothly in q. The trivial example, the Minkowski spacetime as defined in Einstein's special relativity, is $\mathbb{R}^4$ endowed with the flat Minkowski metric:

$$g = \eta = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{1}$$

Taking $x_0 = t$, $x_1 = x$, $x_2 = y$, and $x_3 = z$, we have $\eta_{00} = -c^2$, $\eta_{ii} = 1$ for $i = 1, 2, 3$, and $\eta_{\mu\nu} = 0$ for $\mu \neq \nu$. In mathematical general relativity we often normalize the speed of light, $c = 1$.

Schwarzschild spacetime describes a black hole.

The family of Schwarzschild metrics are solutions of the Einstein vacuum equations that describe spacetimes containing a black hole, where the parameter values are $M > 0$. Taking $r_s = 2GM/c^2$, it has the metric:

$$g = -c^2 \frac{\left(1 - \frac{r_s}{4\rho}\right)^2}{\left(1 + \frac{r_s}{4\rho}\right)^2} dt^2 + \left(1 + \frac{r_s}{4\rho}\right)^4 h, \tag{2}$$

where $\rho^2 = x^2 + y^2 + z^2$, $h = dx^2 + dy^2 + dz^2$, and G denotes the Newtonian gravitational constant. This space is asymptotically flat as $\rho \to \infty$.

The Friedmann-Lemaître-Robertson-Walker spacetimes describe homogeneous and isotropic universes through the metric

$$g = -c^2dt^2 + a^2(t)g_{\chi}, \tag{3}$$

where g_χ is a Riemannian metric with constant sectional curvature, χ, (e.g. a sphere when $\chi = 1$) and $a(t)$ describes the expansion of the universe. The function, $a(t)$, is found by solving the Einstein equations as sourced by fluid matter.

In an arbitrary Lorentzian manifold, M, a vector $X \in T_xM$ is called *null* or lightlike if

$$g_x(X, X) = 0.$$

At every point there is a cone of null vectors called the *null cone*, as in Figure 2. A vector $X \in T_xM$ is called *timelike* if

$$g_x(X, X) < 0,$$

and *spacelike* if

$$g_x(X, X) > 0.$$

In general relativity nothing travels faster than the speed of light, so the velocities of massless particles are null vectors whereas those for massive objects are timelike. A causal curve is a differentiable curve for which the tangent vector at each point is either timelike or null.

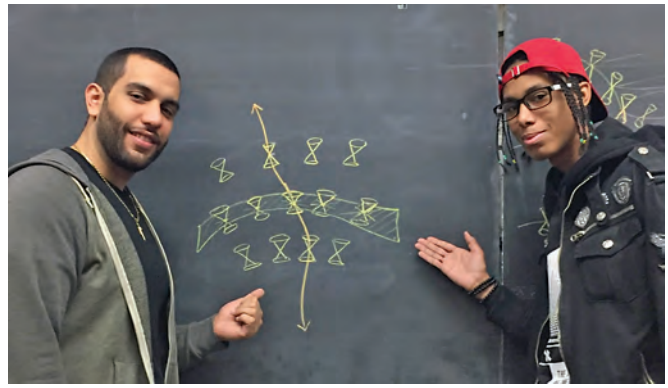

Figure 2. Light cones, a timelike curve, and a spacelike hypersurface as demonstrated by physics majors at Lehman College.

A hypersurface is called *spacelike* if its normal vector is timelike, so that the metric tensor restricted to the hypersurface is positive definite. A *Cauchy hypersurface* is a spacelike hypersurface where each causal curve through any point $x \in M$ intersects $\mathcal{H}$ exactly at one point. A spacetime (M, g) is said to be *globally hyperbolic* if it has a Cauchy hypersurface. In a globally hyperbolic spacetime, there is a time function t whose gradient is everywhere timelike or null and whose level surfaces are Cauchy surfaces. A globally hyperbolic spacetime is causal in the sense that no object may travel to its own past.

As in Riemannian geometry, curves with 0 acceleration are called geodesics. Light travels along null geodesics. Geodesics that enter the event horizon of a black hole as in Figure 3 never leave. Objects in free fall travel along timelike geodesics. They also can never leave once they have entered a black hole. When two black holes fall into each other, they merge and form a single larger black hole.

In curved spacetime, geodesics bend together or apart and the relative acceleration between geodesics is described by the Jacobi equation, also known as the geodesic deviation equation. In particular, the relative acceleration of nearby geodesics is given by the Riemann curvature tensor times the distance between them. The Ricci curvature tensor, $R_{\mu\nu}$, measures the average way in which geodesics curve together or apart. The scalar curvature, R, is the trace of the Ricci curvature.

Einstein's field equations are:

$$R_{\mu\nu} - \tfrac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}\ , \tag{4}$$

where $T_{\mu\nu}$ denotes the energy-momentum tensor, which encodes the energy density of matter. Note that for

[2] *Editor's note: Don't miss the intriguing and most readable final sections of this article.*

Figure 3. The horizon of the black hole is depicted here as a cylinder with inward pointing light cones, as demonstrated by physics majors at Lehman College.

cosmological considerations, one can add $\Lambda g_{\mu\nu}$ on the left-hand side, where Λ is the cosmological constant. However, nowadays, this term is commonly absorbed into $T_{\mu\nu}$ on the right-hand side. Here we will consider the noncosmological setting. One then solves the Einstein equations for the metric tensor $g_{\mu\nu}$. If there are no other fields, then $T_{\mu\nu} = 0$ and (4) reduce to the Einstein vacuum equations:

$$R_{\mu\nu} = 0 \,. \tag{5}$$

Note that the Einstein equation is a set of second order quasilinear partial differential equations for the metric tensor. In fact, when choosing the right coordinate chart (wave coordinates), taking $c = 1$, and writing out the formula for the curvature tensor, $R_{\mu\nu}$, in those coordinates, the equation becomes:

$$\Box_g g_{\alpha\beta} = N_{\alpha\beta} \tag{6}$$

where $\Box_g$ is the wave operator and $N_{\alpha\beta} = N_{\alpha\beta}(g, \partial g)$ denote nonlinear terms with quadratics in ∂g.

Quite a few exact solutions to the Einstein vacuum equations are known. Among the most popular are the trivial solution (Minkowski spacetime) as in (1); the Schwarzschild solution, which describes a static black hole, as in (2); and the Kerr solution, which describes a black hole with spin angular momentum. Note that the exterior gravitational field of any spherically symmetric object takes the form of (2) for $r > r_0$ where $r_0 > r_s$ is the radius of the object, so this model can be used to study the spacetime around an isolated star or planet. However, in order to understand the dynamics of the gravitational field and radiation, we have to investigate large classes of spacetimes. This can only be done by solving the initial value problem (Cauchy problem) for the Einstein equations, which will be discussed in the next section.

If there are matter fields, so that $T_{\mu\nu} \neq 0$, then these fields satisfy their own evolution equations, which have to be solved along with the Einstein field equations (4) as a coupled system. The scale factor, $a(t)$, of the Friedmann-Lemaître-Robertson-Walker cosmological spacetimes in (3) can then be found by solving a second order, ordinary differential equation derived from (4). If a solution has a time where the scale factor vanishes, then the solution is said to describe a cosmos whose early phase is a "big bang."

