Course Notes Fall 1996

DEFORMATION THEORY

Sections

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

References:

General deformation theory:
M. Gerstenhaber and S. D. Schack, *Algebraic cohomology and deformation theory*, Deformation theory of algebras and structures and applications, Kluwer, Dordrecht (1988) 11-264.

Deformation of commutative algebras:
M. Artin, *Deformations of singularities*, Tata Institute of Fundamental Research, Bombay 1976.

1. Heuristic computation of first order deformations of an affine curve.

Throughout, k will denote an algebraically closed field of characteristic zero. An affine algebraic curve C_0 is the locus of zeros of a polynomial

$$f(x,y) = \sum_{i,j} a_{ij} x^i y^j. \tag{1.1}$$

It corresponds naturally to the ring $A = k[x,y]/(f)$. If $\alpha_{ij}(t)$ are differentiable functions of t and if $a_{ij} = \alpha_{ij}(0)$, then we obtain a family of plane curves, defined by the family of polynomials $\phi(x,y) = \sum \alpha_{ij} x^i y^j$. The derivative $\frac{\partial\phi}{\partial t} = \sum \frac{d\alpha_{ij}}{dt} x^i y^j$, evaluated at $t = 0$, defines the first order deformation of f.

To study deformations formally, we work with coefficients in the ring $R = k[t]/(t^2)$. A polynomial in $R[x,y]$ can be written in the form $\phi(x,y) = f(x,y) + g(x,y)t = \sum (a_{ij} + b_{ij}t) x^i y^j$, where $f = \sum a_{ij} x^i y^j$ and $g = \sum b_{ij} x^i y^j$ are in $k[x,y]$. The derivative $\frac{\partial\phi}{\partial t}$ at $t = 0$ is defined to be the polynomial g.

We think of the equation $\phi(x,y) = 0$ as defining an infinitesimal deformation C_1 of the curve C_0. The geometry implicit in this intuitive idea needs to be worked out, but on the level of rings there is no problem: The ring corresponding to C_0 is $A_0 = k[x,y]/(f)$, and we can view the ring $A_1 = R[x,y]/(\phi)$ as its "deformation".

There are two computations we want to make:

(1.2a) Classify the quotients of $R[x,y]$ which are first order deformations of A_0. These are called the *embedded* deformations, because they correspond to infinitesimal families of curves embedded in a plane with fixed coordinates x, y.

(1.2b) Classify the deformations A_0 as algebra, when the generators x, y are not fixed.

It has turned out that deformation theory works best if one fixes all data at "time" $t = 0$ completely. We will do this, for the time being heuristically.

(a) The polynomial $f(x,y)$ is a generator for the ideal $\mathfrak{a}_0 = ker(k[x,y] \longrightarrow A_0)$. Going on the principle that we fix data at $t = 0$, we choose as generator for the ideal $\mathfrak{a}_1$ such that $R[x,y]/\mathfrak{a}_1 = A_1$ a perturbation of the polynomial f, of the form $\phi = f + gt$. (Why $\mathfrak{a}_1$ should be a principal ideal will be discussed later.) A second perturbation $\phi' = f + g't$ generates the same ideal $\mathfrak{a}_1$ if and only if $\phi' = \phi u$, where u is a unit in $R[x,y]$. The units in this ring are the elements $c+ht$, where $c \in k$ is a nonzero constant and h is an arbitrary polynomial in $k[x,y]$. Setting $u = c+ht$, we find $f + g't = fc + (gc+fh)t$. Therefore $c = 1$, and $g' = g + fh$. Note that since h can be arbitrary, fh can be any element of the ideal $\mathfrak{a}_0$. What matters for the deformation is the residue of g, modulo $\mathfrak{a}_0$. Answer:

(1.3) $\{$1st order embedded deformations$\} \approx \{$elements of $A_0\}$.

(b) To classify deformations as algebra, i.e., without chosen coordinates, we must study the effect of a change of coordinates on the defining polynomial ϕ. Again, because we fix everything at time $t = 0$, we will allow only infinitesimal changes of coordinates, of the form $x \longrightarrow x + ut$, $y \longrightarrow y + vt$, where $u, v \in k[x,y]$ are arbitrary. Then

$$(1.4) \qquad f(x+ut, y+vt) + g(x+ut, y+vt)t = f(x,y) + (\frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v)t + g(x,y)t.$$

Thus g can be changed by adding any element of the ideal in $k[x,y]$ generated by the three elements $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$. Let $S = k[x,y]/(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$. Then

(1.5) $\{$1st order deformations$\} \approx \{$elements of $S\}$.

Example 1.6. An algebraic curve $C_0 : f = 0$ is called *smooth* if the partials $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ don't vanish at any point of C_0, i.e., if $f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ have no common zeros. Then the Nullstellensatz tells us that these elements generate the unit ideal in $k[x,y]$, so $S = 0$.

Corollary 1.7. *Every first order deformation of a smooth plane curve is equivalent, via a change of coordinates, to the trivial deformation.*

This may seem strange, because most curves are smooth, and there are families of smooth curves whose intrinsic structure varies. To capture this variation infinitesimally, one has to use projective geometry.

Example 1.8. Let C_0 be the curve $xy = 0$, the union of the two axes. So $f(x,y) = xy$. Then $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, and $S \approx k$. We represent the basis 1 for k by $1 \in k[x,y]$. Every first order deformation A_1 is isomorphic to one defined by a polynomial of the form $xy + ct$, where $c \in k$, and c is uniquely determined by the deformation.

Example 1.9. $f = y^2 - x^3$, and C_0 is a curve with a cusp at the origin. Then $S \approx k[x,y]/(x^2, y)$. So every first order deformation is defined by a unique polynomial of the form $y^2 - x^3 + (a + bx)t$, with $a, b \in k$.

2. Interlude: Algebras A finite dimensional over an algebraically closed field k.

Let A be a finite dimensional k-algebra. The radical J of A is a nilpotent a ideal, and A/J is semisimple. By Wedderburn's theorem, A/J is is a sum of matrix algebras.

Case 1: $\dim A = 1$. Then $A = k$.

Case 2: $\dim A = 2$. If $J = 0$, then A is semisimple, and its dimension is too small to have an $r \times r$ matrix algebra, with $r > 1$, as summand. So $A = k \oplus k$. Otherwise $\dim J = 1$.

If so, then because J is nilpotent, $J^2 = 0$. Taking for x a basis of J, one finds that $A \approx k[x]/(x^2)$.

Case 3: $\dim A = 3$. If $J = 0$, then $A \approx k \oplus k \oplus k$. If $\dim J = 1$, then $A/J = k \oplus k$. Let e_i, $i = 1, 2$, denote the idempotent which projects onto the i-th factor of this sum. We use the Peirce decomposition $A = \sum_{i,j} e_i A e_j$, or in matrix form

$$A = \begin{pmatrix} e_1Ae_1 & e_1Ae_2 \\ e_2Ae_1 & e_2Ae_2 \end{pmatrix}. \tag{2.1}$$

Since $\dim e_i A e_i \geq 1$, at least one of the off-diagonal terms is zero. If $\dim e_1 A e_1 = \dim e_2 A e_2 = 1$, then

$$A \approx \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}. \tag{2.2}$$

The aring of lower triangular matrices is isomorphic. Otherwise, if say $\dim e_1 A e_1 = 2$, and $\dim e_2 A E_2 = 1$, then both off-diagonal terms are zero, and $A \approx k[x]/(x^2) \oplus k$.

The final case is that $\dim J = 2$ and $A/J \approx k$. Then $\dim J^2 \leq 1$. If $\dim J^2 = 1$, we may choose $x \in J$, $x \notin J^2$. Then $A \approx k[x]/(x^3)$. Otherwise, if $J^2 = 0$, then we choose a basis $\{x, y\}$ for J, and find that $A \approx k[x, y]/(x^2, xy, y^2)$.

(2.3) **k-algebras of dimension 3.**

One noncommutative algebra: upper triangular 2×2 matrices.

Four commutative algebras: $k \oplus k \oplus k$, $k[x]/(x^2) \oplus k$, $k[x]/(x^3)$, $k[x, y]/(x^2, xy, y^2)$.

Reference: Benjamin Peirce, *Linear Associative Algebras*, circa 1870.

Exercises: 1. Determine whether or not the ring of upper triangular matrices has a nontrivial first order deformation.

2. Classify algebras of dimension 4 over k.

3. A second heuristic computation of first order deformations.

We start with the commutative ring $A_0 = k[x, y]/(x^2, xy, y^2)$, and we look for commutative deformations. Let $\mathfrak{a}_0$ denote the ideal (x^2, xy, y^2) of $k[x, y]$, and let $R = k[t]/(t^2)$. As before, we try a perturbation of the defining equations, of the form

(3.1)

$$f_1 = x^2 + \alpha t,$$
$$f_2 = xy + \beta t,$$
$$f_3 = y^2 + \gamma t, \quad \text{where } \alpha, \beta, \gamma \in k[x, y].$$

These elements generate an ideal $\mathfrak{a}_1$ in $R[x, y]$ with the property that setting $t = 0$ (which is the tensor product $\otimes_R k$) gives back the defining ideal $\mathfrak{a}_0$ of the ring $A_0 = k[x, y]/\mathfrak{a}_0$. Our hypothetical first order deformation of A_0 is $A_1 = R[x, y]/\mathfrak{a}_1$.

First note $f_1 t = x^2 t$, and that adding a multiple of $f_1 t$ to any f_i does not change the ideal $\mathfrak{a}_1$. Similarly for the other generators. This means that we can add to α, β, γ arbitrary elements of $\mathfrak{a}_0$ without changing $\mathfrak{a}_1$: It is the residues of α, β, γ in A_0 which determine the deformation. We can represent these residues canonically by linear polynomials. Doing so reduces us to the case that

$$\begin{aligned}
&\alpha = a_0 + a_1 x + a_2 y, \\
&\beta = b_0 + b_1 x + b_2 y, \\
&\gamma = c_0 + c_1 + c_2 y, \quad \text{with } a_i, b_i, c_i \in k.
\end{aligned}$$

However, these coefficients can not be chosen arbitrarily. To see why, note that the kernel N of the map $A_1 \longrightarrow A_0 = A_1 \otimes_R k$ is the principal ideal $A_1 t$, and because $t^2 = 0$, multiplication by t annihilates N. So N is the image of the A_1-linear map $A_0 \xrightarrow{t} A_1$. The ideal N is what we think of as the "infinitesimal component" of the deformation, and we do not want it to be smaller than the ring A_0 that we start with. (The extreme example of a bad choice would be $N = 0$ and $A_1 = A_0 = R[x,y]/(x^2, xy, y^2, t)$.) So we must require the map $A_0 \xrightarrow{t} N$ to be injective. In concrete terms this means the following:

(3.2) If $z \in A_0$ and if $zt = 0$ in A_1, then $z = 0$.

This is equivalent with saying that A_1 is a *flat* R-module. (See Section 6.) Flatness is a requirement we will always put on deformations.

Now the three generators (3.1) for the ideal are not independent, so the flatness requirement imposes a compatibility condition on them. Specifically, the monomial $x^2 y$ can be simpified in two ways using the relations $f_i = 0$:

(3.3)

$$\begin{aligned}
&(x^2)y = -\alpha t y = -(a_0 y + a_1 xy + a_2 y^2)t = -a_0 y t, \quad \text{and} \\
&x(xy) = -x\beta t = -x b_0 t.
\end{aligned}$$

Thus $a_0 yt = b_0 xt$ in A_1, and the flatness condition requires that $a_0 y = b_0 x$ in A_0. (The expression "$u = v$ in A" means that the images of u, v in A are equal.) Since no linear polynomial is zero in A_0, this forces $a_0 = b_0 = 0$. Similarly, c_0 must be zero.

We are left with 6 parameters for a first order *embedded deformation* (a deformation with given generators x, y for the algebra):

(3.5)

$$\begin{aligned}
&f_1 = x^2 + a_1 x + a_2 y, \\
&f_2 = xy + b_1 x + b_2 y, \\
&f_3 = y^2 + c_2 x + c_3 y, \quad \text{with } a_i, b_i, c_i \in k.
\end{aligned}$$

Note that an embedding of the "generic" algebra $k \oplus k \oplus k$ of dimension three is given by a surjective homomorphism $k[x,y] \longrightarrow k^3$, which is determined by a set of three maximal ideals, or three points in the affine plane. The deformations also depend on 6 parameters, which correspond to deforming the 6 coordinates of the three points. So it isn't likely that the number of parameters for deformations of our ring can be reduced any more. In fact it can't be.

Exercise: Work out the embedded deformations of k^3 carefully.

To classify deformations of A_0 as k-algebra, i.e., allowing for change of coordinates, we have to study the effect of an infinitesimal change of coordinates, a substitution of the form $x = x + ut$, $y = y + vt$, with $u, v \in k[x, y]$ on the defining relations. Say that $u = u_0 + u_1 x + u_2 y + \cdots$. Then

$$f_1(x + ut, y + vt) = f_1(x, y) + (\frac{\partial f}{\partial x}u + \frac{\partial f}{\partial y}v)t = (x^2 + \alpha t) + 2u_0 xt. \tag{3.6}$$

The defining equation changes by adding $2u_0xt$. Choosing u_0 appropriately, we can use it to eliminate the term a_1x from 3.5. Similarly, the substitution $y = y + v_0t$ can be used to eliminate b_2y. We are left with four essential parameters. A first order deformation can be described by a unique set of relations of the form

(3.7)
$$f_1 = x^2 + a_2yt,$$
$$f_2 = xy + b_1xt + b_2yt,$$
$$f_3 = y^2 + c_1xt,$$

with coefficients a_i, b_i, c_i in k.

4. Moduli:

We consider the general problem of classifying some particular type of algebraic structure. As a simple example, the structure might be that of an algebra with a given presentation as quotient $A = k[x]/\mathfrak{a}$ of a polynomial ring in one variable, and such that A is isomorphic as algebra to a sum of n copies of k. Geometrically, this data corresponds to an unordered n-tuple of points in the affine line $\mathbb{A}^1 = \text{Spec } k[x]$.

In the weakest possible sense, a *moduli space* for such a problem is a scheme M whose points correspond bijectively to the isomorphism classes of structures A of the type under consideration.

In our example, there is a natural candidate for the moduli space. The ideal $\mathfrak{a}$ will be principal, generated by a unique monic polynomial $f(x)$ of degree n. We can try the n-dimensional space which parametrizes the coefficients of f. Let the variable polynomial be $x^n - s_1x^{n-1} + \cdots \pm s_n$. A point in s-space $\mathbb{A}^n = \text{Spec } k[s_1, ..., s_n]$ is given by evaluating the variable coefficients in k, say $s_i = a_i$, to obtain a particular polynomial $f(x) = x^n - a_1x^{n-1} + \cdots \pm a_n$. This evaluation gives us an algebra $A = k[x]/(f)$. The requirement that A be isomorphic to the sum of n copies of k means that the polynomial f must have n distinct roots. Thus the moduli space M is obtained from s-space by deleting the locus $\{\Delta = 0\}$ at which the discriminant $\Delta(s)$ vanishes. Deleting this locus can also be done by inverting Δ in the ring:

$$M = \text{Spec } k[s_1, ..., s_n] - \{\Delta = 0\} = \text{Spec } k[s_1, ..., s_n, \Delta^{-1}]. \tag{4.1}$$

By the way, the space of ordered n-tuples of points in $\mathbb{A}^1$ is parametrized in the obvious way by the n-dimensional affine space Spec $k[u_1, ..., u_n]$, u_i being the coordinate of the i-th point. If we want distinct points, we must omit from this space the various diagonals $\{u_i = u_j\}$, which we can do if desired by adjoining the inverses of $u_i - u_j$ to $k[u]$ for all $i \neq j$. Let U denote the resulting scheme. Then M is the quotient of U by the action of the symmetric group G, which operates by permuting the variables u_i. This is in agreement with the main theorem on symmetric functions: $k[u]^G = k[s_1(u), ..., s_n(u)]$. The inverse image of the discriminant locus $\{\Delta = 0\}$ in Spec $k[s]$ is the union of the diagonals $\{u_i = u_j\}$.

Now a condition on the points of M is not enough to determine its scheme structure. In order to determine the structure of scheme, we need a concept of *family* of structures parametrized by a scheme S, and of *isomorphism between families* of structures. It is enough to have a definition of family for affine schemes $S =$ Spec R, where R is a finitely generated k-algebra. If these concepts are available, then a *strong moduli space* for our classification problem is a scheme M such that there is a bijective correspondence

(4.2) $\{\text{maps } S \longrightarrow M\} \leftrightarrow \{\text{iso. classes of families, param. by } S\}$.

This correspondence should be compatible with pull backs of maps and of families. In particular, a point of M can be identified with a map from the *point* := Spec k to M, so points correspond to isomorphism classes of structures as before.

Going back to our example, if the scheme 4.1 is to be a strong moduli space for unordered n-tuples, then we have no choice in the definition of a *family*. A family must be given by a map $S \longrightarrow M$, or on rings, a map in the other direction $\phi : k[s_1, ..., s_n] \longrightarrow R$ such that $\phi(\Delta)$ is a unit in R. Such a map is determined by the images $\phi(s_i) = a_i \in R$, the only condition on the $\{a_i\}$ being that the discriminant $\Delta(a_i)$ must be a unit in R. Having made this substitution, we may interpret the a_i as coefficients of a polynomial f in $R[x]$. So a family corresponds to a monic polynomial with coefficients in R whose discriminant is invertible (i.e., is nowhere zero on Spec R), or to the associated R-algebra $A = R[x]/(f)$.

The geometric description of a family of n-tuples of points in $\mathbb{A}^1$ is more complicated: If $S =$ Spec R, then a family is a closed subscheme Z of $S \times \mathbb{A}^1$ which is an unramified n-sheeted covering space of S.

Exercise: Consider, by analogy with the above discussion, these three classification problems of classical geometry:
(a) embedded triangles in the plane,
(b) congruence classes of triangles,
(c) similarity classes of triangles.

Let a "family" of triangles mean a continuous family, parametrized by an open set in $\mathbb{R}^n$, in which side lengths vary differentiably. Determine spaces M which parametrize these three problems explicitly, and discuss the extend to which they are strong moduli spaces.

5. First order deformations as tangents to the moduli space:

Let $R = k[t]/(t^2)$. The scheme $V = \text{Spec } R$ plays the role of a point with an attached vector. To see this, note that a map $V \longrightarrow \mathbb{A}^n$ is given by a homomorphism $\phi : k[x_1, ..., x_m] \longrightarrow k[t]/(t^2)$, and such a homomorphism can be defined by assigning the images $\phi(x_i) = a_i + v_i t$ arbitrarily. Then for any polynomial $g(x) \in k[x]$,

$$\phi(g(x)) = g(a) + \sum \frac{\partial g}{\partial x_i} v_i t = g(a) + (\nabla g \cdot v)t. \tag{5.1}$$

Writing $\phi(g) = \phi_0(g) + dgt$, the constant term ϕ_0 is evaluation at the point: $\phi_0(g) = g(a)$, and the linear term d defined by $dg = \nabla g \cdot v)$ is the directional derivative of g in the direction v, which is a ϕ_0-*derivation.* This means that it is linear, and that

$$d(fg) = f(a)\, dg + df\, g(a). \tag{5.2}$$

Next, if $X = \text{Spec } S$, where $S = k[x_1, ..., x_m]/(f_1, ..., f_n)$, then a map $V \longrightarrow X$ is determined by a ring homomorphism $k[x_1, ..., x_m] \longrightarrow k[t]/(t^2)$ such that $\phi(f_\nu) = 0$ for $\nu = 1, ..., n$. Expanding $\phi(f_\nu) = f_\nu(a) + (\nabla f_\nu \cdot v)t$, we see that this means $f_\nu(a) = 0$, and $(\nabla f_\nu \cdot v) = 0$. Translating,

(5.4)(a) the point $x = a$ lies on the locus X, and
(b) the vector v is tangent to X at the point $x = a$.

In fact, the condition $(\nabla f_\nu \cdot v) = 0$ is the definition of tangent vector to the locus $\{f_\nu = 0\}$. Thus a map $V \longrightarrow X$ corresponds to a point of X together with a tangent vector to X at that point.

Now if M is a moduli space, then a map $V \longrightarrow M$ also corresponds to a family of structures, say A_R, parametrized by $R = k[t]/(t^2)$. Given such a family, setting $t = 0$ yields a structure A_k over k, and A_R is a first order deformation of A_k. Thus first order deformations correspond to tangent vectors to M.

The best way to understand deformations intuitively is to imagine that one has a moduli space M, and that one is studying its infinitesimal structure. Unfortunately, it is pretty rare that a strong moduli space exists, and it is fortunate that a good deformation theory exists more frequently. For instance, we have seen that there are exactly 4 isomorphism classes of schemes $Z = \text{Spec } A$, where A is a commutative ring of dimension 3 over k. The only way to make this set into a scheme is as a set of four points. On the other hand, there are also families A_T parametrized by the affine line Spec $k[t]$, such that $A_{t=0} \approx k[x, y]/(x^2, xy, y^2)$ and $A_{t=1} \approx k \oplus k \oplus k$. So the four points should really be connected, in some sense.

Exercise: Write down such a family explicitly.

Even if a strong moduli space M does exist, we may not know how it looks. Indeed, studying the infinitesimal structure is a good way of getting information about M. The typical problem which we want to address is to find a way to compute the deformations of a given structure directly. This requires choosing the right definition for a family of structures. And to complicate our intuition, the definition must work when the parameter space is the spectrum of a finite k-algebra R such as $k[t]/(t^2)$, whose nilradical is not zero. This requires having faith that the algebra can be made to carry over to such situations, where geometric intuition becomes obscured.

6. Flatness.

Proposition 6.1. *("Nilpotent Nakayama Lemma") Let I be a nilpotent ideal of a ring R, and let $R' = R/I$. Let M be a right R-module, and denote $M \otimes_R R' = M/MI$ by M'.*

(i) Let S be a subset of M whose image S' in M' generates M'. Then S generates M.

(ii) If $M' = 0$, then $M = 0$.

(iii) If $\phi : M \longrightarrow N$ is a homomorphism of R-modules such that $\phi' : M' \longrightarrow N'$ is surjective, then ϕ is surjective.

Proof. (i) We may suppose that $I^2 = 0$. Let N denote the submodule of M generated by S. Let $m \in M$, and write $m' = \sum_{i=1}^{n} s_i' r_i'$, with $s_i' \in S'$ and $r_i' \in R'$. Representing r_i' by $r_i \in R$, we obtain an element $n = \sum s_i r_i \in N$ such that $m - n \in MI$. Set $x = m - n$. Then $x = \sum_j m_j a_j$, with $m_j \in M$ and $a_j \in I$. Substituting m_j for m in the above reasoning shows that $m_j = n_j + x_j$ where $n_j \in N$ and $x_j \in MI$. Then $m_j a_j = n_j a_j + x_j a_j$ and $x_j a_j \in MI^2 = 0$. This shows that $m = n + \sum n_j a_j \in N$, as required.

Proposition 6.2. *Let I be an ideal of a ring R such that $I^2 = 0$, and let $R' = R/I$. Let M be a right R-module and set $M' = M \otimes_R R'$. Suppose that M' is flat over R'. Then M is R-flat if and only if the multiplication map $M \otimes_R I \longrightarrow MI$ is injective, i.e., if and only if the sequence*

$$0 \longrightarrow M \otimes_R I \longrightarrow M \longrightarrow M' \longrightarrow 0$$

is exact.

Proof. We tensor the sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R' \longrightarrow 0 \tag{6.3}$$

on the left with M. Since R is a flat left R-module, $\mathrm{Tor}_1^R(M, R) = 0$. So the exact sequence for Tor gives us

$$0 \longrightarrow \mathrm{Tor}_1^R(M, R') \longrightarrow M \otimes_R I \longrightarrow M \longrightarrow M' \longrightarrow 0. \tag{6.4}$$

Thus $\mathrm{Tor}_1^R(M, R') = 0$ if and only if $M \otimes_R I \longrightarrow MI$ is injective.

Now M_R is flat if and only if $\mathrm{Tor}_1^R(M, N) = 0$ for every left R-module ${}_RN$. So if M_R is flat, then $\mathrm{Tor}_1^R(M, R') = 0$ and that the sequence 6.2 is exact. To prove the converse, we must show that if $\mathrm{Tor}_1^R(M, R') = 0$, then $\mathrm{Tor}_1^R(M, N) = 0$ for every N. Since $I^2 = 0$, the left and right terms of the exact sequence

$$0 \longrightarrow IN \longrightarrow N \longrightarrow R' \otimes_R N \longrightarrow 0 \tag{6.5}$$

are R'-modules. The exact sequence for Tor shows that it suffices to verify $\mathrm{Tor}_1^R(M, V') = 0$ for every left R'-module V'. For such a module, we have $M \otimes_R V' \approx (M \otimes_R R') \otimes_{R'} V'$. There is a spectral sequence for composed functors of the form

$$E^2_{p,q} = \mathrm{Tor}_p^{R'}(\mathrm{Tor}_q^R(M, R'), V') \longrightarrow \mathrm{Tor}_p^R(M, V'), \tag{6.6}$$

and since $M \otimes_R R' = M'$ is flat, $\mathrm{Tor}_p^{R'}(M', V') = 0$ for $p > 0$. The spectral sequence shows that $\mathrm{Tor}_1^R(M, V') \approx \mathrm{Tor}_1^R(M, R') \otimes_{R'} V' = 0$.

Proposition 6.7. *Let I be a nilpotent ideal of a ring R, and assume that $M' = M \otimes_R R'$ is flat over R'. Let $F \longrightarrow M$ be a surjective map from a flat module F to M, and let $Rel(M) \subset F$ denote the kernel of this map. Let $Rel(M')$ denote the kernel of the induced map $F' \longrightarrow M'$. Then M is R-flat if and only if the map $Rel(M) \otimes_R R' \longrightarrow Rel(M')$ is injective. If so, then that map is bijective, and $Rel(M)$ is R-flat.*

Proof. Again, we may suppose that $I^2 = 0$. The first assertion results from Proposition 6.2 and from an inspection of the 3×3 diagram obtained by tensoring the two exact sequences

$$0 \longrightarrow Rel(M) \longrightarrow F \longrightarrow M \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow I \longrightarrow R \longrightarrow R' \longrightarrow 0.$$

Tensoring the exact sequence

$$0 \longrightarrow Rel(M) \longrightarrow F \longrightarrow M \longrightarrow 0 \tag{6.8}$$

with R' and using right exactness of tensor product shows that the map $Rel(M) \otimes_R R' \longrightarrow Rel(M')$ is surjective in all cases, hence that $Rel(M) \otimes_R R' \approx Rel(M')$ if M is flat. Finally, if M is flat, then the exact Tor sequence for the sequence 6.8 shows that $Rel(M)$ is flat too.

Proposition 6.9. *Let I be a nilpotent ideal of a ring R, let M_R be a right R-module, and set $M' = M \otimes_R R'$.*

(i) Let

$$\mathcal{F} = \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

be a free resolution of M as R-module. If M is flat over R, then $\mathcal{F} \otimes_R R'$ is a free resolution of $M' = M \otimes_R R'$ as R'-module.

(ii) Assume that M is flat over R. Then any free resolution $\mathcal{F}'$ of M' lifts step by step to a free resolution of M. Moreover, any lifting of $\mathcal{F}'$ to a complex of free modules whose right hand term is M is a resolution of M.

(iii) Let $C = F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ be a complex of free R-modules such that $F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$ is exact. Let $'$ denote $\otimes_R R'$ as usual. Assume that C' is exact and that M' is flat over R'. Then M is flat over R and C is exact.

Proof. Assertions (i) and (ii) follow by induction from the prevous proposition without difficulty. Let us verify (iii). Let $Rel(M)$ be defined as before. We have a complex

$$F_2 \longrightarrow F_1 \xrightarrow{\phi} Rel(M) \longrightarrow 0, \tag{6.10}$$

in which ϕ is surjective. Tensoring with R', we obtain a complex

$$F'_2 \longrightarrow F'_1 \xrightarrow{\phi'} Rel(M) \otimes_R R' \longrightarrow 0, \tag{6.11}$$

and ϕ' is surjective too. This complex maps to

$$F'_2 \longrightarrow F'_1 \longrightarrow Rel(M') \longrightarrow 0, \tag{6.12}$$

which is an exact sequence by hypothesis. Therefore the map $Rel(M) \otimes_R R' \longrightarrow Rel(M')$ is bijective, and by the previous proposition, M is flat.

Since M is flat, so is $N = Rel(M)$. Let $Rel(N) = ker(F_1 \longrightarrow N)$. This is a flat module too, and so $Rel(N) \otimes_R R' \approx Rel(N')$. We have a map $F_2 \longrightarrow Rel(N)$, and when tensored with R' this map becomes surjective. So it is surjective by Nakayama.

Scholium 6.13. Part (iii) of the previous proposition says this: Let M' be a flat R'-module, with a partial resolution

$$F'_2 \xrightarrow{d'_2} F'_1 \xrightarrow{d'_1} F'_0 \longrightarrow M' \longrightarrow 0.$$

In order to obtain a flat R-module M such that $M \otimes_R R' \approx M'$, we may proceed as follows: We represent the map d'_1 as multiplication by a matrix with entries in R' (a presentation matrix for M'). We lift the matrix entries to R, obtaining an R-matrix and a map $F_1 \xrightarrow{d_1} F_0$, and we let $M = coker(d_1)$ be the module presented by this matrix. Then $M \otimes_R R' \approx M'$, and M is flat if and only if the map $ker(d_1) \longrightarrow ker(d'_1)$ is *surjective.*

7. The general set-up for studying infinitesimal deformations.

We will want to study higher order deformations, for example those parametrized by Spec $k[t]/(t^3)$ as well as first order ones, and even truncated polynomial rings are not general enough. So we will allow as parameter the spectrum of an arbitrary finite local k-algebra R. We will use induction to help with the study of structures parametrized by R.

Lemma 7.1. *A finite local k-algebra R which is not k itself contains an ideal I of dimension 1, and $I^2 = 0$.*

Proof. The maximal ideal $\mathfrak{m}$ of R is nonzero and nilpotent. Let $\mathfrak{m}^r$ be the highest power of $\mathfrak{m}$ which is not zero. Then R acts on $\mathfrak{m}^r$ through the residue field $k = R/\mathfrak{m}$, and so every subspace I of $\mathfrak{m}^r$ is an ideal of R. If I has dimension 1, then because $R \neq k$, $I \subset \mathfrak{m}$. So I is nilpotent, which implies that $I^2 < I$ and that $I^2 = 0$.

We choose an ideal I of dimension 1, with basis ϵ, and set $R' = R/I$. Thus we have an exact sequence

$$0 \longrightarrow k \xrightarrow{\epsilon} R \longrightarrow R' \longrightarrow 0. \tag{7.2}$$

This will be our standard notation. We often suppose that we are given a structure $A_{R'}$ parametrized by Spec R', and we study the structures A_R over Spec R which extend it. This allows us in principle to work our way up inductively from k.

By definition, *family of algebras* parametrized by R is a flat R-algebra A_R. If we are given an algebra A_k, a *deformation of A_k, parametrized by R*, or a *flat extension of A_k to R* is a flat R-algebra A_R together with an isomorphism $A_R \otimes_R k \approx A_k$. Similarly, a flat extension of a flat R'-algebra $A_{R'}$ to R is a flat R-algebra A_R together with an isomorphism $A_R \otimes_R R' \approx A_{R'}$.

As we noted in the beginning, it is important to fix the structure A_k completely. That is why we require that a deformation be given with a fixed isomorphism $A_R \otimes_R k \approx A_k$.

The flatness condition for deformations is fundamental. It's heuristic explanation is the same as for the case that $R = k[t]/(t^2)$: A_R is flat if and only if the sequence

(7.3) $$0 \longrightarrow A_k \xrightarrow{\epsilon} A_R \longrightarrow A_{R'} \longrightarrow 0$$

obtained by tensoring A_R with (7.2), which is always right exact, is actually exact (6.2). The image ϵA_k of A_k in A_R represents the infinitesimal part of the extension, and we want it to be the same "size" as the algebra A_k which we are deforming.

The results of the previous section can be applied to this situation. To do so we let P_R denote either the ring $R[x_1, ..., x_n]$ of commutative polynomials or the ring $R\langle x_1, ..., x_n\rangle$ of noncommutative polynomials. We note that P_R is a free R-algebra in either case. The following are corollaries of (6.9).

Corollary 7.4. *Let M be a finite right P_R-module which is a flat R-module. Let*

$$\mathcal{R} = \cdots \longrightarrow P_R^{n_2} \longrightarrow P_R^{n_1} \longrightarrow P_R^{n_0} \longrightarrow M \longrightarrow 0$$

be a free resolution of M as P_R-module. Then $\mathcal{R} \otimes_R R'$ is a free resolution of M' as $P_{R'}$-module.

Corollary 7.5. *Let M be a P_R-module such that $M' = M \otimes_R R'$ is a flat R'-module, with a resolution*

$$\cdots \longrightarrow P_{R'}^{n_2} \longrightarrow P_{R'}^{n_1} \longrightarrow P_{R'}^{n_0} \longrightarrow M' \longrightarrow 0.$$

Then M is R-flat if and only if the first steps of the resolution lift to an exact sequence

$$P_R^{n_1} \longrightarrow P_R^{n_0} \longrightarrow M \longrightarrow 0$$

and a complex

$$P_R^{n_2} \longrightarrow P_R^{n_1} \longrightarrow P_R^{n_0} \longrightarrow M \longrightarrow 0.$$

If so, then the whole resolution can be lifted step by step, and any lifting as a complex is a resolution of M.

8. First order deformations of a commutative algebra.

Here $R = k[t]/(t^2)$. We classify the flat extensions of a commutative k-algebra A_k to R. Say that $A_k = P_k/\mathfrak{a}_k$, where P_k is the polynomial ring $k[x_1, ..., x_m]$. Let

$$P_k^{n_2} \xrightarrow{r} P_k^{n_1} \xrightarrow{f} P_k \longrightarrow A_k \xrightarrow{\pi} 0 \tag{8.1}$$

be a partial resolution of A_k as P-module. So $f = (f_1, ..., f_{n_1})$ is a set of generators for the ideal $\mathfrak{a}_k$ in P_k, and $r = (r_{ij})$ is a complete set of relations among the generators.

Suppose that a flat R-algebra A_R extending A_k is given. Then we have seen (7.5) that (8.1) lifts to a partial resolution of A_R as P_R-module, where $P_R = R[x_1, ..., x_m]$. Let us write the liftings of f and r in the form $f + gt$, $r + st$ respectively. So A_R is presented as $P_R/\mathfrak{a}_R$, where $\mathfrak{a}_R$ is the ideal generated by $\{f_i + g_i t\}$. The lifting $r + st$ of r is not needed to determine A_R. It is needed only to insure that A_R is flat, and for A to be flat, it suffices that the lifting be a complex, i.e., that

$$(f + gt)(r + st) = 0 \tag{8.2}$$

(We write operators on the left.) Thus a first order deformation of A_k is determined by a P_k-vector $g = (g_1, ..., g_{n_1})$ such that there exists a P_k-matrix $s = (s_{ij})$ satisfying 8.2. Expanding, we find $(f + gt)(r + st) = fr + (gr + fs)t = (gr + fs)t$. So the condition on g can be written as

$$gr = -fs \tag{8.3}$$

for some $s = (s_{ij})$. Since s_{ij} can be arbitrary elements of P_k, fs can be any vector whose entries are in $\mathfrak{a}_k$. This means that the condition on our vector g can also be expressed as

$$gr \equiv 0 \ \text{(modulo } \mathfrak{a}_k\text{)}. \tag{8.4}$$

We suppress the subscripts k in what follows, writing $P = P_k, \mathfrak{a} = \mathfrak{a}_k, A = A_k$. The horizontal exact sequences in the diagram

$$\begin{array}{ccccccc} P^{n_2} & \xrightarrow{r} & P^{n_1} & \xrightarrow{f} & \mathfrak{a} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \gamma} & & \\ P^{n_1} & \xrightarrow{f} & P & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array} \tag{8.5}$$

are determined by 8.1. If g is a map satisfying (8.4), then $\pi g r = 0$, from which we deduce that a map γ exists which makes (8.5) into a commutative diagram. So g defines a map $\gamma \in \mathrm{Hom}_P(\mathfrak{a}, A)$.

Since A is an A-module and since $\mathfrak{a} \otimes_P A \approx \mathfrak{a}/\mathfrak{a}^2$,

$$\mathrm{Hom}_P(\mathfrak{a}, A) \approx \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A). \tag{8.6}$$

Proposition 8.7. *First order embedded deformations of* $A_k = P_k/\mathfrak{a}_k$, *i.e., deformations together with chosen generators* $x_1, ..., x_m$, *are classified by* $\mathrm{Hom}_{A_k}(\mathfrak{a}_k/\mathfrak{a}_k^2, A_k)$.

To prove this proposition, we must show three things:

(a) Every element $\gamma \in \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A)$ is obtained as above from a permissible map g,

(b) If g and g' are two permissible maps $P^{n_1} \longrightarrow P$ such that $\gamma = \gamma'$, then the ideals $\mathfrak{a}_R$ and $\mathfrak{a}'_R$ generated by $f + gt$ and $f + g't$ are equal.

(c) If g and g' are two permissible maps and if the ideals $\mathfrak{a}_R$ and $\mathfrak{a}'_R$ in P_R are equal, then $\gamma = \gamma'$.

The only one of these verifications which is not routine is (c). Suppose g and g' are given and that $\mathfrak{a}_R = \mathfrak{a}'_R$. We have two partial resolutions of $A_R = A'_R$, and in order to keep the two copies of $P_R^{n_1}$ which appear in these resolutions apart, let us denote them by F_R and F'_R respectively. So $F_R \xrightarrow{f+gt} P_R$ and $F'_R \xrightarrow{f+g't} P_R$. Setting $t = 0$ yields the same map f, so there is no harm in identifying F_k and F'_k.

We look for a map $F_R \xrightarrow{\phi} F'_R$ such that the diagram

(8.8)
$$\begin{array}{ccc} F'_R & \xrightarrow{f+g't} & \mathfrak{a}'_R \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle id} \\ F_R & \xrightarrow{f+gt} & \mathfrak{a}_R \end{array}$$

commutes, *and* in which ϕ has the form $1 + ht$, i.e., for which $\phi \otimes_R k = \mathit{identity}$. If such a map can be found, then we will have

(8.9)
$$f + g't = (f + gt)(1 + ht) = f + (g + fh)t.$$

So $g' = g + fh$, hence $g' \equiv g$ (modulo $\mathfrak{a}_k$) and $\gamma' = \gamma$.

If $U \longrightarrow Z$ and $V \longrightarrow Z$ are maps of sets, their *fibred product* $X = U \times_Z V$ is the set of pairs $(u, v) \in U \times V$ such that the images of u and v in Z are equal.

Consider the commutative diagram

(8.10)
$$\begin{array}{ccc} F_R & \xrightarrow{f+gt} & \mathfrak{a}_R \\ \downarrow & & \downarrow \\ F_k & \xrightarrow{f} & \mathfrak{a}_k \end{array}$$

of P_R-modules, in which the vertical arrows are given by $\cdot \otimes_R k$. This diagram defines a map from F_R to the fibred product $X = F_k \times_{\mathfrak{a}_k} \mathfrak{a}_R$. Since $F'_k = F_k$ and $\mathfrak{a}'_R = \mathfrak{a}_R$, it follows that $X' = X$. A map $F'_R \xrightarrow{\phi} F_R$ will have the form $1 + ht$ if and only if the diagram

(8.11)
$$\begin{array}{ccc} F'_R & \longrightarrow & X' \\ {\scriptstyle\phi}\downarrow & & \downarrow \\ F_R & \longrightarrow & X \end{array}$$

commutes. And since F'_R is a projective P_R-module, such a lifting will exist provided that the map $F_R \longrightarrow X$ is surjective.

