Quantifier Rank Spectrum of L-infinity-omega

by

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In Part A we will study the quantifier rank spectrum of sentences of $\mathcal{L}_{\omega_1,\omega}$. We will show that there are scattered sentences with models of arbitrarily high but bounded quantifier rank. We will also consider the case of weakly scattered and almost scattered sentences, and we will make some conjectures.

In Part B we will look at a new method of induction in the case of sheaves. We will then use this method to generalize the classical proof of the Suslin-Kleene Separation Theorem to the context of sheaves on a partial Grothendieck topology.

Thesis Supervisor: Gerald Sacks

Title: Professor of Mathematics

Dedication

I would like to dedicate this paper to my parents, Joanne Leedom-Ackerman and Peter Ackerman, as well as to my brother Elliot Ackerman. I couldn't have done this without their support.

I would especially like to thank my advisor Professor Gerald Sacks for his guidance and advice over the last 8 years. He has taught me an enormous amount, and I wouldn't be the mathematician I am today if it wasn't for him.

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Part A

Quantifier Rank Spectrum

Chapter 1

Introduction

1.1 Summary

1.1.1 The Goal

The goal of Part A of this thesis is to study the expressive power of sentences of $\mathcal{L}_{\infty,\omega}$. Specifically we wish to measure a sentences expressive power by looking at the quantifier rank of its models. We will call the set of quantifier ranks of models of a sentence its "quantifier rank spectrum".

However, as we will see in Section 1.2.4, the question of which quantifier rank spectra exist isn't particularly interesting. It is easy to construct a sentence ϕ with any desirable quantifier rank spectrum (so long as it is a set). The problem though is that the quantifier rank of ϕ is the supremum of it's quantifier rank spectrum (at least in the naive construction). The question that we will ask instead is, given a fixed (but small) bound β, what can be said about the quantifier rank spectra of sentences which themselves have quantifier rank $\leq \beta$?

In this paper we will show that given any limit ordinal $\omega * \alpha$ there is a scattered sentence $\phi_{\omega*\alpha}$ of $\mathcal{L}_{\omega_1,\omega}$ such that the quantifier rank spectrum of $\phi_{\omega*\alpha}$ is cofinal in $\omega * \alpha$, has supremum $\omega * \alpha$, and the quantifier rank of $\phi_{\omega*\alpha} \leq \omega$. As it is a well known fact (See Appendix A) that the quantifier rank spectrum of any scattered sentence must be cofinal in its supremum, this is essentially the best possible result for scattered sentences of $\mathcal{L}_{\omega_1,\omega}$.

In addition to our results concerning scattered theories, we will also prove several results concerning weakly scattered theories, as well as make several conjectures.

1.1.2 The Approach

We will prove the existence of our scattered sentences $\phi_{\omega*\alpha}$ by explicitly constructing them from homogeneous trees with nice extra structure. The method of construction we will use is to "glue" two copies of these trees together in such a way as to maintain homogeneity and to bound the quantifier ranks of the models.

In Chapter 2 we will introduce the languages we wish to work in, and we will prove some important results concerning the tools we have in these languages.

After we have presented the necessary background information concerning our trees we will, in Chapter 3, introduce the idea of a "collection of archetypes". A collection of archetypes is supposed to represent a way to indirectly describe, the tree structure of our models. We will show, among other things, that if a theory of trees has such a collection of archetypes then the set of archetypes which are realized completely determine the model.

Once we understand what a collection of archetypes for a sentence is we will, in Chapter 4, take such a sentence and glue together two copies. This will allow us to show that, assuming the original theory was scattered and had models of high enough quantifier rank, our gluing together produces a scattered sentence of whichever quantifier rank spectrum we want.

While a collection of archetypes is a very useful tool for studying a theory there is one thing we have to be careful of. Having a collection of archetypes is such nice property that it is not, apriori, obvious that it is even consistent. As such, in Chapter 5, we will show that in fact it is consistent to have a collection of archetypes. We will do this by showing that (essentially) the theory Θ from Robin Knight's The Vaught Conjecture: A Counterexample ([8]) has such a collection of archetypes. In addition, in Chapter 5 we will prove several other interesting properties of Θ

Once we have dealt with the case of scattered sentences we will turn our attention to the non-scattered case. In Chapter 6 we will introduce other component trees which we believe will be useful.

In Chapter 7 we will prove some results concerning the case of almost scattered sentences which aren't scattered. In particular we will provide a way of turning any scattered sentence into one which is almost scattered and not scattered but has the same quantifier rank. We will then also use the component trees from Chapter 6 to build a theory which looks very similar to the one in Chapter 4 and we will make some conjectures about them.

In Chapter 8 we will consider the case when we glue together an infinite number of copies of our sentences. In this case the sentences we get are not even weakly scattered. But, we believe that this approach will allow us to get some very sharp results concerning the quantifier ranks of the models. We will prove some results concerning these theories and then make some conjectures concerning their quantifier rank spectrum and discuss why we believe them.

Finally, we will provide Appendixes with important information. In Appendix A we will present the basic theory of Vaught trees, their relationship to Quantifier Rank, and we will give references for the theorems which we do not prove.

1.2 Background

1.2.1 $\mathcal{L}_{\infty,\omega}$

1.2.1.1 Introduction

When studying infinite model theory one often wants more expressive power than is provided by 1st order logic. But at the same time would like to retain the many nice properties of 1st order logic. A natural way in which one might hope to find this balance is to loosen the restrictions on what a formula. Over the years people have looked at many very different ways in which the idea of a formulas can be generalized, and one of the most fruitful generalizations has been to what is called $\mathcal{L}_{\infty,\omega}$.

One of the many advantages of $\mathcal{L}_{\infty,\omega}$ is that not only does it have expressive power far greater than 1st order logic, but it is absolute in a way which many other generalizations of 1st order logic aren't. Specifically, when dealing with infinite models we are often interested in how similar they are, or when they "look the same up to a given complexity of formula". When comparing the complexity of two models we would like the result only depends on the models and not on the underlying model of set theory which we are in. And, we find that this is the case with $\mathcal{L}_{\infty,\omega}$.

In addition, if we are only concerned with countable models we will see that we can restrict ourselves to countable formulas. In this case we will also have that $\mathcal{L}_{\infty,\omega}$ (or $\mathcal{L}_{\omega_1,\omega}$ as it is called) preserves several properties of 1st order logic which make it so easy to work with. Including, a form of compactness, a form of completeness, and an omitting type's theorem.

The purpose of this paper is to study the expressive power of $\mathcal{L}_{\infty,\omega}$. By a well known result of Scott [13] every countable model has a sentence of $\mathcal{L}_{\omega_1,\omega}$ which describes it up to isomorphism. However, the complexity of these sentences is at least as great as the complexity of the model which it characterizes. As such, our measure of expressive power isn't what can be characterized by sentences of $\mathcal{L}_{\infty,\omega}$, but rather knowing the complexity of the formula, what can be said about the complexity of the models of that formula.

All of the material in this section is standard in the study of $\mathcal{L}_{\infty,\omega}$ and $\mathcal{L}_{\omega_1,\omega}$. The interested reader can find a more detailed description in [7], [2], or [1].

1.2.1.2 Definitions and Notation

Definition 1.2.1.1. Let $L = \langle R_i : i \in \kappa \rangle$ be a relational language with $\text{arity}(R_i) = n_i \in \omega$.

Let $\mathcal{L}_{\infty,\omega}(L)$ = least set such that

- $R_i(x_1, \ldots x_{n_i}) \in \mathcal{L}_{\infty,\omega}(L)$ for all $i \in \kappa$, and free variables $x_1, \ldots x_{n_i}$.
- If $\phi \in \mathcal{L}_{\omega_1,\omega}(L)$ then $\neg\phi, (\exists y)\phi \in \mathcal{L}_{\infty,\omega}(L)$
- If for some set A, and finite n $\{\phi_i : i \in A\} \subseteq \mathcal{L}_{\infty,\omega}$, and $(\forall i \in A)$Free Variables$\phi_i \subseteq \{x_1, \ldots x_n\}$ then $(\bigwedge_{i\in A} \phi_i) \in \mathcal{L}_{\infty,\omega}(L)$.

We will use $(\forall y)\phi$ as short hand for $\neg(\exists y)\neg\phi$ and $(\bigvee_{i\in A} \phi_i)$ as a shorthand for $\neg(\bigwedge_{i\in A} \neg\phi_i)$. We also will omit mention of L in $\mathcal{L}_{\infty,\omega}(L)$ when it is understood which language we are talking about.

In the case that all conjunctions and disjunctions are countable we say a formula is in $\mathcal{L}_{\omega_1,\omega}$. We will only mention $\mathcal{L}_{\omega_1,\omega}$ when we are considering countable models and, as we will see, in this case we can assume without loss of generality that we are working in $\mathcal{L}_{\omega_1,\omega}$ or in $\mathcal{L}_{\infty,\omega}$. As such, after this section, we will only refer to $\mathcal{L}_{\omega_1,\omega}$ when we wish to highlight that the argument only works for countable models.

There are two ideas worth pointing out explicitly. First the central idea behind $\mathcal{L}_{\infty,\omega}(L)$ and $\mathcal{L}_{\omega_1,\omega}(L)$ is that we allow ourselves arbitrary infinite conjunctions and disjunctions with the one condition that the end result has to have only finitely many free variables.

The second thing worth mentioning is that restricting L to be a relational language is purely for convenience. This is because given any language L' with function and/or constat symbols, it is easy to convert it to a relational language without loosing any of the expressive power (just add a relation for each function and an axiom saying that the relation is the graph of the function (and treat constants as functions of 0 arity)). As such we will from here on assume all languages are relational.

Definition 1.2.1.2. Let $L = \langle R_i : i \in A\rangle$ be a language and let $\phi(\mathbf{x}) \in \mathcal{L}_{\infty,\omega}$. Let M be a model of the language L, $\overline{a} \in M$. We will recursively define what it means for M to model $\phi(\overline{a})$ ($M \models \phi(a)$).

- If $\phi(\mathbf{x}) = \neg\psi(\mathbf{x})$ then $M \models \phi(\overline{a})$ iff $M \not\models \psi(\overline{a})$. If $\phi(\mathbf{x}) = (\exists y)\psi(y, \mathbf{x})$ then $M \models \phi(\overline{a})$ iff there is a $b \in M$ such that $M \models \psi(b, \overline{a})$.
- If $\phi(\mathbf{x}) = (\bigwedge_{i\in A} \psi_i(\mathbf{x}))$ then $M \models \phi(\overline{a})$ iff for all $i \in A$, $M \models \psi_i(\overline{a})$.

We will see shortly that this covers all formula's of $\mathcal{L}_{\infty,\omega}$. But first we want to define some important conventions which we will use.

Definition 1.2.1.3. Let $\psi(y), \phi(x, \mathbf{z}) \in \mathcal{L}_{\infty,\omega}$ where Free Variables(ψ) = $\{y\}$.
Define $\phi(\mathbf{z})^{\psi}(y)$ to be:

- $\phi(z_1, \ldots, z_n)^{\psi} \Leftrightarrow \phi(z_1, \ldots, z_n) \wedge \bigwedge_{i\in n} \psi(z_i)$ if ϕ is an atomic formula.
- $[\bigwedge_{i\in I} \phi(\mathbf{z})]^{\psi} \Leftrightarrow \bigwedge_{i\in I}[\phi(\mathbf{z})^{\psi}]$
- $[\neg\phi(\mathbf{z})]^{\psi} \Leftrightarrow [\neg\phi(\mathbf{z})^{\psi}] \wedge \bigwedge_{i\in n} \psi(z_i)$
- $[(\exists x)\phi(x, \mathbf{z})]^{\psi} \Leftrightarrow (\exists x)(\psi(x) \wedge [\phi(x, \mathbf{z})]^{\psi})$

We say that $\psi \models \phi(\mathbf{z})$ if $(\phi(\mathbf{z}))^{\psi}$ holds.
We also define $\psi(x_1, \ldots, x_n) \leftrightarrow \bigwedge_{i\in n} \psi(x_i)$.

Intuitively we are relativizing the formulas so that we only consider elements which realize ψ.

Definition 1.2.1.4. Let $\psi(\mathbf{y},\mathbf{x})$ be a formula in $\mathcal{L}_{\infty,\omega}$. For each $n \in \omega$ define $\exists^n \mathbf{y}\phi(\mathbf{y},\mathbf{x}) \leftrightarrow (\exists \mathbf{y}_1, \dots \mathbf{y}_n) \bigwedge_{i \leq n} \phi(\mathbf{y}_i, \mathbf{x})$ and $\mathbf{y}_i \cap \mathbf{y}_j = \emptyset$ if $i \neq j$.

The idea is that $\exists^n \phi$ just says there are n distinct tuples which satisfy ϕ.

Finally, unless otherwise specified $\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a}, \overline{b}, \overline{c}, \overline{d}, \overline{e}$ represent tuples (of the appropriate model or of variables) of some arbitrary but fixed arity. Whereas x, y, z, a, b, c, d, e represent individual elements (of the appropriate model) or individual variables. We will often say a finite tuple $\overline{a} \in M$ if M is a model and $\overline{a} \subseteq M$.