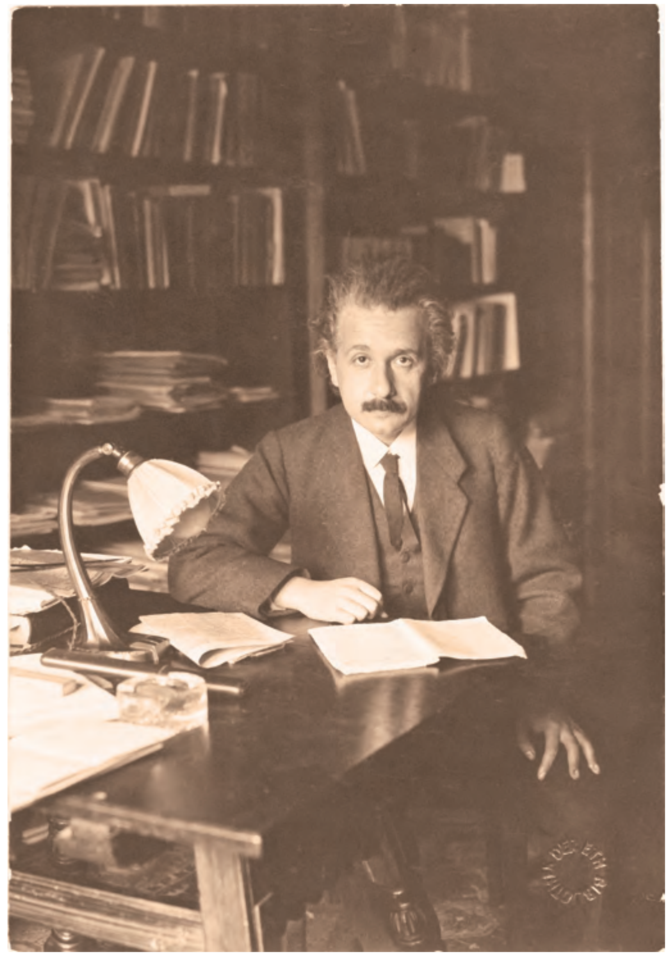

Figure 4. Albert Einstein

The Einstein Equations

Beginnings of Cauchy Problem

In order to study gravitational waves, stability problems, and general questions about the dynamics of the gravitational field, we have to formulate and solve the *Cauchy problem*. That is, we are given as initial data a prescribed Riemannian manifold $\mathcal{H}$ with a complete Riemannian metric $\bar{g}_{ij}$ and a symmetric 2-tensor $\mathcal{K}_{ij}$ satisfying certain consistency conditions called the *Einstein constraint equations*. We then solve for a spacetime (M, g) that satisfies the Einstein equations evolving forward from this initial data set. That is, the given Riemannian manifold $\mathcal{H}$ is a spacelike hypersurface in this spacetime solution M, where $\bar{g}$ is the restriction of g. Furthermore, the symmetric two-tensor $\mathcal{K}_{ij}$ is the prescribed second fundamental form.

All the different methods used to describe gravitational radiation have to be thought of as embedded into the aim of solving the Cauchy problem. We solve the Cauchy problem by methods of analysis and geometry. However, for situations where the geometric-analytic techniques are not (yet) at hand, one uses approximation methods and numerical algorithms. The goal of the latter methods is to produce approximations to solutions of the Cauchy problem for the Einstein equations.

In order to derive the gravitational waves from binary black hole mergers, binary neutron star mergers, or core-collapse supernovae, we describe these systems by asymptotically flat spacetimes. These are solutions of the Einstein equations that at infinity tend to Minkowski space with a metric as in (1). Schwarzschild space is a simple example of such an isolated system containing only a single stationary black hole (2). There is a huge literature about specific fall-off rates, which we will not describe here. The null asymptotics of these spacetimes contain information on gravitational radiation (gravitational waves) out to infinity.

Recall that the Einstein vacuum equations (5) are a system of ten quasilinear, partial differential equations that can be put into hyperbolic form. However, with the Bianchi identity imposing four constraints, the Einstein vacuum system (5) constitutes only six independent equations for the ten unknowns of the metric $g_{\mu\nu}$. This corresponds to the general covariance of the Einstein equations. In fact, uniqueness of solutions to these equations holds up to equivalence under diffeomorphisms. We have just found a core feature of general relativity. This mathematical fact also means that physical laws do not depend on the coordinates used to describe a particular process.

The Einstein equations split into a set of evolution equations and a set of constraint equations. As above, t denotes the time coordinate whereas indices $i, j = 1, \ldots, 3$ refer to spatial coordinates. Taking $c = 1$, the evolution equations read:

$$\frac{\partial \bar{g}_{ij}}{\partial t} = -2\Phi\mathcal{K}_{ij} + \mathcal{L}_X \bar{g}_{ij}, \tag{7}$$

$$\frac{\partial \mathcal{K}_{ij}}{\partial t} = -\nabla_i \nabla_j \Phi + \mathcal{L}_X \mathcal{K}_{ij} + (\bar{R}_{ij} + \mathcal{K}_{ij}\, tr\mathcal{K} - 2\mathcal{K}_{im}\mathcal{K}^m_j)\Phi \tag{8}$$

Here $\mathcal{K}_{ij}$ is the extrinsic curvature of the $t = const.$ surface $\mathcal{H}$ as above. The lapse Φ and shift X are essentially the g_{tt} and g_{ti} components of the metric, and are given by $T = \Phi n + X$ where T is the evolution vector field $\partial/\partial t$ and n is the unit normal to the constant time hypersurface. ∇_i is the spatial covariant derivative and $\mathcal{L}$ is the Lie derivative. However, the initial data $(\bar{g}_{ij}, \mathcal{K}_{ij})$ cannot be chosen freely: the remaining four Einstein vacuum equations become the following constraint equations:

$$\nabla^i \mathcal{K}_{ij} - \nabla_j\, tr\mathcal{K} = 0, \tag{9}$$

$$\bar{R} + (tr\mathcal{K})^2 - |\mathcal{K}|^2 = 0. \tag{10}$$

An *initial data set* is a 3-dimensional manifold $\mathcal{H}$ with a complete Riemannian metric $\bar{g}_{ij}$ and a symmetric 2-tensor $\mathcal{K}_{ij}$ satisfying the constraint equations ((9) and (10)). We will evolve an asymptotically flat initial data set $(\mathcal{H}, \bar{g}_{ij}, \mathcal{K}_{ij})$, that outside a sufficiently large compact set $\mathcal{D}$, $\mathcal{H} \setminus \mathcal{D}$ is diffeomorphic to the complement of a closed ball in $\mathbb{R}^3$ and admits a system of coordinates where $\bar{g}_{ij} \to \delta_{ij}$ and $\mathcal{K}_{ij} \to 0$ sufficiently fast.

It took a long time before the Cauchy problem for the Einstein equations was formulated correctly and understood. Geometry and pde theory were not as developed as they are today, and the pioneers of general relativity had to struggle with problems that have elegant solutions nowadays. The beauty and challenges of general relativity attracted many mathematicians, as for instance D. Hilbert or H. Weyl, to work on general relativity's fundamental questions. Weyl in 1923 talked about a "causally connected" world, which hints at issues that the domain of dependence theorem much later would solve. G. Darmois in the 1920s studied the analytic case, which is not physical but a step in the right direction. He recognized that the analyticity hypothesis is physically unsatisfactory, because it hides the propagation properties of the gravitational field. Without going into details, important work followed by K. Stellmacher, K. Friedrichs, T. de Donder, and C. Lanczos. The latter two introduced wave coordinates, which Darmois later used. In 1939, A. Lichnerowicz extended Darmois' work. He also suggested the extension of the 3 + 1 decomposition with nonzero shift to his student Yvonne Choquet-Bruhat, which she carried out.

Figure 5. Yvonne Choquet-Bruhat proved a local existence and uniqueness theorem for the Einstein equations.

Choquet-Bruhat, encouraged by Jean Leray in 1947, searched for a solution to the nonanalytic Cauchy problem of the Einstein equations, which turned into her famous result of 1952. There are many more players in this game that should be mentioned, but there is not enough space to do justice to their work. These works also built on progress in analysis and pde theory by H. Lewy, J. Hadamard, J. Schauder, and S. Sobolev among many others. Details on the history of the proof can be found in Choquet-Bruhat's survey article published in *Surveys in Differential Geometry 2015: One hundred years of general relativity*, and more historical background (including a discussion between Choquet-Bruhat and Einstein) is given in Choquet-Bruhat's forthcoming autobiography.