Lemma 8.12. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \overline{A} & \longrightarrow & \overline{B} & \longrightarrow & \overline{C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

be an exact commutative diagram of groups. Then the induced map $B \longrightarrow \overline{B} \times_{\overline{C}} C$ is surjective.

We omit the proof of the lemma. To apply it, note that the bottom row of 8.10 is obtained from the top one by tensoring with k. This implies that the map on kernels is surjective.

It remains to consider the deformations of A_k as algebra, without fixed coordinates. Now we are still free to choose a resolution 8.1 of A_k. Then if A_R is a flat extension of A_k to R, we can lift the surjection $P_k \xrightarrow{\pi_k} A_k$ to $P_R \xrightarrow{\pi_R} A_R$, thereby obtaining a lifting of the generators. Having done so, the resolution 8.1 also lifts, and in particular, the defining ideal $\mathfrak{a}_R$ is generated by elements of the form $f + gt$ as before. The difference is in the notion of isomorphism of extensions. An isomorphism of extensions $A'_R \xrightarrow{\psi} A_R$ is any isomorphism of algebras which is compatible with the given isomorphisms $A'_R \otimes_R k \approx A_k \approx A_R \otimes_R k$.

We have a diagram

$$\begin{array}{ccc} P_R & \xrightarrow{\pi_R} & A_R \\ \downarrow & & \downarrow \\ P_k & \xrightarrow{\pi_k} & A_k \end{array} , \tag{8.13}$$

of R-algebras, and Lemma 8.12 shows that the induced map $P_R \longrightarrow X = P_k \times_{A_k} A_R$ is surjective. Using the fact that P_R is a free object in the category of commutative R-algebras, and proceeding as above, one finds that the isomorphism ψ can be lifted to obtain a commutative diagram of algebras

$$\begin{array}{ccc} P_R & \xrightarrow{f+g't} & A'_R \\ \downarrow{\scriptstyle \Psi} & & \downarrow{\scriptstyle \psi} \\ P_R & \xrightarrow{f+gt} & A_R \end{array} \tag{8.14}$$

in which $\Psi \otimes_R k = identity$. Then Ψ is a change of variable of the form $x_i \mapsto x_i + u_i t$ for some $u_i \in P_k$. The commutativity of the diagram means that Ψ carries the defining ideal $\mathfrak{a}_R$ to $\mathfrak{a}'_R$, i.e., that

$$f(x+ut) + g(x+ut)t \tag{8.15}$$

generates the ideal $\mathfrak{a}'_R$. The effect of this change of variable is to change g to

$$\overline{g} = g + \sum_i f_{x_i} u_i = g + (\nabla f \cdot u). \tag{8.16}$$

So $f + g't$ and $f + \overline{g}t$ generate the same ideal. We conclude that if $\mathfrak{a}_R$ is generated by $f+gt$ and if $\mathfrak{a}'_R$ is another ideal defining an isomorphic deformation, then there are elements $u \in P_k$ such that $\mathfrak{a}'_R$ can be generated by $f + g't$, where $g' = g + (\nabla f \cdot u)$. Moreover, for any $u_i \in P_k$, the ideal generated by this $f + g't$ defines an isomorphic deformation.

The results of this computation can be written canonically in terms of differentials. Let $P, A, \mathfrak{a}$ denote $P_k, A_k, \mathfrak{a}_k$ as before. We denote the module of differentials of the k-algebra A by Ω_A. This module comes with a k-derivation $A \xrightarrow{d} \Omega_A$ which is universal for derivations of A into modules. Here are its most important properties:

(8.17)(a) Let $P = k[x_1, ..., x_m]$. Then Ω_P is a free P-module of rank m, generated by elements $dx_1, ..., dx_m$. The map $P \xrightarrow{d} \Omega_P$ sends $x_i \mapsto dx_i$, and if f is an arbitrary polynomial, $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$.

(b) If $A = B/\mathfrak{a}$, then Ω_A is a quotient of Ω_B, and there is an exact sequence of A-modules

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{d} \Omega_B \otimes_B A \longrightarrow \Omega_A \longrightarrow 0. \tag{8.18}$$

Taking $B = P$ in (8.18) and dualizing, we obtain

$$\mathrm{Hom}_A(\Omega_P \otimes_P A, A) \xrightarrow{d^*} \mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A) \longrightarrow T^1_A \longrightarrow 0, \tag{8.19}$$

where "Schlessinger's T^1_A" is the cokernel of d^*, the module defined so as to make this sequence exact. Its definition depends on the chosen presentation, but its universal property shows that it is defined canonically.

Proposition 8.20. *There is a natural bijective correspondence between first order deformations of an algebra A_k and elements of T^1_A.*

Proof. We compute the image of d^*. An A-linear map $\Omega_P \otimes_P A \longrightarrow A$ corresponds to a P-linear map $\phi : \Omega_P \longrightarrow A$. Since Ω_P is freely generated by the dx_i, such a map ϕ is described by assigning arbitrary values $\phi(dx_i) = u_i$. The image of ϕ in $\mathrm{Hom}_A(\mathfrak{a}/\mathfrak{a}^2, A)$ is $\phi\circ d$. If $f = (f_1, ..., f_{n_1})$ generates $\mathfrak{a}$, then $\phi d(f) = \phi(\sum_i \frac{\partial f}{\partial x_i} dx_i) = \sum \frac{\partial f}{\partial x_i} \phi(dx_i) = (\nabla f \cdot u)$.

Changing $\gamma \in \mathrm{Hom}(\mathfrak{a}/\mathfrak{a}^2, A)$ by a map of this form corresponds to a change of coordinates, which agrees with our previous computation (8.16).

9. First order deformations of noncommutative algebras.

We wish to copy the considerations of the previous section over for a noncommutative algebra k. So we replace the polynomial ring $k[x_1, ..., x_m]$ by the free ring, the ring of noncommutative polynomials $k\langle x_1, ..., x_m\rangle$. In this section, P_k will denote the free ring, and similarly, $P_R = R\langle x_1, ..., x_m\rangle$ will denote the free R-algebra of noncommutative polynomials with coefficients in the commutative ring R. The "scalars", the elements of R, commute with the variables, but the variables don't commute with each other.

For the moment, we consider k-algebras, so we drop the subscript k. Let a k-algebra be given with a presentation

$$0 \longrightarrow \mathfrak{a} \longrightarrow P \longrightarrow A \longrightarrow 0, \tag{9.1}$$

where $\mathfrak{a}$ is a two-sided ideal of P. Typically, $\mathfrak{a}$ may be generated by a finite set of elements $f_1, ..., f_m$ of P. However, by generators we mean that $\mathfrak{a}$ is the smallest *two-sided* ideal containing these elements. It is rarely the case that a two-sided ideal in the free ring is finitely generated as a left or as a right module. This complicates our discussion, but only a little bit. We are helped by the fact that the sequence 9.1 is both left and right linear, i.e., it is an exact sequence of homomorphisms of two-sided P-modules, or of P-bimodules.

If A, B are k-algebras, an *A, B-bimodule* $M = {}_AM_B$ is a k-vector space with the structure of left A-module and of right B-module, and such that the actions on left and right commute:

$$(am)b = a(mb). \tag{9.2}$$

Such a bimodule has a canonical structure of right module over the ring $A^\circ \otimes_k B$, where A° is the opposite ring of A, in which multiplication is reversed. The categories of A, B-bimodules and of right $A^\circ \otimes_k B$-modules are equivalent. To be specific, if a bimodule ${}_AM_B$ is given, then $A^\circ \otimes_k B$ operates by

$$m \cdot a \otimes b = amb. \tag{9.3}$$

Thus $a \otimes 1 = \lambda_a$ represents left multiplication by a, and $1 \otimes b = \rho_b$ represents right multiplication by b. The formula $m\lambda_{aa'} = aa'm = (m\lambda_{a'})\lambda_a$ shows that multiplication is reversed.

The ring $E(A) = A^\circ \otimes_k A$ is called the *enveloping algebra* of the k-algebra A. Right modules over the enveloping algebra correspond canonically to A, A-bimodules.

Given elements $a, b \in A$, we will usually write $a^\circ = a \otimes 1$, $b = 1 \otimes b$, and we will suppress the tensor product symbol, writing $a^\circ b$ for $a \otimes b$. When this ambiguity causes problems, it may be advisable to write $\lambda_a\rho_b$ instead. Note that by definition of the tensor product algebra, $(1\otimes b)(a\otimes 1)=a\otimes b=(a\otimes 1)(1\otimes b)$, or $\rho_b\lambda_a=\lambda_a\rho_b$, or

$$ba^\circ = a^\circ b \tag{9.4}$$

for all $a,b\in A$. This corresponds to the fact that the operations of left and right multiplcation on a bimodule are required to commute.

If $A=k[x]$ is a commutative polynomial ring, then $E(A)=k[x^\circ,x]$ is also commutative, with twice the number of variables. On the other hand, if $P=k\langle x_1,...,x_m\rangle$ is the free ring, then $E(P)$ is obtained from the free ring $k\langle x_1^\circ,...,x_m^\circ;x_1,...,x_m\rangle$ by introducing the relations

$$x_jx_i^\circ = x_i^\circ x_j, \quad \text{for all } i,j. \tag{9.5}$$

Every element of $E(P)$ has a unique expression as a linear combination of monomials in which the variables x_i° are to the left of the variables x_i.

Let $P=k\langle x_1,...,x_m\rangle$ denote the free ring, and let $E=E(P)$. Let $S=\{f_1,...,f_{n_1}\}$ be a set of elements of P, and let $\mathfrak{a}$ be the two-sided ideal of P generated by S. Then $\mathfrak{a}$ can also be described as the right E-submodule of P generated by S. The ring $A=P/\mathfrak{a}=k\langle x\rangle/(f)$ has a resolution, as right E-module, of the form

$$E^{n_2}\xrightarrow{r}E^{n_1}\xrightarrow{f}P\longrightarrow A\longrightarrow 0, \tag{9.6}$$

where f denotes left multiplication by the row vector $(f_1,...,f_{n_1})$ whose entries are in P, and r represents left multiplication by some matrix with entries in E. In general, n_1 and n_2 might be infinite.

Following the proof in the commutative case, one finds that the presentation 9.6 of $A=A_k$ can be lifted to a presentation of a first order deformation A_R, where $R=k[t]/(t^2)$. To give such a lifting, one must choose elements $g_i\in P_k$ so that there exist $s_{ij}\in E_k=E(P_k)$ with $(f+gt)(r+st)=0$, or that $gr\equiv 0$ (modulo $\mathfrak{a}_k$).

Lemma 9.7. *There are canonical isomorphisms*

$$\mathfrak{a}/\mathfrak{a}^2\approx\mathfrak{a}\otimes_P A\approx A\otimes_P\mathfrak{a}\approx\mathfrak{a}\otimes_{E(P)}E(A).$$

With this in mind, one finds

Proposition 9.8. *The first order embedded deformations of the algebra A_k presented by 9.6 are in bijective correspondence with elements of* $\mathrm{Hom}_{E(A)}(\mathfrak{a}/\mathfrak{a}^2,A)$.

To carry over the description of deformations as algebra, we use the module $\Omega(A)$ of noncommutative differentials. For any k-algebra A, multiplication defines a surjective map of bimodules $E(A)\longrightarrow A$, which sends $a\otimes b\mapsto ab$. (It is not a ring homomorphism unless

A is commutative.) The kernel of the multiplication map is canonically isomorphic to the bimodule $\Omega(A)$ of *noncommutative differentials* of A:

(9.9) $$0 \longrightarrow \Omega(A) \longrightarrow E(A) \longrightarrow A \longrightarrow 0.$$

A *derivation* from A to an A-bimodule M is a k-linear map d such that

(9.10) $$d(ab) = a\,db + da\,b.$$

There is a canonical derivation $d : A \longrightarrow \Omega(A)$ defined by

$$da = a^\circ - a,$$

and this derivation is universal. In other words, derivations from A to a bimodule M are in bijective correspondence with bimodule homomorphisms, or with homomorphisms of right $E(A)$-modules, $\Omega(A) \longrightarrow M$.

The most important properties of the module of noncommutative differentials are:

(9.11) (a) If $P = k\langle x_1, ..., x_m\rangle$ is a free ring, then $\Omega(P)$ is a free $E(P)$-module generated by $dx_1, ..., dx_m$.

(b) If $A = B/\mathfrak{a}$, then Ω_A is a quotient of Ω_B, and there is an exact sequence of A-modules

$$\mathfrak{a}/\mathfrak{a}^2 \xrightarrow{d} \Omega_B \otimes_{E(B)} E(A) \longrightarrow \Omega_A \longrightarrow 0.$$

Interlude: Noncommutative differentiation.

Let $P = k\langle x, y\rangle$ be the ring of noncommutative polynomials in the variables x and $y = y_1, ..., y_r$. The *partial derivative* f_x of a polynomial $f(x, y) \in k\langle x, y\rangle$ is defined to be the element of $E = E(P)$ such that for a central variable t and a noncommuting variable u,

(9.12) $$f(x + ut, y) = f(x, y) + uf_x t + O(t^2).$$

For example, let h be the monomial xyx^2, then

$$h(x + ut, y) = xyx^2 + (uyx^2 + xyux + xyxu)t + *t^2, \text{ and}$$

(9.13) $$h_x = yx^2 + (xy)^\circ x + (xyx)^\circ = yx^3 + y^\circ x^\circ x + x^\circ y^\circ x^\circ.$$

Taylor 9.14: Let x_i, u_i be noncommuting variables, and let t be a central infinitesimal with $t^2 = 0$. Let $f(x) \in k\langle x_1, ..., x_m\rangle$. Then

$$f(x + ut) = f(x) + \sum_i u_i f_{x_i} t.$$

Differential 9.15: If $f = f(x_1, ..., x_m)$ then $df = dx_1 f_{x_1} + \cdots + dx_m f_{x_m}$.

Similarly, let $f(x,y), g(x,y)$ be polynomials in two variables. Then

$$(df, dg) = (dx, dy)J,$$

where J is the jacobian matrix

$$\frac{\partial(f,g)}{\partial(x,y)} = \begin{pmatrix} f_x & g_x \\ f_y & g_y \end{pmatrix}. \tag{9.16}$$

Product Rule 9.17: $(fg)_x = f_x g + g_x f^\circ$.

For example, let $f = xy$, $g = x^2$, so that $fg = h$ is the monomial considered above. Then $f_x = y$, $g_x = x^\circ + x$, and

$$f_x g + g_x f^\circ = (y)(x^2) + (x^\circ + x)(xy)^\circ = yx^2 + x^\circ y^\circ x^\circ + y^\circ x^\circ x,$$

which agrees with 9.13.

Chain Rule 9.18: Suppose that $x = x(u,v)$, $y = y(u,v)$. Then

$$f_u = x_u f_x + y_u f_y.$$

Exercises: 1. Verify the above formulas.

2. Verify the *Commutator Formula* for polynomials in two variables:

$$[f,g] = [x,y]\,\delta, \quad \text{where } \delta = \text{“}det\ J\text{''} = f_x g_y - g_x f_y.$$

Now going back to the deformation problem, if $f + gt$ generates the defining ideal of a first order deformation A_R, then an infinitesimal change of variable $x \mapsto x + ut$ results in

$$f(x+ut) + g(x+ut)t = f(x) + (\sum_i u_i f_{x_i} + g(x))t, \tag{9.19}$$

which leads as in the previous section to the following. Let $\mathrm{T}^1(A)$ be the left $E(A)$-module which makes the following sequence exact:

$$\mathrm{Hom}_{E(A)}(\Omega(P) \otimes_{E(P)} E(A), A) \xrightarrow{d^*} \mathrm{Hom}_{E(A)}(\mathfrak{a}/\mathfrak{a}^2, A) \longrightarrow \mathrm{T}^1(A) \longrightarrow 0. \tag{9.20}$$

Proposition 9.21. *Isomorphism classes of first order deformations of the algebra A_k presented by 9.6 are in bijective correspondence with elements of $\mathbb{T}^1(A)$.*

Example 9.22: Let A_k be the commutative ring $k[x,y]/(x^2,y^2)$, which has the non-commutative presentation $k\langle x,y\rangle/\mathfrak{a}$, where $\mathfrak{a} = (x^2, y^2, yx - xy)$. The elements $1, x, y, xy$ represent a basis of A. There are two things which simplify the computation of $\mathbb{T}^1$. First, the generators for the ideal are homogeneous, both in x and in y. So A and $M = \mathfrak{a}/\mathfrak{a}^2$ are bigraded, with respect to the degrees in x and y. Second, since A is commutative, the multipication map $E(A) = E \longrightarrow A$ is a k-algebra homomorphism. Therefore $\mathrm{Hom}_E(M, A) \approx \mathrm{Hom}_A(M \otimes_E A, A)$. The tensor product $\overline{M} = M \otimes_E A$ is the quotient of M obtained by forcing left and right multiplications to be equal: $am = ma$. The bigrading carries over to $\overline{M}$.

The module $\overline{M}$ is generated by three elements: m_1, m_2, m_3, the residues of the elements $x^2, y^2, yx - xy$, and direct computation shows that $xm_3 + m_3x = ym_1 - m_1y$ in the free ring, hence in $\overline{M}$. Since left and right multiplications are equal in $\overline{M}$, it follows that $m_3x = 0$ and similarly $m_3y = 0$. On the other hand, using the bigrading one sees that the generators m_1 and m_2 are independent and that $\overline{M} \approx A \oplus A \oplus k$. So if ϕ is a homomorphism $\overline{M} \longrightarrow A$, then $\phi(m_1)$ and $\phi(m_2)$ can be artibtrary, and $\phi(m_3)$ is required to be a multiple of the socle xy.

Bringing this computation back to earth, its implications are that every embedded first order deformation of A can be presented in exactly one way as $R\langle x,y\rangle/\mathfrak{a}_R$, where $\mathfrak{a}_R$ is the two sided ideal generated by

$$\begin{aligned} f_1 &= x^2 + (a_0 + a_1x + a_2y + a_3xy)t, \\ f_2 &= y^2 + (b_0 + b_1x + b_2y + b_3xy)t, \\ f_3 &= yx - xy + cxyt, \quad \text{with } a_i, b_i, c \in k. \end{aligned}$$

A change of variable $x, y \mapsto x + ut, y + vt$, with $u = u_0 + u_1x + u_2y + u_3xy$, and $v = v_0 + v_1x + v_2y + v_3xy$ adds $(2u_0x + 2u_2xy)t$ to f_1 and $(2v_0y + 2v_1xy)t$ to f_2, leaving f_3 unchanged. Thus isomorphism classes of algebra deformations depend on 5 parameters, and can be presented uniquely in the form

$$\begin{aligned} f_1 &= x^2 + (a_0 + a_1x)t, \\ f_2 &= y^2 + (b_0 + b_2y)t, \\ f_3 &= [y, x] + cxyt, \quad \text{with } a_i, b_i, c \in k. \end{aligned}$$

Exercise: Compute the first order commutative deformations of A and compare.

Reference:

H. Cartan and S. Eilenberg, *Homological algebra*, Princeton 1956.

10. Interlude: Gröbner Bases.

Gröbner Bases are tools for computing in an algebra with a given presentation. The method is also often called *Bergman's Diamond Lemma.*

We order monomials lexicographically. This means that if m, m' are monomials, then

(10.1) $m < m'$ if either

(i) $\deg(m) < \deg(m')$, or

(ii) $\deg(m) = \deg(m')$ and m comes before m' in the dictionary.

Given a nonzero polynomial $f \in P = k\langle x_1, ..., x_m\rangle$, we may write

$$f = cm - \phi, \tag{10.2}$$

where m is the largest monomial appearing in f and $c \in k$ is not zero. Thus every monomial appearing in ϕ is smaller than m. For instance, if $f_1 = y^2 - x^2 + y$, then $m_1 = y^2$ and $\phi_1 = x^2 - y$.

Let $S = \{f_i\}$ be a set of elements of $P = k\langle x_1, ..., x_m\rangle$, let $\mathfrak{a}$ be the ideal they generate, and let $A = P/\mathfrak{a}$. We normalize f_i so that its leading coefficient is 1, writing

$$f_i = m_i - \phi_i. \tag{10.3}$$

Then $m_i = \phi_i$ in A. A monomial z is called *reduced* with respect to the set S if none of the monomials m_i appear as a submonomial of z. Similarly, a polynomial g is *reduced* if no m_i is a submonomial of any monomial which appears in g. We can obtain a reduced polynomial from any polynomial in a finite number of steps, each of which consists in replacing a submonomial m_i by the polynomial ϕ_i and expanding. In other words, if u, v are monomials, we replace an occurence of um_iv by $u\phi_iv$. Since all terms of ϕ_i are smaller than m_i, this process terminates.

Example 10.4: The set S consists of the single polynomial $f_1 = y^2 - x^2 + y$. We may reduce $g = y^3$ as follows:

$$y^3 = (y^2)y \longrightarrow (x^2 - y)y = x^2y - y^2 \longrightarrow x^2y - x^2 + y. \tag{10.5}$$

The polynomial on the right is reduced.

Lemma 10.6. *Let g' be polynomial obtained from g by a sequence of replacements, as above. Then $g - g' \in \mathfrak{a}$, i.e., $g = g'$ in A.*

This being so, it is natural to ask whether computation in the quotient ring A can be done by working with reduced polynomials. This is not always the case. The reason is that the process of reducing a polynomial g can often be done in several ways, and the reduced polynomial "g_{red}" which appears at the end need not be uniquely determined by g.

Example 10.7: There is a second way to proceed with the reduction of y^3, namely

$$y^3 = y(y^2) \longrightarrow y(x^2 - y) = yx^2 - y^2 \longrightarrow yx^2 - x^2 + y. \tag{10.8}$$

The term on the right is a different reduced polynomial from the one obtained before. On the other hand, Lemma 10.6 tells us that the two reduced polynomials are equal in A. The same element of A may be represented by several reduced polynomials.

An *overlap* of monomials m, m' is an equality in P of the form $um = m'v$, where u, v are monomials. For example, the monomials $m = yz^2y$ and $m' = xyz$ overlap: $x(yz^2y) = (xyz)zy$. The monomial $m = y^2$ overlaps itself: $ym = my$. Whenever an overlap $um_i = m_jv$ appears as a submonomial in a polynomial g, the replacements $um_i \to u\phi_i$ and $m_jv \to \phi_jv$ provide two ways to proceed with the reduction process. The overlap $um_i = m_jv$ is called *consistent* if these two procedures lead to the same end result. More precisely, consitency means that *there exists* a sequence of reduction steps starting with $h = (u\phi_i - \phi_jv)$, which ends with the zero polynomial. We might write this in shorthand as

$$"(u\phi_i - \phi_jv)_{red} = 0", \tag{10.9}$$

though, since there may be several ways to reduce the polynomial h, not all of which need give the same end result, the terminology is ambiguous.

Once it has been stated, the main result is not difficult to prove:

Proposition 10.10. *The following are equivalent:*

(i) The reduced monomials form a k-basis for A.
(ii) The polynomial g_{red} obtained by reducing a polynomial g is independent of the sequence of reduction steps used.
(iii) Every overlap of the monomials $\{m_i\}$ is consistent.
(iv) Definition: $\{f_i\}$ is a Gröbner basis of the ideal $\mathfrak{a}$.

If we have a Gröbner basis for $\mathfrak{a}$, then we can compute in the ring A by identifying elements of A with reduced polynomials. Addition is polynomial addition, and the product of two reduced polynomials g, h is obtained by reducing the polynomial product: $g \cdot h = (gh)_{red}$.

Besides verifying whether or not one has a Gröbner basis, the process of checking consistency of the overlaps provides a potential method of obtaining a Gröbner basis from any finite set of generators $S = \{f_i\}$ for an ideal $\mathfrak{a}$. The method need not terminate, but it does so in many cases.

The method is as follows:

(10.11)

Step 1: *(Reducing the Replacements)* We order the $S = \{f_1, ..., f_r\}$ so that m_i are in lexicographic order. Then we simplify f_i if possible, by reducing with respect to the set $\{f_1, ..., f_{i-1}\}$. We throw out any zero polynomials. This Step is repeated until no further simplification can be made. Then we proceed to Step 2.

Step 2: *(Checking Consistency of the Overlaps)* We choose an overlap $um_i = m_jv$, and we reduce $u\phi_i - \phi_jv$ in an arbitrary way. If $(u\phi_i - \phi_jv)_{red} = h$ is is not zero, we normalize h, and add it to our set S as $f_{r+1} = m_{r+1} - \phi_{r+1}$. We go back to Step 1 and start over. If h is zero, we choose another overlap. We stop when all overlaps are consistent.

It is most efficient to work with *reduced Gröbner bases*, ones for which this process has been completed.

The reason that this method makes progress is that the new replacement $m_{r+1} \longrightarrow \phi_{r+1}$ introduced in Step 2 changes the overlap $um_i - m_jv$ into a consistent one. For, $f_{r+1} = m_{r+1} - \phi_{r+1}$ reduces to $\phi_{r+1} - \phi_{r+1} = 0$. By construction, $u\phi_i - \phi_j v$ reduces to a constant multiple of f_{r+1}, hence to zero.

Unfortunately, adding m_{r+1} to our list of monomials which are to be replaced is likely to introduce some new overlaps to check. That is why the process may not terminate.

Example 10.12: We begin as before with the set consisting of the single polynomial $f_1 = y^2 - x^2 + y$. There is one overlap $ym_1 = m_1y$, and $y\phi_1 - \phi_1 y = yx^2 - x^2y$ is a reduced polynomial. So the overlap is not consistent. Our method requires us to add a new polynomial $f_2 = yx^2 - x^2y$ to our set. Here $m_2 = yx^2$ and $\phi_2 = x^2y$. This introduces one new overlap $ym_2 = m_1x^2$. We reduce $y\phi_2 - \phi_1 x^2$:

$$(10.13) \quad y(x^2y) - (x^2 - y)x^2 \longrightarrow x^2y^2 - x^4 + x^2y \longrightarrow x^2(x^2 - y) - x^4 - x^2y = 0.$$

This overlap is consistent, and $\{f_1, f_2\}$ is a reduced Gröbner basis.

The case that the polynomials f_i are homogeneous is particularly nice. In that case the reduction process preserves the degree of a homogeneous polynomial. And because an overlap $um_i = m_jv$ will be a monomial of higher degree than either m_i or m_j, the new replacement, if required, has higher degree than either f_i or f_j.

Remark 10.14. *An ideal $\mathfrak{a}$ which is generated by homogeneous elements has a Gröbner basis $\{f_i\}$ in which there are finitely many polynomials of each degree.*

Example 10.15: We start with the single polynomial $f_1 = y^2 - xy$, so that $m_1 = y^2$, $\phi_1 = xy$. There is one overlap $ym_1 = m_1y$, so we reduce $y\phi_1 - \phi_1 y = yxy - xy^2$, obtaining $f_2 = yxy - x^2y$. This introduces three new overlaps: $yxm_1 = m_2y$, $ym_2 = m_1xy$, and $yxm_2 = m_2xy$. Reducing $yx\phi_1 - \phi_2 y = yx^2y - x^2y^2$, we obtain $f_3 = yx^2y - x^3y$. The Gröbner basis obtained by continuing this process contains one polynomial $f_n = yx^{n-2}y - x^{n+1}y$ in every degree $n \geq 2$.

Exercise: Analyze $f_1 = y^2 - xy + x^2$.

It is natural to ask whether Gröbner bases can be used when the ground field k is replaced by an arbitrary ring R. So let us consider an ideal $\mathfrak{a}_R \subset P_R = R\langle x_1, ..., x_m\rangle$ which is generated by some polynomials $f_i \in P_R$. A difficulty arises immediately: If we write $f = cm - \phi$ where m is the leading coefficient, then its coefficient $c \in R$ need not be a unit. And if c isn't a unit, we can't normalize f to eliminate it. This problem is fundamental, because the reduction process would require replacing an occurence of m by $c^{-1}\phi$. It can not be carried out.

On the other hand, it may happen that we are given a set of generators for an ideal $\mathfrak{a}_R$ and that the leading coefficients are all equal to 1, say $\{f_i = m_i - \phi_i\}$. In this special case replacing m_i by ϕ_i is permissible, and Proposition 10.10 carries over without change.

Proposition 10.16. *Let R be a commutative ring and let $S = \{f_i = m_i - \phi_i\}$ be a set of elements of P_R each of whose leading monomials m_i has coefficient 1. The following are equivalent:*

(i) The reduced monomials form an R-basis for $A_R = P_R/\mathfrak{a}_R$.

(ii) The polynomial g_{red} obtained by reducing a polynomial g is independent of the sequence of reduction steps used.

(iii) Every overlap of the monomials $\{m_i\}$ is consistent.

(iv) Definition: $\{f_i\}$ is a Gröbner basis of the ideal $\mathfrak{a}$.

Notice that according to this proposition, a Gröbner basis will not exist unless A_R has a basis of monomials. In particular, if A_R is not flat over R, then there is no Gröbner basis for the ideal $\mathfrak{a}_R$. Also, note that Step 2 of the process 10.11 can't always be carried out, because the polynomial $(u\phi_i - \phi_j v)_{red} = h$ may have a leading coefficient which is not a unit.

Reference:

G. Krause and T.H. Lenagan, *Growth of algebras and Gelfand-Kirillov dimension*, Pitman, London 1985.

11. First order deformations via Gröbner bases.

To apply the discussion of Gröbner bases to deformations, we need to modify the discussion a bit. We suppose that R is a finite local k-algebra with maximal ideal $\mathfrak{m}$, and that $P_R = R\langle x_1, ..., x_m\rangle$ as before. We may write a polynomial $f \in P_R$ in the form $f = u(f) - \eta(f)$ where $u(f)$ is the sum of terms having unit coefficient, and $\eta(f)$ is the sum of the remaining terms. Thus $u(f) \neq 0$ if and only if the residue of f in P_k is not zero. Suppose this is the case, and let m be the largest monomial appearing in $u(f)$. Then f can be written uniquely in the form

$$f = cm - \psi - \eta, \tag{11.1}$$

where c is a unit of R, $\psi = \psi(f)$ is a polynomial with unit coefficients, all of whose monomials are smaller than m, $\eta = \eta(f)$ has coefficients in $\mathfrak{m}$, and where no monomial appears more than once in this expression. We will refer to m as the *leading monomial* of f.

Let $S = \{f_i\}$ be a set of polynomials none of whose residues in P_k is zero, let $\mathfrak{a}_R$ be the ideal they generate in P_R, and let $A_R = P_R/\mathfrak{a}_R$. We normalize so that $f_i = m_i - \psi_i - \eta_i$ as above, and we consider the process of replacing m_i by $\psi_i + \eta_i$ in a given polynomial g. The monomials appearing in η_i may be larger than m_i. Nevertheless this process terminates in a finite number of steps, because the coefficients of η_i are in $\mathfrak{m}$ and $\mathfrak{m}$ is a nilpotent ideal. Specifically, progress is being made when measured using the following partial order on multiples of monomials:

Let $\nu(r)$ denote the largest power $\mathfrak{m}^\nu$ of the maximal ideal which contains an element $r \in R$. We define a partial order on muliples rm of monomials by the rule

(11.2) $rm < r'm'$ if either

(i) $\nu(r) > \nu(r')$, or else

(ii) $\nu(r) = \nu(r')$ and $m < m'$ in lexicographic order.

Lemma 11.3. *(i) The above partial order has the descending chain condition.*

(ii) Let u, v be monomials, and $r \in R$. With the above notation, all terms in the expansion of $ru(\psi_i + \eta_i)v$ are smaller than rum_iv.

This lemma allows us to speak of reduced polynomials and consistent overlaps as before.

Proposition 11.4. *Let R be a finite local k-algebra, and let $S = \{f_i\}$ be a set of elements of P_R none of whose reductions in P_k is zero. Then the conditions (i)-(iv) of 10.16 are equivalent.*

This proposition applies directly to deformations. We suppose given a Gröbner basis $\{f_i = m_i - \phi_i\}$ for the defining ideal $\mathfrak{a}_k$ of $A_k = P_k/\mathfrak{a}_k$, and that $R = k[t]/(t^2)$. We lift the elements $\{f_i\}$ to R, say as $f_i + g_it$, where $g_i \in P_k$. Let $\mathfrak{a}_R$ be the ideal they generate in P_R, and let $A_R = P_R/\mathfrak{a}_R$. We write $f_i + g_it = m_i - \psi_i - \eta_i$ as above. Then η_i is the sum of the terms in g_it which involve monomials that are not present in f_i, and $\psi \equiv \phi$, modulo $\mathfrak{m}$. The leading monomial m_i is the same as for the polynomial f_i. Since the ideal $\mathfrak{a}_R$ depends only on the residues of g_i in A_k, we may assume that g_i are reduced polynomials.

Proposition 11.5. *The algebra A_R is flat over R if and only if $\{f_i + g_it\}$ is a Gröbner basis for $\mathfrak{a}_R$.*

Proof. A_R is flat over R if and only if an arbitrary lifting of a k-basis of A_k to A_R is an R-basis for that ring. Since $\{f_i\}$ is a Gröbner basis for the ideal $\mathfrak{a}_k$, the reduced monomials form a k-basis for A_k. And, the overlaps for $\{f_i + g_it\}$ are consistent if and only if the same reduced monomials form an R-basis for A_R.

Example 11.6 We start with the ideal with Gröbner basis $\{f_1 = y^2 - x^2 + y,\ f_2 = yx^2 - x^2y\}$ considered in the last section. We try a perturbation of the replacements, of the form $y^2 = x^2 - y + gt$, and $yx^2 = x^2y + ht$, and we check the two overlaps $ym_1 = m_1y$ and $ym_2 = m_1x^2$:

$$y(x^2 - y + gt) - (x^2 - y + gt)y \longrightarrow yx^2 - x^2y + (yg - gy)t$$

$$\longrightarrow (x^2y + ht) - x^2y + (yg - gy)t = (h + yg - gy)t.$$

$$y(x^2y + ht) - (x^2 - y + gt)x^2 = yx^2y + yht - x^4 + yx^2 - gx^2t$$

$$\longrightarrow \cdots \longrightarrow (x^2g - gx^2 + h + hy + yh)t.$$

In order to have a flat deformation, we must have $h + yg - gy = 0$ and $x^2g - gx^2 + h + hy + yh = 0$ in A_k. Now since $yx^2 = x^2y$ in A_k, x^2 represents a central element of A_k. So the second relation reduces to $h + hy + yh = 0$. We solve the first relation for $h = gy - yg$

and substitute into the second one, obtaining $gy - yg = 0$ in A_k. Thus deformations are classified by elements $g \in A_k$ such that $gy = yg$.

12. Commutative Gröbner bases and commutative deformations.

Gröbner bases can also be used in the commutative setting. One simply takes the commutative polynomial ring $P_k = k[x_1, ..., x_m]$ and works with lexicographically ordered monomials. Proposition 10.10 carries over without change. There is an important additional fact in the commutative situation, which does not hold for noncommutative rings.

Theorem 12.1. *The procedure 10.11 leads to a finite Gröbner basis in finitely many steps.*

This theorem is mainly of theoretical interest. If the ideal is sufficiently nasty, checking overlaps will cause your computer to crash.

Example 12.2. Take $S = \{f_1, f_2\}$, where $f_1 = xy^3 - 1$, and $f_2 = x^3y - 1$. So $m_1 = xy^3$, $m_2 = x^3y$. There is one overlap: $x^2m_1 = m_2y^2$. Then $x^2(\phi_1) - (\phi_2)y^2 = x^2 - y^2$ is reduced, so we must add $f_3 = y^2 - x^2$ to our list. Then f_1 can be simplified: $f_1 = xy^3 - 1 \longrightarrow x^3y - 1 = f_2$. So this replacement is redundant, and we throw it out, after which we are left with $\{f_3, f_2\}$, in lexicographic order of the leading monomial. There is one overlap: $x^3m_3 = m_2y$, and $x^3(x^2) - (1)y = x^5 - y$. We set $f_4 = x^5 - y$. The relations are now $\{f_3, f_2, f_4\}$, and one additional overlap has been introduced: $m_4y = x^2m_2$. It is consistent, so we have a Gröbner basis.

This algebra has finite dimension over k. To determine its dimension, it suffices to count the number of reduced monomials, which are the monomials not divisible by y^2, x^3y, or x^5. The reduced monomials are: $\{1, x, x^2, x^3, x^4, y, xy, x^2y\}$. So $\dim_k A_k = 8$.

Exercise: 1. The relation $x^8 = 1$ holds in A_k. Show that the Gröbner basis determined from the generators $\{x^8 - 1, x^5 - y\}$ for the ideal is the same as the one just computed.

2. Prove that the reduced Gröbner basis for an ideal is unique.

As in the noncommutative case, first order deformations are classified by perturbations $f_i + g_it$, $i = 3, 2, 4$, where g_i are reduced polynomials in P_k, and such that the overlaps are consistent. In our example, the ring A_k is abstractly isomorphic to $k \oplus \cdots \oplus k$, the sum of 8 copies of the ground field, and the presentation corresponds to embedding eight points in the plane $\mathbb{A}^2$ as the locus $x^8 = 1, y = x^5$. Embedded deformations ought to consist in infinitesimal motions of these points in the plane. An infinitesimal motion of a point is given by a vector having two components.

Exercise: Compute the first order deformations in this case, and show that they depend on 16 parameters, as predicted.

13. Hochschild Cohomology:

Let R be a commutative ring, A an R-algebra which is projective (eg. free) as R-module, and M an A-bimodule. Suppose that the ring of "scalars" R acts centrally on M. Then M is a right module over the enveloping algebra $E = E(A) = A^\circ \otimes_R A$.

The *Hochschild cohomology* of A, M is defined to be

$$H^q(A, M) = \text{Ext}^q_E(A, M), \tag{13.1}$$

in the category of right E-modules.

In principle, Hochschild cohomology can be computed in terms of a free resolution of A as right E-module:

$$\mathcal{R} = \cdots \xrightarrow{d_3} E^{n_2} \xrightarrow{d_2} E^{n_1} \xrightarrow{d_1} E \xrightarrow{d_0} A \longrightarrow 0, \tag{13.2}$$

where d_0 is the multiplication map $a^\circ \otimes b \mapsto ab$. (The exponents n_i might be infinite.) Dropping the augmentation A and taking $\text{Hom}_E(\mathcal{R}., M)$ yields a complex of the form

$$\cdots \xleftarrow{\delta_3} M^{n_2} \xleftarrow{\delta_2} M^{n_1} \xleftarrow{\delta_1} M \longleftarrow 0 \tag{13.3}$$

whose cohomology is $\text{Ext}^{\cdot}_E(A, M)$.

Since the maps d_i of 13.2 are right linear, they can be identified as left multiplication by a suitable matrix P_i with entries in E. Then δ_i is right multiplication by P_i, which sends $M^{n_{i-1}} \longrightarrow M^{n_i}$. Of course, right multiplication by P_i may not commute with right multiplication by elements of E, so δ_i is only R-linear. The Hochschild cohomology $H^i(A, M)$ is an R-module, not an A-module or an E-module.

The next proposition describes Hochschild cohomology in low dimensions. To read it, we need to review a few definitions. Let M be an A-bimodule. The *center* $Z(M)$ is the set of elements which commute with all elements of A:

$$Z(M) = \{m \in M \mid am = ma \text{ for all } a \in A\}. \tag{13.4}$$

An *R-derivation* $d : A \longrightarrow M$ is an R-linear map such that

$$d(ab) = a\,db + da\,b$$

for all $a, b \in A$. The set of R-derivations is an R-module $\text{Der}_R(A, M)$. If $m \in M$ is a fixed element, the map $d : A \longrightarrow M$ defined by $da = am - ma$ is an R-derivation, and the derivations of this form are called *inner derivations*. They form a submodule $\text{InnDer}_R(A, M)$ of $\text{Der}_R(A, M)$ isomorphic to $M/Z(M)$.