1.2.2 Quantifier Rank

1.2.2.1 Introduction

As mentioned in the introduction to the last section one of the most appealing features of $\mathcal{L}_{\infty,\omega}$ is that in the degree to which two models look a like is absolute. The main way in which we will measure how similar two models are will go as follows. First we will assign, in a natural way, an ordinal rank to each formula of $\mathcal{L}_{\infty,\omega}$. Then we say that two models are similar up to $\alpha \in \mathrm{ORD}(\alpha \in \omega_1)$ if they satisfy all the same formula's of rank $< \alpha$. We will also say that two models have relative rank of at least α if the above holds.

Now there are obviously many different ways in which we can assign rank to the formula's of $\mathcal{L}_{\infty,\omega}$ but as it turns out almost all of them are "essentially" the same (i.e. they will be the same on a Closed Unbounded Set of κ for all cardinals κ). We will choose a particularly nice rank named

"Quantifier Rank". Quantifier rank is particularly nice because not only is relative quantifier rank absolute between different models of set theory, but it is also absolute (for countable models) relative to which of $\mathcal{L}_{\infty,\omega}$ or $\mathcal{L}_{\omega_1,\omega}$ we consider.

Despite these very nice properties though, the most useful property of quantifier rank is that there is a very constructive way to determine the relative quantifier rank of two models (via the study of partial isomorphisms between them). Not only will this method allow us to see the absoluteness of relative quantifier rank, but it will prove an invaluable tool for explicitly determining the relative quantifier rank of two models.

All of the material in this section is standard in the study of $\mathcal{L}_{\infty,\omega}$ and $\mathcal{L}_{\omega_1,\omega}$. The interested reader can find a more detailed description in [7], [2], or [1].

1.2.2.2 Formulas

In order to define quantifier rank we will build up $\mathcal{L}_{\infty,\omega}$ (and $\mathcal{L}_{\omega_1,\omega}$) explicitly.

Definition 1.2.2.1. Let $L = \langle R_i : i \in A \rangle$ be a language.
Define $L_{\langle \alpha,n \rangle}$ for $\alpha \in \text{ORD}, n \leq \omega$ as follows:

- $L_{\langle 0,0 \rangle} = \{R_i(\mathbf{x}), \neg R_i(\mathbf{x}) : i \in A\}$.
- $L_{\langle \alpha,n+1 \rangle} = L_{\langle \alpha,n \rangle} \cup \{\bigwedge_{i\in B} \phi_i, \bigvee_{i\in B} \phi_i : \bigcup_{i\in B}$ Free Variables(ϕ_i) is finite, $\{\phi_i : i \in B\} \subseteq L_{\langle \alpha,n \rangle}\}$.
- $L_{\langle \alpha,\omega \rangle} = \bigcup_{j\in\omega} L_{\langle \alpha,j \rangle}$
- $L_{\langle \alpha+1,0 \rangle} = L_{\langle \alpha,\omega \rangle} \cup \{(\exists y)\phi_i, (\forall y)\phi_i : \phi_i \in L_{\langle \alpha,\omega \rangle}, y$ is a free variable$\}$.

- $L_{\langle \omega * \alpha, 0\rangle} = \bigcup_{\gamma < \omega * \alpha} L_{\langle \gamma, 0\rangle}$

Notice that if $\phi \in L_{\langle \alpha, n\rangle}$ then there is a $\psi \equiv_{\neg} \neg\phi$ such that $\psi \in L_{\langle \alpha, n\rangle}$. (Here $A \equiv_{\neg} B$ iff $A/\{\neg\neg\phi \cong \phi : \phi \in \mathcal{L}_{\infty,\omega}\} = B/\{\neg\neg\phi \cong \phi : \phi \in \mathcal{L}_{\infty,\omega}\}$)

The first thing to notice is that $\bigcup_{\alpha \in \text{ORD}} L_{\langle \alpha, 0\rangle} = \mathcal{L}_{\infty,\omega}(L)$. This is because every formula of $\bigcup_{\alpha \in \text{ORD}} L_{\langle \alpha, 0\rangle}$ must be in $\mathcal{L}_{\infty,\omega}$ and $\bigcup_{\alpha \in \text{ORD}} L_{\langle \alpha, 0\rangle}$ is closed under the required operations.

Further notice that if we require all conjunctions and disjunctions to be countable then we have $\bigcup_{\alpha \in \omega_1} L_{\langle \alpha, 0\rangle} = \mathcal{L}_{\omega_1,\omega}(L)$. To see this notice that after stage $\langle \omega_1, 0\rangle$ of the construction any countable conjunction or disjunction we can make from the formula's must already appear (because ω_1 is regular).

Now the least $\langle \alpha, n\rangle$ such that a formula $\phi \in L_{\langle \alpha, n\rangle}$ is a little bit more information than we need. So instead we have:

Definition 1.2.2.2. Let $\phi \in \mathcal{L}_{\infty,\omega}(L)$ (or $\mathcal{L}_{\omega_1,\omega}(L)$). Define the Quantifier Rank of ϕ ($\text{qr}(\phi)$) = least α such that $\phi \in L_{\langle \alpha, \omega\rangle}$.

Notice that if $\phi \in \mathcal{L}_{\omega_1,\omega}(L)$ it doesn't matter if we consider it an element of $\mathcal{L}_{\omega_1,\omega}(L)$ or $\mathcal{L}_{\infty,\omega}(L)$ for the purposes of calculating it's quantifier rank.

Definition 1.2.2.3. Let $\phi, \psi \in \mathcal{L}_{\infty,\omega}(L)$ and ψ is a subformula of ϕ. We then say $\psi \prec_{sub} \phi$.

Definition 1.2.2.4. We say $A \subseteq \mathcal{L}_{\infty,\omega}$ is a fragment iff

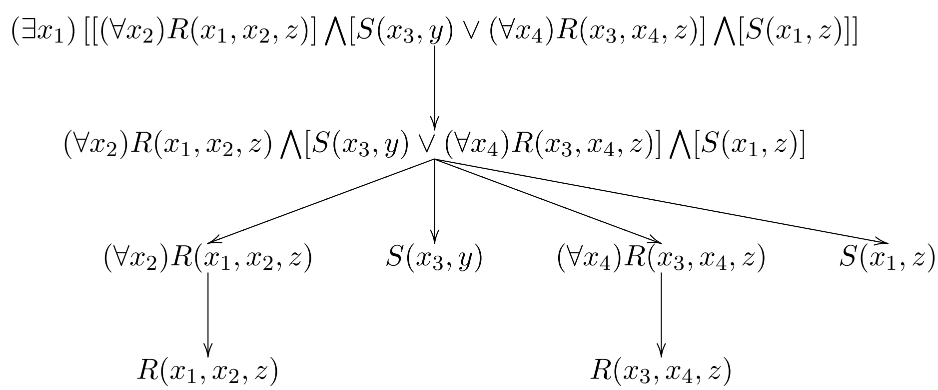
- $A \neq \emptyset$
- $(\forall \phi, \psi \in A) \neg\phi, \psi \wedge \phi, \psi \vee \phi, (\exists x)\phi, (\forall x)\phi \in A$

- $(\forall \phi, \psi \in \mathcal{L}_{\infty,\omega})\phi \in A \wedge \psi \prec_{sub} \phi \rightarrow \psi \in A$

A good way to think of a formula of $\mathcal{L}_{\infty,\omega}(L)$ is as a well founded tree. Specifically the tree is given by $\prec_{sub} |\{\phi : (\exists \alpha)\phi \in L_{\langle \alpha,0\rangle}\}$. I.e. the nodes of the tree are those formulas which have a quantifier outside all conjunctions and disjunctions.
To see this consider the representation of the formula:

$$(\exists x_1)\left[[(\forall x_2)R(x_1,x_2,z)]\bigwedge[S(x_3,y)\vee(\forall x_4)R(x_3,x_4,z)]\bigwedge[S(x_1,z)]\right]$$

This would have the following tree:



We can then think of the quantifier rank of a formula as the height of the wellfounded tree.

1.2.2.3 Models

Now that we have our definition of quantifier rank of a formula, we can begin to compare the complexities of models.

Definition 1.2.2.5. Let M, N be models of a language L. We say M and N are equivalent up to quantifier rank α ($M \equiv_\alpha N$) if and only if

$$(\forall \phi \in \mathcal{L}_{\infty,\omega})[\mathrm{qr}(\phi) \leq \alpha] \Rightarrow [M \models \phi \Leftrightarrow N \models \phi]$$

In other words two models are equivalent up to quantifier rank α if they agree on all formulas of quantifier rank $\leq \alpha$.

Theorem 1.2.2.6. *Let V, W be transitive models of ZFC. Let L be a language such that and $M, N, L \in V \cap W$ where M, N are models of a language L. Then $(M \equiv_\alpha N)^V$ iff $(M \equiv_\alpha N)^W$.*

Proof. See [2] Chapter 7 §5, §6. □

Definition 1.2.2.7. Define $Th_\alpha(M) = \{\phi : M \models \phi, \mathrm{qr}(\phi) \leq \alpha\}$

Theorem 1.2.2.8. *If M, N are countable models then $(M \equiv_\alpha N)^{\mathcal{L}_{\omega_1,\omega}}$ iff $(M \equiv_\alpha N)^{\mathcal{L}_{\infty,\omega}}$*

Proof. See [2] Chapter 7 §5, §6. □

It is this theorem which allows us to assume we are working in $\mathcal{L}_{\omega_1,\omega}$ (without loss of generality) if all our models are countable.

Definition 1.2.2.9. Let M be a model of a language L. We say the Quantifier Rank of M ($\mathrm{qr}(M)$) = least α such that

$$(\forall \text{ models } N)(\forall \beta > \alpha) M \equiv_\alpha N \Leftrightarrow M \equiv_\beta N$$

.

Similarly we define

Definition 1.2.2.10. For $\overline{a} \in M$ define <u>Quantifier Rank</u> of $\overline{a}$ $(\mathrm{qr}(\overline{a})) = \min\{\alpha : \exists\phi(\mathbf{x}), \mathrm{qr}(\phi(\mathbf{x})) = \alpha, M \models \phi(\overline{a}), (\forall\psi(\mathbf{x}) \in \mathcal{L}_{\infty,\omega})[(\forall \text{ models } N, \overline{b} \in N) N \models \phi(\overline{b}) \rightarrow \psi(\overline{b})]$ or $[(\forall \text{ models } N, \overline{b} \in N) N \models \phi(\overline{b}) \rightarrow \neg\psi(\overline{b})]$

We say $\overline{a} \in M$, $\overline{b} \in N$ have the same γ-type ($type^{\gamma}(\overline{a}) = type^{\gamma}(\overline{b})$,($\overline{a} \equiv_{\gamma} \overline{b}$)) if they are equivalent up to formulas of quantifier rank γ. In other words if $(M, \overline{a}) \equiv_{\gamma} (N, \overline{b})$.

We then have

Theorem 1.2.2.11. *For all $\overline{a} \in M$, $qr(\langle M, \overline{a}\rangle) = qr(\overline{a})$.*

Proof. See [2] Chapter 7 §5, §6. □

So in particular we can think of $\mathrm{qr}(M)$ as $\mathrm{qr}(\emptyset_M)$. We then also have.

Theorem 1.2.2.12. *For all M, $qr(M)$ is defined, $qr(M) < |M|^{+}$.*

Proof. See [2] Chapter 7 §5, §6. □

Definition 1.2.2.13. Let M be an L structure. A sentence σ_M of $\mathcal{L}_{\infty,\omega}$ is a <u>Scott Sentence for M</u> if ($\forall N$ an L structure)$N \models \sigma_M \rightarrow N \equiv_{\infty} M$.

Theorem 1.2.2.14. *For all M there is a Scott sentence $\sigma_M \in \mathcal{L}_{\infty,\omega}$. And, if M is countable, $\sigma_M \in \mathcal{L}_{\omega_1,\omega}$.*

Proof. See [2] Chapter 7§6. □

What is more, if M is countable we can get even better results via the Scott Isomorphism Theorem.

Theorem 1.2.2.15. *If M is a countable model and N is any countable model such that $M \equiv_{qr(M)} N$ then $M \cong N$.*

Proof. See [13]. □

Definition 1.2.2.16. Let $\phi \in \mathcal{L}_{\infty,\omega}$. We define the Quantifier Rank Spectrum of ϕ to be $\{\alpha : (\exists M) M \models \phi, \mathrm{qr}(M) = \alpha\}$.

1.2.2.4 Partial Isomorphism

The true value in considering quantifier rank comes not just from its absolute, but from the fact that there is a concrete way to determine if two models are equivalent up to a given rank. This method is by constructing sequences of partial isomorphisms, or what we sometimes call the back and forth method (in deference to the game description which we will discuss at the end of the section). These sequences of partial isomorphisms are also sometimes called Ehrenfeucht-Fraisse (EF) Sequences.