In 1952 Choquet-Bruhat [2] proved a local existence and uniqueness theorem for the Einstein equations, and in 1969 Choquet-Bruhat and R. Geroch [3] proved the global existence of a unique maximal future development for every given initial data set.

Theorem 1 (Choquet-Bruhat, 1952). *Let* $(\mathcal{H}, \bar{g}, \mathcal{K})$ *be an initial data set satisfying the vacuum constraint equations. Then there exists a spacetime* (M, g) *satisfying the Einstein vacuum equations with* $\mathcal{H} \hookrightarrow M$ *being a spacelike surface with induced metric* $\bar{g}$ *and second fundamental form* $\mathcal{K}$.

This was proven by finding a useful coordinate system, called wave coordinates, in which Einstein's vacuum equations appear clearly as a hyperbolic system of partial differential equations. The pioneering result by Choquet-Bruhat was improved by Dionne (1962), Fisher-Marsden (1970), and Hughes-Kato-Marsden (1977) using the energy method.

Global Cauchy Problem

Choquet-Bruhat's local theorem of 1952 was a breakthrough and has since been fundamental for further investigations of the Cauchy problem. Once we have local solutions of the Einstein equations, do they exist for all time, or do they form singularities? And of what type would the latter be? In 1969 Choquet-Bruhat and Geroch proved there exists a unique, globally hyperbolic, maximal spacetime (M, g) satisfying the Einstein vacuum equations with $\mathcal{H} \hookrightarrow M$ being a Cauchy surface with induced metric $\bar{g}$ and second fundamental form $\mathcal{K}$. This unique solution is called the *maximal future development* of the initial data set.

However, there is no information about the behavior of the solution. Will singularities occur or will it be complete? One would expect that sufficiently small initial data evolves forever without producing any singularities, whereas sufficiently large data evolves to form spacetime singularities such as black holes. From a mathematical point of view the question is whether theorems can be proven that establish this behavior. A breakthrough occurred in 2008 with Christodoulou's proof, building on an earlier result due to Penrose, that black hole singularities form in the Cauchy development of initial data, which do not contain any singularities, provided that the incoming energy per unit solid angle in each direction in a suitably small time interval is sufficiently large. This means that a black hole forms through the focussing of gravitational waves. This result has since been generalized by various authors, and the main methods have been applied to other nonlinear pdes.

The next burning question to ask is whether there is any asymptotically flat (and nontrivial) initial data with complete maximal development. This can be thought of as a question about the global stability of Minkowski space. In their celebrated work of 1993, D. Christodoulou and S. Klainerman proved the following result, which here we state in a very general way. The details are intricate and the smallness assumptions are stated for weighted Sobolev norms of the geometric quantities.

Theorem 2 (Christodoulou and Klainerman, 1993). *Given strongly asymptotically flat initial data for the Einstein vacuum equations* (5)*, which is sufficiently small, there exists a unique, causally geodesically complete and globally hyperbolic solution* (M, g)*, which itself is globally asymptotically flat.*

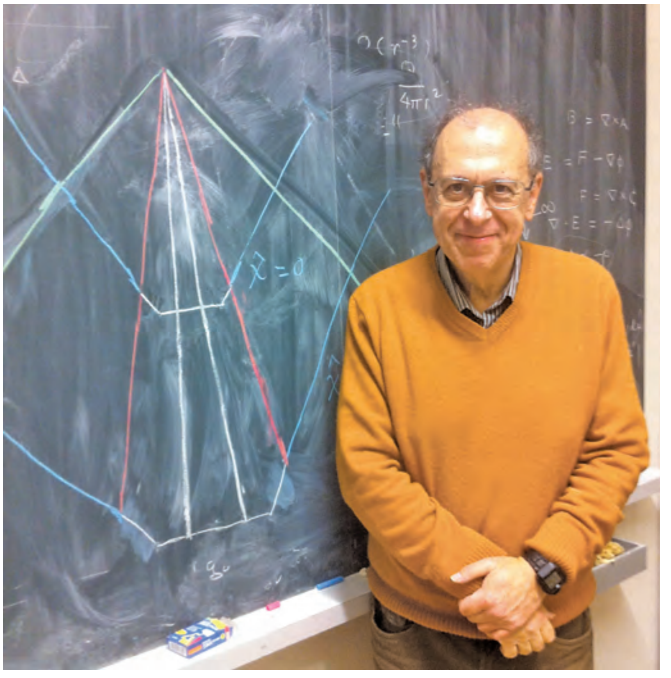

Figure 6. Demetrios Christodoulou proved (with Klainerman) the global nonlinear stability of the Minkowski space in general relativity, and he derived the *null memory effect of gravitational waves*, known as the *Christodoulou effect*.

The proof relies on geometric analysis and is independent of coordinates. First, energies are identified with the help of the Bel-Robinson tensor, which basically is a quadratic of the Weyl curvature. Then, the curvature components are estimated in a comparison argument using the energies. Finally, in a large bootstrap argument with assumptions on the curvature, the remaining geometric quantities are proven to be controlled. The proof comprises various new ideas and features that became important not only for further studies of relativistic problems but also in other nonlinear hyperbolic pdes.

The Christodoulou-Klainerman result of theorem 2 was generalized in 2000 by Nina Zipser for the Einstein-Maxwell equations and in 2007 by Lydia Bieri for the Einstein vacuum equations assuming less on the decay at infinity and less regularity. Thus, the latter result establishes the borderline case for decay of initial data in the Einstein vacuum case. Both works use geometric analysis in a way that is independent of any coordinates.

Next, let us go back to the pioneering results by Choquet-Bruhat and Geroch, and say a few words about

Figure 7. Lydia Bieri generalized the proof of nonlinear stability of Minkowski spacetime in general relativity to borderline decay of the data at infinity, and she has investigated gravitational radiation with memory; among the latter she (with Garfinkle) derived a contribution from neutrino radiation to the null memory effect.

further extensions of these works. A standard result ensures that for an Einstein vacuum initial data set $(\mathcal{H}_0, \bar{g}, \mathcal{K})$ with $\mathcal{H}_0$ allowing to be covered by a locally finite system of coordinate charts with transformations being C^1-diffeomorphisms, and

$$(11) \qquad g_{mn}|_{\mathcal{H}_0} \in H^k_{loc} \ , \quad \partial_0 g_{mn}|_{\mathcal{H}_0} \in H^{k-1}_{loc} \ , \quad k > \frac{5}{2} \, ,$$

there exists a unique globally hyperbolic solution with $\mathcal{H}_0$ being a Cauchy hypersurface. Several improvements followed, including those by Tataru, Smith-Tataru, and then by Klainerman-Rodnianski. The latter proved that for the same problem but with $k > 2$ there exists a time interval $[0, T]$ and a unique solution g such that $g_{mn} \in C^0([0,T], H^k)$ where T depends only on $||g_{mn}|_{\mathcal{H}_0}||_{H^k} + ||\partial_0 g_{mn}|_{\mathcal{H}_0}||_{H^{k-1}}$. Recently the L^2 curvature conjecture was proven by Klainerman-Rodnianski-Szeftel: under certain assumptions they relax the regularity condition such that the time of existence of the solution depends only on the

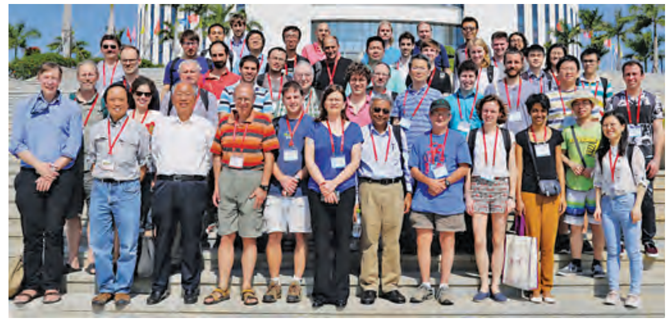

Figure 8. 2016 Conference

L^2-norms of the Riemann curvature tensor and on the gradient of the second fundamental form.