Let M be a (two-sided) ideal of an R-algebra B such that $M^2 = 0$, and let $A = B/M$. Then the left and right actions of B on M make M into a A-bimodule. This leads us to define an *algebra extension* A by an A-bimodule M as an R-*algebra* B which fits into an exact sequence

$$(13.5) \qquad 0 \longrightarrow M \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0,$$

where π is an algebra homomorphism whose kernel M is an ideal such that $M^2 = 0$.

The *trivial extension* is $B = A \times M$, with multiplication law

$$(13.6) \qquad (a, m)(b, n) = (ab, an + mb).$$

Two algebra extensions B, B' are considered isomorphic only in the obvious case, that there is a map of the exact sequence 13.5 to the corresponding one for B' which is the identity on M and on A. The set of algebra extensions will be denoted by $\mathrm{AlgExt}_R(A, M)$. As Proposition 13.8 shows, this set has the structure of an R-module. Here is the reason that algebra extensions arise in deformation theory:

Proposition 13.7. *Let A be a k-algebra. Then $\mathrm{AlgExt}_k(A, A)$ classifies first order deformations of A.*

Proof. We must show that the classifications of deformations and of k-algebra extensions of A by A are equivalent. Let A_R be a deformation of A over $R = k[t]/(t^2)$. Then since A_R is R-flat, the sequence

$$0 \longrightarrow A \xrightarrow{t} A_R \longrightarrow A \longrightarrow 0$$

is exact, and this sequence makes A_R into an algebra extension of A by A.

Conversely, let

$$0 \longrightarrow A \xrightarrow{i} B \xrightarrow{\pi} A \longrightarrow 0$$

be a k-algebra extension, and set $t = i(1)$. Then $t^2 = 0$, and so t generates a subring $R = k[t]/(t^2)$ of B. Because 1 is central in A, it follows that t is central in B, hence that B is an R-algebra. This realizes B as a flat deformation of A. Moreover, any isomorphism $\sigma : B \longrightarrow B'$ between k-algebra extensions fixes the element t, hence it is an isomorphism of deformations.

Proposition 13.8. *$H^0(A, M) = Z(M)$, $H^1(A, M) = \mathrm{Der}_R(A, M)/\mathrm{InnDer}_R(A, M)$, and $H^2(A, M) = \mathrm{AlgExt}_R(A, M)$.*

We will verify the assertions for H^0 and H^1, defering H^2 for a while. By definition, $H^0(A, M) = \mathrm{Hom}_E(A, M)$. A map $\phi \in \mathrm{Hom}_E(A, M)$ is an A-bimodule homomorphism, i.e., it is both left and right A-linear. Let $m = \phi(1)$. Then

$$am = a\phi(1) = \phi(a) = \phi(1)a = ma.$$

This shows that m determines ϕ and that $m \in Z(M)$. Conversely, if $m \in Z(M)$, then the map defined by $\phi(m) = am$ is both left and right linear.

Next, an element of $H^1(A, M) = \mathrm{Ext}^1_E(A, M)$ can be described as an isomorphism class of extensions of right *E-modules*

$$0 \longrightarrow M \xrightarrow{i} N \xrightarrow{\pi} A \longrightarrow 0.$$

Such an extension can also be thought of as an extension of A-bimodules: it is both left A-linear and right A-linear. And since A is a projective left A-module, the sequence of left A-modules splits (forgetting the right module structure). Let $s : A \longrightarrow N$ be such a splitting. So s is left linear, and $\pi s = identity$. But s need not be right linear. So $s(a)b$ may not equal $s(ab)$. However, $s(a)b - s(ab) \in M$ because, since π is right linear,

$$\pi(s(a)b) = (\pi s(a))b = ab = \pi s(ab).$$

Let $x = s(1)$, and define $d : A \longrightarrow M$ by $d(a) = ax - xa$. Then d is an R-derivation:

$$a\,db + da\,b = a(bx - xb) + (ax - xa)b = abx - xab = d(ab).$$

The derivation d depends on the left splitting, and two splittings s, s' differ by a homomorphism of left modules $f : A \longrightarrow M$. Let $m = f(1)$. Then $d'(a) = ax' - x'a = a(x+m) - (x+m)a = d(a)a + am - ma$. Thus d' and d differ by an inner derivation.

Conversely, we can use an R-derivation $d : A \longrightarrow M$ to define an extension of bimodules. To do this, we take the trivial extension of left modules ${}_AN = {}_A(A \times M)$, and we define a right A-module structure on N by the rule:

$$(c, m)a = (ca, c\,d(a) + ma). \tag{13.9}$$

The associative law is verified as follows:

$$((c, m)a)b = (ca, c\,d(a) + ma)b = (cab, ca\,d(b) + c\,d(a)b + mab) = (c, m)(ab). \tag{13.10}$$

Hochschild cohomology can also be described by explicit cochains coming from a standard resolution, and we will use this description to show that H^2 classifies algebra extensions. A Hochschild n-cochain is an R-multilinear map $f : A^n \longrightarrow M$, and the coboundary operator *delta* is defined by

$$\begin{aligned}\delta f(a_1, \dots, a_{n+1}) &= a_1 f(a_2, \dots, a_{n+1}) - f(a_1a_2, a_3, \dots, a_{n+1}) \\ &+ f(a_1, a_2a_3, \dots, a_{n+1}) -, +, \cdots \pm f(a_1, \dots, a_na_{n+1}) \mp f(a_1, \dots, a_n)a_{n+1}.\end{aligned} \tag{13.11}$$

Thus a Hochschild 2-cocycle is an R-bilinear map $f : A \times A \longrightarrow M$ such that

$$af(b, c) - f(ab, c) + f(a, bc) - f(a, b)c = 0, \tag{13.12}$$

and a 2-coboundary is a map δg of the form

$$\delta g(a, b) = ag(b) - g(ab) + g(a)b$$

where $g : A \longrightarrow M$ is R-linear, i.e., a homomorphism of R-modules. We will show below that $H^2(A, M)$ is isomorphic to the quotient group $(2 - cocycles)/(2 - coboundaries)$.

Proposition 13.13. *Let B be an algebra extension 13.5, and denote $\pi(b)$ by $\overline{b}$. Let $*: B \times B \longrightarrow B$ be an R-bilinear map which defines a second algebra structure on B, such that π is an algebra homomorphism for this structure, and that $M * M = 0$. Then $*$ has the form*

$$x * y = xy + \phi(\overline{x}, \overline{y}),$$

where $\phi : A \times A \longrightarrow M$ is an R-bilinear map satisfying the cocycle condition 13.11.

Proof. For $x, y \in B$, set $\Phi(x, y) = x * y - xy$. This symbol is R-bilinear because both $x * y$ and xy are. Since π is a homomorphism for the new algebra law, $\pi(x * y) = \pi(x)\pi(y) = \pi(xy)$. Therefore $\Phi(x, y) \in M$. Next, because $M * M = 0$, $x * m = \overline{x}m = xm$ and $m * x = m\overline{x} = mx$ for all $x \in B$ and $m \in M$. Hence $\Phi(x, m) = \Phi(m, x) = 0$. This shows that $\Phi(x, y)$ depends only on the residues $\overline{x}, \overline{y}$ in A, and that there is an R-bilinear map $\phi : A \times A \longrightarrow M$ such that $\Phi(x, y) = \phi(\overline{x}, \overline{y})$.

Besides being R-bilinear, an algebra law must be associative and have an identity. Actually, the existence of an identity will follow automatically from the fact that A has one, by lifting of idempotents. We omit the verification of this fact, and examine the implications of the associative law on ϕ:

$$x * (y * z) = x * (yz + \phi(\overline{y}, \overline{z})) = x(yz) + \phi(\overline{x}, \overline{yz}) + \overline{x}\phi(\overline{y}, \overline{z}),$$

$$(x * y) * z = (xy + \phi(\overline{x}, \overline{y})) * z = (xy)z + \phi(\overline{xy}, \overline{z}) + \phi(\overline{x}, \overline{y})\overline{z}.$$

Collecting terms and using the fact that the original multiplication on B is associative, we see that ϕ is a 2-cocycle.

To complete the classification of isomorphism classes of algebra, we must decide when there is an isomorphism of algebra extensions σ carrying one of these structures to another. Equivalently, we can study the effect on the multiplication law of an R-linear map $\sigma : B \longrightarrow B$ compatible with the identity maps on M and A. Such a map can be written in the form

$$\sigma(x) = x + \mu(\overline{x}), \tag{13.14}$$

where $\mu : A \longrightarrow M$ is R-linear. Which multiplication law in B we take makes no difference for this computation, so let us take the original multiplication "xy". To compute the new law, call it $x * y$, we must conjugate by σ:

$$x * y = \sigma^{-1}(\sigma(x)\sigma(y)) = \sigma^{-1}(xy + x\mu(\overline{y}) + \mu(\overline{x})y) = xy - \mu(\overline{xy}) + x\mu(\overline{y}) + \mu(\overline{x})y).$$

Multiplication has been changed by the Hochschild coboundary $\delta\mu$.

We must now explain why the explicit cocycle computation is correct, and there are two steps in this explanation.

First, A has an augmented simplicial resolution as an A-bimodule, which in dimension n is $\mathcal{S}_n = A^{\otimes(n+2)}$:

$$\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} A \otimes_R A \otimes_R A \rightrightarrows A \otimes_R A \longrightarrow A\,. \tag{13.15}$$

The term *augmented* refers to the last term A, which is not a part of the simplicial complex. Thus $\mathcal{S}_0 = A \otimes_R A$, $\mathcal{S}_1 = A \otimes_R A \otimes_R A$, etc... There are E-linear face maps ∂_i and degeneracy maps s_i as follows:

The *face maps*

$$\partial_i : \mathcal{S}_n = A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+1)} = \mathcal{S}_{n-1} \tag{13.16}$$

are defined for $i = 0, ..., n$ by

$$\partial_i(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

The *degeneracies* will not be important for us, but they are part of the simplicial structure. They are maps

$$s_i : \mathcal{S}_n = A^{\otimes(n+2)} \longrightarrow A^{\otimes(n+3)} = \mathcal{S}_{n+1} \tag{13.17}$$

defined for $i = 0, ..., n$ by

$$s_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

These maps satisfy certain identities, the defining relations for a simplicial set. They are

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \\ s_i \partial_j &= \partial_{j+1} s_i \\ \partial_i s_j &= s_{j-1} \partial_i \\ s_i s_{j-1} &= s_j s_i \end{aligned} \tag{13.18}$$

for all $i < j$. No one can remember them more than a day, but they are trivial. For us the important thing is to note that the alternating sum of the face maps makes $\mathcal{S}$ into a complex. In other words, if $d_n = \partial_0 - \partial_1 + \cdots \pm \partial_n$, then $d_{n-1} d_n = 0$.

Proposition 13.19. *The sequence*

$$\cdots \mathcal{S}_2 \xrightarrow{d_2} \mathcal{S}_1 \xrightarrow{d_1} \mathcal{S}_0 \xrightarrow{d_0} A \longrightarrow 0$$

defined by $d_0 = \partial_0$, $d_1 = \partial_0 - \partial_1$, $d_1 = \partial_0 - \partial_1 + \partial_2$, *etc., is a resolution of* A.

Proof. We must show that the sequence is exact. Let $\mathcal{S} = \bigoplus \mathcal{S}_n$, and let $d : \mathcal{S} \longrightarrow \mathcal{S}$ denote the graded map of degree -1 which in degree n is d_n. To show that the sequence is exact, we must show that im d = ker d. It suffices to find a "homotopy" h, a graded R-linear map $\mathcal{S} \longrightarrow \mathcal{S}$ of degree $+1$ such that $hd + dh = identity$. For, if h has been found and if if $x \in$ ker d, then $x = hdx + dhx = dhx \in$ im d. The required homotopy is $h(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1}$:

$$hd(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = 1 \otimes [(a_0 a_1 \otimes a_2 \otimes \cdots) - (a_0 \otimes a_1 a_2 \otimes \cdots) + \ ...], \tag{13.20}$$

and

$$\begin{aligned} dh(a_0 \otimes \cdots \otimes a_{n+1}) &= d(1 \otimes a_0 \otimes a_1 \otimes \cdots) \\ &= (a_0 \otimes a_1 \otimes \cdots) - (1 \otimes a_0 a_1 \otimes \cdots) + (1 \otimes a_0 \otimes a_1 a_2 \otimes \cdots) \ ...). \end{aligned}$$

Lemma 13.21. *(i) If P, Q are projective R-modules, then $P \otimes_R Q$ is a projective R-module.*
(ii) If $A \longrightarrow B$ is a ring homomorphism and if P is a projective right A-module, then $P \otimes_A B$ is a projective right B-module.

Proof. To say that a module M is projective means that $\mathrm{Hom}_R(M, \cdot)$ is an exact functor. Assertion (i) follows from the canonical isomorphism

$$\mathrm{Hom}_R(P \otimes_R Q, \cdot) \approx \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q, \cdot)),$$

and (ii) from the isomorphism $\mathrm{Hom}_B(P \otimes_A B, \cdot) \approx \mathrm{Hom}_A(P, \cdot)$.

Corollary 13.22. *Let $\mathcal{S}.$ be the complex 13.19. The Hochschild cohomology $H^{\cdot}(A, M)$ cohomology of the complex $\mathrm{Hom}_E(\mathcal{S}., M)$.*

Proof. We must show that $\mathcal{S}.$ is a projective resolution of the right E-module A. We use the isomorphism of right E-modules

$$A^{\otimes n} \otimes_R E \approx A^{\otimes(n+2)} = \mathcal{S}_n \tag{13.23}$$

which is defined by $(a_1 \otimes \cdots \otimes a_n) \otimes (b^\circ \otimes c) \mapsto b \otimes a_1 \otimes \cdots \otimes a_n \otimes c$. Since A is assumed to be R-projective, so is $A^{\otimes n}$. Hence $\mathcal{S}_n$ is E-projective, as required.

An n cocycle $F \in Hom_E(\mathcal{S}., M)$ is an E-linear map $F : A^{\otimes(n+2)} = \mathcal{S}_n \longrightarrow M$ whose coboundary $\delta F := F \circ d_{n+1} : \mathcal{S}_{n+1} \longrightarrow M$ is zero. Explicitly, the coboundary is

$$F(a_0 \otimes ... \otimes a_{n+2}) \tag{13.24}$$

$$= F(a_0 a_1 \otimes a_2 \otimes a_3 \otimes \cdots) - F(a_0 \otimes a_1 a_2 \otimes \cdots) + F(a_0 \otimes a_1 \otimes a_2 a_3 \otimes \cdots) -, +, \cdots .$$

This is not the form that we want. To reduce it to a usable form, we set $\mathcal{C}_n = A^{\otimes n}$, and we use the isomorphism 13.23 to conclude that

$$\mathrm{Hom}_R(\mathcal{C}_n, M) \approx \mathrm{Hom}_E(\mathcal{S}_n, M) \tag{13.25}$$

We make $\mathrm{Hom}_R(\mathcal{C}., M)$ into a complex using this isomorphism.

The E-linear map $F : \mathcal{S}_n \longrightarrow M$ which corresponds to an R-linear map $f : \mathcal{C}_n \longrightarrow M$ is the map

$$F(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) = a_0 f(a_1 \otimes \cdots \otimes a_n) a_{n+1}, \tag{13.26}$$

and conversely,

$$f(a_1 \otimes \cdots \otimes a_n) = F(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1).$$

Then the cocycle condition on 13.23 on F translates on f to

$$\delta f(a_1 \otimes \cdots \otimes a_{n+1}) = \delta F(1 \otimes a_1 \otimes \cdots \otimes a_{n+1} \otimes 1) \tag{13.27}$$

$$= a_1 f(a_2 \otimes \cdots \otimes a_{n+1}) - f(a_1 a_2 \otimes \cdots \otimes a_{n+1}) + f(a_1 \otimes a_2 a_3 \otimes \cdots \otimes a_{n+1}) -, + \cdots f(a_1 \otimes \cdots a_n) a_{n+1}.$$

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14. An example of an obstructed deformation.

The general setup for studying higher order deformations is as in Section 7. We consider an extension of finite local k-algebras

$$0 \longrightarrow k \xrightarrow{\epsilon} R \xrightarrow{\pi} R' \longrightarrow 0, \tag{14.1}$$

where $k\epsilon$ is an ideal of dimension 1, and we suppose given a flat R'-algebra A'. Thus A' is a deformation of its "fibre" $A_k = A' \otimes_{R'} k$, parametrized by R'. We want to study the extensions of A' to R. Recall that such an extension is a flat R-algebra A together with a chosen isomorphism $A \otimes_R R' \longrightarrow A'$. The problem of classifying these extensions can be split into two parts:

(14.2)
Problem 1: Decide whether or not any extensions of A' to R exist, and identify an *obstruction*, which vanishes if and only if an extension exists.

Problem 2: If extensions of A' to R exist, classify them up to isomorphism.

Only the second of these problems arises when studying first order deformations, because one always has available the *trivial deformation* $A_R = A_k \otimes_k R$ of an k-algebra A_k. So if $R = k[t]/(t^2)$, the extension of A_k to R is unobstructed. But if A' is a nontrivial extension of A_k to some ring R', then it may happen that the further extension to R is impossible.

The problem of deforming maps of a point into a singular scheme X provides an analogue which is easy to visualize. Let's take for X the union of the two coordinate axes in $\mathbb{A}^2$, i.e., $X = \text{Spec } B$ where $B = k[x,y]/(xy)$. A map Spec $B \longrightarrow X$ corresponds, by definition, to a homomorphism $B \longrightarrow R$. Take the point $\phi_0 : B \longrightarrow k$ to be the origin: $x = y = 0$. A first order deformation of this point is a tangent vector, a homomorphism $\phi_1 : B \longrightarrow R'$, where $R' = k[t]/(t^2)$. It will have the form $\phi_1(x) = 0 + a_1 t$, $\phi_1(y) = 0 + b_1 t$, with $a_1, b_1 \in k$. In order to be defined on B, the relation $\phi_1(xy) = 0$ must hold. But since $t^2 = 0$, this is true for all a_1, b_1: every vector is tangent to X at the origin. On the other hand, the extension of ϕ_1 to a map $\phi_2 : B \longrightarrow R$, where $R = k[t]/(t^3)$, is not always possible: If we try to define ϕ_2 by $\phi_2(x) = a_1 t + a_2 t^2$, $\phi_2(y) = b_1 t + b_2 t^2$, then $\phi_2(xy) = a_1 b_1 t^2$. In order to extend to second order, either a_1 or b_1 must be zero, i.e., the tangent vector must be either vertical or horizontal.

A similar thing happens with embedded deformations of the commutative ring $A_k = k[x,y]/(x^2, xy, y^2)$ that we looked at before. Recall from Section 2 that there are exactly five algebras of dimension 3 over k, of which four are commutative and one is the noncommutative algebra Δ_k of upper trianglar matrices. All of these algebras can be obtained by deformation from A_k (though not by infinitesimal deformations). On the other hand, one can show that Δ_k and the "generic" commutative ring $k \oplus k \oplus k$ are rigid, i.e., they have no nontrivial deformations. In particular, neither one can be deformed into the other. So heuristically, the commutative deformations and the noncommutative ones must lie on different loci in the space of all deformations, and this space should present a picture quite like that of the two lines. We will see later that this is actually the case.

We present A_k as a noncommutative ring:

(14.3) $$A_k = k\langle x, y\rangle/(x^2, xy, yx, y^2).$$

The ring of triangular matrices has the presentation

(14.4) $$\Delta_k = k\langle x, y\rangle/(x^2 - x, xy - y, yx, y^2),$$

where $x = e_{11}, y = e_{12}$. In both cases, the given relations form Gröbner bases for the defining ideal.

A first order deformation of A_k will have relations of the form

$$x^2 = \alpha_1 t,\ xy = \beta_1 t,\ yx = \gamma_1 t,\ y^2 = \delta_1 t.$$

They are subject to the requirement that the overlaps remain consistent. As we know, the deformed algebra depends only on the residues of $\alpha_1, ..., \delta_1$ in A_k, so we may assume that they are linear polynomials in x, y, say $\alpha_1 = a_{10} + a_{11}x + a_{12}y$, etc...

There are several overlaps, such as $y(x^2) = (yx)x$, to check. Reducing this overlap we obtain $(a_{10}y - c_{10}x)t$. Flatness of the deformation requires that $a_{10} = c_{10} = 0$, and similarly $b_{10} = d_{10} = 0$. The eight remaining coefficients can be arbitrary.

It is clear which deformations are commutative: they are the ones in which $\beta_1 = \gamma_1$. The infinitesimal deformation defined by

$$x^2 = 0\ ,\ xy = yt\ ,\ yx = 0\ ,\ y^2 = 0$$

is a plausible candidate for an obstructed deformation. It is not commutative, and it seems not to head towards the ring of triangular matrices.

We try to extend this deformation to second order, i.e., to the ring $R = k[t]/(t^3)$, by adding a second order term:

$$x^2 = 0 + \alpha_2 t^2,\ xy = yt + \beta_2 t^2,\ yx = 0 + \gamma_2 t^2,\ y^2 = 0 + \delta_2 t^2.$$

Again, the perturbation terms can be taken linear in x, y, say $\alpha_2 = a_{20} + a_{21}x + a_{22}y$, etc. Checking the overlap $y(x^2) = (yx)x$ shows that the constant term a_{20} must vanish.

We compute the overlap $x(xy) = (x^2)y$:

$$x(xy) \longrightarrow xyt + x\beta_2 t^2 \longrightarrow yt^2 + (b_{20} + b_{21}x + b_{22}y)xt^2 \longrightarrow (y + b_{20}x)t^2,$$

$$(x^2)y \longrightarrow \alpha_2 yt^2 \longrightarrow (a_{21}xy + a_{22}y^2)t^2 \longrightarrow 0.$$

For this overlap to be consistent, we must have $y + b_{20}x = 0$. This is impossible.

15. The obstruction in Hochschild cohomology.

With notation as in the beginning of Section 14, we suppose given a flat R'-algebra A', and ask to extend it to a flat algebra A over R. To analyze the obstruction, we choose a basis for A' as R'-module, and take for A the free R-module with the same basis. This gives us an exact sequence of R-modules

$$0 \longrightarrow A_k \xrightarrow{\epsilon} A \longrightarrow A' \longrightarrow 0.$$

Since A is a free module, so is $A \otimes_R A$. This enables us to lift the multiplication law on A' to an R-linear map $\mu : A \otimes_R A \longrightarrow A$. Let us denote the corresponding R-bilinear map $A \times A \longrightarrow A$ by $\mu(x, y) = x \cdot y$. The only axiom for an algebra law which is not taken care of by the bilinear property is the asociative law. We set

$$f(x, y, z) = (x \cdot y) \cdot z - x \cdot (y \cdot z). \tag{15.1}$$

Let $\mathfrak{m}$ denote the maximal ideal of R. Since $\mathfrak{m}A \approx \mathfrak{m} \otimes_R A \approx \mathfrak{m} \otimes_{R'} A'$, the multiplication law $\cdot$ is determined on $\mathfrak{m}A$. It is given by the structure of A'-module on $\mathfrak{m} \otimes_{R'} A'$: If $m \in \mathfrak{m}$ and $x, y \in A$, then $mx \cdot y = m \otimes x \cdot y = m \otimes x'y'$. Similarly, $x \cdot ym = x'y' \otimes m$. Since multiplication in A' is associative, it follows that $f(x, y, z) = 0$ if any one of the three entries is in $\mathfrak{m}A$. So we may view f as an R-trilinear map $A_k \times A_k \times A_k \longrightarrow A_k$, and we may write $f(x, y, z) = f(\overline{x}, \overline{y}, \overline{z})$, where $\overline{x}$ is the residue of x in A_k, etc. We now check that $f(\overline{x}, \overline{y}, \overline{z})$ is a 3-cocycle. The coboundary formula is

$$\delta f(\overline{a}, \overline{b}, \overline{c}, \overline{d}) = \overline{a} f(\overline{b}, \overline{c}, \overline{d}) - f(\overline{ab}, \overline{c}, \overline{d}) + f(\overline{a}, \overline{bc}, \overline{d}) - f(\overline{a}, \overline{b}, \overline{cd}) + f(\overline{a}, \overline{b}, \overline{c})\overline{d}.$$

We note that $\overline{ab}$ is the residue of $a \cdot b$ and that $\overline{a} f(\overline{b}, \overline{c}, \overline{d}) = a \cdot f(b, c, d)$. Lifting δf back to A gives

$$\begin{aligned} \delta f(a, b, c, d) = a \cdot [(b \cdot c) \cdot d - b \cdot (c \cdot d)] - [((a \cdot b) \cdot c) \cdot d - (a \cdot b) \cdot (c \cdot d)] \\ +[(a \cdot (b \cdot c)) \cdot d - a \cdot ((b \cdot c) \cdot d)] - [(a \cdot b) \cdot (c \cdot d) a \cdot (b \cdot (c \cdot d))] \\ +[(a \cdot b) \cdot c - a \cdot (b \cdot c)] \cdot d = 0. \end{aligned}$$

We may change the chosen map μ to $\mu + h$, where h is an R-bilinear map $A \times A \longrightarrow \epsilon A_k$. Computation shows that the cocycle f changes to $f + dh$, i.e., by a Hochschild coboundary. Thus the obstruction to the existence of an associative multiplication on A which extends the law on A' is the class in the Hochschild cohomology $H^3(A, A)$ which is represented by f.

The existence for an identity element on A is automatic, provided only that A_k has an identity element. This is proved by the familiar "lifting of idempotents" argument:

Lemma 15.2. *Let* $0 \longrightarrow M \longrightarrow A \xrightarrow{\pi} A' \longrightarrow 0$ *be an extension of associative algebras, possibly without unit element, such that* $M^2 = 0$.

(i) If $e' \in A'$ *is an idempotent element, there is a unique idempotent* $e \in A$ *such that* $\pi(e) = e'$.

(ii) If A' *has a unit element, so does* A.

Proof. (i) We start with an arbitrary lifting $e_0 \in A$ of e'. Then $e_0^2 \equiv e_0$ (modulo M). Set $m_0 = e_0^2 - e_0$, and $e_1 = e_0 - m_0$. Then $m_1 := {e_1}^2 - e_1 = 2m_0 - m_0e_0 - e_0m_0$. We try e_1 instead of e_0. There is some progress, because $e_1m_1e_1 = 0$. Then $e_1^8 - e_1^4 = (e_1^4 + e_1^2)(e_1^2 - e_1)(e_1^2 + e_1) = (e_1^4 + e_1^2)m_1(e_1^2 + e_1) = 0$. So $e = e_1^4$ is an idempotent lifting of e'. If $e, f = e + x$ are two idempotent liftings of e', so that $x \in M$, then $e + ex + xe = (e + x)^2 = e + x$, so $ex + xe = x$. Multiplying on the left by e yields $xe = 0$ and similarly $ex = 0$. So $x = 0$ and $e = f$.

(ii) Let e be the idempotent lifting of $1 \in A'$. Let $a \in A$. Then $x = ea - a \in M$, hence e operates on x through A as the identity. Therefore $x = ex = e^2a - ea = 0$ and $ea = a$.

16. The abstract approach, and why first order deformations are linear.

In this section we consider deformations abstractly, as a functor on the category $\mathcal{R}$ of finite local k-algebras, morphisms being homomorphisms of k-algebras. The category $\mathcal{R}$ has a nice structure which helps to describe sufficiently well behaved functors. We denote by $\mathcal{R}_n$ the full subcategory whose objects are finite local k-algebras R with the property that that $\mathfrak{m}_R^{n+1} = 0$.

Since we have made an overall assumption that the ground field k is algebraically closed, k will also be the residue field of any element $R \in \mathcal{R}$. The discussion of this section carries over to arbitrary ground fields if we add as requirement for an object R of $\mathcal{R}$ that $R/\mathfrak{m} = k$.

In what follows, we set

$$\mathbb{V} = k[t]/(t^2).$$

We omit the proof of the following simple lemma:

Lemma 16.1. *(i)* k *is a final object of the category* $\mathcal{R}$.

(ii) Let $f_i : R_i \longrightarrow S$, $i = 1, 2$, *be homomorphisms in* $\mathcal{R}_n$. *The fibred product* $R_1 \times_S R_2 = \{(a, b) | a = b \text{ in } S\}$ *is also in* $\mathcal{R}_n$, *and the projection maps* $\pi_i : R_1 \times_S R_2 \longrightarrow R_i$ *are homomorphisms.*

(iii) The categorical product of two objects R, S *of* $\mathcal{R}_n$ *is represented by the fibred product* $R \times_k S = \{(r, s) | r = s \text{ in } k\}$. *It is an object of* $\mathcal{R}_n$, *and for any object* $P \in \mathcal{R}$,

$$\mathrm{Hom}_{\mathcal{R}}(P, R \times_k S) \approx \mathrm{Hom}_{\mathcal{R}}(P, R) \times \mathrm{Hom}_{\mathcal{R}}(P, S).$$

(iv) $\mathbb{V} \times_k S \approx S \oplus kt$, *and* $\mathbb{V} \times_k \mathbb{V} \approx k[t_1, t_2]/(t_1, t_2)^2$.

Given a k-algebra A_k, we obtain the *deformation functor*

$$\mathrm{Def}(A_k; \cdot) : \mathcal{R} \longrightarrow (\mathrm{Sets}), \tag{16.2}$$

where (Sets) denotes the category of sets, by defining $\mathrm{Def}(A_k; R)$ be the set of isomorphism classes of deformations of A_k, parametrized by R. As before, a flat deformation of A_k is an R-algebra A_R together with a chosen isomorphism $A_R \otimes_R k \longrightarrow A_k$, and an isomorphism of deformations must be compatible with this chosen isomorphism. If $R \longrightarrow R'$ is a homomorphism in $\mathcal{R}$, the associated map $\mathrm{Def}(A_k; R) \longrightarrow \mathrm{Def}(A_k; R')$ is defined by tensor product: $A_R \mapsto A_R \otimes_R R'$. Note that $\mathrm{Def}(A_k; k)$ is a set of one element, the class of $\{A_k\}$.

If A_k is a commutative algebra, then we can of course study either commutative deformations, or noncommutative ones. Therefore there are really at least two different deformation functors. But instead of overloading the notation, we tend to allow the ambiguity and denote them both by Def.

More generally, we may consider an arbitrary functor

$$F : \mathcal{R} \longrightarrow (\text{Sets}) \tag{16.3}$$

such that $F(k)$ is a set of one element. We call such a functor a *local functor*. There are two general classes of local functors to keep in mind:

(16.4)(i) Deformations of an algebra: $F = \mathrm{Def}(A_k;\ \cdot\ )$ as above, and

(ii) Deformations of a point p of a scheme X.

The second type is described as follows: A point of $X = \mathrm{Spec}\, B$ corresponds to a homomorphism $\phi_k : B \longrightarrow k$. If $R \in \mathcal{R}$, then we define $F(R)$ as the set of maps $\mathrm{Spec}\, R \longrightarrow X$ such that p is the image of the underlying point $\mathrm{Spec}\, k \subset \mathrm{Spec}\, R$:

$$F(R) = \{\phi_R : B \longrightarrow R \,|\, \phi_R \otimes_R k = \phi_k\}.$$

This is made into a functor by composition of functions: Given a map $g : R \longrightarrow R'$, the map $F(g) : F(R) \longrightarrow F(R')$ sends $\phi_R \mapsto g \circ \phi_R$.

The *tangent vectors* to a local functor F are defined to be the elements of $F(\mathbb{V})$, which, in case $F = \mathrm{Def}(A_k;\ \cdot\ )$, correspond to first order deformations. In the second example, they correspond to tangent vectors to X at p, as was described in Section 5. In all of these examples, the tangent vectors form a k-vector space, and the reason for this is inherent in the structure of the ring $\mathbb{V}$.

Let $\mathcal{C}$ be a category with products and with a final object p. A *k-vector space object* in $\mathcal{C}$ is an object V together with morphisms $\sum : V \times V \longrightarrow V$, $\zeta : p \longrightarrow V$, and $\lambda_c : V \longrightarrow V$ for $c \in k$ which satisfy the axioms for a vector space if $\sum$ represents the addition law, ζ represents zero, and λ_c represents scalar multiplication by c.

The associative law, for example asserts that the composed map $\sum \circ (id \times \sum)$, which sends $V \times (V \times V) \xrightarrow{id \times \sum} V \times V \xrightarrow{\sum} V$ is equal to $\sum \circ (\sum \times id)$.

The identity property of ζ reads as follows: Let g denote the unique map from V to the final object p, and let $id = id_V$. Then $(\zeta \circ g, id)$ maps $V \longrightarrow V \times V$, and the zero property is $\sum \circ (\zeta \circ g, id) = id$.

Proposition 16.5. *The ring $\mathbb{V}$ has the structure of a vector space object in the category $\mathcal{R}$. Identifying $\mathbb{V} \times_k \mathbb{V}$ as the ring $k[t_1, t_2]/(t_1^2, t_1t_2, t_2^2)$, the addition law $\sum : \mathbb{V} \times_k \mathbb{V} \longrightarrow \mathbb{V}$ is defined by $\sum(t_1) = \sum(t_2) = t$, or $\sum(a + b_1t_1 + b_2t_2) = a + (b_1 + b_2)t$. The zero map $\zeta : k \longrightarrow \mathbb{V}$ is the algebra structure map, and scalar multiplication λ_c by $c \in k$ is defined to be $\lambda_c(a + bt) = a + bct$.*

The proof is routine.

Now if $F : \mathcal{R} \longrightarrow$ (Sets) is any functor, and if $R_i \longrightarrow S$, $i = 1, 2$ are maps in $\mathcal{R}$, then functorality gives us a canonical map of sets

(16.6) $$F(R_1 \times_S R_2) \longrightarrow F(R_1) \times_{F(S)} F(R_2),$$

and a canonical map

(16.7) $$F(R_1 \times_k R_2) \longrightarrow F(R_1) \times F(R_2).$$

Corollary 16.8. *Suppose that a local functor $F : \mathcal{R} \longrightarrow$ (Sets) has the property that the canonical maps 16.7 are bijective for all $R_1, R_2 \in \mathcal{R}_1$. Then $F(\mathbb{V})$ inherits the structure of a k-vector space from the structure of vector space object on $\mathbb{V}$.*

Proof. We define the addition on $F(\mathbb{V})$ to be the map $F(\mathbb{V}) \times F(\mathbb{V}) \approx F(\mathbb{V} \times_k \mathbb{V}) \xrightarrow{F(s)} F(\mathbb{V})$, where $\sum$ is as in 16.5. The zero element in $F(\mathbb{V})$ is the image of $F(k)$ via the amp $F(\zeta)$, and scalar multiplication by $c \in k$ is defined to be $F(\lambda_c)$. The axioms for a vector space object show that we obtain a vector space.

One more item of structure on the category $\mathcal{R}$ will be important. It is that a length one extension

(16.9) $$0 \longrightarrow k \xrightarrow{\epsilon} R \xrightarrow{\pi} R' \longrightarrow 0$$

defines a categorical *operation* of the additive group object $(\mathbb{V}, \sum)$ on R. Since $\mathbb{V} \times_k R \approx R \oplus kt$, we can define the operation $op : \mathbb{V} \times_k R \longrightarrow R$ by

(16.10) $$op(x + at) = x + a\epsilon.$$

This map has the associative property which defines a group operation, that two resulting maps $\mathbb{V} \times \mathbb{V} \times R \longrightarrow R$ are equal: $op \circ (id \times op) = op \circ (s \times id)$, and the identity ζ of $\mathbb{V}$ operates trivially: $[op \circ \zeta \circ g](x + at) = x$.

We review some elementary facts about group operations. Suppose given a map of sets $X \longrightarrow X'$ and an operation of a group G on X, and consider the induced map $G \times X \xrightarrow{(op,pr)} X \times X$, where pr is the second projection $G \times X \longrightarrow X$. Then

(16.11)(i) The operation is free if and only if the map (op, pr) is injective.

(ii) G operates on the fibres of X/X' if and only if the map (op, pr) factors through the fibred product $X \times_{X'} X$.

(iii) G operates transitively (resp. simply and transitively) on the fibres if and only if (ii) holds and the map $G \times X \longrightarrow X \times_{X'} X$ is surjective (resp. bijective).

Part (i) of the next lemma shows that all of these properties hold for the categorical operation defined above.

Lemma 16.12. *Suppose given an extension 16.9.*

(i) The homomorphism $\mathbb{V} \times_k R \xrightarrow{(op,pr)} R \times_k R$ *induces a bijective map* $V \times_k R \longrightarrow R \times_{R'} R$.

(ii) The extension is split if and only if $R \approx \mathbb{V} \times_k R'$.

Scholium 16.13. Here is the interpretation of (i) for maps into R: Suppose given a homomorphism $\phi : S \longrightarrow R$, and denote the composed map $\pi\phi$ by ϕ'. If we are also given a derivation $d : S \longrightarrow k$ (which corresponds to a homomorphism $S \longrightarrow \mathbb{V}$), then we can define a new homomorphism $\psi : S \longrightarrow R$ by $\psi(s) = \phi(s) + d(s)\epsilon$. The maps ψ obtained in this way are precisely those such that $\phi' = \psi'$.

Corollary 16.14. *(i) Let* F *be a local functor with the following properties:*

(i) For every pair of elements $R_i \in \mathcal{R}$, *the map 16.7 is bijective,*

(ii) For every length one extension 16.9, the map 16.6: $F(R \times_{R'} R) \longrightarrow F(R) \times_{F(R')} F(R)$ *is surjective (resp. bijective).*

Then the tangent space $F(\mathbb{V})$ *is a vector space whose underlying additive group operates transitively (resp. simply and transitively) on the fibres of the map* $F(R) \longrightarrow F(R')$.

The proof is similar to the proof of Corollary 16.8. The operation of $F(\mathbb{V})$ on $F(R)$ is defined by $F(\mathbb{V}) \times F(R) \approx F(\mathbb{V} \times_k R) \xrightarrow{F(op)} F(R)$, and the axioms for an operation follow.

17. Universal and versal objects.

Suppose given a functor $F : \mathcal{C} \longrightarrow$ (Sets) and an object $S \in \mathcal{C}$. Then if $\phi : S \longrightarrow R$ is any map, the functor sends $F(S) \xrightarrow{F(\phi)} F(R)$. The image of an element $s \in F(S)$ via this map could be written as $[F(\phi)](s)$, which is a element of $F(R)$. However, this notation is cumbersome, so we will usually say that ϕ sends s to r in this situation, or that r is the image of s in $F(R)$, via ϕ.

Let us also fix an element $s \in F(S)$. Then for every $\phi : S \longrightarrow R$ we obtain an element $r \in F(R)$, the image of s, which varies naturally with the map ϕ. In other words, an element $s \in F(S)$ defines a morphism of functors

$$\mathrm{Hom}(S, \cdot\,) \longrightarrow F(\cdot\,). \tag{17.1}$$

The notation $\mathrm{Hom}(S, \cdot\,)$ can also get a bit cumbersome when dealing with morphisms of functors, so we introduce the shorthand notation

$$S(\cdot\,) = \mathrm{Hom}(S, \cdot\,). \tag{17.2}$$

Then to an element $s \in F(S)$, we have associated a morphism of functors, which yields a map

$$F(S) \longrightarrow \mathrm{Hom}_{fun}(S(\cdot\,), F(\cdot\,)). \tag{17.3}$$

The Yoneda lemma asserts that this correspondence is bijective.