Definition 1.2.2.17. If M, N are models of a language L, $f : M \to N$ is a partial isomorphism from M to N if

- f is one to one.
- $|\mathrm{dom}(f)| < \omega$
- For each relation $R \in L$ and $\overline{a} \subseteq \mathrm{dom}(f), M \models R(\overline{a})$ iff $N \models R(f(\overline{a}))$.

Definition 1.2.2.18. We say $\langle I_i : i < \alpha \rangle$ is a Sequence of Partial Isomorphisms from M to N if

- $I_i \supseteq I_j$ if $i \leq j$
- If $f \in \bigcup_{i \in \alpha} I_i$ then f is a partial isomorphism from M to N.
- If $\beta + 1 \leq \alpha$ and $f \in I_{\beta+1}$ then
  - $(\forall m \in M)(\exists g \in I_\beta) f \subseteq g$ and $m \in \text{dom}(g)$
  - $(\forall n \in N)(\exists g \in I_\beta) f \subseteq g$ and $n \in \text{range}(g)$

One way to think about these partial isomorphism sequences is to consider the following game between two players (called the Ehrenfeucht-Fraisse (or EF) Game). We will call Player I the Spoiler and Player II the Duplicator. The game starts at an ordinal α. At each stage Spoiler plays an ordinal less than the one already played and an element a of either M or N. Duplicator then plays an element of the other model. The game ends once Spoiler has played the ordinal 0. Duplicator wins this game if the sequence produced from M has the same atomic diagrams (i.e. satisfy the same atomic formulas from L) as the sequence produced from N. Spoiler wins otherwise.

The reason why this game is useful is because we can consider a sequence of partial isomorphisms $\langle I_i : i < \alpha \rangle$ as a winning strategy for duplicator in the game starting at α. This is because if Spoiler plays (β, a) then Duplicator simply finds the partial isomorphism in I_β extending the current play by adding a (to the domain or range) and plays what that isomorphism says corresponds to a.

The real importance of the partial isomorphisms with regards to quantifier rank though is in the following theorems.

Theorem 1.2.2.19. *If M, N are models of a language L and $\overline{a} \in M, \overline{b} \in N$ then $(M, \overline{a}) \equiv_\alpha (N, \overline{b})$ if and only f $\exists \langle I_i : i \leq \alpha \rangle$ a partial isomorphism sequence from M to N and $f \in I_\alpha$ with $f(\overline{a}) = \overline{b}$*

If M, N are models of a language L then $M \equiv_\infty N$ iff $\exists I$ such that $\langle I, I \rangle$ is a partial isomorphism sequence from M to N.

Proof. See [2] Chapter 7 §5. □

Corollary 1.2.2.20. *If M, N are models of a language L then $M \equiv_\alpha N$ iff $\exists \langle I_i : i \in \alpha \rangle$ a partial isomorphism sequence from M to N.*

Proof. Immediate from [2] Chapter 7 □

It is because of this theorem more than any other feature of quantifier rank that we have chosen to use it (as opposed to the many other "essentially" equivalent ranks).

1.2.3 Scattered Like Theories

Now that we have some idea of how we are going to be measuring the complexity of the models of a theory (i.e. by looking at the quantifier rank spectrum) we can start to consider special classes of theories we would like to look at. As we will see there are three levels of refinement that we can apply to our theories. For motivation on why these definitions are chosen see Appendix A.

1.2.3.1 Background Notation

Definition 1.2.3.1. Let $T \in \mathcal{L}_{\infty,\omega}(L)$ be a sentence in a language L. Let $Frag(T)$ be the smallest fragment of $\mathcal{L}_{\infty,\omega}$ containing T, and $\mathcal{S}(T) = \{p : p$ is a consistent complete type over $Frag(T)$ extending $T\}$. Let $k(T) = \inf\{\kappa : T \in L_{\kappa,\omega}\}$, $s(T) = \sup\{k(T), |L|\}$.

Lemma 1.2.3.2. $\mathcal{S}(T) = \bigcup_{N \models T}\{\mathit{type}(n)|A(T)\text{: } n \in N\}$

Proof. See [2] Chapter 3 §2. □

1.2.3.2 Weakly Scattered

Definition 1.2.3.3. Let L be a countable language. Let $T \in \mathcal{L}_{\omega_1,\omega}(L)$. We say that T is <u>Weakly Scattered</u> if for all $M \models T, \alpha \in \omega_1$, $\mathcal{S}(Th_\alpha(M) \cup T)$ is countable.

What this says is that if we have any model of T and we look at its theory among sentences of quantifier rank at most α then that theory has at most countably many types over it. This is an important condition because of the following important theorem (called the "Omitting Types Theorem").

Theorem 1.2.3.4. *Let L_A be a countable fragment of $\mathcal{L}_{\infty,\omega}$ and let T be a set of sentences of L_A which has a model. For each n let Φ_n be a set of formulas of L_A with free variables among $v_1, \dots, v_{k_n}$. Assume that for each n and each formula $\psi(v_1, \dots, v_{k_n})$ of L_A: If*

$$T + \exists v_1, \dots, v_{k_n}\psi$$

has a model, so does

$$T + \exists v_1, \dots, v_{k_n}(\psi \wedge \phi)$$

for some $\phi(v_1, \ldots, v_{k_n}) \in \Phi_n$. Given this hypothesis, there is a countable model M of T such that for each $n < \omega$

$$M \models \forall v_1, \ldots, v_{k_n} \bigwedge_{\phi \in \Phi_n} \phi(v_1, \ldots, v_{k_n})$$

Proof. See [2] Chapter III §3.8 □

With the "Omitting Types Theorem" we see that if a theory T is weakly scattered and $M \models T$ then $Th_\alpha(M) \cup T$ has an atomic model (because we can omit all the non-principle types). This is important because, as can be seen in Appendix A, we then can build a tree of theories over T, each of which has an atomic model. In addition, the height of this tree is (approximately) the supremum of (the quantifier rank spectrum of T).

1.2.3.3 Almost Scattered

Definition 1.2.3.5. Let L be a countable language. Let $T \in \mathcal{L}_{\omega_1,\omega}(L)$. We say that T is <u>Almost Scattered</u> if for all $\alpha < \omega_1$ there exists $\mathcal{S}_a(T, \alpha)$, a countable collection of formulas of $\mathcal{L}_{\omega_1,\omega}(L)$, such that

- For all $M \models T$ and $q \in \mathcal{S}(Th_\alpha(M) \cup T)$ there is at least one $p \in \mathcal{S}_a(T, \alpha)$ such that $\models (\forall \mathbf{x})[p(\mathbf{x}) \wedge Th_\alpha(M) \wedge T] \rightarrow q(\mathbf{x})$

- For all $M \models T$ and $p \in \mathcal{S}_a(T, \alpha)$ there is at most one $q \in \mathcal{S}(Th_\alpha(M) \cup T)$ such that $T \models (\forall \mathbf{x})[p(\mathbf{x}) \wedge Th_\alpha(M) \wedge T] \rightarrow q(\mathbf{x})$

The idea behind weakly scattered theories is we are trying to get a very loose bound on what extensions of the theory look like. To be precise, in the case of a weakly scattered theory T, we don't really know what the collection of theories extending T looks like, all we know is that each individual theory extending T is itself weakly scattered (i.e. it and all its extensions have only countably many types).

However, in the context of an Almost Scattered theories, we know a little bit more. In this context we know that not only does each extension only have countably many types over it, but in some sense all extensions at the same level have countably many types over them in a uniform way. That is each extension at the same level has all of its types come from a set of types which is "the same" for all theories at that level of the construction.

Theorem 1.2.3.6. *If T is almost scattered then T is weakly scattered.*

Proof. As $\mathcal{S}_a(T, \alpha)$ is countable and for each $M \models T$, every element of $\mathcal{S}(Th_\alpha(M) \cup T)$ comes from a unique $p \cup Th_\alpha(M) \cup T$ with $p \in \mathcal{S}_a T, \alpha$ we must have $\mathcal{S}(Th_\alpha(M) \cup T)$ is countable as well. $\square$

Almost Scattered theories are important because often the easiest way to show a theory is weakly scattered is to simply show that it is almost scattered. In fact, at the time of writing, the author knows of no example of a weakly scattered sentence which is not almost scattered (although the author is very confident that they exists). In other words, often the easiest way to show a theory is weakly scattered isn't to show that for each individual extension of the theory, at each quantifier rank, that that extension must have only countably many types, but rather to show that the collection of all types over all theories up to the same quantifier rank is itself (essentially) countable. And hence the collection over any particular extension must also be countable.

In fact, as we will see later in Section 7.2, there is a simple way to turn any scattered theory into an almost scattered theory which isn't scattered but which has the same quantifier rank.

1.2.3.4 Scattered

Definition 1.2.3.7. Let L be a countable language. Let $T \in \mathcal{L}_{\omega_1,\omega}(L)$. We say that T is <u>Scattered</u> if for all α $\{Th_\alpha(M) \cup T : M \models T\}$ is countable.

Scattered sentences are especially nice because we find that the Vaught Tree for a scattered sentence is absolute. In other words, the Vaught tree doesn't change as we move from one model of set theory to another. Hence, everything that we want to know about a scattered sentence T we can find out by looking at $L(T)$ (here $L(T)$ is the constructible universe with T at the base). For more information on this see [3].

We also have the following very nice theorem concerning scattered sentences.

Theorem 1.2.3.8. *If T is scattered, then for each $\alpha < \omega_1$ and for all $M \models T$, $\mathcal{S}(Th_\alpha(M) \cup T)$ is countable.*

Proof. Assume $\mathcal{S}(Th_\alpha(M) \cup T)$ is uncountable. Then there must be uncountably many non-principle types. But then each one of those must be realized by some model. Hence $\{Th_{\alpha+1}(M) \cup T : M \models T\}$ must be uncountable. $\Rightarrow\Leftarrow$ $\square$

Corollary 1.2.3.9. *If T is Scattered, T is Almost Scattered.*

Proof. Because for each M, $\mathcal{S}(Th_\alpha(M)\cup T)$ is countable, and because $\{Th_\alpha(M)\cup T : M \models T\}$ is countable we know $\bigcup\{\mathcal{S}(Th_\alpha(M)\cup T) : M \models T\}$ is countable and hence witnesses that T is almost scattered. $\square$

1.2.3.5 General Case

As all of these definitions are essentially absolute (see [3]), we want to define the uncountable case in terms of the countable case.

Definition 1.2.3.10. Let V be the model of ZFC in which all proofs up to this point have been carried out in.

Definition 1.2.3.11. Let L be a language and $T \in \mathcal{L}_{\infty,\omega}(L)$. Then T is (weakly/almost) scattered if for all set generic G, and for all

- $(W, \epsilon) \models ZFC$
- $W \subseteq V[G]$
- $W \models s(T) \leq \omega$

$W \models T$ is (weakly/almost) scattered.

So our definition in the case that T or L is uncountable is that it is (weakly/almost) scattered exactly when it is (weakly/almost) scattered in every universe which makes the theory countable. Once again, this definition isn't a problem because the definability of the Vaught tree (see [3] and Appendix A) means that these ideas are absolute.

1.2.4 Vaught Tree

1.2.4.1 Possible Quantifier Ranks

While the quantifier rank spectrum of a sentence is a good measure of how complex its collection of models the study of the possible quantifier rank spectrums themselves isn't very interesting. This is because of the following theorem.

Theorem 1.2.4.1. *Let $X \subseteq \alpha$ where $|\alpha| = \omega_a$. Then there is a formula ϕ_X of $\mathcal{L}_{\omega_{a+1},\omega}$ such that Quantifier Rank Spectrum of $\phi_X = X$.*

First we need an easy lemma which we will state without proof.

Lemma 1.2.4.2. *For all α there is a model M_α in the language $\langle \leq \rangle$ such that the Scott Sentence of M_α is in $\mathcal{L}_{|\alpha|^+,\omega}$ and $qr(M_\alpha) = \alpha$.*

(to see this consider the well-founded homogeneous tree of height α)

Proof of Theorem. Let $\phi_\alpha =$ Scott sentence of M_α. Then $\phi_X = \bigvee_{\alpha \in X} \phi_\alpha$ □

1.2.4.2 Vaught Tree

As it turns out what we really want to be studying is not the quantifier rank spectrum of a sentence but the "Vaught Tree" of the sentence. The reason for this is that the Vaught tree of a sentence is a very robust object which determines many of the nice properties of the sentence. Further the Vauht tree will also allow us to get a handle on the models of its sentence.

We find that the height of the Vaught tree of a sentence is a good measure for how complex the collection of models of the sentence is relative to the sentence itself. For example, while the sentences ϕ_X in Theorem 1.2.4.1 had arbitrary quantifier rank spectrums, the ϕ_X constructed in the theorem all have Vaught trees of height 1. This corresponds to our intuition that while the collection of models is complicated, there is no simple way to describe the complex structure.