For our purposes, we want to know what the global existence theorem says about the properties of radiation, i.e. the behavior of curvature at large distances. In particular, because we expect gravitational radiation to propagate at the speed of light, we would like to study the behavior at large distances along outgoing light rays. This sort of question was addressed long before Christodoulou and Kainerman. However, these works assume a lot about the spacetimes considered. As a consequence, components of the Riemann curvature tensor show a specific hierarchy of decay in r. The spacetimes of the Christodoulou-Klainerman theorem do not fully satisfy these properties, showing only some of the fall-off but not all. In fact, Christodoulou showed that physical spacetimes cannot fulfill the stronger decay. The results by Christodoulou-Klainerman provide a precise description of null infinity for physically interesting situations.

Gravitational Radiation

In this section, we consider radiative spacetimes with asymptotic structures as derived by Christodoulou-Klainerman. The asymptotic behavior of gravitational waves near infinity approximates how gravitational radiation emanating from a distant black hole merger would appear when observed by aLIGO. Asymptotically the gravitational waves appear to be planar, stretching and shrinking directions perpendicular to the wave's travel direction.

As an example, let us consider the merger of two black holes. Long before the merger, the total energy of the two-black-hole spacetime, the so-called ADM energy or "mass," named for its creators Arnowitt-Deser-Misner, is essentially the sum of the masses of the individual black holes. During the merger, energy and momentum are radiated away in the form of gravitational waves. After the merger, once the waves have propagated away from the system, the energy left in the system, what is known as the Bondi mass, decreases and can be calculated through the formalism introduced by Bondi, Sachs, and Trautman.

Gravitational radiation travels along null hypersurfaces in the spacetime. As the source is very far away from us, we can think of these waves as reaching us (the experiment) at null infinity, which is defined as follows.

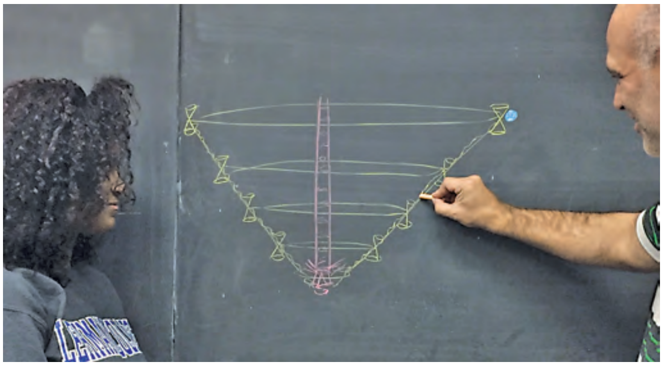

Figure 9. Gravitational waves demonstrated by Luis Anchordoqui of Lehman College. Here the horizon of the merging black holes is depicted in red and then the waves propagate at the speed of light towards Earth located at null infinity.

Definition 1. Future null infinity $\mathcal{I}^+$ is defined to be the endpoints of all future-directed null geodesics along which $r \to \infty$. It has the topology of $\mathbb{R} \times \mathbb{S}^2$ with the function u taking values in $\mathbb{R}$.

A null hypersurface C_u intersects $\mathcal{I}^+$ at infinity in a 2-sphere. To each C_u at null infinity is assigned a Trautman-Bondi mass $M(u)$, as introduced by Bondi, Trautman, and Sachs in the middle of the last century. This quantity measures the amount of mass that remains in an isolated gravitational system at a given retarded time, i.e. the Trautman-Bondi mass measures the remaining mass after radiation through $\mathcal{I}^+$ up to u. The Bondi mass-loss formula reads for $u_1 \leq u_2$

$$M(u_2) = M(u_1) - C \int_{u_1}^{u_2} \int_{S^2} |\Xi|^2 \, d\mu_{\overset{\circ}{\gamma}} \, du \tag{12}$$

with $|\Xi|^2$ being the norm of the shear tensor at $\mathcal{I}^+$ and $d\mu_{\overset{\circ}{\gamma}}$ the canonical measure on S^2. If other fields are present, like electromagnetic fields, then the formula contains a corresponding term for that field. In the situations considered here, it has been proven that $\lim_{u \to -\infty} M(u) = M_{ADM}$.

The effects of gravitational waves on neighboring geodesics are encoded in the Jacobi equation. This very fact is at the heart of the detection by aLIGO and is discussed in the section entitled Gravitational Wave Experiment. From this, we derive a formula for the displacement of test masses, while the wave packet is traveling through the apparatus. This is what was measured by the aLIGO detectors.

Now, there is more to the story. From the analysis of the spacetime at $\mathcal{I}^+$ one can prove that the test masses will go to rest after the gravitational wave has passed, meaning that the geodesics will not be deviated anymore. However, will the test masses be at the "same" position as before the wave train passed or will they be dislocated? In mathematical language, will the spacetime geometry have changed permanently? If so, then this is called the *memory effect* of gravitational waves. This effect was first computed in 1974 by Ya. B. Zel'dovich and A. G. Polnarev in the linearized theory, where it was found to be very small and considered not detectable at that time.

Figure 10. David Garfinkle has worked in many areas of general relativity; lately he has contributed significant results on the memory effect. He showed (with Bieri) that there are two types of memory.

In 1991 D. Christodoulou, studying the full nonlinear problem, showed that this effect is larger than expected and could in principle be measured. Bieri and Garfinkle showed that the formerly called "linear" (now ordinary) and "nonlinear" (now null) memories are two different effects, the former sourced by the difference of a specific component of the Weyl tensor, and the latter due to fields that do reach null infinity $\mathcal{I}^+$. In the case of the Einstein vacuum equations, this is the shear appearing in (12). In particular, the permanent displacement (memory) is related to

$$\mathcal{F} = C \int_{-\infty}^{+\infty} |\Xi(u)|^2 \, du \tag{13}$$

where $\mathcal{F}/4\pi$ denotes the total energy radiated in a given direction per unit solid angle. A very recent paper by P. Lasky, E. Thrane, Y. Levin, J. Blackman, and Y. Chen suggests a method for detecting gravitational wave memory with aLIGO.

Approximation Methods

To compare gravitational wave experimental data to the predictions of the theory, one needs a calculation of the predictions of the theory. It is not enough to know that solutions of the Einstein field equations exist; rather, one needs quantitative solutions of those equations to at least the accuracy needed to compare to experiments. In addition, sometimes the gravitational wave signal is so weak that to keep it from being overwhelmed by noise one must use the technique of matched filtering in which one looks for matches between the signal and a set of templates of possible expected waveforms. These quantitative solutions are provided by a set of overlapping approximation techniques, and by numerical simulations. We will discuss the approximation techniques in this section and the numerical methods in the section entitled Mathematics and Numerics.

Linearized Theory and Gravitational Waves. Since gravitational waves become weaker as they propagate away from their sources, one might hope to neglect the nonlinearities of the Einstein field equations and focus instead on the linearized equations, which are easier to work with. One may hope that these equations would provide an approximate description of the gravitational radiation for much of its propagation and for its interaction with the detector. In linearized gravity, one then writes the spacetime metric as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{14}$$

where $\eta_{\mu\nu}$ is the Minkowski metric as in (1) and $h_{\mu\nu}$ is assumed to be small. One then keeps terms in the Einstein field equations only to linear order in $h_{\mu\nu}$. The coordinate invariance of general relativity gives rise to what is called gauge invariance in linearized gravity. In particular, consider any quantity F written as $F = \bar{F} + \delta F$ where $\bar{F}$ is the value of the quantity in the background and δF is the first order perturbation of that quantity. Then for an infinitesimal diffeomorphism along the vector field ξ, the quantity δF changes by

$$\delta F \to \delta F + \mathcal{L}_\xi \bar{F},$$

where recall that $\mathcal{L}$ stands for the Lie derivative. Recall also that harmonic coordinates made the Einstein vacuum equations look like the wave equation in (6). We would like to do something similar in linearized gravity. To this end we choose ξ to impose the Lorenz gauge condition (not Lorentz!)

$$\partial_\mu \bar{h}^{\mu\nu} = 0,$$

where

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - (1/2)\eta_{\mu\nu} h.$$

The linearized Einstein field equations then become

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{15}$$

where $\Box$ is the wave operator in Minkowski spacetime.