Yoneda Lemma 17.4. *Let $F : \mathcal{C} \longrightarrow$ (Sets) be a functor. For every object $S \in \mathcal{C}$, there is a natural bijective correspondence between elements of $F(S)$ and morphisms of functors $S(\cdot) \longrightarrow F(\cdot)$, where $S(\cdot) = \mathrm{Hom}_{\mathcal{C}}(S, \cdot)$. The bijective correspondence is defined as follows: Given a map of functors $f : S(\cdot) \longrightarrow F(\cdot)$, the corresponding element is $f(id_S) \in F(S)$. In the other direction, given $s \in F(S)$ we must describe a rule associating to every map $\phi \in \mathrm{Hom}_{\mathcal{C}}(S, R)$ an element of $F(R)$. That element is $[F(\phi)](s)$.*

An element $\mathbf{u} \in F(\mathbf{U})$ is called *universal* if the induced map $\mathbf{U}(\cdot) \longrightarrow F(\cdot)$ is bijective. In down to earth terms, $\mathbf{u}$ is universal if and only if it has the following property:

(17.5) For every $R \in \mathcal{C}$ and every element $r \in F(R)$, there is a unique map $\mathbf{U} \longrightarrow R$ which sends $\mathbf{u} \mapsto r$.

Let F be a local functor on the category $\mathcal{R}_n$. Then we can also express the universal property in this way: $\mathbf{u} \in F(\mathbf{U})$ is universal if F is isomorphic to the functor of deformations of the point $p = \mathrm{Spec}\ k \in X = \mathrm{Spec}\ \mathbf{U}$, as was described above.

If $\mathcal{C}$ is a category with fibred products, a functor $F : \mathcal{C} \longrightarrow$ (Sets) is called *left exact* if it commutes with fibred products, i.e., if the induced map

$$F(A \times_B C) \longrightarrow F(A) \times_{F(B)} F(C)$$

is bijective.

Corollary 17.6. *If a functor F has a universal element, then it is left exact.*

This is true because the mapping property of the fibred product shows that the functor $\mathbf{U}(\cdot) = \mathrm{Hom}(\mathbf{U}, \cdot)$ is left exact.

We propose to investigate the question of whether or not a universal object exists for a local functor on $\mathcal{R}_n$. As a preliminary step, we consider functors on the category $\mathcal{V}$ of finite-dimensional k-vector spaces. The relevance of this step to our situation is explained by the next lemma.

Lemma 17.7. *The category $\mathcal{V}$ of finite-dimensional k-vector spaces is equivalent with $\mathcal{R}_1$ via the functor $W \mapsto k \oplus W$.*

Let V denote the vector space k^1. We note that V is made into a k-vector space object in the category $\mathcal{V}$ in the obvious way, by using the actual laws of composition. Therefore, as with the object $\mathbb{V} \in \mathcal{R}$, if $F : \mathcal{V} \longrightarrow$ (Sets) is any sufficiently well-behaved functor, then $X := F(V)$ will be a vector space too.

Proposition 17.8. *Let $F : \mathcal{V} \longrightarrow$ (sets) be a functor such that $F(0)$ is a set of one element.*

(i) Suppose that the canonical map $F(W_1 \times W_2) \longrightarrow F(W_1) \times F(W_2)$ is bijective for all $W_i \in \mathcal{V}$. Then X is a k-vector space.

(ii) If in addition X is a finite-dimensional vector space, then there is an isomorphism of functors $\mathrm{Hom}_k(X^*, \cdot) \longrightarrow F(\cdot)$.

Proof. The proof of (i) is routine. For (ii), we note that by the Yoneda lemma, an element $X \in X = F(V)$ determines a map $W = \mathrm{Hom}(V, W) \longrightarrow F(W)$. This rule gives us a bilinear map $X \times W \longrightarrow F(W)$, hence a functorial map map $\mathrm{Hom}(X^*, W) = X \otimes_k W \longrightarrow F(W)$. This is the morphism of functors $\mathrm{Hom}(X^*, \cdot) \longrightarrow F(\cdot)$. To show that this morphism is an isomorphism, it suffices to check for the case that $W = V$, because every W is isomorphic to k^n for some n, and the functor is compatible with products. Setting $W = V$, we obtain the identity map $X = \mathrm{Hom}(X^*, k) \longrightarrow F(k) = X$.

Under the hypotheses of this proposition, the identity map on X^* corresponds to a universal element of $F(V)$.

On the category $\mathcal{R}_n$, there is a weaker property than universality of an element which is more likely to exist. An element $\mathbf{u} \in F(\mathbf{U})$ is called *versal* if it has the following property. Suppose given a diagram

(17.9)
$$\begin{array}{ccc} & & \mathbf{U} \\ & & \big\downarrow{\scriptstyle \phi'} \\ R & \xrightarrow{\pi} & R' \end{array}$$

Let $r \in F(R)$ be an element with image $r' \in F(R')$. Suppose that $\mathbf{u} \mapsto r'$ via ϕ'. Then there is a map $\mathbf{U} \xrightarrow{\phi} R$ such that $\phi' = \phi \circ \pi$, and such that ϕ sends $\mathbf{u} \mapsto r$.

Corollary 17.10. *If $\mathbf{u} \in F(\mathbf{U})$ is a versal element, then the induced map $\mathbf{U}(R) \longrightarrow F(R)$ is surjective for all R.*

This follows from Lemma 7.1 by induction. However, the versal property is much stronger than surjectivity.

Theorem 17.11. *(Schlessinger's Theorem)*

(a) Let F be a local functor on $\mathcal{R}_n$ which satisfies the following condition:

(i) For all $R, S \in \mathcal{R}_n$, the map

$$F(R \times_k S) \longrightarrow F(R) \times F(S)$$

is bijective.

Then F has a versal element if and only if it also satisfies the following:

(ii) For every extension $R \longrightarrow R'$ of length 1 and every map $S' \longrightarrow R'$, the map

$$F(R \times_{R'} S') \longrightarrow F(R) \times_{F(R')} F(S')$$

is surjective.

(iii) $F(\mathbb{V})$ is a finite dimensional vector space.

(b) A local functor F on $\mathcal{R}_n$ has a universal element if and only if (i),(iii) hold the map (ii) is always bijective.

Proof. The "only if" direction is easy. So assume that (i)-(iii) hold. Since $\mathcal{R}_1$ is equivalent to the category of finite-dimensional vector spaces, Proposition 17.8 implies that this is true when $n = 1$. Suppose that $n > 1$. Then there is an object $\mathbf{U}_1$ of $\mathcal{R}_1$ and an element $\mathbf{u}_1 \in F(\mathbf{U}_1)$ which is *universal* on the restriction $F_1 : \mathcal{R}_1 \longrightarrow$ (Sets) of F to $\mathcal{R}_1$.

Thus $\mathbf{U}_1$ is a local k-algebra and the square of its maximal ideal is zero. Choosing a basis for the maximal ideal, we write $\mathbf{U}_1 \approx k[x_1, ..., x_m]/(x_1, ..., x_m)^2$. Let P denote the truncated polynomial ring $k[x_1, ..., x_m]/(x_1, ..., x_m)^{n+1}$, which is a free object of $\mathcal{R}_n$. We have a surjective map $P \longrightarrow \mathbf{U}_1$, and we try to construct $\mathbf{U}$ as a quotient of P. We consider quotients $\mathbf{U} = P/I$, such that $I \subset (x_1, ..., x_m)^2$, and such that there is an element $\mathbf{u} \in F(\mathbf{U})$ whose image in $F(\mathbf{U}_1)$ is $\mathbf{u}_1$, i.e., such that the element $\mathbf{u}_1$ lifts to $F(\mathbf{U})$. Since P is finite-dimensional, we may choose an $\mathbf{U}$ such that $\mathbf{u}_1$ lifts to $\mathbf{u} \in F(\mathbf{U})$, and which has *maximal dimension* as k-vector space. We will prove that $\mathbf{u}$ is versal.

¿From the data (ii), we obtain a row-exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & k & \xrightarrow{\epsilon} & R \times_{R'} \mathbf{U} & \longrightarrow & \mathbf{U} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \phi'} & & \\ 0 & \longrightarrow & k & \xrightarrow{\epsilon} & R & \xrightarrow{\pi} & R' & \longrightarrow & 0 \end{array} \tag{17.12}$$

Condition (ii) of Schlessinger's theorem implies that there is an element $x \in F(R \times_{R'} \mathbf{U})$ mapping to the pair $(r, \mathbf{u}) \in F(R) \times_{F(R')} F(\mathbf{U})$. Since P is free, the map $P \longrightarrow \mathbf{U}$ lifts to a map $f : P \longrightarrow R \times_{R'} \mathbf{U}$. Since $\mathbf{U}$ was chosen maximal with the property that $\mathbf{u}_1$ lifts to $\mathbf{U}$, the map f can not be surjective. Because the kernel of $R \times_{R'} \mathbf{U} \longrightarrow \mathbf{U}$ has dimension 1 and $P \longrightarrow \mathbf{U}$ is surjective, the image of f must be isomorphic to $\mathbf{U}$. This gives us a splitting of the top row of 17.12, and we obtain a map $\phi : \mathbf{U} \longrightarrow R$ such that $\pi \circ \phi = \phi'$.

Let $s \in F(R)$ be the image of the element $\mathbf{u}$ via ϕ. If $s = r$, then we are done. But this need not be the case, so we still have an adjustment to make. But since $\pi\phi = \phi'$, the image of s in $F(R')$ is r'. So $(s, r) \in F(R) \times_{F(R')} F(R)$, and condition (ii) of the theorem allows us to represent this pair by an element $x \in F(R \times_{R'} R)$. By Lemma 16.12, and condition (ii), $F(\mathbb{V}) \times F(R) \approx F(\mathbb{V} \times_k R) \approx F(R \times_{R'} R)$. The pair in $F(\mathbb{V}) \times F(R)$ which corresponds to x has the form (v, r). Since $\mathbb{V} \in \mathcal{R}_1$, $\mathrm{Hom}(\mathbf{U}, \mathbb{V}) \approx F(\mathbb{V})$. Let $d \in \mathrm{Hom}(\mathbf{U}, \mathbb{V})$ be the corresponding element, and let $\psi : \mathbf{U} \longrightarrow R$ be the homomorphism defined by $\psi(z) = \phi(z) + d(z)\epsilon$. We replace ϕ by ψ, which reduces us to the case that $x = (0, r)$. The image of this element in $F(R) \times_{F(R')} F(R)$ is the pair (r, r). So ψ sends $\mathbf{u} \mapsto r$, as required.

It is quite unusual that a functor on the category $\mathcal{R}$ of all finite local k-algebras has a vandersal or universal element, because there is no limit on the order of nilpotence of the maximal ideals $\mathfrak{m}_R$. Instead, the versal element should be parametrized by a complete local ring, which is not in the category $\mathcal{R}$, and to which infinitesimal methods do not necessarily apply.

Let F be a functor on $\mathcal{R}$, and let $\mathbf{U}$ be a complete local k-algebra with residue field k, and let $\mathbf{U}_n = \mathbf{U}/\mathfrak{m}^{n+1}$. A *formal element* $\hat{\mathbf{u}}$ parametrized by $\mathbf{U}$ is a sequence of elements $\mathbf{u}_n \in F(\mathbf{U}_n)$ such that the projection maps $\mathbf{U}_n \longrightarrow \mathbf{U}_{n-1}$ send $\mathbf{u}_n \mapsto \mathbf{u}_{n-1}$ for all n. A formal element $\hat{\mathbf{u}}$ is *versal*, or *universal* if for every n, $\mathbf{u}_n$ is versal or universal on the restriction of the functor to the category $\mathcal{R}_n$. The previous theorem extends without difficulty to show

Corollary 17.13. *A functor $F : \mathcal{R} \longrightarrow$ (Sets) has a versal, or universal formal element if and only if the conditions of (i)-(iii) of 17.11 hold.*

Proposition 17.14. *(Uniqueness of versal elements) Let F be a local functor on $\mathcal{R}$ satisfying the conditions of Schlessinger's theorem. Let $\mathbf{u} \in F(\mathbf{U})$, $\mathbf{v} \in F(\mathbf{V})$ be versal elements, and assume that the image of $\mathbf{u}_1$ in $F(\mathbf{U}_1)$ is universal. Then*

(i) For some r, there is an isomorphism $\mathbf{V} \approx \mathbf{U}[[x_1, ..., x_r]]$ such that the inclusion $\mathbf{U} \longrightarrow \mathbf{U}[[x]]$ sends $\mathbf{u} \mapsto \mathbf{v}$.

(ii) If the image $\mathbf{v}_1$ of $\mathbf{v}$ in $\mathbf{V}_1$ is also universal, then there is an isomorphism $\sigma : \mathbf{U} \longrightarrow \mathbf{V}$ which sends $\mathbf{u} \mapsto \mathbf{v}$.

The analogous assertions hold for versal elements and functors on $\mathcal{R}_n$.

Proof. Using the versal property of $\mathbf{u}$ and lifting step by step, one obtains a map $\phi : \mathbf{U} \longrightarrow \mathbf{V}$ which sends $\mathbf{u} \mapsto \mathbf{v}$. Similarly, the versal property of $\mathbf{v}$ produces a map $\psi : \mathbf{V} \longrightarrow \mathbf{U}$ sending $\mathbf{v} \mapsto \mathbf{u}$. The composition $\psi\phi$ need not be the identity, but it sends $\mathbf{u} \mapsto \mathbf{u}$. it follows that the induced map $\mathbf{U}_1 \longrightarrow \mathbf{U}_1$ sends $\mathbf{u}_1 \mapsto \mathbf{u}_1$, hence since $\mathbf{u}_1$ is universal, this map is the identity. The next lemma, which follows from Nakayama, shows that $\psi\phi$ is an automorphism.

Lemma 17.15. *Let σ be a map from a complete local ring A to itself, which reduces to the identity modulo $\mathfrak{m}_A^2$. Then σ is an automorphism.*

If $\mathbf{v}_1$ is also universal, then $\phi\psi$ is also an automorphism of $\mathbf{V}$. In that case ϕ and ψ are bijective.

In general, we know only that ϕ is injective. We can write $\mathbf{V}$ as a quotient of a power series ring $P = \mathbf{U}[[x_1, ..., x_r]]$ for some r. Choosing r to be minimal, we will have $P_1 \approx \mathbf{V}_1$. Let $p \in F(P)$ denote the formal element which is the image of $\mathbf{u}$ via the inclusion map $\mathbf{U} \longrightarrow P$. Then the versal property of $\mathbf{v}$ yields a map $\tau : \mathbf{V} \longrightarrow P$ sending $\mathbf{v} \longrightarrow p$, and which is the identity on $\mathbf{V}_1$. By Nakayama, τ is surjective, and by the lemma, the composition of τ with the projection $P \longrightarrow \mathbf{V}$ is an automorphism of $\mathbf{V}$. It follows that τ is an isomorphism.

18. A sample computation of a versal deformation.

The weak point of Schlessinger's theorem is that it does not describe the structure of the ring $\mathbf{U}$ which parametrizes a versal element. We propose to compute it in a simple case, that F is the deformation functor $F(\cdot) = \text{Def}(A_k; \cdot)$, where $A_k = k[x,y]/(x^2, xy, y^2)$, the example we have studied before. It is easier to compute the *commutative* deformations, so we will do that, using the Gröbner basis method.

The first step is to compute the universal first order embedded deformation, and we have done that already. Every first order embedded deformation, parametrized by $\mathbb{V}$, can be described by a unique set of defining equations of the form

$$(18.1) \qquad x^2 = (a_1x + a_2y)t\ ,\ xy = (b_1x + b_2y)t\ ,\ y^2 = (c_1x + c_2y)t.$$

There is a slight conflict of terminology, because up to now we have used the term "first order deformation" to mean a deformation parametrized by $\mathbb{V}$. We now want to use the same term to indicate a deformation parametrized by any ring $R = k \oplus W$ in $\mathcal{R}_1$. The simplest thing will be to allow this ambiguity of terminology.

The universal first order embedded deformation is parametrized by the ring $\mathbf{U}_1 = P/\mathfrak{m}^2$, where $P = k[[a_i, b_i, c_i]]$, and $\mathfrak{m}$ is the maximal ideal at the origin in (a,b,c)-space. The universal element $\mathbf{u}_1 \in F(\mathbf{U}_1)$ is the $\mathbf{U}_1$-algebra whose defining relations are obtained from 18.1 by dropping the symbol t:

$$\begin{aligned}(18.2)\ x^2 &= a_1x + a_2y,\\ xy &= b_1x + b_2y,\\ y^2 &= c_1x + c_2y.\end{aligned}$$

A homomorphism $\mathbf{U}_1 \longrightarrow V$ is described by sending the indeterminates a, b, c to specific multiples of t, thereby yielding defining equations of the form 18.1.

We refer back to the proof of Schlessinger's theorem. To obtain the versal element in higher order n, we look for a minimal ideal $I_n \subset P_n = P/\mathfrak{m}^{n+1}$ such that $\mathbf{u}_1$ lifts to $\mathbf{U}_n = P_n/I_n$. The lifting can be arbitrary. This is an important point: We don't ask that the first order equations 18.2 define a (flat) deformation modulo I_n, but that if we add some undetermined terms of higher order in a, b, c to these relations, then the algebra they define is flat. So we add indeterminate terms to the equation of the general embedded deformation, say

$$\begin{aligned}(18.3)\ x^2 &= a_1x + a_2y + \alpha',\\ xy &= b_1x + b_2y + \beta',\\ y^2 &= c_1x + c_2y + \gamma',\end{aligned}$$

where α', β', γ' are in $P[x,y]$, and terms have degree at least 2 in a, b, c.

As before, all terms of degree ≥ 2 in x, y can be eliminated step by step, using these equations. So we may assume that $\alpha' = \alpha_0 + \alpha_1x + \alpha_2y$, etc...

Next, we make a change of variable in the ring P, replacing a_i by $a_i + \alpha_i$. This eliminates the terms α_1x and α_2y, leaving us with $\alpha' = \alpha_0$, and whose that we may as well assume $\alpha', \beta', \gamma' \in P$.

We ask for the conditions on these elements which imply that the equations 18.3 define a flat algebra. The algebra will be flat if and only if the overlaps are consistent, i.e., if and only if 18.3 is a Gröbner basis. There are two commutative overlaps: $(x^2)y = x(xy)$ and $x(y^2) = (xy)y$. We reduce the first one:

$$(x^2)y \longrightarrow a_1xy + a_2y^2 + \alpha'y \longrightarrow a_1(b_1x + b_2y + \beta') + a_2(c_1x + c_2y + \gamma') + \alpha'y,$$

$$x(xy) \longrightarrow b_1x^2 + b_2xy + \beta'x \longrightarrow b_1(a_1x + a_2y + \alpha') + b_2(b_1x + b_2y + \beta') + \beta'x.$$

The two reduced polynomials must be equal. Extracting coefficients of $1, x, y$, we obtain three relations

(18.4) $\alpha' = a_2b_1 - a_1b_2 + b_1^2 - a_2c_2,$
$\beta' = a_2c_1 - b_1b_2,$
$a_1\beta' + a_2\gamma' = b_1\alpha' + b_2\beta'.$

The second overlap yields three more equations, two of which we can solve for β', γ'. It turns out that the two solutions for β' agree. Substituting back into the third equation of 18.4 yields a relation $f(a,b,c) = 0$ in a_i, b_i, c_i. This relation must hold if the algebra is to be flat. The second overlap computation provides another relation, say $g(a,b,c) = 0$. So the maximal quotient of P to which $\mathbf{u}_1$ lifts is $\mathbf{U} = P/I$, where $I = (f,g)$.

Actually, it happens that f and g are identically zero, so that $\mathbf{U} = P$. The deformation is "unobstructed". The final equations for a versal embedded deformation are:

(18.5) $x^2 = a_1x + a_2y + (a_2b_1 - a_1b_2 + b_2^2 - a_2c_2),$
$xy = b_1x + b_2y + (a_2c_1 - b_1b_2),$
$y^2 = c_1x + c_2y + (b_2c_1 - b_1c_2 + b_1^2 - a_1c_1).$

Using change of coordinates $x \mapsto x + u$, $y \mapsto y + v$ wtih $u, v \in P$ one can eliminate the terms a_1, c_2. The versal unembedded deformation has the form

(18.6) $x^2 = a_2y \qquad\qquad + (a_2b_1 + b_2b_2),$
$xy = b_1x + b_2y + (a_2c_1 - b_1b_2),$
$y^2 = c_1x \qquad\qquad + (b_2c_1 + b_1b_1).$

19. Application to deformations.

Proposition 19.1. *The functor* Def$(A_k;\,\cdot)$ *satisfies conditions (i),(ii) of Schlessinger's theorem.*

Proof. the verification is quite simple. We will verify condition (ii). We set $F(\cdot) =$ Def$(A_k;\,\cdot)$, and we suppose given a length one extension $R \longrightarrow R'$, an arbitrary map $S' \longrightarrow R$, and a pair $(r, s') \in F(R) \times_{F(R')} F(S')$. Set $S = R \times_{R'} S'$. Then r is represented by a deformation A_R of A_k, parametrized by R, and similarly, s' is represented by a deformation $A_{S'}$. Let $A_{R'}$ denote the R'-deformation $A_R \otimes_R R'$ induced by A_R. The hypothesis that the pair is in the fibred product means that $A_{S'} \otimes_{S'} R' \approx A_{R'}$ too. So we have maps $A_R \longrightarrow A_{R'}$ and $A_{S'} \longrightarrow A_{R'}$, and we form the fibred product $A_S = A_R \times_{A_{R'}} A_{S'}$, obtaining a diagram

$$\text{(19.2)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & A_k & \longrightarrow & A_S & \longrightarrow & A_{S'} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & A_k & \longrightarrow & A_R & \longrightarrow & A_{R'} & \longrightarrow & 0 \end{array}$$

which is compatible with the diagram

$$\text{(19.3)} \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & k & \stackrel{\epsilon}{\longrightarrow} & S & \longrightarrow & S' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & k & \stackrel{\epsilon}{\longrightarrow} & R & \longrightarrow & R' & \longrightarrow & 0 \end{array} .$$

The fibred product represents the element of $F(S)$ mapping to the pair (r, s'). To verify this, it suffices to show

(19.4) (i) $A_S \otimes_S S' \approx A_{S'}$,

(ii) A_S is a flat S-algebra, and

(iii) $A_S \otimes_S R \approx A_R$.

The fact that $A_S \otimes_S k \approx A_k$ follows. First, $A_S \otimes_S S' \approx A_S/\epsilon A_S \approx A_S/\epsilon A_k \approx A_{S'}$. Next, this being the case, Proposition 6.2 shows that A_S is flat over S. Finally, to show (ii), we tensor A_S with the bottom row of 19.3, obtaining an exact sequence

$$\text{(19.5)} \qquad 0 \longrightarrow A_k \longrightarrow A_S \otimes_S R \longrightarrow A_S \otimes_S R' \longrightarrow 0.$$

This sequence maps to the bottom row of 19.2. We identify the right hand term as follows: $A_S \otimes_S R' \approx A_S \otimes_S S' \otimes_{S'} R' \approx A_{S'} \otimes_{S'} R' \approx A_{R'}$. Thus the end terms of the sequence 19.5 and the bottom row of 19.2 agree, hence the middle terms agree too.

It is not always true that the third condition of 17.11, that $F(\mathbb{V})$ be finite-dimensional, holds for deformations. The most useful conditions under which condition (iii) does hold are that

19.6 (i) A_k is finite-dimensional over k, or that

(ii) one is studying commutative deformations, and A_k has isolated singularities.

The fact that this is true for finite dimensional algebras follows immediately from the description of first order deformations in terms of Hochschild cohomology. For isolated singularities, one uses the description of $T^1(A_k)$ given in 8.19. An algebra A_k is said to have *isolated singularities* if is is smooth except at finitely many points of Spec A_k.

Smoothness will be discussed below. For a smooth algebra, the sequence 8.18 obtained from a presentation $A = P/\mathfrak{a}$, where $P = k[x_1, ..., x_m]$, is actually a split exact sequence:

$$(19.7) \qquad 0 \longrightarrow \mathfrak{a}/\mathfrak{a}^2 \xrightarrow{d} \Omega_P \otimes_P A \longrightarrow \Omega_A \longrightarrow 0.$$

This implies that Ω_A is projective, and that $T^1(A) = 0$ (see 8.19). Now suppose that A is smooth except at a finite set Δ of points. Suppose for simplicity that A does not have a zero-dimensional embedded component. The sequence 19.7 is exact. except on Δ and it is right exact in any case. So the kernel of the left hand map is finite-dimensional, and can be ignored. Then $T^1(A) \approx \mathrm{Ext}^1_A(\Omega_A, A)$, which is a finite module supported on Δ, hence it is finite-dimensional.

I don't know a useful analogue of 19.6(ii) in the noncommutative case.

Interlude: Smooth maps in commutative algebra.

Let S be a noetherian commutative ring and $A = P/\mathfrak{a}$ an S-algebra, where $P = S[x_1, ..., x_m]$. Then A is called *smooth* over S if the following condition holds locally at every point $p \in \mathrm{Spec}\, A$: There are elements $g_1, ..., g_{m-d} \in \mathfrak{a}$ such that $\mathfrak{a}_p = (g_1, ..., g_{m-d})A_p$, and $\frac{\partial g_i}{\partial x_j}$ has rank $m-d$ at p. The integers d is the relative dimension of A over S. Smoothness is the analogue of a smooth map $f : X \longrightarrow Y$ in analysis. The maximal rank of the jacobian matrix allows one to eliminate $m - d$ variables from the embedding of X, and realize X locally as a product $Y \times Z$ where Z is smooth. But the implicit function theorem is not available in algebra, so solving equations has to remain "implicit".

Theorem 19.8. *(Grothendieck's characterization of smooth algebras) An S-algebra A is smooth if and only if it has the following extension property: Let N be a nilpotent ideal of a commutative S-algebra R, and let π denote the map $R \longrightarrow R' = R/N$. Then every map of S-algebras $\phi' : A \longrightarrow R'$ lifts to an S-algebra map $\phi : A \longrightarrow R$ such that $\phi' = \pi\phi$.*

Proof. Assume that A is smooth over S. Let $(f_1 ..., f_r)$ be generators for the defining ideal $\mathfrak{a}$ of A. To give an algebra map $\phi : A \longrightarrow R$, we have to assign the images a_j of the variables $x_i \in P$, subject to the requirement that $f_i(a) = 0$. In other words we have to solve the system of equations $f_1(x) = \cdots = f_r(x) = 0$ in R. The coefficients of the polynomials are in S, so the algebra structure defines their images in R. To lift the given homomorphism ϕ' from R' to R, we have to lift the solution a' of $f = 0$ to a solution $a \in R$. We may assume that $I^2 = 0$.

We begin by lifting the solution a' arbitrarily to $a \in R$. Then $f(a) \equiv 0$ (modulo I), and we try to adjust a_j by adding elements $h_j \in I$ so as to obtain a solution. By Taylor's formula,

$$(19.9) \qquad f(a + h) = f(a) + Jh + \text{ higher order terms in } h,$$

where $f = (f_1, ..., f_r)^t$, $J = \frac{\partial f_i}{\partial x_j}$ and $h = (h_1, ..., h_m)^t$. Since $h_j \in I$ and $I^2 = 0$, the higher order terms vanish. In order to obtain a solution we must solve

$$(19.10) \qquad Jh = -f(a).$$

Suppose first that R is a local ring. Then R' is also local, and we may factor ϕ' through a suitable localization A_p. Having done this, we solve the system $g_1, ..., g_{n-d}$ given in the definition of smoothness instead. Since the jacobian has maximal rank, this can be done.

In general, we choose a presentation for the ideal $\mathfrak{a}$, of the form

$$P^n \xrightarrow{K} P^r \xrightarrow{f} P^m, \tag{19.11}$$

where $f = (f_1, ..., f_r)^t$ as above, and where K is an $n \times r$ matrix, operating by left multiplication. So, $fK = 0$. Then by the product rule for differentiation,

$$JK \equiv 0 \text{ (modulo } \mathfrak{a}). \tag{19.12}$$

Hence, passing to the ring A, we obtain a complex

$$A^n \xrightarrow{K} A^r \xrightarrow{J} A^m. \tag{19.13}$$

The next lemma, together with the definition of smoothness, shows that this complex is split exact, i.e., that $\ker K = \operatorname{im} J$ is a projective direct summand of A^r. Then to solve 19.10, we note that $fK = 0$, hence $-f(a) \in \ker K = \operatorname{im} J$.

Lemma 19.14. *Let A be a commutative ring, and let*

$$F_2 \xrightarrow{d_2} F_1 \xrightarrow{d_1} F_0$$

be a complex of free A-modules. Let D_i denote the matrix representing the maps d_i by left multiplication. Let $M \subset A$ be the ideal generated by all products $m_1 m_2$, where m_1 is an k rowed minor of D_1 and m_2 is a $n-k$-rowed minor of D_2, k being arbitrary. Then M is the unit ideal in A if and only if $\operatorname{im} d_1 = \ker d_2$, and $F_2/\operatorname{im} d_1$ is projective. If so, then $F_2 = B \oplus C$, where $B = \operatorname{im} d_1 = \ker d_2$.

Proof. Both statements can be verified locally. So if M is the unit ideal, we may assume that we have minors m_2, m_1 of complementary dimensions which are invertible in A. The minor m_1 identifies k basis elements in F_1 which generate a submodule C of rank k on which d_1 is injective. The minor m_2 identifies $n-k$ elements of F_1 which generate a free summand $B \subset F_1$ which is in the image of d_2. Then $C \subset \operatorname{im} d_2 \subset \ker d_1$, and $\ker d_1 \cap C = 0$. It follows that $F_1 = B \oplus C$ and that $B = \operatorname{im} d_2 = \ker d_1$. The converse is proved in a similar way. In fact, it follows that the unit ideal is generated by products of minors $m_1 m_2$ where the columns of m_2 and the rows of m_1 have complementary sets of indices.

20. Parametrizing finite dimensional algebras.

Let R be a commutative algebra, and let $\{a_1, ..., a_n\}$ denote the standard basis of the free module R^n. An algebra law on R^n is a bilinear map $A \times A \xrightarrow{\mu} A$, which is determined when we know its effect on the basis elements. We may assign the products $a_i a_j$ as elements of A, say

(20.1)
$$a_i a_j = \sum_{k=1}^{n} c_{ijk} a_k,$$

with $c_{ijk} \in R$, and extend bilinearly to $A \times A$. The axioms for an algebra, the associative law, existence of identity, and, if we wish, the commutative law, impose polynomial relations on the *structure constants* c_{ijk}. For example, to obtain the relations imposed by the associative law, we write

$$(a_i a_j) a_k = (\sum_{\alpha} c_{ij\alpha} a_\alpha) a_k = \sum_{\alpha,\ell} c_{ij\alpha} c_{\alpha k\ell} a_\ell,$$

$$a_i (a_j a_k) = a_i (\sum_{\alpha} c_{jk\alpha} a_\alpha) = \sum_{\alpha,\ell} c_{jk\alpha} c_{i\alpha\ell} a_\ell.$$

We equate these two expressions and extract coefficients of a_ℓ, obtaining the associativity relations

(20.2)
$$\sum_{\alpha} c_{ij\alpha} c_{\alpha k\ell} = \sum_{\alpha} c_{jk\alpha} c_{i\alpha\ell},$$

which must hold for all i, j, k, ℓ.

The existence of an identity element asserts that there is a linear combination $e = \sum_i \epsilon_i a_i$ such that $e a_i = a_i e = a_i$. A priori, this introduces n new constants ϵ_i which must be included with the structure constants c_{ijk}. However, because the identity element is unique, these constants can be eliminated. An alternate approach is to require that a_1 be the identity. This requirement leads to the relations

(20.3)
$$c_{1jk} = \delta_{j,k} \text{ and } c_{i1k} = \delta_{i,k}.$$

The commutative law, if desired, imposes the relations

(20.4)
$$c_{ijk} = c_{jik}.$$

Other axioms could be introduced.

Exercise: What is the general form of such an axiom?

Example 20.5 Algebras of rank 2. To reduce the dimension a bit, let's take the basis for an algebra of dimension 2 of the form $\{a_1, a_2\} = \{1, a\}$. Then the only product which we must assign is $aa = c_1 a + c_2$. There are two structure constants, and associativity is automatic.

In dimension greater than 2, the explicit form of the relations 20.2,3,4 is not very useful because there are too many of them. So let us write them neutrally as $f_\nu(u_{ijk}) = 0$, where u_{ijk} are commuting indeterminates. The most important thing to notice is that the f_ν are polynomials with integer coefficients. Let $U = k[u_{ijk}]/I$, where I denotes the ideal generated by the polynomials f_ν.

Corollary 20.6. *Let R be a commutative k-algebra. There are bijective correspondences between the sets*
(i) $\mathrm{Hom}_{alg}(U, R)$,
(ii) solutions $\{c_{ijk}\}$ *in* R *of the system of equations* $f_\nu(u) = 0$, *and*
(iii) algebra structures on the free R*-module with basis* $\{a_1, ..., a_n\}$.

More precisely, let let $F : (\mathrm{Alg}) \longrightarrow (\mathrm{Sets})$ be the functor which sends R to the set of algebra structures on the free R-module with basis $\{a_1, ..., a_n\}$. Then

$$\mathrm{Hom}(U, \cdot) \approx F(\cdot). \tag{20.7}$$

The Yoneda Lemma tells us that this isomorphism is given by an element of $F(U)$ - an algebra structure on the free U-module. This algebra A_U is the one whose structure constants are the residues of the variables u_{ijk} in U. It is *universal* in the sense that any R-algebra A_R with chosen basis $\{a_1, ..., a_n\}$ is obtained as $A_R \approx A_U \otimes_U R$ via a unique homomorphism $\phi : U \longrightarrow R$. That is what 20.7 says.

In the case of algebras of rank 2 considered above, $U = k[u_1, u_2]$, and the universal algebra is $A_U = U[a]/(a^2 - u_1 a - u_2)$.

Note that we have left the category of finite local k-algebras here. Our functor F is defined on all algebras R, and Spec U solves the strong moduli problem (see Section 4).

Let $X = \mathrm{Spec}\, U$. By definition, homomorphisms $U \longrightarrow R$ correspond to maps of schemes $\mathrm{Spec}\, R \longrightarrow X$. In algebraic geometry we would say that $A_R = A_U \otimes_U R$ is the algebra obtained by "pullback" from the map $\mathrm{Spec}\, R \longrightarrow X$.

Proposition 20.8. *Let $p \in X$ be a point and let $k(p)$ denote the residue field at p (which isisomorphic to k). Let $\widehat{U}$ denote the completion of U at the point p, and let $A_{\widehat{U}} = A_U \otimes_U \widehat{U}$, $A_k = A_U \otimes_U k(p)$. Then $A_{\widehat{U}}$ is a versal deformation of the algebra A_k.*

Actually, to be a versal deformation requires only that one has a formal element, which in this case would consist of a deformation A_n parametrized by $U_n = U/\mathfrak{m}_P^n$ for every n (see 17.13). But since we are dealing here with a finite algebras, any such sequence defines an algebra over $\widehat{U}$, namely $A_{\widehat{U}} = \mathrm{proj\,lim}\, A_n$. This can be a rather subtle point in other situations. A formal element $\{u_n \in F(U_n)\}$ need not be induced by an element $\overline{u} \in F(\widehat{U})$.

Proof of the proposition. If R is a finite local k-algebra, then maps $\widehat{U} \longrightarrow R$ correspond bijectively to maps $U \longrightarrow R$ such that the composed map $U \longrightarrow R \longrightarrow k$ is the residue field map at the point p. So the completion is relevant only to pin down the underlying point, and we need not involve it further in this proof.

We verify versality: Suppose given a length 1 extension $R \longrightarrow R'$ of finite local k-algebras, a deformation A_R of A_k, and a map $\phi' : U \longrightarrow R'$ such that $A_U \otimes_U R' \approx A_{R'}$. This determines a basis of $A_{R'}$, the image of the given basis of A_U, and we can lift this basis of $A_{R'}$ to a basis of A_R. Having chosen such a lifting, the universal property of A_U tells us that there is a unique map $\phi : U \longrightarrow R$ such that $A_U \otimes_U R \approx A_R$ compatibly with the chosen bases. This map is the lifting of ϕ' required by the definition of versality.

We now ask to eliminate the choice of basis, i.e., to classify algebras without choice of basis. To do this, we consider the operation on X of the general linear group $G = GL_n$ which corresponds to changes of basis in the free module. A change of basis will change the structure constants c_{ijk} in a way that can be computed from the diagram below. The explicit formula is not important. In it, the algebra law μ_A on A is defined by the structure constants c_{ijk}, μ_B is defined by some structure constants d_{ijk}, and the matrix $P \in G$ defines an isomorphism from $A \to B$. This information is summed up in the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\mu_A} & A \\ {\scriptstyle P\otimes P}\downarrow & & \downarrow{\scriptstyle P} \\ B \otimes B & \xrightarrow{\mu_B} & Y \end{array} \tag{20.9}$$

Thus the change of basis carries μ_A to $\mu_B = P \circ \mu_A \circ (P^{-1} \otimes P^{-1})$. Note that d_{ijk} are expressed here as polynomials in c_{ijk}, p_{ij}, and $q = det\, P^{-1}$. These polynomials define a map of schemes $G \times X \longrightarrow X$ sending $(c, p) \mapsto d$, or a map in the other direction: $U \longrightarrow \mathcal{O}_G \otimes U$, where $\mathcal{O}_G = k[p_{ij}, q]$ is the coordinate ring of G.

If we wish to work with bases such that the first element a_1 is the identity of A, the group GL_2 is replaced by the subgroup G consisting of the invertible matrices having a_1 as fixed vector, i.e., with first column equal to $(1, 0, ..., 0)^t$.

Corollary 20.10. *The operation defined above has the property that G-orbits in X correspond bijectively to isomorphism classes of n-dimensional k-algebras A_k.*

The *stabilizer* of a point $p \in X$ also has an interpretation: It is the group of automorphisms of the corresponding algebra. For, the stabilizer is the group of matrices P such that the operation defined by 20.9 sends the structure constants c_{ijk} to themselves, i.e., such that $A = B$. These are the automorphisms. If H is the stabilizer of p, then the orbit is in bijective correspondence with the set G/H of of left cosets of H in G.

Example 20.11. We return to the example 20.5 of algebras of rank 2. There are two isomorphism classes of algebras of dimension 2 over the field k, namely $A_0 = k[a]/(a^2)$ and $A_1 = k[a]/(a^2 - 1) \approx k \oplus k$. Taking the basis $\{1, a\}$ locates them at the points $(0, 0)$ and $(1, 0)$ of the (u_1, u_2)-plane X. Because we have chosen to take the identity element as our first basis element, G is the group of invertible matrices of the form

$$P = \begin{pmatrix} 1 & q \\ 0 & p \end{pmatrix}. \tag{20.12}$$

Since there are two isomorphism classes of algebras, there are two orbits O_0, O_1 for the action of G on X. The group of automorphisms of the algebra A_0 is the subgroup H_0 of matrices P in which $q = 0$. Every left coset of H_0 in G has a unique representative of the form

$$P = \begin{pmatrix} 1 & q \\ 0 & 1 \end{pmatrix}. \tag{20.13}$$

So the coset space G/H_0 is the q-line. It is embedded into the u-plane X as the orbit O_0, the discriminant locus $u_1^2 + 4u_2 = 0$ of the polynomial $a^2 - u_1 a - u_2$. The group H_1 of automorphisms of A_1 is the cyclic group of order 2 acting by permuting the factors of $k \oplus k$, and G/H_1 is mapped bijectively to the complement O_1 of the discriminant locus in X.

The general properties of subgroups and coset spaces in algebraic geometry are reviewed in the next theorem. We omit the proof.

Theorem 20.14. *(i) The group* Aut A_k *of automorphisms of an algebra* A_k *of dimension* n *is a closed algebraic subgroup of* $G = GL_n$.

(ii) An algebraic group over a field of characteristic zero is a smooth variety.