What we would like be able to do, and indeed what we will do in this part of our paper, is show that for arbitrary α there is a scattered formula whose Vaught tree is (approximately) of height α. This intuitively means that will be able find scattered sentences (i.e. well behaved sentences) which can have models far more complicated than the sentence itself.

Unfortunately though there is no nice way know to build up Vaught trees and no nice way know to study individual Vaught trees. However, we are in luck as we will be able to study the Vaught trees by looking at approximations to them given by quantifier ranks.

Specifically we know by Theorem A.1.2.2 and Theorem A.1.2.3 that the height of the Vaught tree of ϕ is (approximately) $\mathrm{qr}(\phi)+\max\{\text{Quantifier Rank Spectrum}(\phi)\}$. So what this means is that if we can find scattered sentences ϕ_α such that $\max\{\text{Quantifier Rank Spectrum}(\phi)\}$ is (approximately) α and $\mathrm{qr}(\phi_\alpha) < \beta$ for all α then these sentences will also have Vaught trees whose height is (approximately) α.

1.2.5 Miscellaneous

Here is some other important notation we will use later.

1.2.5.1 Slant Lines

Definition 1.2.5.1. A slant line is a function $m - \{0\} \rightarrow \mathrm{ORD} \cup \{-\infty\}$ such that for all $n < m - 1$, $f(n) > f(n+1)$. If in addition we have that whenever $f(n) = \gamma + 1$ we also have $f(n+1) = \gamma$ then we call f a slow slant line

If f is a slant line and $f(1) = \gamma * \omega + n$ we say the rank of f is n ($\mathrm{rank}(f) = n$) and the base of f is $\gamma * \omega$ ($\mathrm{base}(f) = \gamma * \omega$).

For two slant lines f, g with the same domain, we say that $f \leq g$ iff $(\forall x) f(x) \leq g(x)$.

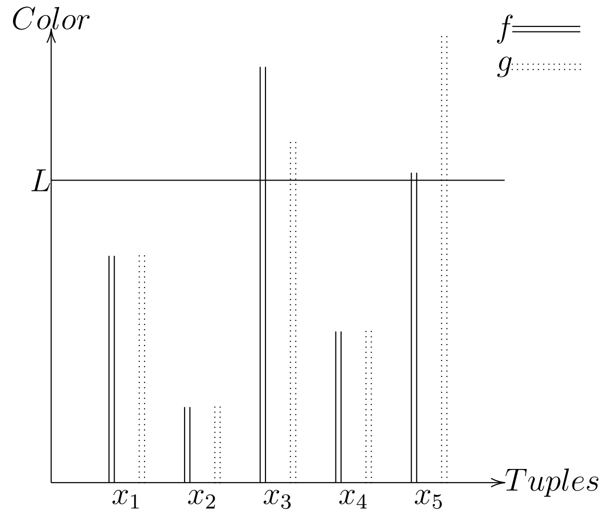
We say that a slant line f is less than an ordinal γ iff $(\forall x) f(x) < \gamma$.

Definition 1.2.5.2. Let $f, g : A \times X^{\leq n} \rightarrow \mathrm{ORD}$ be maps from finite subsets of X indexed by A into an ordinal α. Let $L : \omega \rightarrow \mathrm{ORD}$ be a function with domain $\geq n$.

We say that f is the same as g up to a function K ($f|K = g|K$) if $\forall (a, \mathbf{x}) \in A \times X^{<n}$

- $f(a, \mathbf{x}) \geq K(|\mathbf{x}|)$ iff $g(a, \mathbf{x}) \geq K(|\mathbf{x}|)$
- If $f(a, \mathbf{x}), g(a, \mathbf{x}) < K(|\mathbf{x}|)$ then $f(a, \mathbf{x}) = g(a, \mathbf{x})$

To get an intuitive idea of what it means for two functions to be the same up to a function K, look at the case when K is the constant function at γ. Then the idea is that below γ we have a completely clear view of what f and g are. But, once we have passed γ, things are less focused. In this case, all we can say about the functions is that they have passed the furthest point at which we can distinguish distances, in the same places.



Example of two functions, f, g on tuples which are equivalent up to L

1.2.5.2 Short Hand

Definition 1.2.5.3. Let T be some sentence of $\mathcal{L}_{\infty,\omega}$. If $\sigma(\mathbf{x})$ is some statement about $\mathbf{x}$ and $\varphi(\mathbf{x})$ is some statement then we say $\underline{\sigma(\mathbf{x}) \text{ forces } \varphi(\mathbf{x})}$ ($\sigma(\mathbf{x}) \Vdash_T \varphi(\mathbf{x})$) if $(\forall M \models T)(\forall \mathbf{x} \in M)(\sigma(\mathbf{x}) \to \varphi(\mathbf{x}))$

Similarly, if $\sigma(\mathbf{x}), \tau(\mathbf{y})$ are some statements and $\varphi(\mathbf{x}, \mathbf{y})$ is some statement, then we say $\underline{\sigma(\mathbf{x}), \tau(\mathbf{y}) \text{ forces } \varphi(\mathbf{x}, \mathbf{y})}$ ($\sigma(\mathbf{x}), \tau(\mathbf{y}) \Vdash_T \varphi(\mathbf{x}, \mathbf{y})$) if $(\forall M, N \models T)(\forall \mathbf{x} \in M, \mathbf{y} \in N)([\sigma(\mathbf{x}) \wedge \tau(\mathbf{y})] \to \varphi(\mathbf{x}, \mathbf{y}))$

The idea behind this forcing notation is we want to be able to talk about properties of the models of our theory which aren't necessarily expressible in $\mathcal{L}_{\infty,\omega}$. One such example could be a statement like "$(P, \leq)$ is a well ordering $\Vdash$ $(P, \leq)$ has limit order type". These are ideas which we might know for some external reason are true but which can't necessarily be expressed in $\mathcal{L}_{\infty,\omega}$.

It is also worth pointing out explicitly that if we have something of the form $\sigma(\mathbf{x}), \tau(\mathbf{y}) \Vdash \varphi(\mathbf{x}, \mathbf{y})$ then it is possible that $\mathbf{x}$ and $\mathbf{y}$ are in different models. For example consider the case

> "$\langle P(x, -), \leq\rangle$ is a linear order, $\langle P(y, -), \leq\rangle$ is a linear order $\Vdash \langle P(x, -), \leq\rangle$ is an initial segment of $\langle P(y, -), \leq\rangle$ or vice versa".

Now the point is that what this says is that for all x and y $\langle P(x, -), \leq\rangle$ is an initial segment of $\langle P(y, -), \leq\rangle$ or vice versa, even if x and y are in different models, which is much stronger than just saying

> "$(\forall M) M \models (\forall x, y)\langle P(x, -), \leq\rangle$ is a linear order, $\langle P(y, -) \leq\rangle$ is a linear order $\rightarrow \langle P(x, -), \leq\rangle$ is an initial segment of $\langle P(y, -), \leq\rangle$ or vice versa".

Chapter 2

Component Trees

2.1 Introduction

In this chapter we begin to define the languages and theories we will use as building blocks for the rest of Part A. We will introduce our method of representing trees as well as our method for comparing the heights of trees.

2.2 Basic Trees

2.2.1 Introduction

In this section we will finally begin to define the tree structure on our models. As we will see, much of our later arguments will assume that the trees we are working with are well founded. This poses a problem though as well-foundedness can't be isolated in $\mathcal{L}_{\infty,\omega}$as can be seen from the following well know theorem:

Definition 2.2.1.1. Let L be a language with a binary relational symbol $<$ and possibly other relations. A sentence ϕ pins down an ordinal α if

- $N \models \phi$ implies $N \models <$ is a well ordering
- ϕ has a model N such that $N \models$order type$(<) = \alpha$.

Theorem 2.2.1.2. *Let L be a language with a binary relation $<$. Let $\phi \in \mathcal{L}_{\infty,\omega}(L)$ be a formula which pins down an ordinal. Then there is an ordinal α such that every ordinal pinned down by ϕ is less than α*

Proof. This is an immediate consequence of [2] Chapter 3 Theorem 7.3 □

We therefore know that no matter what we do, non well-founded models trees will be lurking in the background. As a first attempt to contain them we will want to make sure that if we have an ill-founded model, any ill-founded branch must have order type ω, the least possible for an ill-founded tree. There are several ways in which we can do this, but the method we will choose is to build the tree on the finite subsets of the model itself. In other words, we will specify whether or not a finite set $\mathbf{x}$ is in the tree. Any extension of $\mathbf{x}$ in the tree will then be of the form $\mathbf{x}^\wedge a$ for some a in our model. This way we can never have a branch which has length greater than ω as our relations only deal with finite tuples.

As an added bonus, we will see that defining our trees in this way will allow us (in the case our trees are well founded) to compare in a definable way the height of nodes of our tree.

2.2.2 Language of Trees

2.2.2.1 P

In order to define a tree structure on the finite tuples we will need a predicate of each arity which says that the tuple is in the tree. We will then also require that if a tuple is in the tree so are all subtuples. Finally we will also want "being in the tree" to be a property of a finite set and not a specific tuple (i.e an ordered finite set).

Definition 2.2.2.1. Let $L_P = \{P_n : P_n \text{ is an n-ary predicate}\}$.

Definition 2.2.2.2. Let T_P be universal closure of the following L_P sentences:

- For all $i_1, \ldots i_n \in n$ $P^n(x_1, \ldots, x_n) \rightarrow P^n(x_{i_1}, \ldots, x_{i_n})$
- $P^{n+1}(x_0, \ldots, x_n) \rightarrow P^n(x_1, \ldots, x_n)$

We can consider (by abuse of notation) all the predicates P^n to really be one $< \omega$-ary predicate (i.e. a predicate on the finite tuples of the model). Under this abuse of notation, the predicate P satisfies the axioms of being a tree under the partial order $\mathbf{x} \subseteq \mathbf{y}$. (In other words under this partial order the predicate P is upwards closed).

2.2.2.2 Color

Now that we have defined our tree, we can define informally the height of a tree. The name we give for the height of the tree extending a tuple is it's "Color". This is inspired by the definition of color given in [8].

Definition 2.2.2.3. Define the color of $\overline{a}$ ($\|\overline{a}\|$) as follows:

- $\neg P(\overline{a}) \leftrightarrow \|\overline{a}\| = -\infty$
- $P(\overline{a}) \leftrightarrow \|\overline{a}\| \geq 0$
- For all b $\|\overline{a}\| \geq \|\overline{a}b\| + 1$
- $\|\overline{a}\| = \sup\{\alpha : \|\overline{a}\| \geq \alpha\}$ if it exists.
- $\|\overline{a}\| = \infty$ otherwise.

Under this definition if a tuple is not in the tree defined by P it has color $-\infty$. But, if a tuple is in the tree, its color is the height of the tree of tuples extending it, by the usual definition of height on a tree. (here ill founded branches are defined to have height ∞).

Lemma 2.2.2.4. *If* $M \models T_P, \overline{a} \in M, P(\overline{a})$ *then* $\|\overline{a}\| = sup\{\|\overline{a}b\|+1 : b \in M\}$ *(where we consider* $-\infty + 1 = 0$*)*

Proof. We know that $\|\overline{a}\| \geq \sup\{\|\overline{a}b\| + 1 : b \in M\}$. But, for any α the only condition that will allow $\|\overline{a}\| > \alpha$ is that there is a b such that $\|\overline{a}b\| + 1 > \alpha$. So in fact $\|\overline{a}\| \leq \sup\{\|\overline{a}b\| + 1 : b \in M\}$ and hence $\|\overline{a}\| = \sup\{\|\overline{a}b\| + 1 : b \in M\}$. □

It is worth mentioning explicitly that $\| \cdot \|$ is not a predicate in the 1st order language and by Theorem 2.2.1.2 $\| \cdot \|$ is not even definable in $\mathcal{L}_{\infty,\omega}$. $\| \cdot \|$ is just a shorthand we will use to discuss colors in a model.

Definition 2.2.2.5. Let $M \models T_P$. Then Spectrum of M

$$\text{Spec}(M) = \{(\alpha : \exists \overline{a} \in M)\|\overline{a}\| = \alpha\}$$

.

Thus the spectrum of a model is just the collection of colors realized in it. We will see later that in the models of T_P we are interested in, the spectrum essentially determines the model.

Theorem 2.2.2.6. *If* $M \models T_P$*,* $\alpha < \beta$ *and* $\beta \in Spec(M) \cap ORD$ *then* $\alpha \in Spec(M)$

Proof. Assume not and let $\alpha \in \mathrm{Spec}(M)$ be the least such that there is a $\beta < \alpha$ for which the theorem fails. Let $\mathbf{x} \in M, \|\mathbf{x}\| = \beta$. So $\beta = \sup\{\|\mathbf{x}a\| + 1 : a \in M\}$. Therefore, there must exists an $a \in M$ such that $\beta > \|\mathbf{x}a\| > \alpha$ (because the colors of the extensions $\mathbf{x}$ are are cofinal in $\|\mathbf{x}\|$.) But, then we know there is a $c \in M$ such that $\|\mathbf{x}ac\| = \alpha$ because β was assumed to be the least such that the theorem failed. $\Rightarrow\Leftarrow$. □

So in any model M of T_P $\mathrm{Spec}(M) \cap \mathrm{ORD}$ is in fact an ordinal.