In a vacuum one can use the remaining freedom to choose ξ to impose the conditions that $h_{\mu\nu}$ has only spatial components and is trace-free, while remaining in Lorenz gauge. This refinement of the Lorenz gauge is called the TT gauge, since it guarantees that the only two propagating degrees of freedom of the metric perturbation are transverse $\partial^i h_{ij} = 0$ and (spatially) traceless $\eta^{ij} h_{ij} = 0$. The metric in TT gauge has a direct physical interpretation given by the following formula for the linearized Riemannian curvature tensor

$$R_{itjt} = -\frac{1}{2}\ddot{h}^{TT}_{ij}, \tag{16}$$

which sources the geodesic deviation equation, and thus encapsulates how matter behaves in the presence of gravitational waves. Combining Eq. (16) and the Jacobi equation, one can compute the change in distance between two test masses in free fall:

$$\triangle d^i(t) = \frac{1}{2} h^{TT}_{ij}(t) d^j_0$$

where d^j_0 is the initial distance between the test masses.

The TT nature of gravitational wave perturbations allows us to immediately infer that they only have two polarizations. Consider a wave traveling along the z-direction, such that $h^{TT}_{ij}(t-z)$ is a solution of $\Box h^{TT}_{ij} = 0$. The Lorenz condition, the assumption that the metric perturbation vanishes for large r, the trace-free condition, and symmetries imply that there are only two independent propagating degrees of freedom:

$$h_+(t-z) = h^{TT}_{xx} = -h^{TT}_{yy}$$

and

$$h_\times(t-z) = h^{TT}_{xy} = h^{TT}_{yx}.$$

The h_+ gravitational wave stretches the x direction in space while it squeezes the y direction, and vice-versa. The interferometer used to detect gravitational waves has two long perpendicular arms that measure this distortion. Therefore, one must approximate these displacements in order to predict what the interferometer will see under various scenarios.

The Post-Newtonian Approximation

The post-Newtonian (PN) approximation for gravitational waves extends the linearized study presented above to higher orders in the metric perturbation, while also assuming that the bodies generating the gravitational field move slowly compared to the speed of light. The PN approach was developed by Einstein, Infeld, Hoffman, Damour, Deruelle, Blanchet, Will, Schaefer, and many others. In the harmonic gauge $\partial_\alpha(\sqrt{-g} g^{\alpha\beta}) = 0$ commonly employed in PN theory, the expanded equations take the form

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4}\tau^{\alpha\beta}, \tag{17}$$

where $\Box$ is the wave operator and

$$\tau^{\alpha\beta} = -(g) T^{\alpha\beta} + (16\pi)^{-1} N^{\alpha\beta},$$

with $N^{\alpha\beta}$ composed of quadratic forms of the metric perturbation.

These expanded equations can then be solved order by order in the perturbation through Green function methods, where the integral is over the past lightcone of Minkowski space for $x \in M$. When working at sufficiently high PN order, the resulting integrals can be formally divergent, but these pathologies can be bypassed or cured through asymptotic matching methods (as in Will's method of the direct integration of the relaxed Einstein equations) or through regularization techniques (as in Blanchet and Damour's Hadamard and dimensional regularization approach). All approaches to cure these pathologies have been shown to lead to exactly the same end result for the metric perturbation.

The metric perturbation is solved for order by order, where at each order one uses the previously calculated information in the expression for $N^{\alpha\beta}$ and also to find the motion of the matter sources, thus leading to an improved expression for $T^{\alpha\beta}$ at each order. In particular, the emission of gravitational waves by a binary system causes a change in the period of that system, and this change was used by Hulse and Taylor to *indirectly* detect gravitational waves through their observations of the binary pulsar. In this way, the PN iterative procedure provides a perturbative approximation to the solution to the Einstein equations to a given order in the feebleness of the gravitational interaction and the speed of the bodies.

Little work has gone into studying the mathematical properties of the resulting perturbative series. Clearly, the PN approximation should not be valid when the speed of the bodies becomes comparable to the speed of light or when the objects described are black holes or neutron stars with significant self-gravity. Damour, however, has shown that the latter can still be described by the PN approximation up to a given order in perturbation theory. Moreover, recent numerical simulations of the merger of binary black holes and neutron stars have shown that the PN approximation is accurate even quite late in the inspiral, when the objects are moving at close to a third of the speed of light.

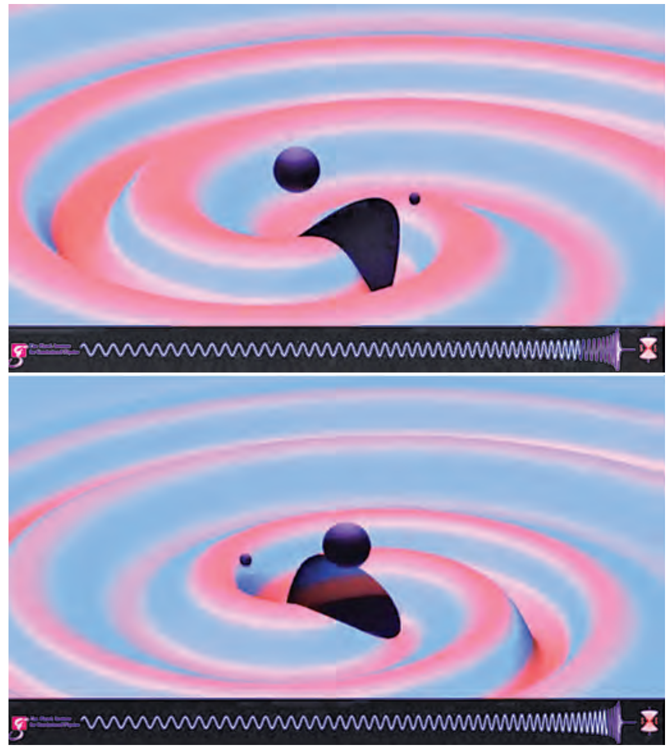

Figure 11. Numerical simulation of the late inspiral of an unequal-mass, black hole binary. Even late in the inspiral, the PN approximation for gravitational waves remains accurate.

Resummations of the PN Approximation

The accuracy of the approximate solutions can be improved by applying resummation techniques: the rewriting of the perturbative expansion in a new form (e.g. a Chebyshev decomposition or a Padé series) that makes use of some physical feature one knows should be present in the exact solution. For example, one may know (through symmetry arguments or by taking certain limits) that some exact result contains a first-order pole at a certain spacetime position, so one could rewrite the approximate solution as a Padé approximant that makes this pole explicit.

A particular resummation of the PN approximation that has been highly successful at approximating numerical solutions is the *effective one-body approach*. Recall that in Newtonian gravity, the motion of masses m_1 and m_2 under their mutual gravitational attraction is mathematically equivalent to the motion of a single mass μ in the gravitational field of a stationary mass M, where $M = m_1 + m_2$ and $\mu = (m_1 m_2)/M$. The effective one body approach similarly attempts to recast the motion of two black holes under their mutual gravitational attraction as the motion of a single object in a given spacetime metric.