(iii) Let H *be a subgroup of a smooth algebraic group* G. *The coset space* G/H *has a natural structure of smooth algebraic variety.*

(iv) Under the hypotheses of (iii), $\dim G = \dim H + \dim G/H$, *where* dim *denotes the dimension of the variety.*

The above description, which identifies isomorphism classes of k-algebras with orbits for a group action, is very satisfactory. However, it is not compatible with *families* of algebras, R-algebras with $R \neq k$. For example, let $R = k[t]$, and let $A_R = R[x]/(x^2 - t)$. Taking the basis $\{1, x\}$ this algebra is obtained from the universal algebra of 20.6 via the map $k[u_1, u_2] \longrightarrow k[t]$ sending $u_2 \mapsto t$ and $u_1 \mapsto 0$. The map Spec $k[t] = T \longrightarrow X =$ Spec $k[u_1, u_2]$ passes through the orbit O_0 at $t = 0$, and all other points of T map to the orbit O_1. Decomposing X into orbits is not compatible with this family.

Taking orbits for a group action solves the covariant problem of finding *cokernel* for two maps $(op, pr) : G \times X \rightrightarrows X$, a map $X \longrightarrow Y$ which is universal for equalizing the pair. On rings, the two maps go in the opposite direction: $U \rightrightarrows O_G \otimes_k U$. The closest approximation on the level of rings is to take the kernel of this pair of homomorphisms, which is the ring of functions on X which are *invariant* with respect to the group operation. The invariant ring U^G does equalize the two maps, so we do obtain a map $X \longrightarrow Y =$ Spec U^G which collapses the orbits to points. And if G is a finite group, then the points of Y correspond bijectively to orbits. But if G has positive dimension there is usually too much collapsing. Indeed, whenever one orbit has another orbit in its closure, as in the above example, the continuity of the map $X \longrightarrow Y$ forces the images of the two orbits to be equal. Thus in our example of algebras of dimension 2, the invariant ring is just k itself, and the two orbits are collapsed.

Exercise: Write down the action of the group G of 20.12 explicitly, and verify that the constants are the only invariant functions.

Example 20.15 Algebras of rank 3. As we know (see 2.3), there are five isomorphism classes of algebras of dimension 3 over k.

$A_0 = k[x, y]/(x, y)^2$,
$A_1 = k[x]/(x^3)$,

$A_2 = k[x]/(x^2) \oplus k$,
$A_3 = k \oplus k \oplus k$,
A_4, the algebra of upper triangular matrices.

In order to cut down the number of variables a bit, let's choose bases of the form $\{1, a_1, a_2\}$. Even so, there are four products $a_i a_j$, $i, j = 1, 2$ to determine, each of which is a linear combination of the three basis elements, hence 12 structure constants c_{ijk}. The associativity relations are too cumbersome to be of much use. So the ring U and its spectrum X are not so easy to compute directly.

The group which operates on X is the group G of dimension 6 of 3×3 matrices whose first column is $(1, 0, 0)^t$. The group of automorphisms H_i of A_i are as follows:

$H_0 \approx GL_2$,
H_1 is the group of substitutions $x \mapsto ax + bx^2$, a commutative group,
H_2 is the multiplicative group $x \mapsto ax$,
$H_3 \approx S_3$,
H_4 operates as $e_{11} \mapsto e_{11} + be_{12}$, $e_{12} \mapsto ae_{12}$.

The dimensions are $4, 2, 1, 0, 2$ respectively. So X is the union of five G-orbits O_i, and their dimensions are $2, 4, 5, 6, 4$ respectively. According to 17.4 and 20.8, one obtains a minimal versal deformation of the algebra A_0 by taking a slice of dimension 4 across the orbit O_0. This decreases dimensions by 2. Let Y denote this slice, and let $Y_i = O_i \cap Y$. Working locally, Y_0 is reduced to a point, and Y has dimension 4.

We saw in Section 18 that the versal commutative deformation of the algebra A_0 depends on 4 independent variables. So the union of the local schemes $Y_0, ..., Y_3$ is smooth and of dimension 4. It is analyically equivalent to the scheme whose coordinates are the parameters a_2, b_1, b_2, c_1 which appear in the equations 18.6. The closure of the scheme Y_4 is the union $Y_0 \cup Y_4$, which has dimension 2. It turns out that this scheme is also smooth. To show this, it suffices to find a flat deformation of A_0 whose first order terms are not commutative, and which depends on two independent parameters. The versal property of Y will define a formal map to Y which, since $\dim(Y_0 \cup Y_4) = 2$, must be bijective. The family is the following one, in which u, v are the parameters. A Gröbner basis computation shows that it is a flat family.

(20.16)
$x^2 = v^2$,
$xy = uv + ux - vy$,
$yx = uv - ux + vy$,
$y^2 = u^2$.

Thus X is a union $V \cup W$, where V,W are smooth schemes of dimensions 6 and 4 repectively. The intersection of these two schemes is the orbit O_0, which has dimension 2, and the intersection is transversal.

On the other hand, the union $Y_0 \cup Y_1$, which also has dimension 2, is not smooth. It is a cone over a twisted cubic curve.

21. Groupoids.

A general classification problem can be described this way: We are given a collection of *objects*, and a notion of *isomorphism* between two objects. The standard problem asks to describe the set of *isomorphism classes* of objects. But as the example of finite algebras shows, this becomes problematic when want the classification to apply also to *families* of objects, parametrized by a commutative k-algebra R, or when we want to put an algebraic structure on the set of isomorphism classes. The most serious obstruction to doing this arises when two objects which can be put into a connected family have groups of automorphisms of different dimensions. When this happens, it is better to ask for an algebraic structure which encodes the objects and the isomorphisms between them at the same time. The fact that two isomorphisms $x \xrightarrow{\sigma} y$ and $y \xrightarrow{\tau} z$ can be composed if the range of the first one is the domain of the second gives us the structure of a *groupoid* on the pair $\{isomorphisms,\ objects\}$.

We have the model for such a structure in the case of finite algebras: We may take the scheme X which parametrizes algebras of rank n with given basis. Then the isomorphisms between them are described as change of basis, by the operation of the general linear group $G = GL_n$.

One definition of a *groupoid* is as a category in which all the maps are isomorphisms. As an algebraic structure, a groupoid consists of the following data:

(i) a pair of sets X, Y, the "objects", and the "isomorphisms",
(ii) a pair of arrows $\pi_0, \pi_1 : Y \longrightarrow X$, the "domain" and "range", and
(iii) a map "composition" $P \xrightarrow{c} Y$, where $P = Y\,{}_1{\times}_0\,Y$ denotes the fibred product which completes the diagram below:

(21.1)
$$\begin{array}{ccc} P & \xrightarrow{f} & Y \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \pi_0} \\ Y & \xrightarrow{\pi_1} & X \end{array}$$

Thus P is the set of pairs $(\alpha, \beta) \in Y \times Y$ such that $\pi_1(\alpha) = \pi_0(\beta)$. This data is required to satisfy the following axioms, in which $Q = Y\,{}_1{\times}_0\,Y\,{}_1{\times}_0\,Y$ denotes the set of triples (α, β, γ) such that $\pi_1(\alpha) = \pi_0(\beta)$ and $\pi_1(\beta) = \pi_0(\alpha)$.

(21.2)
(i) *(associative law)* The two maps

$$c \circ (c \times id_Y) \text{ and } c \circ (id_Y \times c),$$

which send $Q \longrightarrow P \longrightarrow Y$, are equal.

(ii) *(identity)* There is a map $e : X \longrightarrow Y$, such that $\pi_0 e = \pi_1 e = id_X$, and such that the two maps

$$c \circ (e\pi_0, id_Y) \text{ and } c \circ (id_Y, e\pi_1),$$

which send $Y \longrightarrow P \longrightarrow Y$, are equal to id_Y.

(iii) *(inverses)* There is a map $\iota : Y \longrightarrow Y$ such that $\pi_0 \iota = \iota \pi_1$ and $\pi_1 \iota = \iota \pi_0$, and

$$c \circ (\iota, id_Y) = e\pi_1, \quad \text{and} \quad c \circ (id_Y, \iota) = e\pi_0.$$

Examples 21.3. (i) *(groups)* Let $Y = G$ be a group and let $X = \{\cdot\}$ be the set of one element. Then $P = G \times G$, and the multiplication law in G makes this pair into a groupoid.

(ii) *(equivalence relations)* Let X be a set and let $Y \subset X \times X$ be an equivalence relation. An element of Y is a pair $(u, v) \in X \times X$ such that $u \equiv v$. The associative law. identity, and inverses translate to the transitive, reflexive, and symmetric axioms for an equivalence relation. This is the case that the objects of the groupoid have no automorphisms except the identity.

(iii) *(group actions)* If a group G operates on X, then setting $Y = G \times X$, $\pi_0 = pr_2$, and $\pi_1 = op$ yields an operation. Here P is the set of pairs of pairs $((g, u), (h, v))$ such that $gu = v$, and the composition sends this pair to (g, v).

As we go along, it will be notationally simpler for us to relabel the sets which arise in the definition of a groupoid as follows: We label the sets X, Y, P which appear there as X_0, X_1, X_2 respectively. They form a truncated simplicial set

$$X_2 \overrightarrow{\overrightarrow{\rightarrow}} X_1 \rightrightarrows X_0. \tag{21.4}$$

The maps $d_i : X_1 \longrightarrow X_0$, $i = 0, 1$ which are required in the definition of a simplicial set are $d_i = \pi_{i+1}$, and the maps $d_i : X_2 \longrightarrow X_1$, $i = 0, 1, 2$ are f, c, g respectively (see 21.1). The identity element $e : X_0 \longrightarrow X_1$ is the degeneracy operator s_0, and the degeneracies $s_0, s_1 : X_1 \longrightarrow X_2$ are $s_0 = e \times id$ and $s_1 = id \times e$. They satisfy the requirements of a simplicial set, up to dimension 1, which are, if operators are written on the right,

$$d_0 d_0 = d_1 d_0, \quad d_2 d_0 = d_0 d_1, \quad d_1 d_1 = d_2 d_1, \tag{21.5}$$

$$s_0 s_0 = s_0 s_1, \quad s_1 d_0 = d_0 s_0, \quad s_0 d_2 = d_1 s_0,$$

$$s_0 d_0 = s_0 d_1 = s_1 d_1 = s_1 d_2 = identity.$$

This truncated simplicial set can be extended to a simplicial set by a canonical "coskeleton" construction. When this is done, then the triple product Q which appears in 20.1 becomes the set of simplices X_3.

22. The groupoid associated to a family of algebras.

There are several closely related schemes which parametrize isomorphisms of algebras which are free modules of finite rank, and we will begin by describing four of them. The last is the one we are really after.

(22.1) *Isomorphisms between two S-algebras.*

Let S be a commutative k-algebra and let A_S, B_S be two free S-algebras. We define a functor $\underline{\text{Isom}}^{(1)}(A_S, B_S; \cdot)$. If R is another commutative k-algebra and if $\phi : S \longrightarrow R$ is a homomorphism, we set $A_B = A_S \otimes_S R$, it being understood that R is made into an S-algebra via the map ϕ.

We let $\underline{\text{Isom}}^{(1)}(A_S, B_S; R)$ denote the set of pairs (ϕ, σ), where $\phi : S \longrightarrow R$ is a k-algebra homomorphism, and $\sigma : A_R \longrightarrow B_R$ is an isomorphism of R-algebras. This is made into a functor as follows: If $f : R \longrightarrow R'$, then the map $\underline{\text{Isom}}^{(1)}(A_S, B_S; R) \longrightarrow \underline{\text{Isom}}^{(1)}(A_S, B_S; R')$ sends $(\phi, \sigma) \mapsto (f\phi, \sigma \otimes_R R')$.

A *point* of this functor, which by definition is an element of $\underline{\text{Isom}}^{(1)}(A_S, B_S; k)$, is given by choosing a point p of $X = \text{Spec } S$ (a map $S \longrightarrow k = k(p)$), and an isomorphism between the two "fibres" $A_{k(p)} \approx B_{k(p)}$.

There is a universal element for this functor, which is constructed as follow: We choose bases $\mathbf{a} = \{a_i\}$, $\mathbf{b} = \{b_i\}$ for A_S, B_S. Then an isomorphism $A \longrightarrow B$ is determined by an invertible matrix $P = (p_{ij})$: the matrix such that $\mathbf{b}P = \mathbf{a}$. This matrix must be compatible with the laws of composition on A and B, i.e., the diagram 20.9 must be commutative. When written out explicitly, the commutativity condition imposes some polynomial relations among the variables which represent the structure constants c_{ijk}, d_{ijk} for A and B, the matrix entries p_{ij}, and $q = \det P^{-1}$, say $G_\nu(c_{ijk}, d_{ijk}, p_{ij}, q) = 0$. The polynomials G_ν have integer coefficients.

In our situation, the structure constants c_{ijk}, d_{ijk} are given elements of S. We substitute these elements into the polynomials G_ν, obtaining polynomial relations $g_\nu(p_{ij}, q) = 0$ with $g_\nu \in S[p_{ij}, q]$, which hold among the variables p_{ij} if and only if P defines an isomorphism. To obtain the universal isomorphism, we set $T = S[p_{ij}, q]/(g_\nu)$. The map $\phi_T : S \longrightarrow T$ is the canonical one, and σ_T is the isomorphism defined by the matrix P_T whose entries are the residues of the p_{ij}. Any element $(\phi, \sigma) \in \underline{\text{Isom}}^{(1)}(A_S, B_S; R)$ is obtained from a unique map $T \longrightarrow R$.

(22.2) *Automorphisms of an S-algebra.*

The case that $A_S = B_S$ in the above construction yields the functor $\text{Aut}(A_S; \cdot)$ of *automorphisms* of the algebra A_S. A point of this functor consists of a point of $X = \text{Spec } S$ and an automorphism of the corresponding algebra $A_{k(p)}$. Here the fact that automorphisms form a group puts a group structure on the functor. However, because there are many points, it is not a single group, but a *family* of groups, parametrized by S. Specifically, two elements (ϕ, σ) and (ψ, τ) of $\text{Aut}(A_S; R)$ can be composed only if $\phi = \psi$.

Let T be the universal automorphism, constructed as above, and let $G = \text{Spec } T$. The universal element (ϕ_T, σ_T) provides a map $G \longrightarrow Y$, and it can be shown that composition of σ_T with itself defines the group structure $G \times_X G \longrightarrow G$.

(22.3) *Isomorphisms between algebras over different rings.*

If A_{S_1} and B_{S_2} are algebras over two commutative k-algebras S_1, S_2, we can modify the construction 22.1 as follows: We define $\underline{\text{Isom}}^{(2)}(A_{S_1}, B_{S_2}; R)$ to be the set of triples (ϕ_1, ϕ_2, σ), where $\phi_i : S_i \longrightarrow R$ are k-algebra homomorphisms, and $\sigma : A_R \longrightarrow B_R$ is an isomorphism between the induced algebras over R (i.e., $A_R = A_{S_1} \otimes_{S_1} R$ and $B_R = B_{S_2} \otimes_{S_2} R$).

Let $X_i = \text{Spec } S_i$. A point of this functor, an element of $\underline{\text{Isom}}^{(2)}(A_{S_1}, B_{S_2}; k)$, consists of two points $p_i \in X_i$ and an isomorphism between the fibres $A_{k(p_1)} \longrightarrow B_{k(p_2)}$.

Again, there is a universal object for this functor. In fact, this case can be reduced to 22.1 by the following method. We set $P = S_1 \otimes_k S_2$, which is the coproduct of S_1, S_2 in the category of commutative k-algebras. Then a pair of maps $\phi_i : S_i \longrightarrow R$ corresponds to a map $\phi : P \longrightarrow R$. We set $A_P = A_{S_1} \otimes_k S_2$ and $B_P = S_1 \otimes_k B_{S_2}$. Then $\underline{\text{Isom}}^{(2)}(A_{S_1}, B_{S_2}; \cdot\,) \approx \underline{\text{Isom}}^{(1)}(A_P, B_P; \cdot\,)$.

(22.4) *The isomorphism functor when a single algebra is given.*

We set $S_1 = S_2 = S$ and $A_S = B_S$ in 22.3, obtaining a functor $\underline{\text{Isom}}(A_S; \cdot\,)$. Thus an element of $\underline{\text{Isom}}(A_S; R)$ is a triple (ϕ_1, ϕ_2, σ), where $\phi_i : S \longrightarrow R$ are k-algebra homomorphisms, and $\sigma : A_R^{(1)} \longrightarrow A_R^{(2)}$ is an isomorphism. Here $A_R^{(i)}$ denotes the push out $A_S \otimes_S R$ using the homomorphism ϕ_i.

A point of this functor consists in a pair of points p_1, p_2 of $X = \text{Spec } S$ and an isomorphism $\sigma : A_{k(p_1)} \longrightarrow A_{k(p_2)}$.

Because it is a special case of the previous functor, this one has a univeral element too. Here if T is the universal ring, the universal element gives us two maps $S \rightrightarrows T$, hence two maps Spec $T = Y \rightrightarrows X$, and composition makes this pair into a *groupoid*. The structure map for a groupoid is an algebraic map. This groupoid encodes at the same time the algebras which can be obtained from A_S by extension of scalars, and the isomorphisms between the resulting algebras.

When X is the universal algebra with chosen basis considered in Section 20, then the groupoid is the one obtained from the group operation 20.9, i.e., $Y = G \times X$. But this will not be the case for most algebras A_S.

23. The Amitsur complex.

The *Amitsur complex* $\mathcal{A}(S/R)$ is a cosimplicial complex associated to an arbitrary ring homomorphism $\theta : R \longrightarrow S$. The word "cosimplicial" means that the arrows go in the opposite direction from those in a simplicial complex. Roughly speaking, the Amitsur complex is obtained from the simplicial complex 13.15 which defines Hochschild cohomology by switching the roles of boundary and degeneracy operations. The degeneracy and face operations of S become, respectively, the coface and codegeneracy operations in $\mathcal{A}$.

Since the word "coface" is ugly and "codegeneracy" is long, we will refer to them as the face and degeneracy maps of the cosimplicial complex.

The unadorned tensor product stands for $\otimes_R$ here and in what follows. In dimension n, $\mathcal{A}_n = S \otimes \cdots \otimes S = S^{\otimes^{n+1}}$. The face maps $d^i : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$, $i = 0, ..., n+1$, are defined by

(23.1) $$x_0 \otimes \cdots\cdots \otimes x_n \mapsto x_0 \otimes \cdots \otimes x_i \otimes 1 \otimes x_{i+1} \otimes \cdots \otimes x_n,$$

and the degeneracies s^i, $i = 0, ..., n-1$, by

(23.2) $$x_0 \otimes \cdots\cdots \otimes x_n \mapsto x_0 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n.$$

These maps satisfy certain *standard identities* which are obtained from the simplicial identities by reversing the arrows. The standard identities tell us when compositions of two face or degeneracy operators are equal, and when such a composition is the identity. Exactly how they come out depends on whether the operations are written on the left or on the right. In order to put composition of functions in the natural order, we will write operators on the right. So, for example, we write $(x_0 \otimes x_1)d^2 = x_0 \otimes x_1 \otimes 1$, and $(x_0 \otimes x_1 \otimes x_2)s^0 = x_0 x_1 \otimes x_2$. Three of the standard identities are:

$$s^1 d^0 = d^2 s^0 \ , \ d^1 d^2 = d^1 d^1 \ , \ d^1 s^0 = i.$$

Here and in what follows, i denotes the appropriate identity map.

In the most general situation, the $\mathcal{A}_n$ are S-bimodules. The face and degeneracy maps are S-bimodule homomorphisms, except for d^0 and $d^n : \mathcal{A}_n \longrightarrow \mathcal{A}_{n+1}$. On the left, d^0 is only R-linear, and similarly, d^n is only R-linear on the right. In more special situations, for instance if R, S are commutative rings, one can make $\mathcal{A}_n$ into rings in such a way that the face maps are ring homomorphisms.

The map $\theta : R \longrightarrow S$ is called an *augmentation* of the Amitsur complex because the composed maps are equal:

(23.3) $$\theta d^0 = \theta d^1.$$

We will often use the shorthand notation $S \otimes S = SS$ and $S \otimes S \otimes S = SSS$. Similarly, if M is a left R-module, we may write $S \otimes M = SM$, etc...With this notation, the face operators in the augmented Amitsur complex look like this:

(23.4) $$R \to S \rightrightarrows SS \mathrel{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}} SSS \cdots .$$

This cosimplicial complex yields a *complex* of R-modules

(23.5) $$0 \longrightarrow R \xrightarrow{\theta} S \xrightarrow{\delta^0} SS \xrightarrow{\delta^1} SSS \cdots ,$$

where $\delta^n = d^0 - d^1 + \cdots \pm d^n$.

Definition 23.6. A homomorphism $\theta : R \longrightarrow S$ is called *left flat* if: for every exact sequence

(23.7) $$M \longrightarrow M' \longrightarrow M''$$

of left R-modules, the sequence

(23.8) $$SM \longrightarrow SM' \longrightarrow SM''$$

is exact. The map θ is *faithfully flat* if: a sequence 23.7 is exact if and only if 23.8 is exact.

Lemma 23.9. *A map $R \longrightarrow S$ is faithfully flat if and only if it is flat and the map* Spec $S \longrightarrow$ Spec R *is surjective.*

Exercise: Assume that $R \longrightarrow S$ is left faithfully flat. Show that if M is and R-module such that $S \otimes M = 0$, then $M = 0$. Show that if $\phi : M \longrightarrow M'$ is a homomorphism of R-modules such that $S \otimes M \longrightarrow S \otimes M'$ is an isomorphism, then ϕ is an isomorphism.

Theorem 23.10. *Let $\theta : R \longrightarrow S$ be a left faithfully flat ring homomorphism. Then the Amitsur complex is a resolution of R, i.e., 23.5 is exact. Moreover, this complex remains exact when tensored on the right by an arbitrary left R-module M:*

$$M \longrightarrow SM \rightrightarrows SSM \overset{\longrightarrow}{\rightrightarrows} \cdots .$$

Proof. Grothendieck's trick is to note that 23.5 is a sequence of left R-modules. To prove it exact, it suffices to show that the sequence obtained by tensoring on the left with S is exact. When we tensor 23.5 on the left with S, we obtain the Amitsur complex, except that the face d^0 and degeneracy s^0 are missing, $i \otimes \theta = d^1$, and all indices in the remaining face maps are increased by 1. So $\overline{\delta} := i \otimes \delta = d^1 - d^2 + \cdots$.

This new complex is homotopically trivial, the homotopy being given by the missing degeneracy: $h = s^0$. In other words, $h\overline{\delta} + \overline{\delta}h = identity$. This is checked directly:

$$(y \otimes x_0 \otimes \cdots \otimes x_n)s^0\overline{\delta} = (yx_0 \otimes x_1 \otimes \cdots \otimes x_n)(d^1 - d^2 + \cdots)$$

$$= yx_0 \otimes 1 \otimes x_1 \otimes \cdots \ - \ yx_0 \otimes x_1 \otimes 1 \otimes x_2 \otimes \cdots \ + \ \cdots ,$$

while

$$(y \otimes x_0 \otimes \cdots \otimes x_n)\overline{\delta}s^0 = (y \otimes 1 \otimes x_0 \otimes x_1 \otimes \cdots \ - \ y \otimes x_0 \otimes 1 \otimes x_1 \otimes \cdots + \cdots)s^0$$

$$= y \otimes x_0 \otimes x_1 \otimes \cdots \ - \ yx_0 \otimes 1 \otimes x_1 \otimes \cdots \ + \ \cdots .$$

This shows that 23.5 is exact (see 13.20), and if we tensor on the right by an arbitrary left R-module M, the same proof shows that the resulting sequence is exact.

Corollary 23.11. (Descent for elements of a module) *Let M be a left R-module, and let $N = S \otimes M$, where $R \longrightarrow S$ is left faithfully flat. An element $x \in N$ has the form $x = 1 \otimes y$ for some unique $y \in M$ if and only if $xd^0 = xd^1$.*

This corollary follows immediately from the exactness of the augmented Amitsur complex. It is usually stated this way: *An element $x \in N$ lies in M if and only if $xd^0 = xd^1$.* In terms of tensors, suppose that we write $x = \sum_\nu a_\nu \otimes m_\nu \in SM$. Then $xd^0 = xd^1$ reads

$$\sum_\nu 1 \otimes a_\nu \otimes m_\nu = \sum_\nu a_\nu \otimes 1 \otimes m_\nu .$$

Proposition 23.12. (Descent for maps of modules) *Let $\theta : R \longrightarrow S$ be a faithfully flat ring homomorphism, let M, M' be R-modules, and let $g : SM \longrightarrow SM'$ be an S-linear map. Then g has the form $i \otimes \gamma$ for a unique R-linear map $\gamma : M \longrightarrow M'$ if and only if the following diagram commutes:*

$$\begin{array}{ccc} SM & \xrightarrow{d^1} & SSM \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle i\otimes g} \\ SM' & \xrightarrow{d^1} & SSM' \end{array} .$$

Proof. The diagram obtained by replacing d^1 by d^0 commutes in all cases, i.e., $gd^0 = d^0(i \otimes g)$. Suppose that $gd^1 = d^1(i \otimes g)$ as well. Let $x \in M$. We identify M with $\ker(d^0 - d^1)$, so that $xd^0 = xd^1$. Then

$$(xg)d^0 = xd^0(i \otimes g) = xd^1(i \otimes g) = (xg)d^1.$$

Therefore $xg \in \ker(d^0 - d^1) = M'$. This shows that g defines a map $\gamma : M \longrightarrow M'$ by restriction to M. Since g is S-linear, its restriction to the R-module M is R-linear. Then g and $i \otimes \gamma$ are S-linear maps $SM \longrightarrow SM'$ which agree on M, so they are equal.

Conversely, if $g = i\otimes\gamma$, then $i\otimes g = i\otimes i\otimes\gamma$. It is true that $gd^1 = (i\otimes\gamma)d^1 = d^1(i\otimes i\otimes\gamma)$.

This proposition can be stated by saying that the sequence

$$\text{(23.13)} \qquad \mathrm{Hom}_R(M, M') \longrightarrow \mathrm{Hom}_S(SM, SM') \rightrightarrows \mathrm{Hom}(SM, SSM')$$

is exact, where the double arrow stands for the pair of maps sending $g \mapsto gd^1$ and $g \mapsto d^1(i \otimes g)$.

24. Descent via a faithfully flat ring homomorphism.

The descent problem for modules is the following: Given a left faithfully flat ring homomorphism $\theta : R \longrightarrow S$, what information on a left S-module N is needed to present N as tensor product $S \otimes M$ from some R-module M? The point, of course, is to solve this problem without mentioning R explicitly, and the necessary information will be called *descent data* for the module.

We use the Amitsur complex. If M is a left R-module, Theorem 23.10 provides us with a cosimplicial resolution of M. We set $N = S \otimes M$, obtaining a cosimplicial complex

$$\text{(24.1)} \qquad N \rightrightarrows S \otimes N \overset{\longrightarrow}{\rightrightarrows} SS \otimes N.$$

This cosimplicial complex has the information necessary to recover M, because Theorem 23.10 tells us that $M = \ker(S \rightrightarrows S \otimes N)$. Thus the cosimplicial complex provides descent data for the module M.

There are several problems which need to be addressed: First, how do we describe this descent data when M is not present? Second, the cosimplicial module is terribly redundant, and we must remove this redundancy. Third, and this is very important, we must provide a useful interpretation of the descent data. We will treat the first two problems in this section, and we will discuss the third one in special cases in the following sections.

The first step in removing redundancy is to truncate the complex at dimension 2, i.e., to keep only the part shown in 24.1, together with the appropriate degeneracies on this truncation.

If an S-module N is given, we already have part of the truncated complex. For instance, we have the face map $d^0 : N \longrightarrow S \otimes N$ and the degeneracy $s^0 : N \leftarrow S \otimes N$. The missing maps are $d^1 : N \longrightarrow S \otimes N$ and $d^2 : S \otimes N \longrightarrow SS \otimes N$.

Our first definition of descent data is as follows: Let N be a left S-module. Descent data for the module N consists of a map $\phi : N \longrightarrow S \otimes N$ such that, if we set $d^1 = \phi$ and $d^2 = i \otimes \phi$, we obtain a truncated cosimplicial complex 24.1.

Suppose that a map $\phi : N \longrightarrow S \otimes N$ is given. To check that one obtains a truncated cosimplicial complex, one must check the standard identities. The ones which don't involve ϕ or $i \otimes \phi$ hold automatically. The important new ones are

(24.2)

(a) $d^1 d^2 = d^1 d^1$, or $\phi(i \otimes \phi) = \phi d^1$,

(b) $d^2 s^0 = s^0 d^1$, or $(i \otimes \phi) s^0 = s^0 \phi$,

(c) $d^1 s^0 = i$, or $\phi s^0 = i$.

There are two other new identities: $d^0 d^2 = d^0 d^1$ and $d^2 s^1 = i$. However, the first of these always holds, and the second is a consequence of (c). Note also that the identity $d^1 s^0 = i$ appears twice, once at N and once at $S \otimes N$. Only the one at N is new.

Lemma 24.3. *(i) The identity (b) is equivalent with the S-linearity of ϕ.*
(ii) If (a) and (b) hold, then (c) is equivalent with injectivity of the map ϕ.

Proof. (i) $(a \otimes x)(i \otimes \phi) s^0 = (a \otimes (x\phi)) s^0 = a(x\phi)$, and $(a \otimes x) s^0 \phi = (ax)\phi$.
(ii) That $\phi s^0 = i$ implies ϕ injective is clear. Suppose that $\phi = d^1$ is injective. Then to prove that $d^1 s^0 = i$, it suffices to show $d^1 s^0 d^1 = d^1$. Using the available simplicial identities, we find

$$d^1 s^0 d^1 = d^1 d^2 s^0 = d^1 d^1 s^0 = d^1 i.$$

Note that the identity $d^1 s^0 = i$ we used here is on $S \otimes N$. This isn't the one we are proving.

We can sum up our conclusions as follows:

Definition 24.4. *Descent data for a left S-module N is given by an injective S-linear map $\phi : N \longrightarrow SN$ which satisfies the cocycle condition*

$$\phi(i \otimes \phi) = \phi d^1.$$

Exercise: Write the cocycle condition out explicitly in terms of bases in the case that S is a free R-module and that N is a free S-module of rank 1. (It is not easy to give a useful formula for the cocycle condition in terms of elements in the general case.)

We can construct an R-module from descent data, as $M = \ker(N \xrightarrow{d^0-\phi} S \otimes N)$. Here ϕ is S-linear, but d^0 is only R-linear. So the kernel is an R-module.

Theorem 24.4. *Let $\theta : R \longrightarrow S$ be a left faithfully flat ring homomorphism, and let $\phi : N \longrightarrow S \otimes N$ be descent data. Let*

$$M = \ker(N \xrightarrow{d^0-\phi} S \otimes N).$$

Then M is an R-module, and the canonical map $S \otimes M \longrightarrow N$ is an isomorphism. Moreover, the cosimplicial complex 24.1 is canonically isomorphic to the truncation of $\mathcal{A} \otimes M$, where $\mathcal{A}$ is the Amitsur complex for S/R.

Proof. We use Grothendieck's trick again. To show that $S \otimes M$ is isomorphic to N, we set $\phi = d^1$ and we tensor the exact sequence

$$0 \longrightarrow M \xrightarrow{u} N \xrightarrow{d^0-d^1} S \otimes N \tag{24.5}$$

on the left with S, obtaining an exact sequence

$$0 \longrightarrow SM \xrightarrow{i \otimes u} S \otimes N \xrightarrow{d^1-d^2} SS \otimes N. \tag{24.6}$$

It suffices to show that this sequence is isomorphic to

$$0 \longrightarrow N \xrightarrow{d^1} S \otimes N \xrightarrow{d^1-d^2} SS \otimes N, \tag{24.7}$$

i.e., that N is the kernel of the map $d^1 - d^2$. The injectivity of the map d^1 is a hypothesis. We verify that $h = s^0$ is a homotopy, i.e., that $hd^1 + (d^1 - d^2)h = i$, from which it will follow that the sequence 24.7 is exact at $S \otimes N$. This is a simplicial identity:

$$s^0 d^1 + (d^1 - d^2)s^0 = d^2 s^0 + i - d^2 s^0 = i,$$

as required. We omit the verification of the last assertion.

Scholium 24.8. Let M be an R-module. The descent data on SM which produces the module M is, of course, obtained by the canonical map $d^2 : SM \longrightarrow SSM$. If N is an

S-module which is isomorphic to SM by an S-isomorphism $g : N \longrightarrow SM$, this canonical descent data carries over, and it gives descent data on N, as is exhibited in the diagram below:

$$\begin{array}{ccccccc} & & N & \rightrightarrows & SN & \overset{\longrightarrow}{\rightrightarrows} & SSN \\ & & \downarrow g & & \downarrow i\otimes g & & \downarrow i\otimes i\otimes g \\ M & \longrightarrow & SM & \rightrightarrows & SSM & \overset{\longrightarrow}{\rightrightarrows} & SSSM \end{array} \tag{24.9}$$

The descent data on N is $\phi = gd^1(i \otimes g)^{-1}$.

Exercise: Work out the fact that this does give descent data by verifying the equalities

$$\phi d^1 = gd^1(i \otimes g)^{-1}d^1 = gd^1d^1(i \otimes i \otimes g)^{-1} = gd^1d^2(i \otimes i \otimes g)^{-1} = \phi(i \otimes \phi).$$

Critique 24.10. The problem with the description 24.4 of descent data is that it is difficult to interpret in a useful way. However, Nuss has given an interpretation in terms of connections.

References:

M.A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in Math. 389, Springer, 1974.

Philippe Nuss, *Noncommutative descent*, preprint.

25. Descent when the tensor products $S \otimes \cdots \otimes S$ are rings.

Let $\theta : R \longrightarrow S$ be a left faithfully flat ring homomorphism. In this section we suppose that $S \otimes S$ and $S \otimes S \otimes S$ have ring structures with these properties:

(25.1)

(i) The face maps d^i which appear in the truncated Amitsur complex are ring homomorphisms,

(ii) for all $b, c \in S$, $(b \otimes 1)(1 \otimes c) = b \otimes c$ in $S \otimes S$.

This includes, for example, the case that R is commutative and that S is an R-algebra. We do not assume given a formula for evaluating the product $(1 \otimes c)(b \otimes 1)$, nor that the degeneracies s^i are ring homomorphisms.

The main case in which the hypotheses 25.1 hold is that $\theta : R \longrightarrow S$ is generated by centralizing elements. The centralizer of an R bimodule M is

$$Z_R(M) = \{m \in M \mid am = ma \text{ for all } a \in R\}. \tag{25.2}$$

A ring homomorphism $\theta : R \longrightarrow S$ is called an *extension* if S is generated by $Z_R(S)$. If S is finitely generated over R, this amounts to saying that S is a quotient of a noncommutative polynomial ring $R\langle x_1, ..., x_n\rangle$, in which the elements of R commute with the variables x_i.

Proposition 25.3. *If $\theta_\nu : R \longrightarrow S_\nu$, $\nu = 0, 1$ are extensions, then the tensor product $S_1 \otimes_R S_0$ has a unique structure of ring with these properties:*
(i) The maps $d^\nu : S_\nu \longrightarrow S_0 \otimes S_1$ are ring homomorphisms.
(ii) $(z \otimes 1)(1 \otimes w) = z \otimes w$ for all $z \in S_0$ and $w \in S_1$.

Proof. Note that if $S_0 = R\langle x_1, ..., x_m\rangle$ and $S_1 = k\langle y_1, ..., y_n\rangle$, then the tensor product module $S_1 \otimes_R S_0$ is isomorphic to the bimodule of the noncommutative polynomial ring $R\langle y_1, ..., y_n; x_1, ..., x_n\rangle$, where the variables commute with elements of R, and where x_i commutes with y_j. We take the standard ring structure in that case. In general, we write S_i as quotients, say $S_0 = R\langle x\rangle/I$ and $S_1 = R\langle y\rangle/J$. Then as bimodule, $S_0 \otimes_R S_1 \approx R\langle y; x\rangle/(J \otimes_R S_0 + S_1 \otimes_R I)$. We note that $(J \otimes_R S_0 + S_1 \otimes_R I)$ is a two-sided ideal. So the tensor product inherits a ring structure. This ring structure has the property that $(z \otimes 1)(1 \otimes w) = z \otimes w$ for all $z \in S_1$ and $w \in S_0$, and that $(1 \otimes w)(z \otimes 1) = z \otimes w$ if either $w \in Z_R(S_1)$ or $z \in Z_R(S_0)$, but not otherwise. We omit the rest of the proof.

The only difficulty in what follows is keeping track of the way scalars move past the tensor product symbol. So all tensor products which are not over the ring R must be labeled clearly.

We will need to use the face maps and also of the three compositions of face maps $d^{01}, d^{02}, d^{12} : S \longrightarrow SSS$, defined by

(25.4)
$$z\, d^{01} = 1 \otimes 1 \otimes z, \quad z\, d^{02} = 1 \otimes z \otimes 1, \quad z\, d^{12} = z \otimes 1 \otimes 1.$$

The relations between them:

(25.5)
$$d^{01} = d^0 d^1 = d^0 d^0, \quad d^{02} = d^0 d^2 = d^1 d^0, \quad d^{12} = d^1 d^2 = d^1 d^1.$$

In general we use subscripts on the tensor symbol to indicate that a tensor product is via one of these maps. So if N is an S-module, then $SS \otimes_0 NN$ and $SS \otimes_1 N$ will denote the extension of scalars via d^0 and d^1 respectively. Then for $b, c \in S$ and $x \in N$,

$$b \otimes c \otimes_0 x = b \otimes 1 \otimes_0 cx. \tag{25.6}$$

We don't have an analogous formula for the element $b \otimes c \otimes_1 x$, but we do have

$$c \otimes 1 \otimes_1 x = 1 \otimes 1 \otimes_1 cx. \tag{25.7}$$

Similarly, we use the notation $SSS \otimes_{\alpha\beta} N$ to denote extension of scalars by the map $d^{\alpha\beta}$ of 25.4.

We may also write $N^\nu = SS \otimes_\nu N$ and $N^{\alpha\beta} = SSS \otimes_{\alpha\beta} N$. Then 25.5 shows that we have canonical isomorphisms

(25.8)
$$N^{01} \approx SSS \otimes_1 N^0 \approx SSS \otimes_0 N^0,$$
$$N^{02} \approx SSS \otimes_2 N^0 \approx SSS \otimes_0 N^1,$$
$$N^{12} \approx SSS \otimes_2 N^1 \approx SSS \otimes_1 N^1.$$

Let $f : N^1 \longrightarrow N^0$ be an SS-linear map, and denote by f^ν the map obtained from f by extension of scalars via $d^\nu : SS \longrightarrow SSS$. According to 25.5, the domains and ranges of the maps f^ν are as follows:

(25.9) $\quad f^0 : N^{02} \longrightarrow N^{01}, \quad f^1 : N^{12} \longrightarrow N^{01}, \quad f^2 : N^{12} \longrightarrow N^{02}.$

So the composition $f^2 f^0$ is defined, and it has the same domain and range as f^1.

Definition 25.10. *Descent data for a module N consists of an isomorphism of SS-modules $f : N^1 \longrightarrow N^0$ which satisfies the cocycle condition $f^2 f^0 = f^1$.*

Proposition 25.11. *Descent data 24.4 for a module N is equivalent to descent data 25.10, and hence descent data 25.10 for N is equivalent with a presentation of N as tensor product SM.*

The descent property for homomorphisms 23.11 can also be restated in this situation:

Corollary 25.12. *With hypotheses as at the beginning of this section, let M, M' be R-modules. There is an exact sequence*

$$\mathrm{Hom}_R(M, M') \longrightarrow \mathrm{Hom}_S(SM, SM') \rightrightarrows \mathrm{Hom}_{SS}(SSM, SSM').$$

If available, the form 25.10 of descent data may be preferable to 24.4 because it is exhibited as an isomorphism between modules obtained from N by extension of scalars.