Lemma 2.2.2.7. *There exists* $\varphi^{\geq}_{\alpha}(\boldsymbol{x}) \leftrightarrow \|\boldsymbol{x}\| \geq \alpha, qr(\varphi^{\geq}_{\alpha}) = \alpha$*.*

Proof. Let $\varphi^{\geq}_0(\mathbf{x}) = P(\mathbf{x})$ then

$$\varphi^{\geq}_0(\mathbf{x}) \leftrightarrow \|\mathbf{x}\| \geq 0, \mathrm{qr}(\varphi^{\geq}_0) = 0)$$

and $\mathrm{qr}(\varphi^{\geq}_0(\mathbf{x})) = 0$.
Let $\varphi^{\geq}_{\gamma}(\mathbf{x}) = \bigwedge_{\alpha<\gamma}(\exists x)(\varphi^{\geq}_{\alpha}(\mathbf{x}, x))$ then

$$\begin{aligned} \varphi^{\geq}_{\gamma}(\mathbf{x}) &\leftrightarrow \bigwedge_{\alpha<\gamma} \|\mathbf{x}x\| \geq \alpha \\ &\leftrightarrow \bigwedge_{\alpha<\gamma} \|\mathbf{x}\| \geq \alpha + 1 \\ &\leftrightarrow \|\mathbf{x}\| \geq \alpha \end{aligned}$$

and $\mathrm{qr}(\varphi^{\geq}_{\gamma}) = \sup\{\mathrm{qr}(\varphi^{\geq}_{\alpha}) + 1 : \alpha < \gamma\} = \gamma$. □

Corollary 2.2.2.8. *There exists* $\varphi^{=}_{\alpha}(\boldsymbol{x}) \leftrightarrow \|\boldsymbol{x}\| = \alpha, qr(\varphi^{=}_{\alpha}) = \alpha + 1, \varphi^{=}_{\alpha} \in \mathcal{L}_{|\alpha|,\omega}$.

Proof. $\varphi^{=}_{\alpha}(\mathbf{x}) = \varphi^{\geq}_{\alpha}(\mathbf{x}) \wedge \neg\varphi_{\alpha+1}(\mathbf{x})$ □

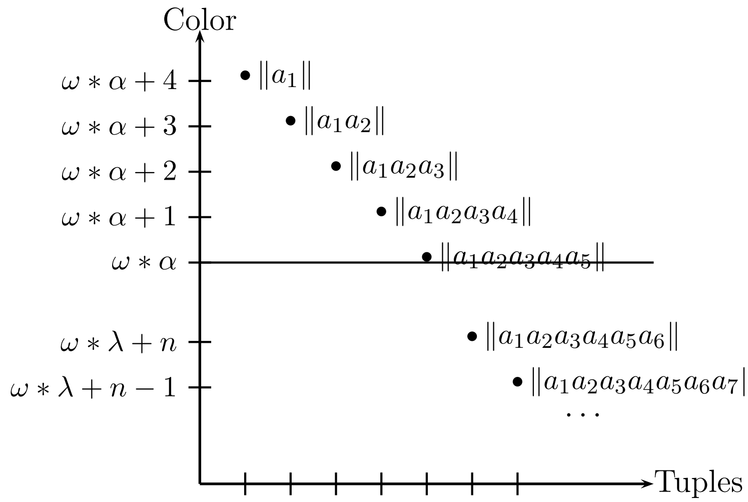
2.2.2.3 Slant Lines

Now that we understand the definition of color we can begin to understand why slant lines are important. First though we will need a definition.

Definition 2.2.2.9. Let $M \models T_P$, $\overline{a} \in M$, f is a slant line such that $\mathrm{dom} f \geq |\overline{a}|$. Then we say $\overline{a}$ is on the slant line f if $f(|\overline{a}|) = \|\overline{a}\|$.

The purpose of slant lines is to allow us (in some sense) to compare the information contained in tuples of different lengths. To see this notice that in a well-founded model, as we extend a tuple, the color must decrease. This is because in any well-founded tree the height of any node is always less than the height of it's predecessor.

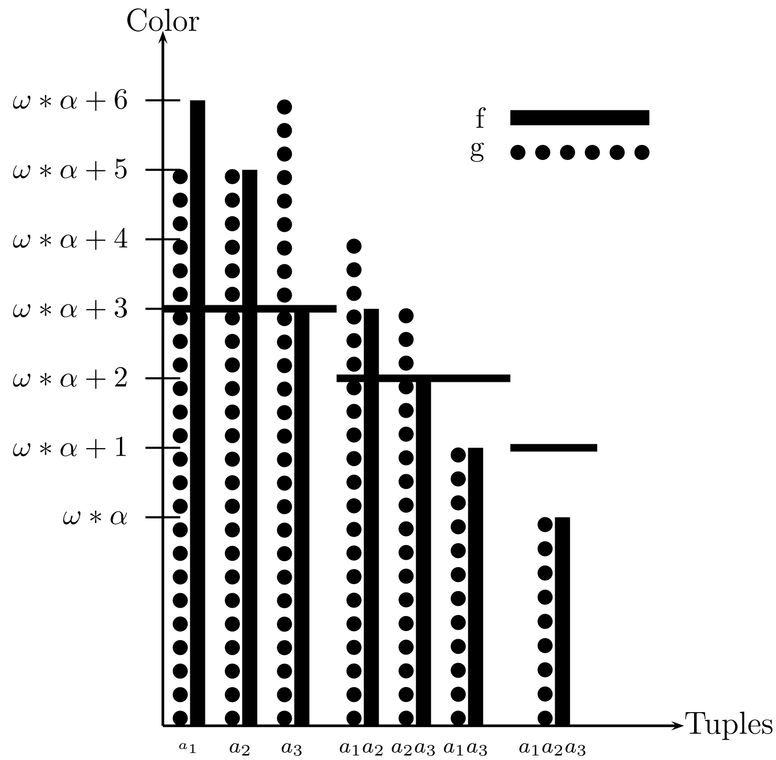
Intuitively when we think about slant lines we want to be thinking about slow slant lines. This is because if we have a slow slant line f and we have two tuples $\overline{a}, \overline{b}$ of arity n, m such that $\|\overline{a}\| = f(n)$, $\|\overline{b}\| = f(m)$ and $f(0) = \gamma + \max\{m, n\}$ then $\overline{a}$ and $\overline{b}$ differ by the minimum amount possible. So, if there was a way to minimally extend them to be of the same arity then they would have the same color. Hence when looking at the color of a tuple, it isn't the actual color which matters as much as the slant lines the tuple/color is on.



Now that we know the right information to consider is the slant line of a tuple and not its color, we will be interested in comparing tuples when the color functions are the same up to a slant line (see Definition 6.1.1.7). By only comparing tuples up to slant lines we know that, assuming enough homogeneity, we will be able to extend the tuples we are comparing and not gain any new information about them up to the slant line from the extension. This construction will be very useful for building partial isomorphisms. To see why we can find extensions which don't add information up to slant lines, but why we can't in general find extensions that preserve information up to an ordinal, lets look at an example.

Lets assume we are in a model such that if a tuple has a color then it has extensions by single elements which have all lesser colors (this is a very mild homogeneity requirement which we will almost always have). Now consider a tuple of color ω and one of color $\omega + 1$. Intuitively we would think that we shouldn't be able to distinguish between these tuples up to ω, and in fact we can't. But now lets look at the extensions. We know that the tuple of color $\omega + 1$ has an extension which is different up to ω from all the extensions of the tuple of color ω. Hence by looking at the extensions we can distinguish between the two tuples. This is bad if we want to say when two sequences contain the same information (as will be necessary for back and forth arguments).

Now lets look at the case where we only preserve information up to a slant line. For each slant line, and each extension of the tuple of color $\omega + 1$, there is an extension of the tuple of color ω which looks the same up to the slant line. This is what we want.



Example of two functions the same up to a slant line.

2.2.2.4 Color Archetypes

Definition 2.2.2.10. Define the Color Archetype of a tuple $\mathbf{x} = x_0 \ldots x_{n-1}$

$$(\text{ctype}(\mathbf{x})) = \{\langle (i_0, \ldots, i_m), \|x_{i_0} \ldots x_{i_m}\| \rangle : (i_0, \ldots, i_m) \subseteq n\}$$

The idea behind the color archetype is that it contains all the information about the color of a tuple and its subtuples. . We will see that in the models we are interested in this in fact is all the information we have about a tuple in L_P.

Before we continue lets consider what an extension of a color archetype C on $\mathbf{x}$ might look like (and which ones are possible). Say we are looking at an extension $\mathbf{x}b$ and we want to know what possible color archetypes it might have. Well we know that any color restricted to a subtuple of $\mathbf{x}$ must be the same as the color witnessed by C. The only new information is the color of the new subtuples. But, because we are only extending the tuple by one element, we know that the only new subtuples are of the form $\mathbf{y}b$ where $\mathbf{y} \subseteq \mathbf{x}$. And, because of the nature of the definition of color, the only requirement limiting the value $\|\mathbf{y}b\|$ is that we must have $\|\mathbf{y}b\| < \|\mathbf{y}\|$, and $\|\mathbf{z}b\| < \|\mathbf{y}b\|$ if $z \subseteq b$ (Here we consider $\infty < \infty$.)

It is also worth mentioning explicitly that while order of the tuple doesn't matter with regards to the color predicate, it does matter with regards to the color archetype.

Definition 2.2.2.11. We say that two tuples $\mathbf{x} = (x_0 \ldots x_{n-1}), \mathbf{y} = (y_0 \ldots y_{n-1})$ have the same color archetype up to a slant line f $(\text{ctype}(\mathbf{x})|f = \text{ctype}(\mathbf{y})|f)$ if the they are the same when the color archetypes are considered as functions from $\mathbf{x}$, $\mathbf{y}$ to ORD. We say that two tuples $\mathbf{x} = (x_0 \ldots x_{n-1}), \mathbf{y} = (y_0 \ldots y_{n-1})$

have the same color archetype up to a limit ordinal γ if for all slant lines $(f < \gamma)$ $\text{ctype}(\mathbf{x})|f = \text{ctype}(\mathbf{y})|f$

Lemma 2.2.2.12. *Let $\boldsymbol{x}, \boldsymbol{y}$ be tuples. Then $\boldsymbol{x}, \boldsymbol{y}$ have the same color archetype up to γ if and only if $ctype(\boldsymbol{x})|c_\gamma = ctype(\boldsymbol{y})|c_\gamma$*

(where here we consider the color archetypes as functions and c_γ as the constant function with value γ (and domain ω)).

Proof. The implication from right to left is immediate from the definitions. The implication left to right follows from the fact that if we have a finite collection of ordinals less than a limit ordinal (like the colors of the subtuples of $\mathbf{x}, \mathbf{y}$) then there is an ordinal greater than all of them. Hence with respect to $\mathbf{x}, \mathbf{y}$ if we choose a slant line with values large enough below γ it is just as good as choosing γ. □

2.3 Comparing Color

Now that we have defined the tree structure we are putting on our model, as well as the color of the tuples in the tree structure, we will want to be able compare the colors of different tuples in the same model.

Of course by considering all formulas of $\mathcal{L}_{\infty,\omega}$ it is easy to tell what the difference between two colors is (as we are able to say with a single formula if a tuple has exactly color α for any particular α). But as we are concerned about the quantifier ranks of the models, we would like a way to compare colors that is definable in a 1st order manner. However, being able to compare the colors of tuples in a 1st order definable way isn't enough either. We will want our definition to be unique as well. In other words we want our method of comparing colors to be the true one and to completely define the relationships between the colors of two tuples.

We will need our method of comparison to be unique because we are also worried about how many models and types our theories will have. If there are several different ways to extend our model of T_P so that we have a model of our new theory (which allows us to compare colors), we very quickly could find ourselves with to many models and types.

As it turns out though, this is impossible. It is impossible because if we could really come up with an extension that satisfied all those properties then we could define what it means to be well-founded (i.e. all extensions have color strictly less than you). And, as we mentioned earlier (Theorem 2.2.1.2) it is a well known fact that $\mathcal{L}_{\infty,\omega}$ can't define what it means to be wellfounded. Fortunately though this is the only stumbling block. In the models we are looking at (i.e. the trees which are very homogeneous in a strong sense that will be described later), our theory T_R does correctly and completely define the relationships between colors, so long as our tree is well founded, i.e. there are no tuples of color ∞ in the model.

While in homogeneous trees the color of a tuple does completely determine the structure of the tree extending it, in the non-homogeneous case there is more going on. As such, there will be times when we want a predicate which doesn't just talk about the color of a tuple, but also talks about the tree extending the tuple. This is where the theory T_S comes in. We will find that this theory (in the case there are no tuples of color ∞) correctly and completely says when two tuples of the same arity have isomorphic trees extending them.