More precisely, one recasts the two-body problem onto the problem of an effective body that moves on an effective external metric through an energy map and a canonical transformation. The dynamics of the effective body are then described through a (conservative) improved Hamiltonian and a (dissipative) improved radiation-reaction force. The improved Hamiltonian is resummed through two sets of square-roots of PN series, in such a way so as to reproduce the standard PN Hamiltonian when Taylor expanded about weak-field and slow-velocities. The improved radiation-reaction force is constructed from quadratic first-derivatives of the gravitational waves, which in turn are product-resummed using the Hamiltonian (from knowledge of the extreme mass-ratio limit of the PN expansion) and a field-theory resummation of certain tail-effects.

Once the two-body problem has been reformulated, the Hamilton equations associated with the improved Hamiltonian and radiation-reaction force are solved numerically, a significantly easier problem than solving the full Einstein equations. This resummation, however, is not enough because the improved Hamiltonian and radiation-reaction force are built from finite PN expansions. The very late inspiral behavior of the solution can be corrected by adding calibration coefficients (consistent with PN terms not yet calculated) to the Hamiltonian and the radiation-reaction force, which are then determined by fitting to a set of full, numerical relativity simulations.

The calibrated effective-one-body waveforms described above are incredibly accurate representations of the

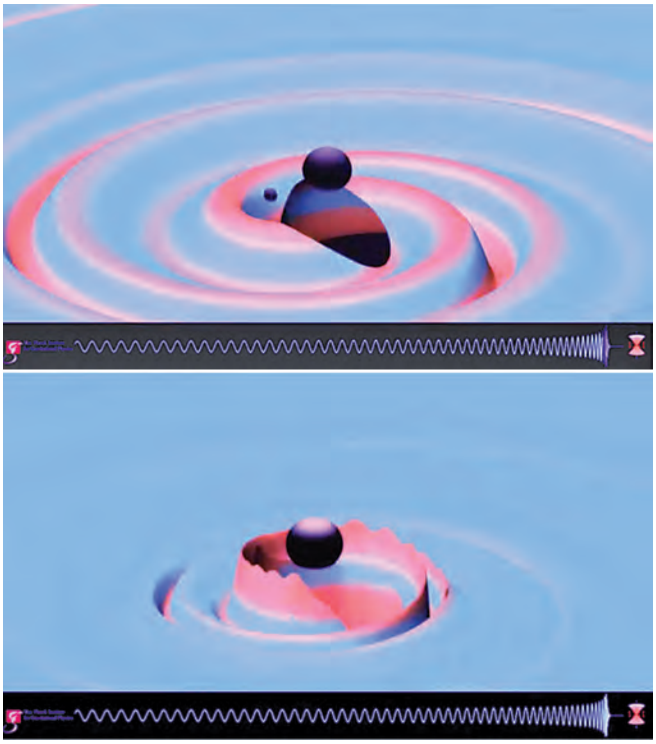

Figure 12. Numerical simulation of the merger of an unequal-mass, black hole binary. The effective-one-body approximation remains highly accurate almost up to the moment when the black holes horizons touch, when full numerical simulations are required.

gravitational waves emitted in the inspiral of compact objects, up to the moment when the black holes merge. They become accurate after the merger by adding on information from black hole perturbation theory that we describe next.

Perturbations About a Black Hole Background

After the black holes merge, they form a single distorted black hole that sheds its distortions by emitting gravitational waves and eventually settling down to a Kerr black hole. This "ringdown" phase is described using perturbation theory with the Einstein vacuum equations linearized around a Kerr black hole background. Teukolsky showed how to obtain a wave type equation for these perturbed Weyl tensor components from the Einstein vacuum equations. The result of the Teukolsky method is that the distortions can be expanded in modes, each of which has a characteristic frequency and exponential decay time. The ringdown is well approximated by the most slowly decaying of these modes. This ringdown waveform can be stitched to the effective-one-body inspiral waveforms to obtain a complete description of the gravitational waves emitted in the coalescence of black holes.

Mathematics and Numerics

In numerical relativity, one creates simulations of the Einstein field equations using a computer. This is needed when no other method will work, in particular when gravity is very strong and highly dynamical (as it is when two black holes merge).

The Einstein field equations, like most of the equations of physics, are differential equations, and the most straightforward of the techniques for simulating differential equations are finite difference equations. In the one-dimensional setting, one approximates a function $f(x)$ by its values on equally spaced points

$$f_i = f(i\delta) \text{ for } i \in \mathbb{N}.$$

One then approximates derivatives of f using differences

$$f' \approx (f_{i+1} - f_{i-1})/(2\delta)$$

and

$$f''(i) \approx (f_{i+1} + f_{i-1} - 2f_i)/\delta^2.$$

For any pde with an initial value formulation one replaces the fields by their values on a spacetime lattice, and the field equations by finite difference equations that determine the fields at time step $n+1$ from their values at time step n. Thus the Einstein vacuum equations are written as difference equations where the step 0 information is the initial data set.

One then writes a computer program that implements this determination and runs the program. Sounds simple, right? So what could go wrong? Quite a lot, actually. It is best to think of the solution of the finite difference equation as something that is supposed to converge to a solution of the differential equation in the limit as the step size δ between the lattice points goes to zero. But it is entirely possible that the solution does not converge to anything at all in this limit. In particular, the coordinate invariance of general relativity allows one to express the Einstein field equations in many different forms, some of which are not strongly hyperbolic. Computer simulations of these forms of the Einstein field equations generally do not converge.

Another problem has to do with the constraint equations. Recall that initial data have to satisfy constraint equations. It is a consequence of the theorem of Choquet-Bruhat that if the initial data satisfy those constraints then the results of evolving those initial data continue to satisfy the constraints. However, in a computer simulation the initial data only satisfy the finite difference version of the constraints and therefore have a small amount of constraint violation. The field equations say that data with zero constraint violation evolve to data with zero constraint violation. But that still leaves open the possibility (usually realized in practice) that data with small constraint violation evolve in such a way that the constraint violation grows rapidly (perhaps even exponentially) and thus destroys the accuracy of the simulation.

Finally there is the problem that these simulations deal with black holes, which contain spacetime singularities. A computer simulation cannot be continued past a time where a slice of constant time encounters a spacetime singularity. Thus either the simulations must only be run

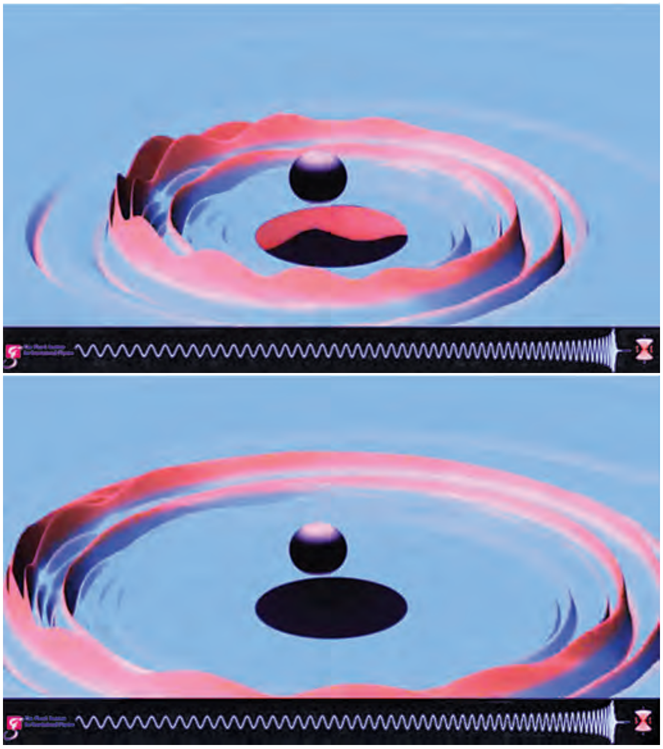

Figure 13. Numerical simulation of the ringdown after the merger of an unequal-mass, black hole binary. In the ringdown phase, perturbation theory provides an excellent approximation to the waveform.

for a short amount of time, or the time slices inside the black hole must somehow be "slowed down" so that they do not encounter the singularity. But then if the time slice advances slowly inside the black hole and rapidly outside it, this will lead to the slice being stretched in such a way as to lead to inaccuracies in the finite difference approximation.