Scholium 25.13. Suppose that $N = SM$. The two maps θd^0 and θd^1, which send $R \longrightarrow SS$, are equal. So the functors $SS \otimes_\nu S \otimes \cdot$ are canonically isomorphic, for $\nu = 0, 1$. The isomorphism $N^1 \longrightarrow N^0$ which provides descent data for N is the canonical one.

Proof of the proposition. Define

(25.14) $$j : N \longrightarrow N^1 \;, \text{ by } xj = 1 \otimes 1 \otimes_1 x, \text{ and}$$

(25.15) $$\epsilon : SN \longrightarrow N^0 \;, \text{ by } (b \otimes x)\epsilon = b \otimes 1 \otimes_0 x.$$

These maps are d^1-linear, j is injective, and ϵ is bijective. For example, the d^1-linearity of ϵ is verified as follows:

(25.16) $$(ab \otimes x)\epsilon = ab \otimes 1 \otimes_0 x = (a \otimes 1)(b \otimes 1 \otimes_0 x) = (ad^1)((b \otimes x)\epsilon).$$

Lemma 25.17. *There is a bijective correspondence* $\mathrm{Hom}_{SS}(N^1, N^0) \longrightarrow \mathrm{Hom}_S(N, S \otimes N)$ *defined by* $f \mapsto jf\epsilon^{-1}$.

Proof. The isomorphism ϵ identifies $\mathrm{Hom}_S(N, S \otimes N)$ with $\mathrm{Hom}_S(N, N^0)$, where N^0 is given the structure of S-module via d^1. Then the assertion becomes the adjointness of restriction and extension of scalars via d^1.

This lemma provides the correspondence between the two types of descent data. Note that if f is bijective, then $jf\epsilon^{-1}$ is injective, as required by 24.4.

The main point is to verify the equivalence of the two cocycle conditions. Since the maps f^ν are SSS-linear, the cocycle condition $f^2f^0 = f^1$ holds if and only if the two sides take the same values on elements of the form $z = 1 \otimes 1 \otimes 1 \otimes_{12} x$. On such elements, $zf^1 = (1 \otimes 1 \otimes_1 x)fd^1$ and $zf^2 = (1 \otimes 1 \otimes_1 x)fd^2$. We write $x\phi = \sum_i c_i \otimes y_i$. Then

$$(1 \otimes 1 \otimes_1 x)f = \sum c_i \otimes 1 \otimes_0 y_i,$$

$$zf^1 = \sum c_i \otimes 1 \otimes 1 \otimes_{01} y_i \quad , \quad \text{and} \quad zf^2 = \sum c_i \otimes 1 \otimes 1 \otimes_{02} y_i.$$

Next, we write $y_i\phi = \sum_j b_{ij} \otimes z_{ij}$. Since

$$(c_i \otimes 1 \otimes 1 \otimes_{02} y_i)f^0 = (c_i \otimes 1 \otimes 1)((1 \otimes 1 \otimes 1 \otimes_{02} y_i)f^0), \quad \text{and}$$

$$(1 \otimes 1 \otimes 1 \otimes_{02} y_i)f^0 = (1 \otimes 1 \otimes_1 y_i)fd^0 = \sum_j 1 \otimes b_{ij} \otimes 1 \otimes_{01} z_{ij}.$$

We have

$$(1 \otimes 1 \otimes 1 \otimes_{12} x)f^2f^0 = \sum_{i,j} c_i \otimes b_{ij} \otimes 1 \otimes_{01} z_{ij}. \tag{25.18}$$

The cocycle condition $f^2f^0 = f^1$ reads

$$\sum_i c_i \otimes 1 \otimes 1 \otimes_{01} y_i = \sum_{i,j} c_i \otimes b_{ij} \otimes 1 \otimes_{01} z_{ij}. \tag{25.19}$$

On the other hand, with the same notation, we have

$$(x)\phi(i \otimes \phi) = (\sum_i c_i \otimes y_i)(i \otimes \phi) = \sum_{i,j} c_i \otimes b_{ij} \otimes z_{ij}.$$

So the cocycle condition $\phi d^1 = \phi(1 \otimes \phi)$ reads

$$\sum_i c_i \otimes 1 \otimes y_i = \sum_{i,j} c_i \otimes b_{ij} \otimes z_{ij}. \tag{25.20}$$

There is a bijective map $\epsilon' : SSN \longrightarrow SSS \otimes_{01} N$ defined by $a \otimes b \otimes x \mapsto a \otimes b \otimes 1 \otimes_{01} x$, from which it follows that the two cocycle conditions are equivalent.

This lemma completes the proof of the proposition, except for the verification that if f is obtained from descent data ϕ, then f is an isomorphism. This last fact follows from the description of the map f in the case that $N = SM$ for some R-module M. In that case, ϕ is the canonical map $d^1 : SM \longrightarrow SSM$, and f is the canonical isomorphism

$$SS \otimes_1 SM \approx SSM \approx SS \otimes_0 SM.$$

Since the descent data N provides us with the module M, this proves that f is an isomorphism. I'm too confused to figure out whether or not the fact that f is an isomorphism follows from the simplicial relations alone, i.e., if it is true without the assumption that $f : R \longrightarrow S$ is faithfully flat.

26. Interpretation of descent for extensions of commutative rings.

In the next three sections we assume that $\theta : R \longrightarrow S$ is a faithfully flat map of commutative rings. Then all of the maps appearing in the Amitsur complex $\mathcal{A}(S/R)$ are ring homomorphisms, and one can rewrite the descent data in a slightly simpler form. We interpret a left S-module N as a bimodule on which the actions on left and right are the same. Then in addition to the canonical isomorphism 25.5 $S \otimes N \approx N^0 = SS \otimes_0 N$, we also have a natural isomorphism $N \otimes S \approx N^1 = SS \otimes_1 N$, defined by $x \otimes b \mapsto 1 \otimes b \otimes_1 x$.

Corollary 26.1. *Let $R \longrightarrow S$ be a faithfully flat map of commutative rings. Descent data for an S-module N is given by an isomorphism of S-modules $f : N \otimes S \longrightarrow S \otimes N$ satisfying the following cocycle condition: The composition*

$$N \otimes S \otimes S \xrightarrow{f^2 = f \otimes i} S \otimes N \otimes S \xrightarrow{f^0 = i \otimes f} S \otimes S \otimes N$$

is equal to the map $f^1 : N \otimes S \otimes S \longrightarrow S \otimes S \otimes N$ obtained from f by tensoring with S in the middle.

The most important benefit of working with commutative rings is that many constructions on modules are compatible with extension of scalars. Let M, M' be R-modules, and set $N = S \otimes M$, and $N' = S \otimes M'$. Then there are canonical isomorphisms

(26.2)
(i) $N \oplus N' \approx S \otimes (M \oplus M')$,
(ii) $N \otimes_S N' \approx S \otimes (M \otimes M')$,
(iii) $\mathrm{Hom}_S(N, N') \approx S \otimes \mathrm{Hom}_R(M, M')$.

Here (i) is true for any homomorphism of rings, commutative or not, and any left modules M, M'. (ii) is true whenever R, S are commutative, and (iii) requires that R, S are commutative, M is a finitely presented module and that θ is a flat map. Since this is slightly less trivial, we provide a proof. Both (ii) and (iii) become problematic when the ground rings aren't commutative, because they require moving scalars past elements of the module.

Lemma 26.3. *Let $\theta : R \longrightarrow S$ be a flat map of commutative rings. Let M, M' be R-modules, and assume that M is finitely presented. Let $N = S \otimes M$ and $N' = S \otimes M'$. Then $S \otimes \mathrm{Hom}_R(M, M') \approx \mathrm{Hom}_S(N, N')$.*

Proof. We choose a presentation for M, say

(26.4) $$R^{n_2} \longrightarrow R^{n_1} \longrightarrow M \longrightarrow 0.$$

We note that $\mathrm{Hom}_R(R^n, M') \approx M'^n$. Because Hom is left exact and contravariant in the first variable, we obtain an exact sequence

(26.5) $$0 \longrightarrow \mathrm{Hom}_R(M, M') \longrightarrow M'^{n_1} \longrightarrow M'^{n_2}.$$

Tensoring 26.4 on the left with S yields a resolution

$$S^{n_2} \longrightarrow S^{n_1} \longrightarrow N \longrightarrow 0,$$

hence an exact sequence

(26.6) $$0 \longrightarrow \mathrm{Hom}_R(N, N') \longrightarrow N'^{n_1} \longrightarrow N'^{n_2}.$$

The assertion follows by comparing this sequence with the one obtained by tensoring 26.5 on the left with S.

Because algebraic structures are usually compatible with extension of scalars, descent can be applied to them when $\theta : R \longrightarrow S$ is a faithfully flat map of commutative rings. Let's take the example of descent of an algebra: The starting point is to note that if A is an R-algebra, then $B = S \otimes A$ has a natural structure of S-algebra. This is the point which becomes problematic when the "coefficient rings" R, S are not commutative. To be specific, the structure of R-algebra on A is given by a linear map $A \otimes A \xrightarrow{\mu} A$ satisfying some axioms such as associativity. The strucure of S-algebra on $B = S \otimes A$ is obtained by tensor product, using the canonical isomorphism

(26.7) $$B \otimes_S B = (S \otimes A) \otimes_S (S \otimes A) \approx S \otimes A \otimes A.$$

Via this isomorphism, we obtain the algebra law on B by tensor product:

(26.8) $$B \otimes_S B \approx S \otimes A \otimes A \xrightarrow{S \otimes \mu} S \otimes A.$$

Scholium 26.9. There is an important principle which states roughly: *If a module M is unique, then it exists.* Similarly, *if an element of a module is unique, then it exists.* Taken literally, these assertions are of course either trivial or false. What is meant is that if a module N is sufficiently well described, one may be able to deduce a priori that the two modules N^0, N^1 obtained by extension of scalars are naturally isomorphic. In that case, the natural isomorphism provides descent data. Or, if an element of SM is described canonically, then the descent of the element may be automatic.

For example, let A be an R-algebra, possibly without unit element. Suppose that $R \longrightarrow S$ is a faithfully flat map such that the algebra SA has a unit element. Then because the unit element is unique, A has a unit element as well. More precisely, the unit element e of SA lies in A. To make this clear, we first note that if A is an algebra with unit element e and if $R \longrightarrow S$ is a homomorphism, then $1 \otimes e$ is a unit element of $S \otimes A$, because $(1 \otimes e)(s \otimes a) = s \otimes ea = s \otimes a$. We apply this to the two maps $d^0, d^1 : S \longrightarrow SS$. If e is a unit element of SA, we obtain two unit elements e^0, e^1 of SSA. Because unit elements are unique, $e^0 = e^1$, and 23.11 shows that $e \in A$.

A more trivial example of this principle is with the canonical constructions. If we are given the modules M, M', then the module $N \otimes_S N'$ is canonically isomorphic to $S \otimes M \otimes M'$. So there is natural descent data for the module $N \otimes_S N'$ such that the R-module obtained from this data is isomorphic to $M \otimes M'$. We don't need to carry the canonical isomorphisms along to verify this.

Exercise: Let A be an R-algebra and let $R \longrightarrow S$ be faithfully flat. Prove:
(i) If $S \otimes A$ is a field, then A is a field.
(ii) If SA is left noetherian, then A is left noetherian.
(iii) If SA is a finitely generated S algebra, then A is a finitely generated R-algebra.
(iv) Let M be an R-module, and let $R \longrightarrow S$ be faithfully flat. Prove that if SM is a finitely generated (or finitely presented) S-module, then so is M.

Suppose that an S-algebra B is given, which we wish to present as $S \otimes A$, for some R-algebra A.

Proposition 26.11. *Descent data for an S-algebra B consists of an $S \otimes S$-**algebra** isomorphism $f : B^1 \longrightarrow B^0$ which satisfies the cocycle condition $f^2 f^0 = f^1$. In other words, this data corresponds bijectively to isomorphism classes of R-algebras A together with an isomorphism $S \otimes A \longrightarrow B$.*

Proof. Here B^ν denotes the module obtained from B by the extension of scalars $d^\nu : S \longrightarrow SS = S \otimes S$. So B^ν inherits the structure of SS-algebra from the algebra structure on B, as above. The hypothesis that f is an algebra isomorphism makes sense.

Next, if A is an R-algebra and if $B = S \otimes A$, then B^1 and B^2 are canonically isomorphic as modules, and also as algebras, to $SS \otimes A$, and the descent data for the underlying S-module B is the canonical isomorphism.

Now suppose that descent data $f : B^1 \longrightarrow B^0$ is given. We have to construct the R-algebra A. Since it is an isomorphism of algebras, f is in particular an isomorphism of SS-modules. Therefore it gives us descent data for the underlying module of B. We use that descent data to determine an R-module A and an isomorphism $S \otimes A \longrightarrow B$. We replace B by the isomorphic module SA. Then the desent data f becomes the canonical isomorphism $SS \otimes_1 SA \approx SSA \approx SS \otimes_0 SA$.

We must show that the algebra structure on SA descends to A. Let us write $P = A \otimes A$. Then SP is canonically isomorphic to $SA \otimes_S SA$. We carry the algebra law over to an S-linear map $g : SP \longrightarrow SA$. The criterion 25.12 tells us when such a map descends: It does so if and only if the two maps g^1, g^0 obtained from g by extension of scalars to SS are equal. Now when we identify B with SA via the canonical isomorphism, the descent data f carries over to the identity map on SSA. Similarly, when $B \otimes_S B$ is identified with the canonically isomorphic module SP, the descent data $f \otimes f$ for $B \otimes_S B$ carries over to the identity map on SSP. The statement that f is an isomorphism of algebras carries over to the commutativity of the diagram

(26.12)
$$\begin{array}{ccc} SSP & \xrightarrow{\ i\ } & SSP \\ {\scriptstyle g^1}\big\downarrow & & \big\downarrow{\scriptstyle g^0} \\ SSA & \xrightarrow{\ i\ } & SSA \end{array} .$$

This diagram implies that $g^1 = g^0$, as required.

Finally, we must verify the axioms for an algebra. The associative law asserts that two maps $A \otimes A \otimes A \longrightarrow A$ are equal, or that their difference is zero. Assume that $B = SA$ is an associative algebra. Then the two corresponding maps $SA \otimes SA \otimes SA \longrightarrow SA$ are equal. Since $A \subset SA$, the associativity for A follows. We have already seen that the unit element descends.

27. Forms of a structure.

A problem which is closely related to descent is that of classifying structures over a commutative ring R which become isomorphic after a faithfully flat commutative ring extension $R \longrightarrow S$. Let's take the case of algebras as our example. Suppose given two algebras A, A' over R such that SA and SA' are isomorphic S-algebras. We choose an isomorphism $g : SA \longrightarrow SA'$. Extending scalars to SS via the two maps $d^\nu : S \longrightarrow SS$, we obtain two isomorphisms $g^1, g^0 : SSA \longrightarrow SSA'$. They are equal if and only if g has the form $i \otimes \gamma$ for some isomorphism $\gamma : A \longrightarrow A'$, but this need not be the case.

We define an automorphism of SSA by

(27.1)
$$f = (g^1)(g^0)^{-1}.$$

(The parentheses are only for clarity.) The formulas 25.5 show that $f^2 = (g^{12})(g^{02})^{-1}$, $f^0 = (g^{02})(g^{01})^{-1}$, and $f^1 = (g^{12})(g^{01})^{-1}$. Thus the cocycle condition

(27.2)
$$f^2 f^0 = f^1$$

holds. This means that f provides descent data for the algebra SA. Working out the definitions, one finds that the descended algebra is isomorphic to A'.

The cocycle f depends on the choice of the isomorphism g. If $h : SA \longrightarrow SA'$ is another isomorphism, then $h = \alpha g$, where α is an automorphism of SA. The cocycle obtained from αg is $(\alpha^1) f (\alpha^0)^{-1}$.

Proposition 27.3. *Let $R \longrightarrow S$ be a faithfully flat ring homomorphism and let A be an R-algebra. The algebras A' such that SA' is isomorphic to SA are in bijective correspondence with equivalence classes of cocycles: automorphisms f of SSA such that $f^2 f^0 = f^1$, where two such cocycles $f, \tilde{f}$ are equivalent if there is an automorphism α of SA such that $\tilde{f} = (\alpha^1) f (\alpha^0)^{-1}$.*

Given an R-algebra A, the R-algebras A' such that SA' is isomorphic to SA for some faithfully flat ring homomorphism $R \longrightarrow S$ are called *forms* of the algebra A. A *trivial form* is one isomorphic to A. One says that a form A' *splits* over S if $SA' \approx SA$.

Analogous results hold for any type of structure to which the descent formalism carries over. For example, let M be an R-module. Then the isomorphism classes of R-modules M' such that $SM \approx M'$ are in bijective correspondence with automorphisms f of the SS-module SSM which satisfy the cocycle condition, modulo equivalence.

28. Sheaves and cohomology.

Let R be a commutative ring and denote by $\mathcal{R}$ the category of commutative R-algebras. It is convenient to introduce the *functor* of automorphisms of an R-algebra

$$\underline{\mathrm{Aut}}(A) : \mathcal{R} \longrightarrow (groups)$$

defined as follows: Given $R \longrightarrow S$, we let $\underline{\mathrm{Aut}}(A; S)$ denote the group of automorphisms of the S-algebra SA.

A functor $F : \mathcal{R} \longrightarrow (sets)$ is called a *sheaf* (for the flat topology) if it has the following two properties:

(28.1)
(ii) for every pair of rings $R_1, R_2 \in \mathcal{R}$,

$$F(R_1 \oplus R_2) \approx F(R_1) \times F(R_2).$$

(i) For every faithfully flat map $R' \longrightarrow S'$ in $\mathcal{R}$, the sequence

$$F(R') \longrightarrow F(S') \rightrightarrows F(S' \otimes_{R'} S') \tag{28.2}$$

is exact, meaning that the map $F(R') \longrightarrow F(S')$ is injective, and its image is the set of elements of $F(S')$ whose images in $F(S' \otimes_{R'} S')$ under the two maps $F(d^0), F(d^1)$ are equal.

A *sheaf of groups* is a functor $G : \mathcal{R} \longrightarrow (groups)$ which satisfies the sheaf axiom.

Proposition 28.3. *Let Y be a scheme over $X = \mathrm{Spec}\ R$, for example, $Y = \mathrm{Spec}\ A$, where C is a commutative R-algebra. The functor $Y(\cdot)$ defined by $Y(S) = \mathrm{Maps}_X(\mathrm{Spec}\ S, Y)$ is a sheaf.*

Proof. We will give a proof in the case that $Y = \mathrm{Spec}\ C$ is affine. Say that $C = R[x_1, ..., x_n]/(f_1, ..., f_k)$. If S is an R-algebra, then an element of $\mathrm{Maps}_X(\mathrm{Spec}\ S, Y)$ is a map Spec $S \longrightarrow Y$ which is compatible with the maps of the two schemes to X. Such maps correspond bijectively to solutions of the system of polynomial equations $f(x) = 0$ in the algebra S. The sheaf axiom reads as follows: Let $R' \longrightarrow S'$ be faithfully flat. Then solutions $a \in R'$ of the system of equations $f = 0$ correspond bijectively to solutions in S' such that the two solutions obtained by extension of scalars to $S' \otimes_{R'} S'$ via d^0, d^1 are equal. This follows from descent of elements 23.11.

Examples 28.4 The proposition tells us that algebraic groups over R define sheaves of groups. Three of the most important ones are: the *additive group* $\mathbb{G}_a$, defined by $\mathbb{G}_a(S) = S^+$, the *multiplicative group* $\mathbb{G}_m$ over R, defined by $\mathbb{G}_m(S)$ is the group of units in S, and the *general linear group* GL_n defined by $GL_n(S)$ = the group of invertible matrices with values in S. If we wish, we can think of $\mathbb{G}_m$ as the group GL_1.

Proposition 28.5. *Let A be an R-algebra. The functor of automorphisms $\underline{\mathrm{Aut}}(A;\cdot)$ is a sheaf.*

Proof. This is descent of homomorphisms.

Let G be a sheaf of groups, and let $R \longrightarrow S$ be a faithfully flat homomorphism of commutative rings. A 1-*cocycle* with values in G is an element $f \in G(S \otimes S)$ such that $f^2 f^0 = f^1$, where f^i denotes the image of f via the map $G(d^i)$. Two cocycles $f, \tilde{f}$ are *cohomologous* if there exists an element $g \in G(S)$ such that $\tilde{f} = (g^1)f(g^0)^{-1}$. The set of cohomology classes of 1-cocycles is called the 1-*cohomology* of G and is denoted by $H^1(S/R, G)$. This is a set with a distinguished element "0" called the trivial class, the class of the cocycle $1 \in G(S \otimes S)$. The statement $H^1(S/R, G) = 0$ means that the trivial element is the only element of the set. The union of the cohomology groups $H^1(S/R, G)$ over all faithfully flat maps $R \longrightarrow S$ is denoted by $H^1(R, G)$.

One also defines the 0-*cohomology* to be $H^0(R, G) = G(R)$.

Corollary 28.6. *The forms of an algebra A are classified by $H^1(R, \underline{\mathrm{Aut}}(A))$. Those which become trivial over a faithfully flat extension S of R are classified by $H^1(S/R, \underline{\mathrm{Aut}}(A))$. The trivial class corresponds to the trivial form.*

Notice that the cohomology is defined in terms of the diagram of groups obtained by applying the functor F to the Amitsur complex:

$$F(R) \to F(S) \rightrightarrows F(SS) \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} F(SSS) \cdots . \tag{28.7}$$

If the sheaf F takes its values in the category of abelian groups, say written additively, then one obtains a complex by applying F to the alternating sums $\delta = d^0 - d^1 + \cdots$ of the face maps. The cohomology of the resulting complex is called the *Cech cohomology* $H^q(S/R, F)$. This is an abelian group in all dimensions, and it agrees with the previously defined groups in dimensions $0, 1$.

Corollary 28.8. *(i) $H^q(S/R, \mathbb{G}_a) = 0$ for all $q > 0$.*
(ii) $H^1(S/R, GL_n)$ classifies R-modules M such that $S \otimes M$ is isomorphic to the free module S^n.
(iii) If R is a local ring, then $H^1(S/R, GL_n) = 0$ and $H^1(S/R, \mathbb{G}_m) = 0$.

Proof. (i) Since $\mathbb{G}_a(S) = S^+$, this follows from the exactness of the Amitsur complex 23.10.

(ii) This follows from the descent formalism because the group of automorphisms of the free S-modules of rank n is $GL_n(S)$, so GL_n is the sheaf of automorphims of the free module.

(iii) Let M be an R-module such that $S \otimes M$ is free. The next lemma shows that M is projective. If R is a local ring, then every projective R-module is free. Therefore every form of the free module is free, and this shows that $H^1(S/R, GL_n)$, which classifies such forms, is zero.

Lemma 28.9. *Let $R \longrightarrow S$ be a faithfully flat map of noetherian rings. If M is a finitely generated R-module and if SM is a projective S-module, then M is a projective R-module.*

Proof. Let $R^n \xrightarrow{\phi} M$ be a surjective map. If SM is projective, then $i \otimes \phi : S^n = s \otimes R^n \longrightarrow SM$ splits. Then the map $\mathrm{Hom}_S(SM, S^n) \longrightarrow \mathrm{Hom}_S(SM, SM)$ is surjective. This map is obtained from the map $\mathrm{Hom}_R(M, R^n) \longrightarrow \mathrm{Hom}_R(M, M)$ by tensoring with S 26.2. Hence ϕ splits too.

There is one point which should be mentioned, and that is the definition of a surjective map of sheaves. The *kernel* K of a map $F \longrightarrow G$ of sheaves of groups is defined to be the functorial kernel:

$$K(S) = \ker\left(F(S) \longrightarrow G(S)\right)$$

for every S. This does give a sheaf.

On the other hand, suppose that $F \longrightarrow G$ is a map of sheaves of abelian groups. The functor $\mathcal{C}(S) = \mathrm{coker}(F(S) \longrightarrow G(S))$ is usually not a sheaf. So the sheaf cokernel must be defined as the solution to the universal problem of constructing a cokernel in the category of sheaves. This is called the *associated sheaf* to the functorial cokernel $\mathcal{C}$.

This leads to the following characterization of short exact sequences of sheaves: A sequence of sheaves

$$0 \longrightarrow E \xrightarrow{i} F \xrightarrow{j} G \longrightarrow 0 \qquad (28.10)$$

of sheaves of groups is *exact* if

(28.11) (i)

$$0 \longrightarrow E(S) \longrightarrow F(S) \longrightarrow G(S)$$

is exact for every R-algebra S, and

(ii) for every R-algebra R' and every elements $g \in G(R')$, there is a faithfully flat map $R' \longrightarrow S'$ and an element $f \in F(S')$ such that the images of f and g in $G(S')$ are equal.

The second condition is usually expressed by saying that the element g lifts to F over S', or that g lifts to F *locally*. This definition is forced by sheaf axiom.

Suppose given an exact sequence 28.10. Substituting R for R' into (ii) and using the definition of H^0, we obtain a left exact sequence

$$0 \longrightarrow H^0(R,E) \longrightarrow H^0(R,F) \longrightarrow H^0(R,G) \tag{28.12}$$

If $R \longrightarrow S$ is a faithfully flat extension such that an element $g \in H^0(R,G)$ lifts to F over ~~S, then there is an element of $H^1(S/R,R)$ which is zero if and only if g lifts to F over R.~~ We say that the obstruction to lifting over R is an element of $H^1(S/R,E)$. To obtain this element, we take a lifting $f \in F(S)$. Via the two maps $F(S) \longrightarrow F(SS)$, we obtain two elements f^1, f^0 of $F(SS)$. By the sheaf axiom, these two elements are equal if and only if $f \in F(R)$. Since $g \in G(R)$, the two images of f^ν in $G(SS)$ are equal. So $e = (f^1)(f^0)^{-1}$ is an element of $E(SS)$. This element is a cocycle whose cohomology class represents the obstruction in $H^1(S/R,E)$.

The exact sequence 28.10 also provides sequences of 1-cohomology

$$H^1(S/R,E) \longrightarrow H^1(S/R,F) \longrightarrow H^1(S/R,G). \tag{28.13}$$

The properties of this sequence of pointed sets are discussed in the following works:

References:

Cohomology of sheaves of abelian groups:
J. Milne, *Etale cohomology*, Princeton University Press 1980.

nonabelian cohomology:
J.-P. Serre, *Cohomologie galoisienne*, Springer Lecture Notes No. 5, 1973.

J.-P. Serre, *Local Fields*, Hermann, Paris 1979.

29. Azumaya algebras.

In this section we apply descent to study forms of two algebras: the polynomial ring $K[x]$ when K is a field, and the algebra $M_n(R)$ of $n \times n$ matrices.

Lemma 29.1. *Let K be a field, and let $A = K[x]$ be the polynomial algebra in one variable over K. An automorphism of A is given by a substitution of the form $x \mapsto ax+b$, where $a,b \in K$ and $a \neq 0$. The group of automorphisms is isomorphic to the subgroup $\mathcal{G}(K)$ of $GL_2(K)$ of matrices of the form*

$$m(a,b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}. \tag{29.2}$$

Proof. Let $\phi : A \longrightarrow A$ be an automorphism. The image of x is a polynomial, say $\phi(x) = p(x) = a_m x^m + \cdots + a_0$. Similarly, the image of x via the inverse function is a polynomial, say $q(x) = b_n x^n + \cdots + b_0$. Then $x = p(q(x)) = a_m(b^n x^n)^m + \cdots$. Clearly, we must have $m = n = 1$, and this shows that $p(x) = a_1 x + a_0$, as required.

For any ring S, the group of S-automorphisms of the polynomial ring $S[x]$ contains the group $\mathcal{G}(S)$ of matrices $m(a,b)$ with $a, b \in S$ and a invertible. But if S contains nilpotent elements, there will be other automorphisms. For example, suppose that $s \in S$ and that $s^2 = 0$. Then the map defined by $x \mapsto p(x) = x + tx^2$ is an automorphism. Its inverse function sends $x \mapsto x - tx^2$.

Anyway, the group $\mathcal{G}(S)$ fits into an exact sequence of groups

$$(29.3) \qquad 0 \longrightarrow S^+ \longrightarrow \mathcal{G}(S) \longrightarrow S^* \longrightarrow 0$$

defined by $b \mapsto m(1,b)$ and $m(a,b) \mapsto a$, and we have a corresponding exact sequence of sheaves

$$(29.4) \qquad 0 \longrightarrow \mathbb{G}_a \longrightarrow \mathcal{G} \longrightarrow \mathbb{G}_m \longrightarrow 0.$$

Denoting by $\underline{\mathrm{Aut}}(A)$ the sheaf of automorphisms of the algebra A, we know that $\mathcal{G} \subset \underline{\mathrm{Aut}}(A)$, and that $\mathcal{G}(K) = \underline{\mathrm{Aut}}(A; K)$ if K is a field.

Proposition 29.5. *If K is a perfect field, then every form of the polynomial algebra $A = K[x]$ is trivial. In other words, if B is a K-algebra and if $S \otimes B \approx S[x]$ for some faithfully flat map $K \longrightarrow S$, then $B \approx K[x]$.*

Proof. We are helped by the fact that, since K is a field, every homomorphism $K \longrightarrow S$ is faithfully flat, provided only that $S \neq 0$. Let B be a form of A which becomes isomorphic to A after a faithfully flat extension $K \longrightarrow S$. We may assume that S is a finitely generated ring extension of K. (This is not an interesting point, so we omit the verification.) Then we replace S by one of its residue fields, to reduce to the case that $S = L$ is a finite field extension of K. Extending L further as necessary, we may assume that L/K is a Galois extension.

The form B corresponds to a 1-cohomology class in $H^1(L/K, \underline{\mathrm{Aut}}(A))$, say the class of the cocycle β, and such a cohomology class is an automorphism of $L \otimes L \otimes A = LL[x]$. Because L/K is Galois, $L \otimes L$ is isomorphic to a sum of copies of L. Then because $\mathrm{Aut}(L[x]) \approx \mathcal{G}(L)$, we also have $\mathrm{Aut}(LL[x]) \approx \mathcal{G}(LL)$. So our form is represented by a class in $H^1(L/K, \mathcal{G})$. The exact sequence 28.13, together with the facts that $H^1(L/K, \mathbb{G}_a) = 0$ and $H^1(L/K, \mathbb{G}_m) = 0$ 28.8, show that $H^1(L/K, \mathcal{G}) = 0$. Specifically, let $\overline{\beta}$ denote the cocycle with values in $\mathbb{G}_m$ which is the image of β. This cocycle represents the trivial class. So there is an element $\overline{\alpha} \in \mathbb{G}_m(L)$ such that $(\overline{\alpha}^1)\overline{\beta}(\overline{\alpha}^0)^{-1} = 1$. We lift $\overline{\alpha}$ to $\alpha \in \mathcal{G}(L)$, and replace β by $(\alpha^1)\beta(\alpha^0)^{-1}$, thereby reducing ourselves to the case that $\overline{\beta} = 1$. When this is so, β is a cocycle in the kernel $\mathbb{G}_a$, and so it represents the trivial class in $H^1(L/K, \mathbb{G}_a)$ and in $H^1(L/K, \mathcal{G})$. Note that the above argument on cohomology classes does not depend on the particular sheaves, but only on the existence of the exact sequence.

We turn next to Azumaya algebras:

Proposition 29.6. *The following are equivalent conditions on an R-algebra A:*
(i) There is a faithfully flat extension $R \longrightarrow S$ such that $S \otimes A$ is isomorphic to the matrix algebra $M_n(S)$.
(ii) The canonical map $A^{opp} \otimes_R A \longrightarrow End_R(A)$, which sends $a \otimes b \mapsto \lambda_a \circ \rho_b$ is bijective.
(iii) Definition: *A is an Azumaya algebra over R.*

Proposition 29.7. (Skolem-Noether Theorem) *If R is a local ring, then every automorphism of the matrix algebra $A = M_n(R)$ is an inner automorphism.*

We defer the proofs of these propositions. According to 29.6, an Azumaya algebra B over R is a form of the matrix algebra $A = M_n(R)$. If $R \longrightarrow S$ is a faithfully flat map which splits this form, i.e., so that $SB \approx SA = M_n(S)$, then B is represented by a cohomology class in $H^1(S/R, \underline{\mathrm{Aut}}(A))$. The Skolem-Noether theorem allows us to analyze the functor $\underline{\mathrm{Aut}}(A)$.

Corollary 29.8. *There is an exact sequence of sheaves*

$$0 \longrightarrow \mathbb{G}_m \xrightarrow{j} GL_n \xrightarrow{c} PGL_n \longrightarrow 0,$$

where $j(u) = uI$ and where $PGL_n = \underline{\mathrm{Aut}}(A)$.

Proof. The units in the matrix algebra $M_n(R)$ are the invertible matrices, i.e., they are the elements of $GL_n(R)$. The map c sends an invertible matrix P to the automorphism ϕ of conjugation by P: $\phi(X) = PXP^{-1}$. The kernel of this map consists of the invertible matrices in the center of the matrix algebra. These are the matrices of the form uI, where $u \in R^* = \mathbb{G}_m(R)$. This explains the sequence except for the surjectivity of the map c. Surjectivity of the map is defined in 28.11. If $\phi \in PGL_n(R')$, the Skolem-Noether Theorem tells us that ϕ becomes inner in every local ring of R'. Taking for S' the sum of a suitable bunch of localizations, ϕ becomes inner over S'. This shows that c is a surjective map of sheaves.

The projective general linear group can be represented as an algebraic subgroup of GL_{n^2}. But this representation of the group is not very nice because n^2 is so large in relation to n. The representation abstractly as quotient of GL_n is usually more convenient.

Exercise: Let A be an R-algebra which is a free module of rank m over R. Show that $\underline{\mathrm{Aut}}(A)$ is a closed algebraic subgroup of GL_m.

Corollary 29.9. *Let B be an Azumaya algebra over R, which becomes isomorphic to the matrix algebra over the faithfully flat extension $R \longrightarrow S$. Say that $g : M_n(S) \longrightarrow SB$ is an isomorphism, and say that $\alpha \mapsto \beta$ by this isomorphism. The coefficients of the characteristic polynomial of α are independent of S and of g, and they lie in R.*

This corollary allows us to define the characteristic polynomial of β as the characteristic polynomial of α. In particular, trace β and $det\,\beta$ are defined. These elements are often referred to as the "reduced trace" and the "reduced norm" respectively.

Proof. This proof follows the principle that if an element is unique then it exists. We know that the characteristic polynomial of any matrix α and of a conjugate matrix $P\alpha P^{-1}$ are equal. Skolem-Noether implies that for any automorphism ϕ of $M_n(S)$, α and $\phi(\alpha)$ have the same characteristic polynomial, provided that S is a local ring. But since the characteristic polynomial is compatible with localization, this is true whether S is local or not: two elements which become equal in every local ring are equal.

We note without proof that the exact sequence 29.8 yields an interesting exact cohomology sequence

$$(29.10)\ 0 \longrightarrow R^* \xrightarrow{j^0} GL_n(R) \xrightarrow{c^0} PGL_n(R) \xrightarrow{\delta^0}$$
$$H^1(R,\mathbb{G}_m) \xrightarrow{j^1} H^1(R,GL_n) \xrightarrow{c^1} H^1(R,PGL_n) \xrightarrow{\delta^1} H^2(R,\mathbb{G}_m).$$

Interpreting the terms, we have

$$(29.11)\ 0 \longrightarrow (units) \longrightarrow (inv.\ mat.) \longrightarrow (autos.) \longrightarrow$$
$$(inv.\ modules) \longrightarrow (loc.\ free\ modules) \longrightarrow (Azumaya\ alg.) \longrightarrow H^2.$$

Reading this sequence at $PGL_n(R)$: Let ϕ be an automorphism of the matrix algebra. Then $\phi\delta^0$ is the class of an invertible R-module. This module is the free module of rank 1 if and only if ϕ is an inner automorphism.

At $H^1(R,PGL_n)$: Given a locally free R-module V of rank n (a form of the free module), the algebra $\text{End}_R(V)$ is an Azumaya algebra. The map c^1 sends the class $[V]$ of V to the class $[\text{End}_R(V)]$. If we are given an Azumaya algebra A, the coboundary of its class $[A]\delta^1$ is a class in $H^2(R,\mathbb{G}_m)$ which is zero if and only if $A \approx \text{End}_R(V)$ for some V.

The behavior at $H^1(R,\mathbb{G}_m)$ is a bit more complicated. It can be shown that, because $\mathbb{G}_m$ is in the center of GL_n, the group $H^1(R,\mathbb{G}_m)$ *operates* on $H^1(R,GL_n)$. Let L be an invertible R-module and V a locally free R-module. Then $L \otimes V$ is also a locally free module, and the operation is $[L][V] = [L \otimes V]$. The fibres of the map c^1 are the orbits for this action.

Proof of the Skolem Noether Theorem. Let ϕ be an automorphism of the matrix algebra A over the local ring R, and let V denote the space R^n of column vectors. To find a matrix such that ϕ is conjugation by that matrix is equivalent with finding an invertible map $p : V \longrightarrow V$ which is equivariant with repsect to ϕ, i.e., such that $\phi(a)p(v) = p(av)$, or that the diagram

$$(29.12)\qquad \begin{array}{ccc} A \otimes V & \xrightarrow{mult} & V \\ {\scriptstyle \phi\otimes p}\downarrow & & \downarrow{\scriptstyle p} \\ A \otimes V & \xrightarrow{mult} & V \end{array}$$

commutes. If so, then $p^{-1}\phi(a)p = a$, so ϕ is conjugation by the matrix representing p. The converse also holds.

Let e_{ij} denote the matrix units in A, and let u_i be the standard basis of V. Set $\epsilon_{ij} = \phi(e_{ij})$. Then ϵ_{ii} are orthogonal idempotents. So $V = \bigoplus \epsilon_{ii}V$, and $V_i = \epsilon_{ii}V$, being a summand of V, is a projective R-module, hence is free. It is easily seen that V_i must be free of rank 1. Let $\{v_1\}$ be a basis of V_1, and set $v_i = \epsilon_{i1}v_1$. Then defining $p(u_i) = v_i$, we have $\phi(e_{ij})p(u_k) = \epsilon_{ij}v_k = p(e_{ij}u_k)$, and p is invertible.

Proof of Proposition 29.6. Let A be an R-algebra. First of all, the conditions that A be flat and that the isomorphism $m : A^{opp} \otimes_R A \longrightarrow \mathrm{End}_R(A)$ are compatible with change of scalars, i.e., if they hold for A and if $R \longrightarrow S$ is any map, then they hold for SA. Moveover, if $R \longrightarrow S$ is faithfully flat and if these conditions hold for SA, then they hold for A too. Since the conditions do hold for the matrix algebra, it follows that 29.6(i) implies (ii). To prove the converse, we suppose that A satisfies (ii). In this case we will find a faithful and *smooth* map $R \longrightarrow S$ such that SA is a matrix algebra over A. (Let's omit the verification of the technical point that smooth maps are flat.)

Suppose first that $R = K$, where K is an algebraically closed field. Then $\mathrm{End}\, A$ is a matrix algebra over K, which is a simple ring. The fact that $A^{opp} \otimes A$ is simple shows that A is simple too. By Wedderburn's theorem, A is a matrix algebra over K.

Next, suppose that A contains an idempotent element e, and let $e' = 1 - e$. We use the Peirce decomposition

$$A = \begin{pmatrix} eAe & eAe' \\ e'Ae & e'Ae' \end{pmatrix}. \qquad (29.13)$$

The terms, being summands of A, are projective R-modules.