2.3.1 $R_{\leq}$

2.3.1.1 Definitions

We will compare the color of nodes by mimicking one half of a back and forth construction. We will do this by saying that two tuples "look the same" if for every extension of one we can find an extension of the other such that the extensions "look the same".

Definition 2.3.1.1. Let $L_{R_{\leq}} = L_R = L_P \cup \{R_{\leq}^{i,j} : R_{\leq}^{i,j} \text{ is } i+j \text{ ary}, i, j \in \omega\}$.

For notational convenience we will treat $R_{\leq}^{i,j}$ as a predicate of two arguments (one i-ary, one j-ary). Further abusing notation (in a similar way as we did with P) we will consider $R_{\leq}$ as a two argument predicate on finite tuples.

We also define in L_R for notational convenance

$$R_{=}(\overline{a}, \overline{b}) \leftrightarrow R_{\leq}(\overline{a}, \overline{b}) \wedge R_{\leq}(\overline{b}, \overline{a})$$

and

$$R_{<}(\overline{a}, \overline{b}) \leftrightarrow R_{\leq}(\overline{a}, \overline{b}) \wedge \neg R_{=}(\overline{a}, \overline{b})$$

Now before we continue we want to point out the intended interpretation of $R_{\leq}(\overline{a}, \overline{b})$ is $\|\overline{a}\| \leq \|\overline{b}\|$.

Definition 2.3.1.2. Let T_R be universal closure of the following L_R sentences:

- T_P
- $R_{\leq}(\mathbf{x}, \mathbf{y}) \leftrightarrow [[\neg P(\mathbf{x})] \vee [P(\mathbf{x}) \rightarrow P(\mathbf{y}) \wedge (\forall a)(\exists b) R_{\leq}(\mathbf{x}a, \mathbf{y}b)]]$

The idea behind the predicate $R_{\leq}$ is that if we know for every extension of the first argument there is an extension of the second argument with color at least as great as the first, then we know that the second argument must have color at least as great (by the definition of color).

2.3.1.2 Correctness

In this section we will show that if $R_{\leq}$ holds then $R_{\leq}$ accurately describes the relationship between the colors of its arguments. Further we will show that if our model has no tuple of color ∞ then we have $R_{\leq}(\bar{a}, \bar{b})$ iff $\|\bar{a}\| \leq \|\bar{b}\|$

Theorem 2.3.1.3. *If* $M \models T_R$ *then* $M \models (\forall \bar{a}, \bar{b}) R_{\leq}(\bar{a}, \bar{b}) \rightarrow \|\bar{a}\| \leq \|\bar{b}\|$

Proof. Assume $\|\bar{a}\| = -\infty$

$(\forall \bar{b}) R_{\leq}(\bar{a}, \bar{b})$ by the definition of $R_{\leq}$, and $(\forall \|\bar{a}\| \leq \|\bar{b}\|$.

Assume [if $\|\mathbf{x}\| < \alpha < \infty$ then $R_{\leq}(\mathbf{x}, \mathbf{y}) \Rightarrow \|\mathbf{x}\| \leq \|\mathbf{y}\|$] and let $\|\bar{a}\| = \alpha$

Then $R_{\leq}(\bar{a}, \bar{b}) \rightarrow (\forall a')(\exists b') R_{\leq}(\bar{a}a', \bar{b}b')$ and hence $[(\forall a')(\exists b')\|\bar{a}a'\| \leq \|\bar{b}b'\|]$ by the induction hypothesis.

Therefore $R_{\leq}(\bar{a}, \bar{b}) \rightarrow [\|\bar{a}\| = \sup\{\|\bar{a}a'\| + 1 : a' \in M\} \leq \sup\{\|\bar{b}b'\| + 1 : b' \in M\} = \|\bar{b}\|]$. So $R_{\leq}(\bar{a}, \bar{b}) \rightarrow \|a\| \leq \|b\|$.

By induction our theorem holds for any $\bar{a}$ such that $\|\bar{a}\| < \infty$.

Assume $\|\bar{a}\| = \infty$.

Let $\overline{a}, a_0, a_1, \ldots$ be an infinite sequence such that $P(\overline{a}, a_0, \ldots, a_n)$ for all n (this exists by the definition of color ∞). Therefore there must exist a sequence $\overline{b}, b_0, b_1, \ldots$ such that $R_{\leq}(\overline{a}a_0 \ldots a_n, \overline{b}b_0 \ldots b_n)$ for all n. But then we have (by the definition of $R_{\leq}$), $P(\overline{b}, b_0, \ldots, b_n)$ for all n. Hence, $\|b\| = \infty$ by the definition of color ∞.

□

Theorem 2.3.1.4. *If $M \models T_R$, $\overline{a}, \overline{b} \in M$ and $\|\overline{a}\| \leq \|\overline{b}\| < \infty$ then $M \models R_{\leq}(\overline{a}, \overline{b})$.*

Proof. First notice that if $\|\overline{a}\| = -\infty$ then this is trivially true.
Now assume for all $\mathbf{x}, \mathbf{y} \in M \models T_R$ if $\|\mathbf{x}\| < \alpha < \infty$ and $\|\mathbf{x}\| \leq \|y\| < \infty$ then $M \models R_{\leq}(\mathbf{x}, \mathbf{y})$. Then let $\|\overline{a}\| = \alpha$.

First off we know that $P(\overline{a}) \rightarrow P(\overline{b})$ by the definition of color. We also know by the definition of $(\forall a')(\exists b')$ such that $\|\overline{a}a'\| \leq \|\overline{b}b'\|$. Further, by the inductive hypothesis, we then have $R_{\leq}(\overline{a}a', \overline{b}b')$. But then by the definition of $R_{\leq}$ we then have $R_{\leq}(\overline{a}, \overline{b})$ and we are done.

□

Theorem 2.3.1.5. *Let $M \models T_P$ and has no tuples of color ∞. Then there is a unique extension of M to a model of L_R.*

Proof. This is immediate from Theorem 2.3.1.4 and Theorem 2.3.1.3. □

Theorem 2.3.1.6. *For each n there is a n+n ary formula $E_a^n(\overline{a}, \overline{b}) \in \mathcal{L}_{\infty,\omega}(L_{R_{\leq}})$ such that $T_R \models E_a^n(\overline{a}, \overline{b}) \rightarrow ctype(\overline{a}) = ctype(\overline{b})$.*

Proof. Let

$$E_a^n(x_1, \ldots, x_n, y_1, \ldots, y_n) \leftrightarrow \bigwedge_{S \subseteq n} R_{=}(\{x_i : i \in S\}, \{y_i : i \in S\})$$

Then the theorem follows by Theorem 2.3.1.3 and Theorem 2.3.1.4. □

Corollary 2.3.1.7. *If* $M \models R_=(\bar{a}, \bar{b}) \leftrightarrow \|\bar{a}\| = \|\bar{b}\|$ *then* $E^n_a(\bar{a}, \bar{b}) \leftrightarrow ctype(\bar{a}) = ctype(\bar{b})$

Proof. By the definition of archetypes. □

When the context is clear, we will leave of the superscript of E^n_a.

2.3.2 $S_=$

In this chapter we will introduce a theory which will allow us to tell when the trees extending two tuples of the same arity are isomorphic. We will do this by mimicking a back and forth argument in a similar manner to what we did in Section 2.3.1. However, this time instead of looking at only one side of the back and forth argument at a time, we will look at both sides simultaneously.

2.3.2.1 Definitions

Definition 2.3.2.1. Let $L_{S_=} = L_P \cup \{S^i_= : S^i_= \text{ is } 2i \text{ ary}, i \in \omega\}$.

For notational convenience we will treat $S^i_=$ as a predicate of two arguments (each i-ary).

Definition 2.3.2.2. Let T_S be universal closure of the following $L_{S_=}$ sentences:

- T_P

- $S_=(\mathbf{x},\mathbf{y}) \leftrightarrow [[\neg P(\mathbf{x}) \wedge \neg P(\mathbf{y})] \vee [P(\mathbf{x}) \wedge P(\mathbf{y}) \wedge (\forall a)(\exists b)S_=(\mathbf{x}a,\mathbf{y}b) \wedge (\forall b)(\exists a)S_=(\mathbf{x}a,\mathbf{y}b)]]$

There is one subtle point in this definition that is worth stressing. This point is that unlike with $R_\leq$ we can't compare the color tuples of different sizes. This is because if we could compare colors of tuples of different sizes then we could find a formula which would allow us to say when one tuple had greater color than another. Specifically $\varphi_{R_=}(\mathbf{x},\mathbf{y}) := (\exists z)R_=(\mathbf{x},\mathbf{y}z)$ would imply $\|\mathbf{y}\| > \|\mathbf{x}\|$ and hence we would have the same information (in the case of homogeneous trees) as if we actually had defined $R_\leq$.

2.3.2.2 Correctness

In this section we will show that if $S_=$ hold between two tuples then the tuples do have the same color.

Theorem 2.3.2.3. *If $M \models T_S$ then $M \models (\forall \bar{a},\bar{b})S_=(\bar{a},\bar{b}) \rightarrow \|\bar{a}\| = \|\bar{b}\|$*

Proof. Assume $\|\bar{a}\| = -\infty$

$S_=(\bar{a},\bar{b}) \Rightarrow M \models \neg P(\bar{b}) \Rightarrow \|\bar{b}\| = -\infty$

Assume if $\|\mathbf{x}\| < \alpha < \infty$ then $S_=(\mathbf{x},\mathbf{y}) \Rightarrow \|\mathbf{x}\| = \|\mathbf{y}\|$ and let $\|\bar{a}\| = \alpha$

Then $S_=(\bar{a},\bar{b}) \rightarrow [(\forall a')(\exists b')\|\bar{a}a'\| = \|\bar{b}b'\|]$ by the induction hypothesis.

Therefore $S_=(\bar{a},\bar{b}) \rightarrow [\|\bar{a}\| = \sup\{\|\bar{a}a'\| + 1 : a' \in M\} \leq \sup\{\|\bar{b}b'\| + 1 : b' \in M\} = \|\bar{b}\|]$

And similarly we have $S_=(\bar{a},\bar{b}) \rightarrow [(\forall b')(\exists a')\|\bar{a}a'\| = \|\bar{b}b'\|]$ and hence $[\|\bar{b}\| = \sup\{\|\bar{b}b'\| + 1 : b' \in M\} \leq \sup\{\|\bar{a}a'\| + 1 : a' \in M\} = \|\bar{a}\|]$.

So $S_=(\overline{a}, \overline{b}) \to \|\overline{a}\| = \|\overline{b}\|$.

And by induction works for any $\overline{a}$ such that $\|\overline{a}\| < \infty$.
Assume $\|\overline{a}\| = \infty$.

Then let $\overline{a}, a_0, a_1, \ldots$ be an infinite sequence such that $P(\overline{a}, a_0, \ldots, a_n)$ for all n (this exists by the definition of color ∞). Therefore there must exist a sequence $\overline{b}, b_0, b_1, \ldots$ such that $S_=(\overline{a}a_0 \ldots a_n, \overline{b}b_0 \ldots b_n)$ for all n. But then we have (by the definition of $S_=$), $P(\overline{b}, b_0, \ldots, b_n)$ for all n. So in particular $\|b\| = \infty$ by the definition of color ∞.

□

Theorem 2.3.2.4. *Let $M \models T_P$ and have no tuples of color ∞. Then there is a unique extension of M to a model of T_S.*

Proof. We will prove this by induction on the color of the tuples of M.
Base Case: $\|\overline{a}\| = -\infty$
In this case we have $S_=(\overline{a}, \overline{b})$ iff $\|\overline{b}\| = -\infty$.
Inductive Case: Assume that for all tuples $\mathbf{x}$ of color $< \alpha$ there is only one consistent way to define $S_=(\mathbf{x}, \mathbf{y})$. Now let $\|\overline{a}\| = \alpha$.

Assume we have two extensions of M to T_S, M_0, M_1 and suppose $M_0 \models S_=(\overline{a}, \overline{b})$. We therefore know that $M_0 \models (\forall c)(\exists d) S_=(\overline{a}c, \overline{b}d) \wedge (\forall d)(\exists c) S_=(\overline{a}c, \overline{b}d)$. But, $\|\overline{a}c\| < \|\overline{a}\| = \alpha$ and so we know that we also must have $M_1 \models (\forall c)(\exists d) S_=(\overline{a}c, \overline{b}, d) \wedge (\forall d)(\exists c) S_=(\overline{a}c, \overline{b}d)$ (by the inductive assumption). So we also have $M_1 \models S_=(\overline{a}, \overline{b})$. Hence, there is a unique way to extend M to T_S on tuples of color $\leq \alpha$.

So in particular we know by induction that there is a unique way to extend M to a model of $L_{S_=}$. □

This is a very important theorem when dealing with $L_{S_=}$ because it says that in the case where there are no tuples of color ∞ (the case we care about) $T_{S_=}$ is a conservative extension of T_P.