Before 2005 these three difficulties were insurmountable, and none of the computer simulations of colliding black holes gave anything that could be used to compare with observations. Then suddenly in 2005 all of these problems were solved by Frans Pretorius, who produced the first fully successful binary black hole simulation. Then later that year the problem was solved again (using completely different methods!) by two other groups: one consisting of Campanelli, Lousto, Marronetti, and Zlochower and the other of Baker, Centrella, Choi, Koppitz, and van Meter. Though the methods are different, both sets of solutions can be thought of as consisting of the ingredients *hyperbolicity, constraint damping*, and *excision*, and we will treat each one in turn.

Hyperbolicity. Since one needs the equations to be strongly hyperbolic, one could perform the simulations in harmonic coordinates. However, one also needs the time coordinate to remain timelike, so instead Pretorius used generalized harmonic coordinates (as first suggested by Friedrich) where the coordinates satisfy a wave equation with a source. The other groups implemented hyperbolicity by using the BSSN equations (named for its inventors: Baumgarte, Shapiro, Shibata, and Nakamura). These equations decompose the spatial metric into a conformal factor and a metric of unit determinant and then evolve each of these quantities separately, adding appropriate amounts of the constraint equations to convert the spatial Ricci tensor into an elliptic operator.

Constraint damping. Because the constraints are zero in exact solutions to the theory, one has the freedom to add any multiples of the constraints to the right-hand side of the field equations without changing the class of solutions to the field equations. In particular, with clever choices of which multiples of the constraints go on the right-hand side, one can arrange that in these new versions of the field equations small violations of the constraints get smaller under evolution rather than growing. Carsten Gundlach showed how to do this for evolution using harmonic coordinates, and his method was implemented by Pretorius. The BSSN equations already have some rearrangement of the constraint and evolution equations. The particular choice of lapse and shift (Φ and X from eqns. (7–8)) used by the other groups (called 1+log slicing and Gamma driver shift) were found to have good constraint damping properties.

Excision. Because nothing can escape from a black hole, nothing that happens inside can have any influence on anything that happens outside. Thus in performing computer simulations of colliding black holes, one is allowed to simply excise the black hole interior from the computational grid and still obtain the answer to the question of what happens outside the black holes. By excising, one no longer has to worry about singularities or grid stretching. Excision was first proposed by Unruh and Thornburg, and first implemented by Seidel and Suen, and used in Pretorius' simulations. The other groups essentially achieve excision by other methods. They use a "moving puncture method" that involves a second asymptotically flat end inside each black hole, which is compactified to a single point that can move around the computational grid. The region between the puncture and the black hole event horizon undergoes enormous grid stretching, so that effectively only the exterior of the black hole is covered by the numerical grid.

Since 2005, many simulations of binary black hole mergers have been performed, for various black hole masses and spins. Some of the most efficient simulations are done by the SXS collaboration using spectral methods instead of finite difference methods. (SXS stands for "Simulating eXtreme Spacetimes" and the collaboration is based at Cornell, Caltech, and elsewhere.) Spectral methods use the grid values f_i to approximate the function $f(x)$ as an expansion in a particular basis of orthogonal functions. The expansion coefficients and the derivatives of the basis functions are then used to compute the derivatives of $f(x)$. Compared to finite difference methods, spectral methods can achieve a given accuracy of the derivatives with significantly fewer grid points.

Gravitational Wave Experiment

The experimental search for gravitational waves started in the 1960s through the construction of *resonant bar detectors*. These essentially consist of a large (meter-size) cylinder in a vacuum chamber that is isolated from vibrations. When a gravitational wave at the right frequency interacts with such a bar, it can excite the latter's resonant mode, producing a change in length that one can search for. In 1968, Joseph Weber announced that he had detected gravitational waves with one such resonant bar. The sensitivity of Weber's resonant bar to gravitational waves was not high enough for this to be possible, and other groups could not reproduce his experiment.

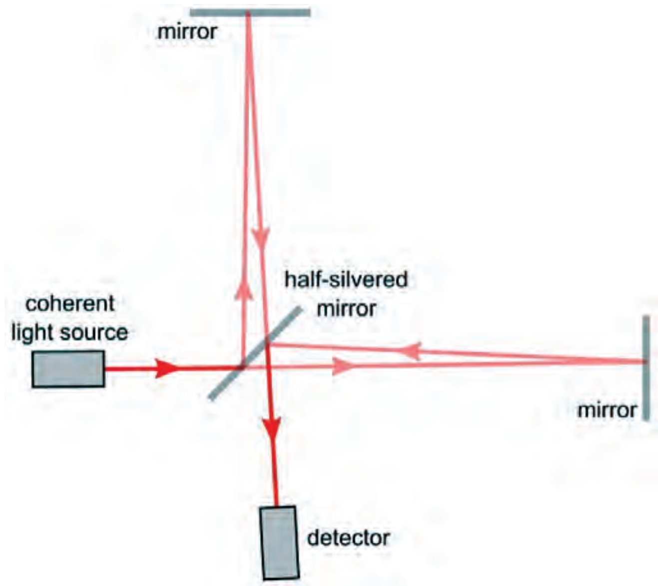

Figure 14. Schematic diagram of a Michelson interferometer, which is at the heart of the instrumental design used by aLIGO. In the diagram, a laser beam is split into two sub-beams that travel down orthogonal arms, bounce off mirrors, and then return to recombine.

In the late 1960s and early 1970s, the search for gravitational waves with laser interferometers began through the pioneering work of Rainer Weiss at MIT and Kip Thorne and Ronald Drever at Caltech, among many others. The basic idea behind interferometry is to split a laser beam into two sub-beams that travel down orthogonal arms, bounce off mirrors, and then return to recombine. If the light travel time is the same in each sub-beam, then the light recombines constructively, but if a gravitational wave goes through the detector, then the light travel time is not the same in each arm and interference occurs. Gravitational wave interferometers are devices that use this interference process to measure small changes in light travel time very accurately so as to learn about the gravitational waves that produced them, and thus, in turn, about the properties of the source of gravitational waves.

The initial Laser Interferometer Gravitational-Wave Observatory (iLIGO) was funded by the National Science

Figure 15. (from top to bottom) Ronald Drever, Kip Thorne, and Rainer Weiss pioneered the effort to detect gravitational waves with laser interferometers.

Foundation in the early 1990s and operations started in the early 2000s. There are actually two LIGO facilities (one in Hanford, Washington, and one in Livingston, Louisiana) in operation right now, with an Italian counterpart (Virgo) coming online soon, a Japanese counterpart (KAGRA) coming online by the end of the decade, and an Indian counterpart (LIGO-India) coming online in the 2020s. The reason for multiple detectors is to achieve redundancy and increase the confidence of a detection by observing the signal by independent detectors with uncorrelated noise. Although iLIGO was over four orders of magnitude more sensitive than Weber's original instrument in a wide frequency band, no gravitational waves were detected.

Figure 16. One of the two aLIGO facilities, this one in Livingston, Louisiana, where the interference pattern associated with a gravitational wave produced in the merger of two black holes was recorded within days of the first science run.