Suppose that eAe has rank 1. Then the map $R \longrightarrow eAe$ is an isomorphism. (Why is this so?) Consider the decomposition $A = eA \oplus e'A$ into projective right A-modules. The isomorphism $\mathrm{End}\, A \approx A^{opp} \otimes A$ shows that $\mathrm{End}\, eA \approx (eAe)^{opp} \otimes A \approx R \otimes A = A$. This shows that $A \approx \mathrm{End}\, eA$. Since eA is projective, it is locally free. Then A is locally isomorphic to the ring of endomorphisms of a free module, hence it is locally a matrix algebra.

Now to split the Azumaya algebra A, it suffices to find a fatihfully flat map $R \longrightarrow S$ and an idempotent $e \in SA$ such that $e(SA)e$ is an S-module of rank 1. We look for the universal solution to finding such an idempotent. We may assume that A is a free R-algebra. We choose a basis $\{a_\nu\}$ for A, and write a hypothetical idempotent as a linear combination $e = \sum x_\nu a_\nu$. The condition $e^2 = e$ can be expressed in terms of the structure constants c_{ijk} for the algebra (20.1):

$$\sum_k x_k a_k = e = e^2 = \sum_{i,j} x_i x_j a_i a_j = \sum_{i,j,k} x_i x_j c_{ijk} a_k. \qquad (29.14)$$

Collecting coefficients, the requirement on the indeterminates x_ν is

$$x_k = \sum_{i,j} x_i x_j c_{ijk} \qquad (29.15)$$

for all k. These relations define an algebra U, a quotient of the polynomial ring $R[x_\nu]$, such that idempotent elements of SB correspond bijectively to maps of R-algebras $U \longrightarrow S$, or to solutions of the system of equations 29.15 in S. In particular, we have the universal solution by the residues of the variables in U. Roughly speaking, we plan to set $S = U$.

Three things remain to be done: First, we must show that the map $R \longrightarrow U$ is smooth. To do this, we verify the infinitesimal criterion for smoothness: Let $S \longrightarrow S'$ be a length one extension of finite local R-algebras, and suppose given a map of algebras $f' : U \longrightarrow S'$. We have to lift this map to a map $f : U \longrightarrow S$. We interpret f' as giving an idempotent e' in $S'A$. Then the problem becomes that of lifting that idempotent to $e \in SA$. This can be done 15.2.

Second, there is the question of the rank of eAe, when an idempotent is found. Now because of the Peirce decomposition, eAe will be a projective module in any case. In particular, if $\tilde{e}$ is the universal idempotent in $UA = \widetilde{A}$, then $\tilde{e}\widetilde{A}\tilde{e}$ is a projective U-module. As such, its rank is a locally constant function on Spec U. This means that Spec U is a disjoint union of closed subsets, and that the rank is constant on each piece. Correspondingly, U splits into a direct sum according to this rank. We replace U by the summand on which $\tilde{e}\widetilde{A}\tilde{e}$ has rank 1. The summand is also smooth.

Finally, we must show that $R \longrightarrow U$ is faithfully flat, so we must show that the map Spec $U \longrightarrow$ Spec R is surjective. We choose a residue field K of R. Let $\overline{K}$ be the algebraic closure of K. We have already shown that $\overline{K}A$ is a matrix algebra. So this ring contains an idempotent e_{11} with the required property on rank. The idempotent corresponds to a map of R-algebras $U \longrightarrow \overline{K}$, hence to a point of Spec U. This point lies over Spec $K \in$ Spec R.

30. Noncommutative deformations of commutative algebras.

The general notation for the remaining sections will be similar to that introduced in Section 7: R denotes a finite local k-algebra with nilpotent maximal ideal N. Let r be such that $N^{r+1} = 0$ but $N^r \neq 0$. We set $R' = R/N^r$ and $R'' = R/N^{r-1}$.

All rings considered will be R-algebras. If A is an R-algebra, then $A' = A \otimes_R R'$, etc... The notation $\Gamma_1(A)$ will stand for the ideal of A generated by N and by the commutators $[A, A] = \{xy - yx | x, y \in A\}$, and $\Gamma_2(A)$ for the ideal of A generated by Γ_1^2 and by the double commutators $[[A, A], A]$.

Let $\mathcal{A}_R$ denote the category of R-algebras A such that $A_k = A \otimes_R k$ is commutative. The kernel of the map $A \to A_k$ is the nilpotent ideal NA. So A_k is commutative if and only if $[A, A] \subset NA$. Thus an R-algebra A is in $\mathcal{A}_R$ if and only if $\Gamma_1(A) = NA$. If so, then $\Gamma_2(A) = N^2A$ too.

Note that for $A \in \mathcal{A}_R$, Spec $A \approx$ Spec A_k.

Let R_1 be a quotient of R. As before, if $A_1 \in \mathcal{A}_{R_1}$, we call an R-algebra A together with an isomorphism $A \otimes_R R_1 \to A_1$ an *extension* of A_1 to R. Similarly, suppose that $A \in \mathcal{A}_R$ is given, that $A_1 = A \otimes_R R_1$, and that we are given a homomorphism $\phi_1 : A_1 \to B_1$ in $\mathcal{A}_{R_1}$. An *extension* of ϕ_1 to R is an R-homomorphism $\phi : A \to B$ together with an isomorphism

$\phi \otimes_R R_1 \approx \phi_1$. Unless flatness is stated explicitly, an extension needn't be flat over R, though as always, deformations of algebras are required to be flat.

We leave the proof of the following propositions as exercises.

Proposition 30.1. *Let $A \in \mathcal{A}_R$.*
(i) A is noetherian if and only if A_k is noetherian.
(ii) Every idempotent $e \in A$ is central.
(iii) Suppose that $N^{r+1} = 0$. For any $x, y \in A$, there is an element $z \in A$ such that $x^{r+1}y = zx$.
(iv) If $a, b \in A$ and if $ab = 1$, then $ba = 1$.
(v) Let $s \in A$, and let $S = \{s^n\}$. Then S is an Ore set in A. Moreover, $S^{-1}A = AS^{-1}$ depends only on the residue of s in A_k.
(vi) With the notation of (v), let $t \in A$ be another element, and let $T = \{t^n\}$, and $U = \{(st)^n\}$. There are canonical isomorphisms

$$(S^{-1}A) \otimes_A (T^{-1}A) \approx T^{-1}(S^{-1}A) \approx U^{-1}A.$$

(vii) Let $A \to B$ be a homomorphism in $\mathcal{A}_R$, and let $S = \{s^n\}$ be as before. There is a canonical bijective map $\phi : (S^{-1}A) \otimes_A B \to S^{-1}B$.

Proposition 30.2. *(i) A map $\phi : A \longrightarrow B$ in $\mathcal{A}_R$ is left flat if and only if it is right flat.*
(ii) A flat map $\phi : A \longrightarrow B$ in $\mathcal{A}_R$ is faithfully flat if and only if $\phi_k : A_k \longrightarrow B_k$ is faithfully flat, and this is the case if and only if the map Spec $B \longrightarrow$ Spec A *is surjective.*

Let $A \in \mathcal{A}_R$ be a noetherian ring. The *support* of a finite right A-module M is the set of $p \in$ Spec $A =$ Spec A_k such that $M \otimes k(p) \neq 0$, where $k(p)$ denotes the (commutative) residue field at the prime ideal p. Thus the supports of M and of $M \otimes_R k$ are equal, and in particular, the support is a closed subset of Spec A.

Proposition 30.3. *Let $A \in \mathcal{A}_R$ and let M be a finite right A-module supported on a closed set $Z \subset X =$* Spec A.
(i) Let I denote the annihilator of M. Then Spec $A/I = Z$.
(ii) If Z is the union of two disjoint closed subsets C_1, C_2, then there is a unique splitting $M = M_1 \oplus M_2$, where M_i is supported on C_i.
(iii) Let $A \longrightarrow B$ be a map in $\mathcal{A}_R$, and let M be a finite right B-module. Let $Z \subset Y =$ Spec B *denote the support of M. If the map $Z \longrightarrow X =$* Spec A *is a finite morphism, then M is a finite A-module.*

31. Interlude: solving polynomial equations.

A *polynomial* with coefficients in an R-algebra A will mean a left linear combination of monomials: $f(y) = \sum_i a_i y^i$. Its derivative is defined by the usual formula:

$$f'(y) = \sum_i i a_i y^{i-1}.$$

If $f(y)$ and $g(y) = \sum_j b_j y^j$ are two polynomials, $f * g$ will denote the polynomial which represents the product in the polynomial ring $A[y]$, in which scalars commute with the variable: $f * g = \sum_{i,j} a_i b_j y^{i+j}$. In the free R-algebra extension $A\langle y\rangle$,

$$fg = \sum_{i,j} a_i y^i b_j y^j = f * g + O([y, A]), \tag{31.1}$$

where the notation $O(S)$ stands for an element in the ideal generated by S. In $A\langle x, y\rangle$,

$$[x, y^n] = \sum_{\nu=1}^{n} y^{\nu-1}[x, y]y^{n-\nu} = ny^{n-1}[x, y] + O([[x, y], y]). \tag{31.2}$$

Also,

$$[x, f(y)] = [x, \sum_i a_i y^i] = \sum [x, a_i]y^i + \sum a_i [x, y^i],$$

and using 31.2, this gives

$$[x, f(y)] = \sum_i [x, a_i]y^i + f'(y)[x, y] + O([[x, y], y]). \tag{31.3}$$

Let (z) denote the two-sided ideal generated by z in $A\langle y, z\rangle$, and let $f(y) = \sum_i a_i y^i$ as before. Then

$$f(y + z) = f(y) + f'(y)z + O((z)^2 + [y, z]). \tag{31.4}$$

Lemma 31.5. *Let I be an ideal of a ring A. The following conditions are equivalent:*
(i) A/I is commutative, and I/I^2 is a central A/I-bimodule.
(ii) $[A, A] \subset I$ and $[A, I] \subset I^2$.
If these conditions hold, then $[A, I^r] \subset I^{r+1}$.

The next proposition is a noncommutative version of the implicit function theorem.

Proposition 31.6. *Let $I \subset A$ be a nilpotent ideal satisfying the conditions of 31.5, and let $A_0 = A/I$. Let $f(y), g(y), u(y), v(y)$ be polynomials with coefficients in A such that $uf + vf' = g$ in $A_0[y]$. Let $a_0 \in A_0$ be an element such that $f(a_0) = 0$ and $g(a_0)$ is invertible. There exists a unique representative a of a_0 in A such that $f(a) = 0$.*

Proof. From the equation $uf + vf' = g$ in $A_0[y]$ we conclude that in $A\langle y\rangle$,

$$u * f + v * f' = g + \alpha,$$

where $\alpha \in IA\langle y\rangle$. Then 31.1 shows that

$$uf + vf' = g + \alpha + \sigma, \tag{31.7}$$

where $\sigma = O([y, A])$.
We solve the equation $f(y) = 0$ by Newton's method. Say that we have found $a \in A$ such that $f(a) \equiv 0$ (*modulo* I^r), and that $r \geq 1$. We look for an element $h \equiv 0$ (*modulo* I^r) so that $f(a + h) \equiv 0$ (*modulo* I^{r+1}). We may suppose that $I^{r+1} = 0$. We substitute $y = a$ into 31.7, finding that

$$v(a)f'(a) = g(a) - \epsilon,$$

where $\epsilon \equiv 0$ (*modulo* I). Since $g(a_0)$ is invertible, so are $g(a)$, $g(a - \epsilon)$ and $f'(a)$. We can solve the equation

$$f(a) + f'(a)h = 0$$

uniquely for $h \equiv 0$ (*modulo* I^r). We substitute $y = a$ and $z = h$, where h is as above, into 31.4. Then $(h)^2 = 0$ and $[a, h] = 0$. So in A,

$$f(a + h) = f(a) + f'(a)h = 0.$$

This shows that a solution exists. Moreover, since h is uniquely determined, the uniqueness also follows.

32. Rigidity of etale maps.

A map $A \longrightarrow B$ of commutative R-algebras is called *etale* if it is smooth and of relative dimension zero (see Section 19). Thus, if $P = A[x_1, ..., x_m]$ is a polynomial ring and if $B = P/\mathfrak{b}$, then B is etale over A if for every point $p \in \operatorname{Spec} B$, there are elements $g_1, ..., g_m \in \mathfrak{b}$ which generate the ideal $\mathfrak{b}_p$ in the local ring B_p, and such that $\frac{\partial g_i}{\partial x_j}$ is invertible at p.

Proposition 32.1. *Let $\phi : A \longrightarrow B$ be a flat map of finitely generated R-algebras in $\mathcal{A}_R$. The following two assertions are equivalent. If they hold, then ϕ will be called etale, and B will be called etale over A.*
(i) $\phi_k : A_k \longrightarrow B_k$ is etale,
(ii) The multiplication map $B \otimes_A B \xrightarrow{m} B$ has a two-sided splitting. In other words, there is an element $s = \Sigma u_i \otimes v_i \in B \otimes_A B$ with $\Sigma u_i v_i = 1$, such that $bs = sb$ for all $b \in B$.

Proof. Note that $(B \otimes_R B) \otimes_R k \approx B_k \otimes_k B_k$ and that $(B \otimes_A B) \otimes_R k \approx B_k \otimes_{A_k} B_k$. In particular, $E(B) = B^\circ \otimes_R B \in \mathcal{A}_R$, and if m splits, then $m \otimes k$ does too. It is a standard fact of commutative algebra that (i) and (ii) are equivalent for a map of commutative rings. This proves that (ii) implies (i).

Conversely, suppose that $\phi_k : A_k \longrightarrow B_k$ is etale. Let $X = \operatorname{Spec} A = \operatorname{Spec} A_k$ and $Y = \operatorname{Spec} B = \operatorname{Spec} B_k$. The splitting of m_k corresponds to a decomposition of $Y \times_X Y$ into disjoint closed sets Δ, Γ, Δ being the diagonal. The $E(B)$-bimodule $B \otimes_A B$ is supported on $\Delta \cup \Gamma$, and so it splits accordingly, say $B \otimes_A B \approx M \oplus N$, where M is the part supported on Δ. Then M is a finite right B-module.

Consider the map of bimodules $\alpha : M \longrightarrow B$ induced by m. The induced map $\alpha \otimes k$: $M \otimes k \longrightarrow B \otimes k$ is bijective, hence by the Nakayama Lemma, α is surjective. Since ${}_B B$ is projective, the map splits as a map of left modules, and it follows, again from Nakayama, that $M \approx B$. Therefore m splits, as required.

It seems plausible that (ii) gives a good definition of etale maps in a more general context. Here is an example of such a map, which also shows that $B \in \mathcal{A}_R$ is not a consequence of $A \in \mathcal{A}_R$ and (ii).

Example 32.2. A flat map which satisfies (ii), where $A = A_k$ is commutative, but $B = B_k$ isn't. Let u, v, y be indeterminates, let $A = k[u, u^{-1}, v]$, and let $B = k\langle y, y^{-1}, v\rangle$, where $yv = -vy$. Then A embeds as a subring of B by $u = y^2$. The element $e = \frac{1}{2}(1 \otimes 1 + y^{-1} \otimes y) \in B \otimes_A B$ centralizes B, hence sending $1 \mapsto e$ defines a splitting of $B \otimes_A B \xrightarrow{m} B$.

For $A \in \mathcal{A}_R$, let $\mathrm{Et}(A)$ denote the category of etale maps $A \longrightarrow B$. Grothendieck showed that when A is commutative, the functor $\cdot \otimes_R k$ induces an equivalence of categories $\mathrm{Et}(A) \longrightarrow \mathrm{Et}(A_k)$, and the object of this section is to prove the same result for the category $\mathcal{A}_R$. This is done in Theorem 32.4.

A *standard etale* algebra over a commutative k-algebra A_k is a commutative A_k-algebra B_k such that there exist polynomials $f, g, u, v \in A_k[y]$ with $g = uf + vf'$, and an isomorphism $B_k \approx A_k[y, z]/(f, zg - 1)$. We omit the proof of the next lemma.

Lemma 32.3. *(i) Let $A_k \to B_k$ be an etale map of commutative k-algebras, and let $q \in \mathrm{Spec}\, B_k$. There is an element $t \in B_k$ such that $q \in \mathrm{Spec}\, B_k[t^{-1}]$ and that $A_k \to B_k[t^{-1}]$ is a standard etale map.*
(ii) If $A_k \to B_k$ is a standard etale map and if $s \in A_k$, $t \in B_k$, then $A_k[s^{-1}] \to B_k[(st)^{-1}]$ is also a standard etale map.

Theorem 32.4. *Let A be a finitely generated algebra in $\mathcal{A}_R$. For every etale map $\phi_k : A_k \to B_k$, there exists an etale extension $\phi : A \to B$ of ϕ_k to R such that*

$$\mathrm{Hom}_A(B, C) \approx \mathrm{Hom}_{A_k}(B_k, C_k) \tag{32.4.1}$$

for all $C \in \mathcal{A}_R$. In particular, the functor $\cdot \otimes_R k$ defines an equivalence of categories $\mathrm{Et}(A) \longrightarrow \mathrm{Et}(A_k)$.

Proposition 32.5. *Let A be an object of $\mathcal{A}_R$, and let $\phi_k : A_k \to B_k$ be a standard etale homomorphism. There is an extension $\phi : A \to B$ of ϕ_k to R which has the universal property 32.4.1.*

This proposition is a first step in the proof of the theorem. The remainder of the proof, including the flatness of the universal extension, will be given in Section 38.

Proof. Let f, g, u, v be polynomials with coefficients in A whose residues in $A_k[y]$ yield the presentation of the standard etale algebra B_k so that, as above, $B_k \approx A_k[y, z]/(f, zg - 1)$, and let b_0 denote the residue of y in B_k. Let $\psi : A \to C$ be a map such that ψ_k factors through B_k, say $\psi_k = \xi_k \phi_k$. Let $c_0 = \xi_k(b_0)$. Proposition 31.6 shows that c_0 can be lifted uniquely to a solution c of $f(y) = 0$ in C. This lifting provides a map $A\langle y, z\rangle \to C$ sending

$y \mapsto c$ and $z \mapsto g(c)^{-1}$. Let $Q = A\langle y, z\rangle/(f, zg-1, gz-1)$. The map $A\langle y, z\rangle \to C$ factors through the quotient Q, and the uniqueness of c shows that $\mathrm{Hom}_A(Q, C) \approx \mathrm{Hom}_{A_k}(B_k, C_k)$ if $C \in \mathcal{A}_R$. But $Q \notin \mathcal{A}_R$. The universal extension we seek is a quotient of Q.

Lemma 32.6. *Let $\gamma : A \to NQ$ be an arbitrary function, and let J denote the ideal of Q generated by the elements $[a, y] - \gamma(a)$, $a \in A$. Then $Q/J \in \mathcal{A}_R$ and $A \to Q/J$ extends ϕ_k to R.*

Proof. We have $A\langle y, z\rangle \otimes_R k \approx A_k\langle y, z\rangle$, hence $Q \otimes_R k = Q_k \approx A_k\langle y, z\rangle/(f, zg-1, gz-1)$. Let J_k denote the image of J in Q_k. Because $\gamma(a) \in NQ$, J_k is generated by the elements $[a, y]$, $a \in A_k$. Thus $[a, y] = 0$ in $Q_k/J_k \approx (Q/J) \otimes_R k$. Then since z inverts g in Q, it follows that $[y, z] = 0$ and that $[a, z] = 0$ in Q_k/J_k for all $a \in A_k$. So $Q_k/J_k \approx A_k[y, z]/(f, zg-1) = B_k$.

The kernel of the map $Q \to B_k$ is $\Gamma_1(Q)$. Let $q \in \Gamma_1(Q)$. If there is an element $\beta \in NQ$, such that $q = \beta$ in C for *every* homomorphism $\psi : Q \to C$ with $C \in \mathcal{A}_R$, we call β a ***natural reduction*** of q.

Lemma 32.7. *(i) If a universal extension $\phi : A \to B$ of ϕ_k exists, then every element $q \in \Gamma_1(Q)$ has a natural reduction.*
(ii) Conversely, to prove the proposition, it suffices to show that for every $a \in A$, the commutator $[a, y]$ has a natural reduction.
(iii) Let $\phi : A \longrightarrow B$ be a universal extension of ϕ_k, and define $\gamma : A \longrightarrow NQ$ so that $\gamma(a)$ is a natural reduction of $[a, y]$. Then with the notation of Lemma 32.6, $B \approx Q/J$.

Proof. (i) Assume that ϕ exists, and let $p \in \Gamma_1(Q)$ have image $b \in \Gamma_1(B) = NB$. The canonical map $Q \to B$ is surjective because $Q \to B_k$ is surjective, so we can represent b by an element $\beta \in NQ$. The universal property of ϕ shows that β is a natural reduction of p. (ii),(iii) follow from the previous lemma.

Now to prove the proposition, we use induction on the nilradical. With the standard notation, we may assume that a universal extension $\phi' : A' \to B'$ of ϕ_k to $R' = R/N^r$ exists. Note that $Q' \approx A'\langle y, z\rangle/(f, zg-1, gz-1)$. Hence B' is a quotient of Q'.

Lemma 32.8. *Under the inductive hypothesis, every element $q \in \Gamma_2(Q)$ has a natural reduction $p \in N^2Q$.*

Proof. Let $q \in \Gamma_1(Q)$. Because B' is universal, the image q' of q in $\Gamma_1(Q')$ has a natural reduction $\beta' \in NQ'$, which has the property that

$$q' = \beta' \text{ in } C' \tag{32.9}$$

for every map $\psi' : Q' \to C'$ such that $C' \in \mathcal{A}_{R'}$. We can represent β' by an element $\beta \in NQ$. Then if $\psi : Q \to C$ is a map, with $C \in \mathcal{A}_R$, 32.9 implies that $\psi(q) \equiv \psi(\beta)$ (*modulo* N^r). Hence we may write $\psi(q) = \psi(\beta) + \epsilon$, with $\epsilon \in N^rC$.

Now if q_1q_2 is a product of elements of Γ_1 and if β_1, β_2 are elements of NQ described as above, then $\psi(q_1q_2) = (\psi(\beta_1) + \epsilon_1)(\psi(\beta_2) + \epsilon_2) = \psi(\beta_1\beta_2)$. This shows that $\beta_1\beta_2$ is a natural reduction for q_1q_2. Thus every element of Γ_1^2 has a natural reduction in N^2Q.

Finally, suppose that $p = [[u,v],w]$ is a double commutator. Set $q = [u,v]$, and define β as above. We may write $\beta = \sum n_i\gamma_i$, with $n_1 \in N$ and $\gamma_i \in Q$. Then $[\beta,w] = \sum n_i[\gamma_i,w] \in \Gamma_1^2$. By what has been shown, $[\beta,w]$ has a natural reduction $\eta \in N^2Q$. This is the required natural reduction of p.

To show that $[x,y]$ has a natural reduction for every element $x \in A$, we use the fact that in Q, $[x,f(y)] = 0$. Substituting into 31.3, we find $0 = \sum a_i[x,y^i] + f'(y)[x,y] + O(\Gamma_2)$. Since $a_i, x \in A$, $q := -\sum[x,a_i]y^i$ is a well-determined element of NQ. Thus

$$f'(y)[x,y] = q + \eta, \tag{32.10}$$

where $\eta \in \Gamma_2$.

We use the expansion 31.7 to conclude that in Q, $vf' = g - \epsilon$, or $zvf' = 1 - z\epsilon$, with $\epsilon \in \Gamma_1$. Multiplying 32.10 by zv, we obtain $[x,y] = zvq + zv\eta + z\epsilon[x,y]$. By Lemma 32.8, $zv\eta + z\epsilon[x,y]$ has a natural reduction in N^2Q, call it β. Then $zvq + \beta$ is a natural reduction of $[x,y]$. This completes the proof of Proposition 32.5.

Example 32.11. Going back to Example 32.2, it is not difficult to find deformations of the commutative algebra $A_k = k[u,u^{-1},v]$ which do not extend to the algebra $B_k = k\langle y,y^{-1},v\rangle$ defined by $yv = -vy$ and $y^2 = u$. Let $R = k[t]/(t^2)$. We consider a deformation $A_R = R\langle u,u^{-1},v\rangle$ with $vu = uv + \gamma t$, where $\gamma \in A_k$, and we look for a deformation B_R of the form $k\langle u,u^{-1},v\rangle$, with $yv = -vy + \beta t$, $\beta \in B_k$. The map $A_R \longrightarrow B_R$ will be given by a substitution of the form $u = y^2 + \delta t$, $\delta \in B_k$. Then $(y^2)v = uv - \delta vt$, while reduction of $y(yv)$ yields $uv + \gamma t + v\delta t + [y,\beta]t$. Thus we must have

$$\gamma + [v,\delta] + [y,\beta] = 0$$

in B_k. Now since y and v skew commute in B_k, $[y,\beta]$ is an odd polynomial in v. So the above equation implies that v divides γ. If we start with γ not divisible by v, for instance with the deformation A_R defined by the relation $vu = uv + t$, then there is no compatible extension B_R.

On the other hand, Theorem 32.4 tells us that if, in the above computation, we replace B_k by the commutative ring $k[y,y^{-1},v]$, then every deformation of A_k extends to B_k. Needless to say, this can be checked directly.

33. Noncommutative deformations of commutative polynomial rings.

The most direct way to compute the deformations of a polynomial ring $A_k = k[x_1,...,x_m]$ uses Gröbner bases. Let P_R denote the free ring $R\langle x_1,...,x_n\rangle$. The defining relations for A_k in P_k are $x_jx_i = x_ix_j$, $i < j$, so to obtain a flat deformation we add a perturbation term $\alpha_{ij} \in NP_R$ to those relations:

$$x_jx_i = x_ix_j + \alpha_{ij}. \tag{33.1}$$

In order for the ring $A = P/\mathfrak{a}$ defined by these relations to be R-flat, it is necessary and sufficient that the overlaps be consistent. The overlaps are $(x_k x_j)x_i = x_k(x_j x_i)$, $i < j < k$. The monomial $(x_k x_j)x_i$ reduces as follows:

$$x_k x_j x_i \longrightarrow x_j x_k x_i + \alpha_{jk} x_i \longrightarrow x_j x_i x_k + x_j \alpha_{ik} + \alpha_{jk} x_i \longrightarrow$$

$$x_i x_j x_k + \alpha_{ij} x_k + x_j \alpha_{ik} + \alpha_{jk} x_i,$$

and $x_k(x_j x_i)$ reduces similarly to

$$x_i x_j x_k + x_k \alpha_{ij} + \alpha_{ik} x_j + x_i \alpha_{jk}.$$

The end results here need not be reduced, but we can't continue the reduction process without knowing the terms α_{ij} explicitly. In any case, consistency of the overlap requires that

$$(\alpha_{ij} x_k + x_j \alpha_{ik} + \alpha_{jk} x_i)_{red} = (x_k \alpha_{ij} + \alpha_{ik} x_j + x_i \alpha_{jk})_{red}.$$

This equation can be rewritten in the form

$$(33.2) \qquad [x_i, \alpha_{jk}]_{red} - [x_j, \alpha_{ik}]_{red} + [x_k, \alpha_{ij}]_{red} = 0.$$

Recapitulating, the relations 33.1 hold in A, and they hold in P if and only if A is flat over R. Note that the defining relations 33.1 show that $\alpha_{ij} = [x_j, x_i]$ in A. So 33.2 becomes the Jacobi identity when carried over to the ring A. For this reason, we call 33.2 the *Jacobi condition.*

As an example, we consider the case of three variables: $A_k = k[x, y, z]$. Changing a sign, we write a deformation in the form

$$(33.2) \qquad zy = yz + \alpha \;,\; zx = xz - \beta \;,\; yx = xy + \gamma.$$

The Jacobi condition is that

$$(33.3) \qquad [x, \alpha] + [y, \beta] + [z, \gamma]$$

must reduce to zero.

Note that if $\alpha \in At$, then $[x, \alpha] \in At^2$. So there is no requirement on α, β, γ for a deformation over $k[t]/(t^2)$. The Jacobi condition is a second order condition, an obstruction to extending such a deformation to $k[t]/(t^3)$.

We may apply the commutator formula (see Section 9) for noncommutative differentiation in the free ring to write

$$[x, \alpha] = [x, y]\alpha_y + [x, z]\alpha_z,$$

etc. Then 33.3 becomes

$$[x, y](\alpha_y - \beta_x) + [z, x](\gamma_x - \alpha_z) + [y, z](\beta_z - \gamma_y),$$

or

$$\alpha(\beta_z - \gamma_y) + \beta(\gamma_x - \alpha_z) + \gamma(\alpha_y - \beta_x). \tag{33.4}$$

Say that we work in $R = k[t]/(t^3)$. Write $(\alpha, \beta, \gamma) = (at, bt, ct)$, with $a, b, c \in k[x, y, z]$. We view $F = (a, b, c)$ as a vector field on Spec $k[x, y, z]$. The Jacobi condition is that 33.4 must reduce to zero modulo the relations 33.2, and since each term of 33.4 is divisible by t^2, the relations 33.2 act on 33.4 as the commutative relations in x, y, z. Cancelling t^2 and making 33.4 commutative, we obtain, in calculus notation,

$$F \cdot (\nabla \times F) = 0. \tag{33.5}$$

This condition has a geometric interpretation. Wherever $F \neq 0$, it is equivalent to saying that F is orthogonal to an analytic foliation, or that there are local analytic functions g, h such that $F = h\nabla g$.

34. Deforming smooth algebras.

An algebra $A \in \mathcal{A}_R$ is called *smooth* if it is flat over R and if A_k is smooth. Thus a smooth algebra A over R is the same thing as deformation of the smooth commutative k-algebra A_k. The symbol $\otimes$ denotes tensor product over R in this section.

Let $A' = A \otimes R'$, where $R' = R/N^r$ and $N^{r+1} = 0$. An R-automorphism of A which reduces to the identity on A' has the form $\phi = id + d$, where d is an R-derivation $A \longrightarrow N^r A$. Now because A is flat over R, $N^r A \approx N^r \otimes_R A$, and because $N^{r+1} = 0$, the action of A on N^r is through A_k. So $N^r \otimes A \approx N^r \otimes A_k$. It follows that an R-derivation $d : A \longrightarrow N^r A$ annihilates NA, and defines a k-derivation $A_k \longrightarrow N^r \otimes A_k$:

$$\mathrm{Aut}(A; A') \approx \mathrm{Der}_k(A_k, N^r \otimes A_k). \tag{34.1}$$

This group is the same, whether or not A is commutative.

Grothendieck's characterization of smoothness shows that every commutative deformation of a smooth algebra A_k is trivial, i.e., isomorphic to $A \otimes_k R$. For, let A_1, A_2 be two smooth R-algebras and let $\phi' : A'_1 \longrightarrow A'_2$ be an isomorphism. Substituting A_1, A_2, R for A, R, S into (19.8) shows that ϕ' lifts to an R-homomorphism $\phi : A_1 \longrightarrow A_2$, which is surjective by the Nakayama lemma. Because A_2 is flat, the Nakayama lemma applied to ker ϕ shows that ϕ is an isomorphism.

The object of this section is to generalize this fact to noncommutative deformations, by showing that a smooth algebra $A \in \mathcal{A}_R$ is determined by its *commutator*, the map $A \times A \longrightarrow A$ sending $x, y \mapsto [x, y] = xy - yx$. The commutator is the zero map if and only if A is commutative, in which case the deformation is trivial.

To state a precise theorem, we must remove direct reference to A from the definition of the commutator map.

Lemma 34.2. *Let $A \in \mathcal{A}_R$.*
(i) There are natural A'-bimodule isomorphisms

$$NA \approx N \otimes A \approx N \otimes A'.$$

(ii) There is a canonical exact sequence exact sequence

$$0 \to N^r \otimes A_k \to N \otimes A' \xrightarrow{\epsilon} NA' \to 0.$$

The commutator on A is a map $A \times A \to NA$. Since $N^{r+1} = 0$, every element of $N^r A$ is central, so the value of the commutator $[u, v]$ depends only on the residues of u, v in A'. Taking into account 34.2i, we can view the commutator as a map

$$A' \times A' \xrightarrow{\alpha} N \otimes A'. \tag{34.3}$$

This is a good way to write the commutator, because the algebra A has been eliminated from the notation.

The map α has the following properties:

Lemma 34.4. *(i) α is an R-bilinear map $A' \times A' \longrightarrow N \otimes A'$.*
(ii) The diagram

$$\begin{array}{ccc} A' \times A' & \xrightarrow{\alpha} & N \otimes A' \\ {\scriptstyle id}\downarrow & & \downarrow \\ A' \times A' & \xrightarrow{[\,,\,]} & NA' \end{array}$$

commutes, where $[\ ,\ ]$ denotes the commutator of A'.
(iii) α is skew-symmetric, and a derivation in each variable.

A map α satisfying these conditions will be called a *bracket* on A'.

For $x, y, z \in A$, the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{34.5}$$

holds. Carrying this identity over to A' via α, we find

$$[x, \alpha(y, z)] + [y, \alpha(z, x)] + [z, \alpha(x, y)] = 0.$$

Proposition 34.6. *Let A' be a flat R'-algebra in $\mathcal{A}_{R'}$.*
(i) There exists an R-bilinear map $\alpha : A' \times A' \to N \otimes A'$ which extends the commutator on A', i.e., which satisfies 34.4i,ii.
(ii) Let α be such a map. The function $\gamma : A' \times A' \times A' \longrightarrow N \otimes A'$ defined by

$$\gamma(x, y, z) = [x, \alpha(y, z)] + [y, \alpha(z, x)] + [z, \alpha(x, y)] \tag{34.6.1}$$

takes values in $N^r \otimes A_k$, and it vanishes when any one of the variables is in NA'. So it induces a map

$$A_k \times A_k \times A_k \xrightarrow{\gamma} N^r \otimes A_k,$$

which we will also denote by γ and which we call the Jacobi function of A'.
(iii) The Jacobi function is independent of the choice of α. It is an alternating function, and is a derivation in each variable.

We say that the *Jacobi identity* holds in A' if γ is identically zero. Since the identity 34.5 for A' is a tautology, this should not cause confusion.

Lemma 34.7. *Given R, let A' be a smooth R'-algebra and let α satisfy 34.4i,ii.*
(i) For any $x, y, z, w \in A'$, $\alpha(x, [y, z]w) = [x, \alpha(y, z)w]$.
(ii) For any $x, y, z, w \in A'$, $[x, y]\alpha(z, w) = \alpha(x, y)[z, w]$.

Proof. (i) Write $\alpha(y, z) = \sum n_i \otimes a_i$, so that $[y, z] = \sum n_i a_i$, and also, for each i, write $\alpha(x, a_i) = \sum m_{ij} \otimes b_{ij}$. Then

$$[x, \alpha(y, z)w] = \sum n_i \otimes [x, a_i] = \sum n_i \otimes m_{ij} b_{ij} = \sum n_i m_{ij} \otimes b_{ij},$$

while

$$\alpha(x, [y, z]) = \alpha(x, \sum n_i a_i) = \sum n_i \alpha(x, a_i) = \sum n_i m_{ij} \otimes b_{ij}.$$

(ii) Write $\alpha(x, y) = \sum m_i \otimes a_i$ and $\alpha(z, w) = \sum n_j \otimes b_j$, where $m_i, n_j \in N$ and $a_i, b_j \in A'$. Then $[x, y]\alpha(z, w) = (\sum m_i a_i)(\sum n_j \otimes b_j) = \sum_{i,j} m_i n_j \otimes a_i b_j$, and $\alpha(x, y)[z, w]$ has the same expansion.

Proof of Proposition 34.6. (i) The commutator can be viewed as an R'-linear map $A' \otimes A' \longrightarrow NA'$ which we must lift to $N \otimes A'$. Since A' is R'-flat, so is $A' \otimes A'$. Since R' is a finite local k-algebra, a flat R'-module is free, hence projective. So the map does lift.

(ii) This follows from the Jacobi identity in A' and from 34.2ii.

(iii) The first assertion follows because α extends the commutator on A' and N annihilates γ. Since α extends the commutator, it is congruent to a derivation modulo $N^r \otimes A$, and so it behaves like a skew derivation inside the bracket $[x, \alpha(y, z)]$. In other words, we have $[x, \alpha(uv, z)] = [x, u\alpha(v, z) + \alpha(u, z)v]$ and $[x, \alpha(z, y)] = [x, -\alpha(y, z)]$. It follows that γ is alternating. To show that γ is a derivation in the first variable, we set $x = uv$ in 34.6.1:

$$\gamma(uv, y, z) = [uv, \alpha(y, z)] + [y, \alpha(z, uv)] + [z, \alpha(uv, y)].$$

Obviously, $[x, \alpha(y, z)]$ is a derivation in x. We expand $[y, \alpha(z, uv)]$, obtaining

$$[y, u\alpha(z, v)] + [y, \alpha(z, u)v] = [y, u]\alpha(z, v) + u[y, \alpha(z, v)] + [y, \alpha(z, u)]v + \alpha(z, u)[y, v].$$

The sum of the second and third terms is a derivation, and we are left with

$$[y, u]\alpha(z, v) + \alpha(z, u)[y, v].$$

the same computation for $[z, \alpha(uv, y)]$ leads to a derivation plus the unwanted terms

$$[z, u]\alpha(v, y) + \alpha(u, y)[z, v],$$

and 35.7ii shows that the sum of these four terms is zero.

Theorem 34.8. *(i) Let A' be a flat algebra in $\mathcal{A}_{R'}$. If there exists a flat extension A of A' to R, then the Jacobi identity holds.*
(ii) Let $A' \in \mathcal{A}_{R'}$ be smooth, and assume that the Jacobi identity holds. There exists a bracket α on A', and for any α, there is a smooth extension A of A' to R, unique up to isomorphism, whose commutator is α.

The first assertion was derived above. The second will be proved in Section 38.

35. Interlude: flatness of the completion.

Theorem 35.1. *Let A be a noetherian algebra in $\mathcal{A}_R$, and let $\mathfrak{m}$ be a maximal ideal of A. The completion $\widehat{A}$ of A at $\mathfrak{m}$ is flat over A, and for any finite A-module M, the canonical map $\widehat{A} \otimes_A M \to \widehat{M}$ is bijective.*

Corollary 35.2. *(i) If $A \in \mathcal{A}_R$ is noetherian, then the completion $\widehat{A}$ at a maximal ideal $\mathfrak{m}$ is also a noetherian algebra in $\mathcal{A}_R$.*
(ii) An algebra $A \in \mathcal{A}_R$ is a complete, noetherian, local ring if and only if A_k is.

Proof of the Corollary. It follows from the theorem that $\widehat{A} \otimes_R k$ is the completion of A_k. So $\widehat{A} \in \mathcal{A}_R$, and (i) follows from Proposition 30.1i. It also follows from the theorem that A_k is complete if A is. Conversely, suppose that A_k is a complete local ring. Then A is local. Its completion $\widehat{A}$ has the property that $\widehat{A} \otimes k \approx A \otimes k$. The fact that $A \approx \widehat{A}$ follows by induction on the nilradical N.

Proof of the Theorem. For a finite module M, denote by M_n the module $M/\mathfrak{m}^n M$. If

$$0 \to V \to W \to U \to 0$$

is an exact sequence of finite modules, then the sequences

$$V_n \to W_n \to U_n \to 0$$

are exact, and since W_n etc. have finite length, the inverse limit sequence

$$\widehat{V} \to \widehat{W} \to \widehat{U} \to 0$$

is also exact. To prove the proposition it suffices to show the following:

Lemma 35.3. *If $V \subset W$ are finite left A-modules, then for any r, $\mathfrak{m}^r V \supset (\mathfrak{m}^n W) \cap V$, if $n \gg 0$.*

This will show both that completion of a module is an exact functor, and then that it is isomorphic to $\widehat{A} \otimes_A \cdot$.