Theorem 2.3.2.5. *For each n there is a n+n ary formula $E^n_a(\overline{a}, \overline{b}) \in \mathcal{L}_{\infty,\omega}(L_{S_=})$ such that $T_S \models E^n_a(\overline{a}, \overline{b}) \to ctype(\overline{a}) = ctype(\overline{b})$.*

Proof. Let

$$E^n_a(x_1, \ldots, x_n, y_1, \ldots, y_n) \leftrightarrow \bigwedge_{S \subseteq n} S_=(\{x_i : i \in S\}, \{y_i : i \in S\})$$

By Theorem 2.3.2.3 □

Corollary 2.3.2.6. *If $M \models S_=(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\| = \|\overline{b}\|$ then $E_a(\overline{a}, \overline{b}) \leftrightarrow ctype(\overline{a}) = ctype(\overline{b})$*

Proof. By the definition of color archetypes. □

The purpose of Definition 2.3.2.5 is that we will want later to prove theorems which only rely on whether or not two tuples look the same with respect to $S_=$, or $R_=$. Hence we will use this definition which will allow us to prove the theorems for both T_R and T_S simultaneously.

Chapter 3

Collections of Archetypes

3.1 Definitions

The language we will be using for this section is an extension of L_P.

Definition 3.1.0.7. Let $L_K = L_P \bigcup \{K_i : i \in \kappa\}$, arity of $K_i = n_i$.

3.2 Weak Collection of Archetypes

We are now ready to define the extra structure we want. This structure is going to come in four parts. The most important part will be the archetypes. The idea is that we want an archetype to tell us everything we need to know about a tuple. One way to think about an archetype is as a generalized $\mathcal{L}_{\infty,\omega}$ infinity type (i.e. a formula of $\mathcal{L}_{\infty,\omega}$ which completely determines what other formulas of $\mathcal{L}_{\infty,\omega}$ hold). The difference though between an archetype and an $\mathcal{L}_{\infty,\omega}$ type is that we don't require an archetype to actually be a formula. The only thing we require of an archetype is that it is an "abstract property" of tuples.

In addition to the archetypes we are going to want a collection of "consistent pairs of consistent archetypes". Eventually we are going to want to "glue" together two copies of our theory with this extra structure and so the consistent pairs of archetypes are going to tell us how we can "glue" tuples together.

Before we describe the other elements of the extra structure we are going to define a "Weak Collection of Archetypes". The only purpose of this definition is to break up the definition of "Collection of Archetypes" into two parts as it is very long.

Definition 3.2.0.8. Let T be a sentence of $\mathcal{L}_{\infty,\omega}(L_K)$ such that $\models T \rightarrow T_P$. Let $\mathrm{AT}(T)$ be a collection of "abstract properties" on elements of models of T. Further let $\langle 2-\mathrm{AT}(T), \leq \rangle$ be a collection of pairs of elements from $\mathrm{AT}(T)$ with $\leq$ a partial order. We say that $\mathrm{AT}(T)$ is a Weak Collection of Archetypes (with $\langle 2-\mathrm{AT}(T), \leq \rangle$ the Consistent Pairs of Archetypes) if we have

(Truth on Atomic Formulas)

If $\phi \in \mathrm{AT}(T)$ then $\phi(x_1, \ldots, x_n), \phi(y_1, \ldots, y_n) \Vdash \bigwedge_{S \subseteq n}(\forall$atomic formula $\theta)[\theta(\langle x_i : i \in S\rangle) \leftrightarrow \theta(\langle y_i : i \in S\rangle)]$

(Truth on Color)

If $\phi \in \mathrm{AT}(T)$ then $\phi(\mathbf{x}), \phi(\mathbf{y}) \Vdash \mathrm{ctype}(\mathbf{x}) = \mathrm{ctype}(\mathbf{y})$

(Restriction of Arity for Archetypes)

For each $\phi(x_1, \ldots, x_n) \in \mathrm{AT}(T)$ and each $S \subseteq n$ there is a $\phi_S(\langle x_i : i \in S\rangle) \in \mathrm{AT}(T)$ such that $\phi(x_1, \ldots, x_n) \Vdash \phi_S(\langle x_i : i \in S\rangle)$. We say $\phi_S(\langle x_i : i \in S\rangle) =$

$\phi|\langle x_i : i \in S\rangle$

(Restriction of Arity for Consistent Pairs of Archetypes)

If $(\tau_0, \tau_1), (\sigma_0, \sigma_1) \in 2 - \mathrm{AT}(T)$ and $(\tau_0, \tau_1)(\mathbf{x}, y_1, \ldots, y_n) \leq (\sigma_0, \sigma_1)(\mathbf{x})$ then

- $(\tau_0, \tau_1)|\langle \mathbf{x}, \{y_i : i \in S\}\rangle \in 2 - \mathrm{AT}(T)$
- $(\tau_0, \tau_1)|\langle \mathbf{x}, \{y_i : i \in S\}\rangle \leq (\sigma, \sigma')(\mathbf{x})$

for each $S \subseteq n$

(Completeness for Archetypes)

$\phi(\mathbf{x}) \Vdash (\forall \mathbf{y}) \bigvee_{\psi(\mathbf{x},\mathbf{y}) \Vdash \phi(\mathbf{x})} \psi(\mathbf{x}, \mathbf{y})$

(Amalgamation for Archetypes)

For each $\phi, \psi, \zeta \in \mathrm{AT}(T)$ if $\phi(\mathbf{x}, \mathbf{y}) \Vdash \zeta(\mathbf{y})$ and $\psi(\mathbf{y}, \mathbf{z}) \Vdash \zeta(\mathbf{y})$ then there is a $\eta \in \mathrm{AT}(T)$ such that

$\eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Vdash (\phi(\mathbf{x}, \mathbf{y}) \wedge \psi(\mathbf{y}, \mathbf{z})) \wedge (\emptyset \neq \mathbf{x}' \subseteq \mathbf{x}, \emptyset \neq \mathbf{z}' \subseteq \mathbf{z}, \mathbf{y}' \subseteq \mathbf{y}) \rightarrow \neg P(\mathbf{x}', \mathbf{y}', \mathbf{z}')$

(Amalgamation for Consistent Pairs of Archetypes)

For each $(\phi_0, \phi_1), (\psi_0, \psi_1), (\zeta_0, \zeta_1) \in 2 - \mathrm{AT}(T)$ if

- $(\phi_0, \phi_1)(\mathbf{x}, \mathbf{y}) \leq (\zeta_0, \zeta_1)(\mathbf{y})$
- $(\psi_0, \psi_1)(\mathbf{y}, \mathbf{z}) \leq (\zeta_0, \zeta_1)(\mathbf{y})$

then there is a $(\eta_0, \eta_1) \in 2 - \mathrm{AT}(T)$ such that

- $(\eta_0, \eta_1)(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq (\phi_0, \phi_1)(\mathbf{x}, \mathbf{y})$

- $(\eta_0, \eta_1)(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq (\psi_0, \psi_1)(\mathbf{y}, \mathbf{z})$
- $\eta_0 \Vdash (\forall x', z', y')(\emptyset \neq \mathbf{x'} \subseteq \mathbf{x}, \emptyset \neq \mathbf{z'} \subseteq \mathbf{z}, \mathbf{y'} \subseteq \mathbf{y}) \neg P(\mathbf{x'}, \mathbf{y'}, \mathbf{z'})$
- $\eta_1 \Vdash (\forall x', z', y')(\emptyset \neq \mathbf{x'} \subseteq \mathbf{x}, \emptyset \neq \mathbf{z'} \subseteq \mathbf{z}, \mathbf{y'} \subseteq \mathbf{y}) \neg P(\mathbf{x'}, \mathbf{y'}, \mathbf{z'})$

<u>(Consistency of Color)</u>
If $(\phi, \phi') \in 2 - \mathrm{AT}(T)$ then

$$\phi(x_1, \ldots, x_n), \phi'(y_1, \ldots, y_n) \Vdash \bigwedge_{S \subseteq n} \|\{x_i : i \in S\}\| \leq \|\{y_i : i \in S\}\|$$

<u>(Consistency of $\leq$)</u>
If $(\phi_0, \phi_1), (\psi_0, \psi_1), (\zeta_0, \zeta_1) \in 2 - \mathrm{AT}(T)$, $(\zeta_0, \zeta_1)(\mathbf{x}, \mathbf{y}, z) \leq (\phi_0, \phi_1)(\mathbf{x}, \mathbf{y})$ and $(\phi_0, \phi_1)(\mathbf{x}, \mathbf{y}) \leq (\psi_0, \psi_1)(\mathbf{x})$ then

- $(\zeta_0, \zeta_1)(\mathbf{x}, \mathbf{y}, z) \leq (\psi_0, \psi_1)(\mathbf{x})$
- $\phi_0(\mathbf{x}, \mathbf{y}) \Vdash \psi_0(\mathbf{x})$
- $\phi_1(\mathbf{x}, \mathbf{y}) \Vdash \psi_1(\mathbf{x})$

<u>(Extension of 0-Colors)</u>
Suppose $(\sigma, \sigma') \in 2-\mathrm{AT}$. Further assume that $\tau'(\mathbf{x}, \mathbf{y}) \Vdash \sigma'(\mathbf{x})$. Let $\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x}) \wedge \|\mathbf{x'}\mathbf{y'}\| = -\infty$ (if $\emptyset \neq \mathbf{x'} \subseteq \mathbf{x}, \emptyset \neq \mathbf{y'} \subseteq \mathbf{y}$). Then $(\tau, \tau') \in 2 - \mathrm{AT}$ and $(\tau, \tau')(\mathbf{x}, \mathbf{y}) \leq (\sigma, \sigma')(\mathbf{x})$.

Lets go step by step through each part of this definition. The idea behind (Truth on Atomic Formulas) is that we want each archetype to determine the atomic type of the tuple it is meant to describe (i.e what atomic formulas hold on subtuples). Similarly (Truth on Color) says that the archetype should completely determine the color of the tuple it is mean to describe as well as the color of all its subtuples. As we will see, the intuitive purpose behind the axioms for a collection of archetypes is to ensure that the only information which is important about a tuple is it's ctype (i.e the colors of its subtuples), and its atomic type.

Now (Restriction of Arity for Archetypes) and (Completeness for Archetypes) guarantee that every tuple has an archetype which describes it. This is a simplifying assumption which will make our arguments much cleaner.

Similarly (Restriction of Arity for Consistent Pairs Archetypes) says that if we have a consistent pair of archetypes $(\tau, \tau') \leq (\sigma, \sigma')$ and we restrict (τ, τ') to an element of its domain which still contains the domain of (σ, σ') then we still have a consistent pair of archetypes which extends (σ, σ'). This way don't have to worry that $\leq$ only defines extensions on certain arities (which would be annoying)

(Amalgamation for Archetypes) says that if we have two archetypes $\phi(\mathbf{x}, \mathbf{y})$ and $\psi(\mathbf{y}, \mathbf{z})$ which agree on the intersection of their domains, then we can amalgamate them by putting every new tuple (i.e. tuples which overlap with both domains and aren't in the intersection) at $-\infty$.

Definition 3.2.0.9. Let $\sigma(\mathbf{x}, \mathbf{y}) \Vdash \zeta(\mathbf{y}), \tau(\mathbf{y}, \mathbf{z}) \Vdash \zeta(\mathbf{y})$ and $\eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Vdash (\sigma(\mathbf{x}, \mathbf{y}) \wedge \tau(\mathbf{y}, \mathbf{z}))$ and $(\emptyset \neq \mathbf{x}' \subseteq \mathbf{x}, \emptyset \neq \mathbf{z}' \subseteq \mathbf{z}, \mathbf{y}' \subseteq \mathbf{y}) \rightarrow \neg P(\mathbf{x}', \mathbf{y}', \mathbf{z}')$. We then say that $\eta(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \text{Trivial}(\sigma, \tau, \mathbf{y})$ is a Trivial Amalgamation of σ and τ around $\mathbf{y}$.

(Amalgamation for Consistent Pairs of Archetypes) is almost identical to (Amalgamation for Archetypes) except it deals with consistent pairs of archetypes. Specifically it says that if we have two consistent pairs of archetypes $(\phi_0, \phi_1)(\mathbf{x}, \mathbf{y})$ and $(\psi_0, \phi_1)(\mathbf{y}, \mathbf{z})$ each of which which agree on the intersection of their domains, then we can amalgamate them by putting every new tuple (i.e. tuples which overlap with both domains and aren't in the intersection) at $-\infty$ on both archetypes in the consistent pairs of archetypes.

Definition 3.2.0.10. Let

- $(\sigma_0, \sigma_1), (\tau_0, \tau_1) \in 2 - \mathrm{AT}(T)$
- $(\sigma_0, \sigma_1)(\mathbf{x}, \mathbf{y}) \leq (\zeta_0, \zeta_1)(\mathbf{y})$
- $(\tau_0, \tau_1)(\mathbf{y}, \mathbf{z}) \leq (\zeta_0, \zeta_1)(\mathbf{y})$
- $\eta_i(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathrm{Trivial}(\sigma_i, \tau_i, \mathbf{y})$

Then we say (η_0, η_1) is a Trivial Amalgamation of (σ_0, σ_1) and (τ_0, τ_1) around $\mathbf{y}$.