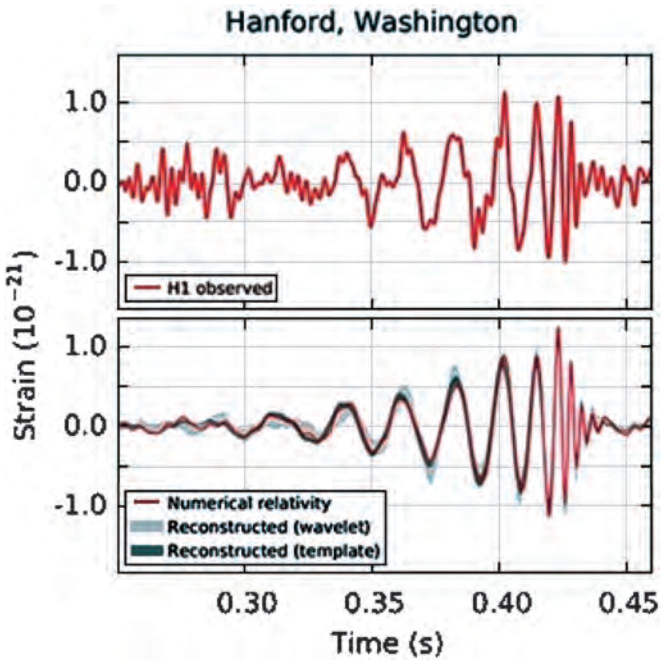

Figure 17. Top: Filtered GW strain as a function of time detected at the Hanford location of aLIGO. Bottom: Best fit reconstruction of the signal using a numerical relativity simulation (red), an analytical waveform template (gray), and a set of Morley wavelets. The latter two are shown as 90 percent confidence regions, while the simulation is a particular run with a choice of parameters within this the 90 percent confidence region.

In the late 2000s, upgrades to convert iLIGO into advanced LIGO (aLIGO) commenced. These upgrades included an increase in the laser power to reduce quantum noise, larger and heavier mirrors to reduce thermal and radiation pressure noise, better suspension fibers for the mirrors to reduce suspension thermal noise, among many other improvements. aLIGO commenced science operations in 2015 with a sensitivity roughly 3–4 times greater than that of iLIGO's last science run.

Within days of the first science run, the aLIGO detectors recorded the interference pattern associated with a gravitational wave produced in the merger of two black holes 1.3 billion light years away. The signal was so loud (relative to the level of the noise) that the probability that the recorded event was a gravitational wave was much larger than 5σ, meaning that the probability of a false alarm was much smaller than 10^{-7}. There is no doubt that this event, recorded on September 14, 2015, as well as a second one, detected the day after Christmas of that same year, were the first direct detections of gravitational waves.

In order to understand how gravitational waves are detected, we must understand how the waves affect the motion of the parts of the interferometer. The mirrors are suspended from wires like a pendulum, but this means that for short time motion in the horizontal direction, the motion of each mirror can be treated as a spacetime geodesic. But the interferometer measures distance between the mirrors, so what we want to know is how this distance changes under the influence of a gravitational wave. The answer to this question comes from the Jacobi equation: the relative acceleration of nearby geodesics is equal to the Riemann tensor times the separation of those geodesics.

Thus if at any time we want to know the separation, we need to integrate the Jacobi equation twice with respect to time. However, the Riemann tensor is the second derivative of the TT gauge metric perturbation. Thus, by using this particular gauge we can say that LIGO directly measures the metric perturbation by using laser interferometry to keep track of the separation of its mirrors.

Listen to the universe with gravity.

Astrophysics and Fundamental Physics

Up until now, we have created a picture of the universe from the information we have obtained from amazing telescopes, such as Chandra in the X-rays, Hubble in the optical, Spitzer in the infrared, WMAP in the microwave, and Arecibo in the radio frequencies. This information was provided by light that traveled from astrophysical sources to Earth. Every time humankind built a new telescope that gave us access to a new frequency range of the light spectrum, amazing discoveries were made. The discovery of accretion disk signatures of black holes using X-ray astronomy is a case in point. This expectation is especially true for gravitational wave detectors, which do not just open a new frequency range, but rather aim to *listen to the universe* in an entirely new way: with gravity instead of light.

This new type of astrophysics has an immense potential to truly revolutionize science because gravitational waves can provide very clean information about their sources. Unlike light, gravitational waves are very weakly coupled to matter, allowing gravitational waves to go right through the intermediate matter (which would absorb light) and provide a clean picture (or soundtrack) of astrophysical sources that until now had remained obscure. Of course, this is a double-edged sword because the detection of gravitational waves is extremely challenging, requiring the ability to measure distances that are as small as 10^{-3} times the size of a proton over a 4 km baseline.

The aLIGO detectors achieved just that, providing humanity with not only the first direct detection of gravitational waves, but also the first direct evidence of the existence of black hole binaries and their coalescence. As of the writing of this article, aLIGO had detected two events, both of which correspond to the coalescence of binary black hole systems in a quasi-circular orbit. Fitting the hybrid analytic and numerical models described in the sections "Approximation Methods" and "Mathematics and Numerics" to the data, the aLIGO collaboration found that the first event consisted of two black holes with masses

$$(m_1, m_2) \approx (36.2, 29.1)M_\odot,$$

where $M_\odot$ is the mass of our sun, colliding at roughly half the speed of light to produce a remnant black hole with mass

$$m_f \approx 62.3M_\odot$$

and dimensionless spin angular momentum

$$|\vec{S}|/m_f^2 \approx 0.68,$$

located 420 mega-parsecs away from Earth (roughly 1.3 billion times the distance light travels in one year). The second event consisted of lighter black holes, with masses

$$(m_1, m_2) \approx (14.2, 7.5)M_\odot$$

that collided to produce a remnant black hole with mass

$$m_f \approx 20.8M_\odot$$

and dimensionless spin angular momentum

$$|\vec{S}|/m_f^2 \approx 0.74,$$

located 440 mega-parsecs away from Earth. In both cases, the peak luminosity radiated was in the range of 10^{56} ergs/s with the systems effectively losing $3M_\odot$ and $1M_\odot$ respectively in less than 0.1 seconds. Thus, for a very brief moment, these events produced more energy than all of the stars in the observable universe put together.

Perhaps one of the most interesting inferences one can draw from such events is that black holes (or at the very least, objects that look and "smell" a lot like black holes) truly do form binaries and truly do merge in nature within an amount of time smaller than the age of the universe. Until now, we had inferred the existence of black holes by either observing how other stars orbit around supermassive ones at the center of galaxies or by observing enormous disks of gas orbit around stellar mass black holes and the X-rays emitted as some of that gas falls into the black hole. The aLIGO observations are the first direct observation of radiation produced by binary black holes themselves through the wave-like excitations of the curvature they generate when they collide. Not only did the aLIGO observation prove the existence of binary black holes, but even the first observation brought about a surprise: the existence and merger of black holes in a mass range that had never been observed before.

Figure 18. Nicolás Yunes and his team explore mathematically the extreme gravity of black holes and neutron stars, as well as the gravitational waves they emit when they inspiral into each other and collide. The goal of his research program is to construct analytic models that enable the extraction of the most astrophysics and theoretical physics information from future astrophysics and gravitational wave observations, thus allowing us to test Einstein's theory in the essentially unexplored extreme gravity regime.

The aLIGO observations have demonstrated that general relativity is not only highly accurate at describing gravitational phenomena in the solar system, in binary pulsar observations, and in cosmological observations, but also in the late inspiral, merger, and ringdown of black hole binaries. Gravity is truly described by Einstein's theory even in the most *extreme gravity* scenarios: when the gravitational interaction is strong, highly nonlinear, and highly dynamical. Such consistency with Einstein's theory has important consequences on theories that modify gravity in hopes of arriving at a quantum gravitational completion. Future gravitational wave observations will allow us to verify many other pillars of Einstein's theory, such as that the gravitational interaction is parity invariant, that gravitational waves propagate at the speed of light, and that it only possesses two transverse polarizations.

The detection of gravitational waves is not only a spectacular confirmation of Einstein's theory, but also the beginning of a new era in astrophysics. Gravitational waves will provide the *soundtrack* to the movie of our universe, a soundtrack we had so far been missing with telescopes. No doubt that they will be a rich source for new questions and inspiration in physics as well as mathematics. We wait anxiously for the unexpected beauty this music will provide.

Acknowledgments

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