Lemma 35.4. *Let $V \subset W$ be given. To prove 35.3 for the modules $\mathfrak{m}^s V \subset \mathfrak{m}^s W$ for all $s \geq 0$, it is enough to prove it for all s and for $r = 1$.*

Proof. Induction on r. Say that 35.3 has been proved for $\mathfrak{m}^s V \subset \mathfrak{m}^s W$ for all s and for $r = k-1$, $k > 1$. Setting $r = 1$ and $s = k-1$, we obtain $\mathfrak{m}(\mathfrak{m}^{k-1}V) \supset \mathfrak{m}^{n_1}(\mathfrak{m}^{k-1}W) \cap \mathfrak{m}^{k-1}V$. Setting $r = k-1$ and $s = 0$, we obtain $\mathfrak{m}^{k-1}V \supset \mathfrak{m}^{n_2}W \cap V$. Combining, we obtain $\mathfrak{m}^k V \supset \mathfrak{m}^{n_3} W \cap V$, which is the desired assertion in the case $s = 0$ and $r = k$. The other values of s are obtained by substitution.

Lemma 35.5. *To prove 35.3 for $V \subset W$ and for given r, it is enough to prove 35.3 for $\overline{V} = V/\mathfrak{m}^r V \subset \overline{W} = W/\mathfrak{m}^r V$. In this case, we must show that $\mathfrak{m}^n \overline{W} \cap \overline{V} = 0$ for $n \gg 0$.*

Proof. Suppose that 35.3 holds for $\overline{V} \subset \overline{W}$, and let $n \gg 0$. If $x \in \mathfrak{m}^n W \cap V$, then $\overline{x} \in \mathfrak{m}^n \overline{W} \cap \overline{V} = 0$, hence $x \in \mathfrak{m}^r V$.

Lemma 35.6. *If $n \gg 0$, then $\mathfrak{m}^r V + NW \supset \mathfrak{m}^n W \cap V$, where as always, N is the maximal ideal of R.*

~~*Proof.* The statement 35.3 is true for A_k-modules, because A_k is commutative.~~ Let $W_0 = W \otimes_R k$, let $\pi : W \to W_0$ denote the canonical map, and and let V_0 denote the image of V in W_0. Then $\mathfrak{m}^r V_0 \supset \mathfrak{m}^n W_0 \cap V_0$ if $n \gg 0$, and $\pi^{-1}(\mathfrak{m}^r V_0) = \mathfrak{m}^r V + NW$. Moreover, $\pi^{-1}(\mathfrak{m}^n W_0 \cap V_0) \supset \mathfrak{m}^n W \cap V$. The assertion follows.

Lemma 35.7. *The assertion 35.3 is true when $V = NW$, i.e., $\mathfrak{m}^r NW \supset \mathfrak{m}^n W \cap NW$ if $n \gg 0$.*

Proof. We first treat the case $r = 1$. By Lemma 35.5, we may assume that $\mathfrak{m}NW = 0$. Then $\mathfrak{m}W$ is an A_k-module, and since $N \subset \mathfrak{m}$, NW is an A_k-submodule of $\mathfrak{m}W$. Therefore 35.3 is true for the modules $NW \subset \mathfrak{m}W$, and $\mathfrak{m}NW \supset \mathfrak{m}^n(\mathfrak{m}W) \cap NW$ for large n.

Now proceeding by induction, we suppose the lemma has been proved for r. We have $\mathfrak{m}^r NW = N\mathfrak{m}^r W \subset \mathfrak{m}^r W$. By the case $r = 1$, $\mathfrak{m}N(\mathfrak{m}^r W) \supset \mathfrak{m}^{n_1}\mathfrak{m}^r W \cap N\mathfrak{m}^r W$. Hence

$$\mathfrak{m}^{r+1} NW \supset \mathfrak{m}^{n_1+r} W \cap \mathfrak{m}^r NW \supset \mathfrak{m}^{n_1+r} W \cap (\mathfrak{m}^{n_2} W \cap NW) \supset \mathfrak{m}^{n_3} W \cap NW,$$

which is 35.3 for the case $V = NW$.

We now prove 35.3 in the general case $V \subset W$. We use induction on the integer k such that $N^k W = 0$. Hence we may assume that 35.3 is true for $NW \cap V = V_1 \subset W_1 = NW$. Moreover, Lemma 35.4 shows that we need only prove the assertion in the case $r = 1$, i.e, to prove that $\mathfrak{m}V \supset \mathfrak{m}^n W \cap V$ if $n \gg 0$. By Lemma 35.5, we may assume that $\mathfrak{m}V = 0$.

Then $\mathfrak{m}V_1 = 0$ too, so $0 = \mathfrak{m}^{n_1} W_1 \cap V_1 = \mathfrak{m}^{n_1} NW \cap NW \cap V$, i.e., $\mathfrak{m}^{n_1} NW \cap V = 0$. Also, by Lemma 35.7, the assertion is true for the inclusion $NW \subset W$, hence $\mathfrak{m}^{n_1} NW \supset \mathfrak{m}^{n_2} W \cap NW$. By Lemma 35.6, $NW = \mathfrak{m}V + NW \supset \mathfrak{m}^{n_3} W \cap V$. Hence $0 = \mathfrak{m}^{n_1} NW \supset \mathfrak{m}^{n_2} W \cap (\mathfrak{m}^{n_3} W \cap V)$, or $0 = \mathfrak{m}^{n_4} W \cap V$, as required.

36. Deformations of a commutative power series ring.

To work with the ring $\widehat{P} = R\langle\langle x_1, ..., x_n \rangle\rangle$ of formal noncommutative power series, we use the *power series ordering* on monomials: $m < m'$ if either $deg(m) > deg(m')$, or if $deg(m) = deg(m')$ and m is earlier in lexicographic order.

Let $f \in \widehat{P}$ be a power series whose residue in $\widehat{P}_k = k\langle\langle x_1, ..., x_n \rangle\rangle$ is not zero. Then as in Section 11, f can be written uniquely in the form $f = cm - \psi - \eta$, where m is the highest monomial with invertible coefficient c, η is the negative sum of the terms in f whose coefficients are in N, and ψ is the negative sum of the remaining terms of f.

Proposition 36.1. *Let $\{f_i\}$ be a set of elements of $\widehat{P}$ none of whose residues in $\widehat{P}_k$ is zero. Let $\widehat{I}$ denote the closure of the ideal generated by this set in the (x)-adic topology on $\widehat{P}$. Set $\widehat{A} = \widehat{P}/\widehat{I}$.*
(i) Let $p \in \widehat{P}$. If done in decreasing power series order, the substitutions $m_i = c^{-1}(\psi_i + \eta_i)$ provide a sequence which converges to a reduced power series p', in which no monomial containing any m_i as submonomial appears.
(ii) The reduced monomials form a topological R-basis of $\widehat{A}$ if and only if all overlaps are consistent. If the overlaps are consistent, then $\widehat{A}$ is R-flat.

A smooth complete R-algebra is defined to be a deformation of the commutative power series ring $\widehat{A}_k = k[[x_1, ..., x_n]]$, a flat extension $\widehat{A}$ of $\widehat{A}_k$ to R.

Let $((S))$ denote the closure of the ideal of $\widehat{P} = R\langle\langle x\rangle\rangle$ generated by a subset S.

Proposition 36.2. *(i) Let $\widehat{\alpha}_{ij}$ be elements of $N\widehat{P}$ for $i < j$, and let $\widehat{A} = \widehat{P}/(([x_j, x_i] - \widehat{\alpha}_{ij}))$. Then $\widehat{A} \otimes k \approx k[[x]]$.*
(ii) Let $\widehat{A}$ be an R-algebra such that $\widehat{A}_k \approx k[[x]]$. Then $\widehat{A}$ is flat over R if and only if the ordered monomials form a topological basis. This is true if and only if the series obtained by reducing

(36.2.1) $$[x_i, \widehat{\alpha}_{jk}] - [x_j, \widehat{\alpha}_{ik}] + [x_k, \widehat{\alpha}_{ij}]$$

are zero, for $i < j < k$.

Proof. (i) is elementary.
(ii) To show this, we use induction as usual. We assume that $\widehat{A}$ is flat and that the ordered monomials form a topological R'-basis for $\widehat{A}'$. Then every element $f \in \widehat{A}$ is congruent to an ordered power series g, modulo N^r, and so $f - g \in N^r A = N^r \otimes A_k$. The elements of $N^r \otimes A_k$ can be identified with ordered series with coefficients in N^r. So f is an ordered series with coefficients in R. The same reasoning shows uniqueness of the representation.

Conversely, if the ordered monomials form a topological basis, then every element of $N^r\widehat{A}$ has a unique expression as a polynomial with coefficients in N^r. So the map $N^r \otimes A_k \longrightarrow N^r A$ is bijective, which shows by induction that $\widehat{A}$ is flat.

The last assertion follows from 36.1ii because 36.2.1 is obtained by substituting into the overlap $(x_i x_j)x_k - x_i(x_j x_k)$.

Suppose given a smooth complete R'-algebra $\widehat{A}'$. The Jacobi function is defined as in 34.10:

$$\widehat{A}_k \times \widehat{A}_k \times \widehat{A}_k \xrightarrow{\gamma} N^r \otimes \widehat{A}_k.$$

Proposition 36.3. *Let $\widehat{A}'$ be a smooth complete R'-algebra, and assume that the Jacobi identity $\gamma = 0$ holds.*

(i) Let $\widehat{A}'$ be a flat extension of $\widehat{A}_k = k[[x]]$ to R'. For $i < j$, let $\alpha_{ij} \in N \otimes \widehat{A}'$ be elements which represent the commutators $[x_i, x_j] \in N\widehat{A}'$. There is a unique continuous bracket α on A' such that $\alpha(x_i, x_j) = \alpha_{ij}$.
(ii) For any continuous bracket α, there is a flat extension $\widehat{A}$ of $\widehat{A}'$ to R whose commutator is α, and this extension is unique up to isomorphism. If the Jacobi identity does not hold, then $\widehat{A}'$ has no flat extension to R.

Proof. (i) Suppose that elements $\alpha_{ij} \in N \otimes \widehat{A}'$ for $1 \leq j < i \leq n$ are given. We extend to all pairs of indices by setting $\alpha_{ji} = -\alpha_{ij}$ and $\alpha_{ii} = 0$, and we use these elements to define a skew derivation as follows: Let $y = x_{i_1}...x_{i_r}$ and $z = x_{j_1}...x_{j_s}$ be two lexicographically ordered monomials. We can use the fact that the commutator is a derivation in each varialbe to expand $[y, z] = [x_{i_1}...x_{i_r}, x_{j_1}...x_{j_s}]$, beginning with y. The result is

$$[y,z] = \sum_{\mu,\nu} x_{i_1}...x_{i_{\mu-1}}x_{j_1}...x_{j_{\nu-1}}[x_{i_\mu}, x_{j_\nu}]x_{j_{\nu+1}}...x_{j_s}x_{i_{\mu+1}}...x_{i_r}.$$

So we set

$$\alpha(y,z) = \sum_{\mu,\nu} x_{i_1}...x_{i_{\mu-1}}x_{j_1}...x_{j_{\nu-1}}\alpha_{i_\mu j_\nu}x_{j_{\nu+1}}...x_{j_s}x_{i_{\mu+1}}...x_{i_r}, \tag{36.4}$$

and we extend bilinearly and continuously to $\widehat{A}' \times \widehat{A}'$. This definition does extend the commutator on A', i.e., 34.4i,ii hold. We must verify 34.4iii.

Note that the formula 36.4 has the property that if $y = uv$ is an ordered product, then

$$\alpha(uv,z) = u\alpha(v,z) + \alpha(u,z)v. \tag{36.5}$$

So to show that α is a derivation in the first variable, it suffices to show that 36.5 holds also when u, v are monomials which are not in lexicographic order. We use induction. Since the permutations $(1 \dots n)$ and $(1\,2)$ generate the symmetric group, it suffices to show the following:

Lemma 36.6. *Let α be an R'-bilinear extension of the commutator α' of A', and let $u, v, w, z \in A'$.*
(i) If $\alpha(uv, z) = u\alpha(v, z) + \alpha(u, z)v$, then $\alpha(vu, z) = v\alpha(u, z) + \alpha(v, z)u$.
(ii) If $\alpha(uvw, z) = uv\alpha(w, z) + \alpha(uv, z)w$, then $\alpha(vuw, z) = vu\alpha(w, z) + \alpha(vu, z)w$.

Proof. (i) Taking into account 34.7i, the Jacobi identity can be written as

$$[u, \alpha(v,z)] - [v, \alpha(u,z)] = \alpha([u,v], z).$$

The assertion follows by expanding the two sides.
(ii) By 34.7i,i we have

$$\alpha([u,v]w, z) = [\alpha(u,v)w, z] = \alpha(u,v)[w,z] + [\alpha(u,v), z]w = [u,v]\alpha(w,z) + \alpha([u,v],z)w.$$

Again, the assertion follows by expansion.

To show that α is alternating, we show that the order in which 36.4 has been expanded can be reversed, beginning with expansion with respect to the first variable. Then α will be a derivation in the second variable, and the alternating property will follow from the choice of α_{ij}. Proceeding by induction, we break up the two monomials in some fashion, say $y = uv$ and $z = pq$. Then with our definition 36.4,

$$\alpha(uv, pq) = up\alpha(v,q) + u\alpha(v,p)q + p\alpha(u,q)v + \alpha(u,p)qv,$$

while with the opposite order of expansion, we would have

$$\alpha(uv, pq) = pu\alpha(v,q) + u\alpha(v,p)q + p\alpha(u,q)v + \alpha(u,p)vq.$$

In order to show that these two expansions are equal, we must show that

$$[u,p]\alpha(v,q) = \alpha(u,p)[v,q].$$

This follows from Lemma 34.7ii.

(ii) By 36.1ii, every element of $\widehat{A}'$ can be written uniquely as a lexicographically ordered series. Similarly, every element of $N \otimes \widehat{A}'$ can be expressed uniquely as a series $\sum_i n_i \otimes x^i$, where $n_i \in N$. For $1 \leq j < i \leq n$, let $\widehat{\alpha}_{ij} \in N \otimes \widehat{P}$ be the lexicographically ordered series $\alpha(x_i, x_j) \in N \otimes \widehat{A}'$. Set $\widehat{A} = \widehat{P}/(([x_i, x_j] - \widehat{\alpha}_{ij}))$. By (i), $\widehat{A} \in \mathcal{A}_R$. Also, let $\widehat{\alpha}'_{ij}$ denote the residue of $\widehat{\alpha}_{ij}$ in $N \otimes \widehat{P}'$. Then the image of $\widehat{\alpha}'_{ij}$ in NA' is $[x_i, x_j]$, and it follows that $\widehat{A}'$ is a quotient of $\widehat{P}'/(([x_i, x_j] - \widehat{\alpha}'_{ij}))$. Because $\widehat{A}'$ is flat, these two rings are isomorphic. So $\widehat{A}$ extends $\widehat{A}'$ to R.

According to 36.1ii, $\widehat{A}$ is flat over R if and only if the lexicographically ordered series obtained by reducing 36.2.1 in $\widehat{P}$ is zero, for all $j < i$. Let $\widehat{\gamma}_{ijk}$ denote the unordered series 36.2.1, which we view as lying in $N \otimes R'\langle\langle x\rangle\rangle$. Its image $\overline{\gamma}_{ijk} \in N \otimes \widehat{A}'$ is unchanged by the reduction process, because because $[x_i, x_j] - \alpha_{ij}$ is true in $\widehat{A}'$, and on the other hand, the image is the Jacobi function $\overline{\gamma}(x_i, x_j, x_k)$. Hence it is zero, and $\widehat{A}$ is flat.

By construction, $\alpha_{ij} = \alpha(x_i, x_j)$. Since these values determine the commutator, it follows that the commutator of $\widehat{A}$ is α, etc...

37. Deforming smooth schemes.

We now consider the problem of deforming a smooth scheme X_k. Since localization poses no difficulty in the category $\mathcal{A}_R$, we can define a *scheme* X_R in $\mathcal{A}_R$ to be a commutative scheme X_k, together with an extension of its structure sheaf $\mathcal{O}_{X_k}$ to a sheaf of rings $\mathcal{O}_{X_R}$ in $\mathcal{A}_R$, compatibly with localization. This sheaf will then be called the structure sheaf of X_R. As in the affine case, X_R will be called *smooth* if it is flat, and if X_k is smooth. Let us write $\mathcal{O}$ for $\mathcal{O}_{X_R}$, Ω^1 for $\Omega^1_{X_k}$, and T for the tangent sheaf $\Omega^{1\vee}_{X_k}$, the sheaf of derivations on X_k. Also, let T^q denote the exterior power $\Lambda^q T_k$.

Proposition 37.1. *(i) Let $A \in \mathcal{A}_R$ be a flat R-algebra with $X_k = \operatorname{Spec} A_k$. The group of R-automorphisms of A which reduce to the identity on A' is* $\operatorname{Hom}_{X_k}(\Omega^1, N^r \otimes \mathcal{O}_k) \approx N^r \otimes T^1$.
(ii) The skew derivations $A_k \times A_k \to A_k$ are in bijective correspondence with elements of $\operatorname{Hom}_{X_k}(T\Omega^2, \mathcal{O}_k) \approx N^r \otimes T^2$.
(iii) Let $A' \in \mathcal{A}_{R'}$ be flat. The difference $\alpha - \beta$ of two brackets is a skew derivation, an element of $N^r \otimes T^2$.
(iv) The Jacobi function γ is an element of $\operatorname{Hom}_{A_k}(\Omega^3, N^r \otimes \mathcal{O}_k) \approx N^r \otimes T^3$.

Proof. The first two assertions are elementary. (iii) follows from 34.2ii and the commutativity of 34.4ii. (iv) follows from 34.8.

The constructions considered in the last section are compatible with localization, so they globalize without difficulty. A *bracket* on a scheme $X_{R'}$ over R' is a skew derivation $\alpha : \mathcal{O}' \times \mathcal{O}' \to N \otimes \mathcal{O}'$ which extends the commutator, as in 34.4iii, and so on. Standard descent arguments yield the following proposition.

Proposition 37.2. *Let $X_{R'}$ be a smooth scheme over R'.*
(i) The Jacobi function is an element $\gamma \in H^0(X_k, T^3) \otimes N^r$. If $\gamma \neq 0$, then no bracket exists.
(ii) If $\gamma = 0$, then a bracket exists locally on X_R. The obstruction to existence of a global bracket α on $X_{R'}$ is an element $\kappa \in H^1(X_k, T^2) \otimes N^r$. If $\kappa = 0$, then the set of brackets on $X_{R'}$ is a principal homogeneous space under $H^0(X_k, T^2) \otimes N^r$.
(iii) Let α be a global bracket. Then a smooth extension of $X_{R'}$ to R with commutator α exists locally. The obstruction to global existence is an element $\eta \in H^2(X_k, T^1) \otimes N^r$. If $\eta = 0$, then the set of isomorphism classes of extensions X_R with commutator α is a principal homogeneous space under $H^1(X_k, T^1) \otimes N^r$.
(iv) Let X_R be a smooth extension of $X_{R'}$ to R. The group $\underline{\operatorname{Aut}}(\mathcal{O}; \mathcal{O}')$ of R-automorphisms of $\mathcal{O}$ which reduce to the identity on $\mathcal{O}'$ is isomorphic to $H^0(X_k, T^1) \otimes N^r$.

Corollary 37.3. *(i) Every deformation of a smooth algebraic curve is commutative.*
(ii) Deformations of smooth affine algebraic surfaces are unobstructed.

Proof. The sheaf T is locally free of rank d on a smooth commutative scheme X_k of dimension d. Hence $T^q = 0$ if $q > d$. If X_k is a curve, then $\gamma = 0$, and α is uniquely determined by $X_{R'}$. By induction, the commutator on $X_{R'}$ is zero, and so the zero bracket is the unique extension of this commutator. This shows that X_R is commutative.
(ii) This is true because when the dimension is two, $T^3 = 0$. Hence the Jacobi identity holds automatically, and Theorem 34.8 implies that an extension of $X_{R'}$ to X_R is always possible.

For projective algebraic surfaces, considerations are simplified by the classification of surfaces. The reason is that one can list the surfaces for which $H^0(X, T^2) \neq 0$. Such a surface is rational, ruled, abelian, or a K3 surface. (Not all rational or ruled surfaces have $H^0(X, T^2) \neq 0$ either.) If X_k is not one of these surfaces, then induction as in the proof of 37.3i shows that every deformation X_R is commutative.

As an example, consider the case that $X_k = \mathbb{P}^2_k$ is the projective plane. In that case, $T^2 \approx \mathcal{O}_k(3)$ is the invertible sheaf of homogeneous functions of degree 3. The cohomology of the tangent sheaf T^1 can be computed using a canonical exact sequence

$$0 \longrightarrow \mathcal{O}_k \longrightarrow \mathcal{O}_k(1)^3 \longrightarrow T^1 \longrightarrow 0.$$

Using these facts, one finds

$$\dim H^0(X_k, T^1) = 8 \ , \ \ \dim H^0(X_k, T^2) = 10,$$

and that the higher cohomology vanishes. Consequently the obstructions to the existence of a global bracket and to the construction of a global extension of $X_{R'}$ To R vanish, and moreover X_R is uniquely determined by the choice of a bracket. The bracket α, being a section of $\mathcal{O}_k(3)$, corresponds to a homogeneous cubic form on $\mathbb{P}^2$, which, if not identically zero, vanishes on a cubic divisor $Y \subset \mathbb{P}^2$. Thus the noncommutative first order deformations of X_k are determined up to scalar factor by a cubic divisor, and they depend on 10 parameters. However, because the group PGL operates on X_k, it also operates on the first order deformations, and the essentially distinct deformations correspond to orbits for this operation.

38. Proofs.

We now proceed with the proof of Theorem 32.4.

Lemma 38.1. *Suppose that $\overline{\phi} : A \to \overline{B}$ is a flat extension of the etale map $\phi_k : A_k \to B_k$ to R, and that $\phi : A \to B$ is an extension of ϕ_k which has the universal property 32.4.1. Then the map $\pi : B \to \overline{B}$ defined by 32.4.1 is an isomorphism.*

Proof. By induction on the nilradical, we may assume that $B' \approx \overline{B}'$. Tensoring with the exact sequence

$$0 \to N^r A \to A \to A' \to 0,$$

we obtain a commutative diagram

$$\begin{array}{ccccccccc} & & B \otimes_A N^r A & \longrightarrow & B & \longrightarrow & B' & \longrightarrow & 0 \\ & & \downarrow \tau & & \downarrow \pi & & \downarrow \pi' & & \\ 0 & \longrightarrow & \overline{B} \otimes_A N^r A & \longrightarrow & \overline{B} & \longrightarrow & \overline{B}' & \longrightarrow & 0 \end{array}$$

in which the rows are exact. The zero at the left in the bottom row results from the fact that $\overline{B}$ is flat. By induction, π' is bijective. Also, $B \otimes_A N^r A \approx B_k \otimes_{A_k} N^r A \approx \overline{B} \otimes_A N^r A$, so τ is bijective. It follows that π is bijective too.

Lemma 38.2. *The theorem is true for a standard etale map $\phi_k : A_k \to B_k$.*

Proof. Proposition 32.6 shows that there exists a universal extension $\phi : A \to B$. So by Lemma 38.1, it suffices to show that this universal extension is also A-flat.

We proceed by induction on the nilradical as usual, and we suppose that B' has been shown to be A'-flat. We will show that the canonical surjective map $\pi : N^r A \otimes_A B \to N^r B$ is an isomorphism. It will follow that $\mathrm{Tor}_1^A(A', B) = 0$, hence that B is A-flat (see 6.2).

Let $p \in \operatorname{Max} B$ have image $q \in \operatorname{Max} A$. It suffices to show that $ker(\pi) = 0$ locally at q. We claim that the completions $\widehat{B}$ and $\widehat{A}$ at the points in question are isomorphic. This is true for the completions $\widehat{A}_k$ and $\widehat{B}_k$. By induction on the nilradical, we may assume that $\widehat{A}' \approx \widehat{B}'$. Consider the diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N^r\widehat{A} & \longrightarrow & \widehat{A} & \longrightarrow & \widehat{A}' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\widehat{\phi}} & & \downarrow{\scriptstyle\widehat{\phi}'} & & \\
0 & \longrightarrow & N^r\widehat{B} & \longrightarrow & \widehat{B} & \longrightarrow & \widehat{B}' & \longrightarrow & 0.
\end{array}
\tag{38.3}
$$

By induction, $\widehat{\phi}'$ is bijective. Also, $N^r\widehat{A}$ is an $\widehat{A}' \approx \widehat{B}'$-module generated by N^r, and the same is true of $N^r\widehat{B}$. Therefore γ is surjective, and it follows that $\widehat{\phi}$ is surjective. We apply 32.4.1 to extend the map $B_k \to \widehat{A}_k$ to a map $B \to \widehat{A}$, hence to a map $\widehat{B} \to \widehat{A}$. This map inverts $\widehat{\phi}$.

Now since $\widehat{A}$ is A-flat and B-flat,

$$N^r A \otimes_A \widehat{B} \approx N^r A \otimes_A \widehat{A} \approx N^r \widehat{A} \approx N^r \widehat{B},$$

from which it follows that the completion of $ker(\pi)$ is zero. Since the completion is $\cdot \otimes_B \widehat{B}$, $ker(\pi)$ is zero locally at p.

Lemma 38.4. *Let $\phi : A \to B$ be a universal extension to R of an etale map $\phi_k : A_k \to B_k$, and let $t \in B$. Then $\phi' : A \to B\langle t^{-1}\rangle$ is a universal extension of the etale map $\phi' : A_k \to B_k[t^{-1}]$.*

Proof. This follows directly from the mapping property of rings of fractions.

Lemma 38.5. *If $\phi : A \to B$ is etale, then ϕ is a universal extension of ϕ_k to R.*

Proof. By Lemma 32.3, there are localizations $B\langle s^{-1}\rangle$ of B such that $\phi_k\langle s^{-1}\rangle : A_k \to B_k[s_0^{-1}]$ is standard etale. Then by Lemma 38.4, $\phi\langle s^{-1}\rangle$ is universal. Thus ϕ is locally universal. Suppose given a map $\psi : A \to C$ and a factorization $\psi_k = \xi_k\phi_k$ through B_k. We must extend ξ_k to a unique map $\xi : B \to C$.

If the element s has invertible image in some localization C' of C, then we obtain a map $\xi_k[s_0^{-1}] : B_k[s_0^{-1}] \to C'$. This map extends uniquely to $\xi\langle s^{-1}\rangle : B\langle s^{-1}\rangle \to C'$, which, by composition, provides the unique extension $\xi' : B \to C'$.

Now the sheaf axiom for $\mathrm{Hom}_A(B, \cdot\,)$ shows that the problem of extending ξ_k is local on $\operatorname{Spec} C$. So these local extensions found above glue to give a unique extension globally.

Now to complete the proof of 32.4, let $\phi_k : A_k \to B_k$ be an arbitrary etale map. By Lemma 32.3, there are localizations B_{ki} of B_k, $i = 1, ..., n$, such that the induced maps $\phi_{ki} : A_k \to B_{ki}$ are standard etale maps, and that $B_k \to \prod_i B_{ki}$ is faithfully flat. Let $C_k = \prod_i B_{ki}$, and

$$C_k^{(n)} = C_k \otimes_{B_k} \cdots \otimes_{B_k} C_k.$$

Each of these rings is a product of standard etale A_k-algebras, hence it has a universal extension $A \to C^{(n)}$. Except for its augmentation, the Amitsur complex $B_k \to C_k$ extends ~~to give a cosimplicial complex~~

(38.6) $$C^{(1)} \rightrightarrows C^{(2)} \overset{\longrightarrow}{\rightrightarrows} C^{(3)} \cdots .$$

Define B to be the kernel of the pair of maps $C^{(1)} \rightrightarrows C^{(2)}$.

We use induction on the nilradical. With the usual notation, tensoring with the exact sequence

$$0 \to N^r A \to A \to A' \to 0$$

yields a three term exact sequence of cosimplicial complexes whose term in degree n is

$$0 \to N^r \otimes_A C^{(n)} \to C^{(n)} \to C'^{(n)} \to 0.$$

This sequence is exact because $C^{(n)}$ is A-flat. By induction, the third term is a resolution of the universal extension B'. Also, the first term is a sequence of B'-algebras, and it is a resolution of $N^r A \otimes_A B'$ 30.2. Therefore the middle sequence is a resolution of B, and we have an exact sequence

$$0 \to N^r \otimes_A B' \to B \to B' \to 0,$$

which shows that B is A-flat (6.2). Moreover, $B \otimes_A A' \approx B'$, hence $\phi : A \to B$ extends ϕ_k. By Lemma 38.5, ϕ is a universal extension. This completes the proof of the Theorem 32.4.

We now pass to the proof of Theorem 34.12. A smooth commutative k-algebra can be realized locally as an etale algebra over a polynomial ring $k[x] = k[x_1, ..., x_n]$. Hence it has a local standard etale presentation, as in Section 32. We call a smooth commutative k-algebra B_k *standard* if there is a standard etale extension $k[x] \to B_k$, i.e., if there are polynomials $f, g, u, v \in k[x, y]$, such that $uf + vf' = g$ and that B_k is isomorphic to $k[x, y, z]/(f, zg - 1, gz - 1)$.

We replace A by B in the statement of the theorem. Our first step is to prove the existence of a universal extension of a standard smooth algebra B_k to R. This is the analogue of Proposition 32.6.

The minimal data necessary to describe an extension of a standard smooth algebra to R are elements α_{ij}, $1 \leq j < i \leq n$ in the free R-algebra $P = R\langle x; y, z\rangle$, which are congruent 0, (modulo N). Given such elements, we set $[x_i, x_j] = \alpha_{ij}$.

Proposition 38.7. *Let $P = R\langle x, y, z\rangle$, let f, g, u, v be polynomials in y with coefficients in $R\langle x\rangle$, whose residues in $k[x, y]$ define a standard smooth algebra $B_k = k[x, y, z]/(f, zg - 1, gz - 1)$, and let $\alpha_{ij} \in NP$ for $1 \leq j < i \leq n$. Let $Q = P/(f, zg - 1, gz - 1, [x_i, x_j] - \alpha_{ij})$. There is a universal quotient B of Q in $\mathcal{A}_R$, such that $\mathrm{Hom}_R(Q, C) \approx \mathrm{Hom}_R(B, C)$ for all $C \in \mathcal{A}_R$.*

Proof. The polynomial ring $k[x, y, z]$ is the quotient of the free ring $k\langle x, y, z\rangle$ by the relations $[x_i, x_j] = [x_j, y] = [x_j, z] = [y, z] = 0$. So in order to define a quotient B of Q which extends B_k, we must introduce relations of the form $[x_j, y] = \beta_j$, $[x_j, z] = \gamma_j$, $[y, z] = \delta$, with $\beta_j, \gamma_j, \delta \in NQ$.

The kernel of the map $Q \to B_k$ is $\Gamma_1 = \Gamma_1(Q)$. Let $q \in \Gamma_1$. A *natural reduction* of q is an element $\beta \in NQ$ such that $q = \beta$ in C for every map $Q \to C$ with $C \in \mathcal{A}_R$. As in Lemma 32.8, we have

Lemma 38.8. *(i) If a universal quotient B of P extending B_k exists, then every element $q \in \Gamma_1$ has a natural reduction.*
(ii) To show that a universal quotient exists, it suffices to show that the commutators $[x_j, y]$, $[x_j, z]$, $[y, z]$ have natural reductions.
(iii) If $\beta_j, \gamma_j, \delta$ are natural reductions of $[x_j, y]$, $[x_j, z]$, $[y, z]$ respectively, and if J is the ideal generated by $[x_j, y] - \beta_j, [x_j, z] - \gamma_j, [y, z] - \delta$, then Q/J is the universal quotient B.

We proceed by induction. With the usual notation, we may assume that a universal quotient B' of Q' exists. We copy the proof of Proposition 32.6.

Lemma 38.9. *Under the inductive hypothesis, every element $q \in \Gamma_2(Q)$ has a natural reduction $\gamma \in N^2Q$.*

Proof. If $q \in \Gamma_1$, then its image $q' \in Q'$ has a natural reduction $\beta' \in NQ'$. We represent β' by $\beta \in NQ$. If $\psi : Q \to C$ is a map with $C \in \mathcal{A}_R$, then $\psi(q) \equiv \psi(\beta)$ (modulo N^rC), hence we may write $\psi(q) = \psi(\beta) + \epsilon$, with $\epsilon \in N^rC$. The remainder of the proof can be carried over verbatim.

As in the proof of Theorem 32.4, we write $f(y) = \sum_i a_i y^i$, where $a_i \in R\langle x\rangle$. We use 31.3:

$$0 = [x_j, f(y)] = \sum_i [x_j, a_i]y^i + f'(y)[x_j, y] + O([[x_j, y], y]).$$

Since $a_i \in R\langle x\rangle$, the brackets $[x_j, a_i]$ are well determined elements of NQ. So as before, $0 = q + f'(y)[x_j, y] + \eta$, where $q = \sum[x_j, a_i]y^i \in NQ$, and $\eta \in \Gamma_2$.

By 32.11, we have $vf' = g - \epsilon$, or $zvf' = 1 - z\epsilon$, where $\epsilon \in \Gamma_1$. Thus

$$[x_j, y] = zvq + z\epsilon[x_j, y] + \eta \in \Gamma_2.$$

Here $zvq \in NQ$, and $z\epsilon[x_j, y] + \eta$ has a natural reduction β_j by the previous lemma. It follows that $zvq + \beta_j$ is a natural reduction of $[x_j, y]$.

Since $gz = 1$ in Q, we have, $0 = [y, zg]z = [y, z] + z[y, g]z$, so $[y, z] = -z[y, g]z$. Since g is a polynomial in y with coefficients in $R\langle x\rangle$, $[y, g]$ has a natural reduction, and so does $[y, z]$. The same is true of $[x_j, z]$. This completes the proof of 38.7

Lemma 38.10. *Let B' be a smooth extension of the standard smooth algebra B_k to R', and suppose that the Jacobi identity holds in B'. Write $B' = P'/J'$, where where $P' = R'\langle x,y,z\rangle$ and $J' = (f, zg-1, gz-1, [x_i,x_j]-\alpha'_{ij}, [z_j,y]-\beta'_j, [x_j,z]-\gamma'_j, [y,z]-\delta')$ for some elements $\alpha'_{ij}, \beta'_j, \gamma'_j, \delta \in NR'\langle x,y,z\rangle$. Further, for $1 \le j < i \le n$, let $\alpha_{ij} \in NP \approx N \otimes P'$ represent α'_{ij}. Then the universal quotient B of Proposition 38.7 is a smooth extension of B' to R.*

Proof. The image of α_{ij} in $N \otimes B''$ is the commutator $\alpha'(x_i,x_j)$. Therefore B maps to B', and since B' is flat, $B' \approx B \otimes_R R'$, i.e., B extends B' to R. So it suffices to show that B is flat over R. To show this, it suffices to show that the completion $\widehat{B}$ at p is flat, for every point $p \in \operatorname{Spec} B_k$. By a change of coordinates, we may assume that the point lies over the origin $x = y = 0$ in affine space $\operatorname{Spec} k[x,y]$.

By Proposition 31.6, we can solve the equation $f(y) = 0$ uniquely for $\overline{y} \in R\langle\langle x\rangle\rangle$, extending the solution $y = 0$ at p. Let $\overline{z} = g(\overline{y})^{-1}$, and let $\overline{\alpha}_{ij}$ be the power series obtained by substituting $y = \overline{y}$, $z = \overline{z}$ into α_{ij}. The flatness of the completion will follow from Proposition 36.1 and from the next lemma, in which $((S))$ denotes the closure of the ideal generated by S, as before.

Lemma 38.11. *With the above notation, the completion $\widehat{B}$ of B at p is isomorphic to $\widehat{C} = R\langle\langle x\rangle\rangle/(([x_i,x_j]-\overline{\alpha}_{ij}))$*

Proof. Note that $\widehat{C} \in \mathcal{A}_R$. The map $R\langle x,y,z\rangle/(f, zg-1, gz-1) \to \widehat{C}$ defined by $\overline{y}, \overline{z}$ factors through P by definition of $\overline{\alpha}_{ij}$, hence through B. This factorization defines a homomorphism $\widehat{B} \to \widehat{C}$, so that we have canonical maps $R\langle\langle x\rangle\rangle \to \widehat{B} \to \widehat{C}$.

Lemma 38.12. *Let c be the value of z at p, and set $w = z - c$. The maps $R\langle\langle x\rangle\rangle \to R\langle\langle x,y\rangle\rangle/((f)) \to R\langle\langle x,y,w\rangle\rangle/((f, zg-1, gz-1))$ are isomorphisms.*

Proof. Since $g(y) = c^{-1}$ at p, it is an invertible element of $R\langle\langle x,y\rangle\rangle$. The fact that the second map is bijective follows easily. We use the solution $\overline{y}$ of $f(y) = 0$ to define a splitting of the first map, which shows that the first map is injective. Let $\overline{y}$ also denote the image of that element in $R\langle\langle x,y\rangle\rangle/((f))$. It suffices to show that $y = \overline{y}$ in this ring. This follows by induction on the maximal ideal $\mathfrak{m} = ((x,y))$. Write $y = \overline{y} + z$, and suppose that $z = O(\mathfrak{m}^r)$. Then

$$f(y) = f(\overline{y}) + f'(\overline{y})z + O(((z))^2 + (([\overline{y}, z]))),$$

and $f'(\overline{y})$ is invertible. Hence $z = O(\mathfrak{m}^{r+1})$.

This lemma shows that $\alpha_{ij} = \overline{\alpha}_{ij}$ in $R\langle\langle x,y\rangle\rangle/((f))$, which completes the proof of Lemma 38.11.

Now if a bracket α on B' is given, then for $1 \le j < i \le n$, we represent $\alpha(x_i,x_j) \in N \otimes B'$ by elements $\alpha_{ij} \in NF \approx N \otimes F'$, where $F' = R'\langle x,y,z\rangle$, and we construct the universal quotient B as in Proposition 38.7, which is smooth by Lemma 38.10. By construction, α_{ij} represents the value $\alpha(x_i,x_j)$ of the bracket, and it also represents the commutator of B. Hence the commutator agrees with α on these evaluations. The next lemma shows that these two brackets are the same.

Lemma 38.13. *Let B' be a smooth extension of the standard smooth algebra B_k, presented as in 38.10. Assume that the Jacobi identity holds in B'. Then every choice of $\alpha(x_i, x_j) \in N \otimes B'$ representing the commutator $[x_i, x_j]$ in B', $1 \leq j < i \leq n$, extends uniquely to a bracket on B'.*

Proof. The existence of the bracket follows from the existence of the smooth extension B. The uniqueness follows from a direct computation similar to the proof of 38.9.

It remains to show that the smooth extension B whose commutator is α is unique up to isomorphism. Let C be another smooth extension of B' to R whose commutator is α. Writing C as a quotient of F, the equation $[x_i, x_j] = \alpha_{ij}$ holds in C. Hence B maps to C, and since C is flat, this map is bijective.

The last step in the proof of 34.12 is to prove the existence of a universal extension for any smooth algebra B'. Lemma 38.10 shows that a smooth extension B of B' to R can be constructed locally on Spec B_k, and that it is locally unique up to isomorphism. Proposition 34.13i shows that the obstruction to globalizing B is an element of $H^2(\text{Spec } B_k, T^1_{B_k} \otimes_k N^r)$, and that the obstruction to existence of an isomorphism between two globalizations lies in $H^1(\text{Spec } B_k, T^1_{B_k} \otimes_k N^r)$. Both of these cohomology groups vanish because Spec B_k is affine and because the sheaves are coherent. The fact that a globalization of local smooth models is itself smooth is proved as in 38.5. So it suffices to construct B locally. Similar reasoning shows that it suffices to contruct α locally.