What (Consistency of Color) says is that if we have a consistent pairs of archetypes (σ, σ') then any color which σ' forces on a subtuple must be at least as big as the corresponding color that color σ forces on the same subtuple.

(Consistency of $\leq$) is the axiom which say that $\leq$ behaves like extension should. In other words if (τ, τ') is an extension of an extension of (σ, σ') then (τ, τ') is an extension of (σ, σ') and (τ, τ') determines the archetype of (σ, σ') (i.e. if $(\sigma, \sigma')(\mathbf{x}) \geq (\tau, \tau')(\mathbf{x}, \mathbf{y})$ and $(\sigma_*, \sigma'_*)(\mathbf{x}) \geq (\tau, \tau')(\mathbf{x}, \mathbf{y})$ then we know $(\sigma, \sigma')(\mathbf{x}) = (\sigma_*, \sigma'_*)(\mathbf{x})$). There is one point to mention explicitly though. We are not assuming though that just because if $\tau_i(\mathbf{x}, \mathbf{y}) \Vdash \sigma_i(\mathbf{x})$ that $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \leq (\sigma_0, \sigma_1)(\mathbf{x})$.

The idea is that we want $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \leq (\sigma_0, \sigma_1)(\mathbf{x})$ to hold not just when (τ_0, τ_1) forces (σ_0, σ_1) to hold but if in addition whenever there is a consistent pair of archetypes which "look like" (σ_0, σ_1) there is a consistent pair of archetypes extending them which "looks like" (τ_0, τ_1) (Here "looks like" means have the same consistent pairs of base predicates).

Once we have our consistent pairs archetypes we will want to make sure that they have enough extensions so as to get models of our original theory T on each of the "components". (Extension of 0-Colors) is the first of the axioms which will guarantee such an extension exists. What it says is that if we have a consistent pair of archetypes (σ, σ') and we have some archetype τ' extending σ' then we can extend σ by "adding no new information" and get a consistent pair extending the original.

Before we give the rest of the conditions we are going to want to define what it means for two archetypes to be the same up to a slant line. Intuitively two archetypes are the same up to a slant line if they satisfy the same atomic formulas and the color archetypes look the same up to the slant line. This is an important idea because we want our archetypes to (in some sense) allow us to say that the only thing which is important about a tuple is its color and which atomic formulas it satisfies.

Definition 3.2.0.11. Let T be a sentence of $\mathcal{L}_{\infty,\omega}$ with a weak collection of archetypes $\mathrm{AT}(T)$. Let $\sigma, \tau \in \mathrm{AT}(T)$ and let sl be a slant line. We then say σ and τ are the same up to sl ($\sigma|sl = \tau|sl$) if

- $\sigma(\mathbf{x}), \tau(\mathbf{y}) \Vdash \mathrm{ctype}(\mathbf{x})|sl = \mathrm{ctype}(\mathbf{y})|sl$
- atomic diagram($\mathbf{x}$) =atomic diagram($\mathbf{y}$)

3.3 Collection of Archetypes

We still have some more information that we want to add to our structure T. Specifically there are two other pieces, the "Extra Information" function, and the "Base Predicates"

The archetype realized by a a tuple is intended to completely describe the tuple. At the same time though, we want to be able to get a handle on the archetypes inside our language (otherwise they aren't very useful). This is where "Base Predicates" come in. The "Base Predicates" are a subset of $L_K - L_P$ which we will use to "talk about" archetypes. In addition, we will require find that every archetype will force exactly one basic predicate to hold (up to equivalence) on its domain. As such it will be very useful to extend the idea of a consistent pair of archetypes to a consistent pair of base predicates.

Definition 3.3.0.12. We say that (A_0, A_1) is a Consistent Pairs of Base Predicates ($(A_0, A_1) \in 2 - \mathrm{BP}$) if there is $(\sigma_0, \sigma_1) \in 2 - \mathrm{AT}(T)$ such that $\sigma_i(\mathbf{x}) \Vdash A_i(\mathbf{x})$.

We say that $(B_0, B_1) \leq (A_0, A_1)$ if there are $(\sigma_0, \sigma_1), (\tau_0, \tau_1) \in 2 - \mathrm{AT}(T)$ such that

- $\sigma_i(\mathbf{x}) \Vdash A_i(\mathbf{x})$
- $\tau_i(\mathbf{x}, \mathbf{y}) \Vdash B_i(\mathbf{x})$
- $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \leq (\sigma_0, \sigma_1)(\mathbf{x})$

In other words, (A_0, A_1) is a consistent pairs of base predicates if there is a consistent pairs of archetypes which witness it and similarly $(B_0, B_1) \leq$

(A_0, A_1) if there are consistent pairs of archetypes which witness this.

However there is one more thing we have to worry about. It is possible that some archetypes are only realized in some "types" of models and so, we will need a way to say this. The method we will use will be an "Extra Information" function. Intuitively each model and each archetype the extra information function returns the "types of tuples" which are forced to exist in the model or forced by the archetype. As it turns out the "type of tuple" (in this context) will be completely determined by its color, its arity and one other bit of information. Note that this is a very loose definition of "type of tuple" as we are only interested in collections of them and not individual tuples (as there is obviously more information that can be said about a tuple than just its arity and color).

The reason why we need this is that it turns out in the case we are interested in, the spectrum of a model won't quite be enough to completely determine it. Specifically when we have a model whose spectrum is not a limit ordinal then the tuples with color above the highest limit ordinals come in two types. In this case, just knowing the color of the tuple isn't enough. We also need to know a little more. In addition we will want to ensure that if we insist an archetype is realized in a model then that archetype won't try and put a tuple on a color where it couldn't go. So the "Extra Information" is there to keep track of this.

With this notation we are now ready to define our Archetype Collection.

Definition 3.3.0.13. Let T be a sentence of $\mathcal{L}_{\infty,\omega}(L_K)$. Let $\mathrm{AT}(T), 2-\mathrm{AT}(T)$ be a weak collection of archetypes for T. We then say $\mathrm{AT}(T), 2-$

AT(T) is a <u>Collection of Archetypes for T</u> if there is a collection of <u>Base Predicates</u> predicates, BP(T) $\subseteq L_K$, and a function $\mathrm{EI}_T : \{M : M \models T\} \cup \{\phi \in \mathrm{AT}(T)\} \rightarrow$Power Set(ORD $\times X$) for some set X such that

<u>(Prediction)</u>

For all $\sigma, \tau \in \mathrm{AT}(T)(\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x}))$ there is a $\eta_\tau(\overline{a})$ such that

- $(\forall M \models T) EI(\tau) \subseteq EI(M) \Leftrightarrow EI(\eta_\tau) \subseteq EI(M)$
- $EI(\eta_\tau) \subseteq EI(M) \rightarrow M \models (\exists \overline{a}) \eta_\tau(\overline{a})$

and there is a base predicate $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})$ such that

- $A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a}) \Vdash A_\eta(\overline{a})$ (where $\eta_\tau(\overline{a}) \Vdash A_\eta(\overline{a})$)
- For all $M \models T$ $M \models \eta_\tau(\overline{a}) \wedge \sigma(\mathbf{x}) \wedge A_{\sigma,\tau}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a}) \rightarrow \tau(\mathbf{x}, \mathbf{y})$.

<u>(Prediction up to a Slant Line)</u>

$(\forall sl$ a slant line with $\omega * \zeta \leq sl(1), sl(|\mathbf{x}, \mathbf{y}|) < \omega * (\zeta + 1) \leq \mathrm{Spec}(M)$ or $sl = \infty)$

For all $\sigma, \sigma', \tau \in \mathrm{AT}(T)(\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x}))$ there is a $\eta_{t|sl}(\overline{a})$ such that if $\sigma | sl = \sigma' | sl$ then

- $(\forall M \models T) EI(\sigma') \cap EI(\sigma) \subseteq EI(M) \Leftrightarrow EI(\eta_{t|sl}) \subseteq EI(M)$
- If $sl = \infty$ then $(\exists M \models T) EI(M) \supseteq EI(\sigma) \cup EI(\sigma')$
- $M \models \sigma'(\mathbf{x}) \rightarrow M \models (\exists \overline{a}) \eta_{\tau|sl}(\overline{a})$

and there is a base predicate $A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a})$ such that

- $A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a}) \Vdash A_\eta(\overline{a})$ (where $\eta_{\tau|sl}(\overline{c}) \Vdash A_\eta(\overline{c})$)

- $(\forall M \models T) M \models \eta_{\tau|sl}(\mathbf{x}, \overline{a}) \wedge A_{\sigma|sl,\tau|sl}(\mathbf{x}, \mathbf{y}, \mathbf{z}, \overline{a}) \wedge \tau'(\mathbf{x}, \mathbf{y}) \rightarrow \tau'|sl = \tau|sl$.

<u>(Truth on Atomic Formulas)</u>

$(\forall A \in \mathrm{BP}(T))[A(x_1, \ldots, x_n), A(y_1, \ldots, y_n)] \Vdash \bigwedge_{S \subseteq n} (\forall \text{atomic formula } \theta)\theta(\langle x_i : i \in S\rangle) \leftrightarrow \theta(\langle y_i : i \in S\rangle)$

<u>(Amalgamation for Base Predicates)</u>

For each $A, B, Z \in BP(T)$ if $A(\mathbf{x}, \mathbf{y}) \Vdash Z(\mathbf{y})$ and $B(\mathbf{y}, \mathbf{z}) \Vdash Z(\mathbf{y})$ then there is a $C \in BP(T)$ such that $C(\mathbf{x}, \mathbf{y}, \mathbf{z}) \Vdash (A(\mathbf{x}, \mathbf{y}) \wedge B(\mathbf{y}, \mathbf{z})) \wedge (\emptyset \neq \mathbf{x}' \subseteq \mathbf{x}, \emptyset \neq \mathbf{z}' \subseteq \mathbf{z}, \mathbf{y}' \subseteq \mathbf{y}) \rightarrow \neg P(\mathbf{x}', \mathbf{y}', \mathbf{z}')$

<u>(Homogeneity for Base Predicates)</u>

If $A \in BP(T)$ and $A(\mathbf{x}, \mathbf{y}) \Vdash B(\mathbf{x})$ then for each n, $B(\mathbf{x}) \Vdash (\exists^n \mathbf{y}) A(\mathbf{x}, \mathbf{y})$

<u>(Extension of 1-Colors)</u>

Suppose $(\sigma, \sigma') \in 2 - \mathrm{AT}(T)$, $\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x})$ and $EI(\sigma') \cup EI(\tau) \subseteq EI(M)$. Then there is an archetype τ' such that $(\tau, \tau') \in 2 - \mathrm{AT}(T)$ and $EI(\tau') \subseteq EI(M)$

<u>(Homogeneity of Consistent Pairs of Archetypes)</u>

Suppose

- $(\sigma, \sigma'), (\tau, \tau'), (\eta, \eta') \in 2 - \mathrm{AT}$
- $(\eta, \eta')(\mathbf{x}, \mathbf{y}) \Vdash (\sigma, \sigma')(\mathbf{x})$
- $(\eta, \eta')(\mathbf{x}, \mathbf{y}) \Vdash (B, B')(\mathbf{x}, \mathbf{y})$
- $(\tau, \tau')(\mathbf{x}), (\sigma, \sigma')(\mathbf{y}) \Vdash (A, A')(\mathbf{x}) \wedge (A, A')(\mathbf{y})$

(where $A, A', B, B' \in \mathrm{BP}$). Then there is a $(\zeta, \zeta') \in 2 - \mathrm{AT}(T)$ such that

- $(\zeta, \zeta')(\mathbf{x}, \mathbf{y}) \leq (\tau, \tau')(\mathbf{x})$
- $(\zeta, \zeta')(\mathbf{x}, \mathbf{y}) \Vdash (B, B')(\mathbf{x}, \mathbf{y})$
- $(\forall M \models T) EI(M) \supseteq EI(\tau) \leftrightarrow EI(M) \supseteq EI(\zeta)$
- $(\forall M' \models T) EI(M') \supseteq EI(\tau') \leftrightarrow EI(M') \supseteq EI(\zeta')$

(Completeness of Extra Information)
If $M \models T$ $\phi \in \mathrm{AT}(T)$, then $M \models (\exists \mathbf{x})\phi(\mathbf{x})$ iff $\phi \subseteq EI(M)$.

(Completeness of Consistent Pairs of Base Predicate)
If (A, A') is a consistent pair of base predicates and $\sigma, \sigma' \in \mathrm{AT}(T)$ such that

- $\sigma(\mathbf{x}) \Vdash A(\mathbf{x})$
- $\sigma'(\mathbf{x}) \Vdash A'(\mathbf{x})$
- $\sigma(x_1, \ldots, x_n), \sigma'(y_1, \ldots, y_n) \Vdash \bigwedge_{S \subseteq n} \|\{x_i : i \in S\}\| \leq \|\{y_i : i \in S\}\|$

Then $(\sigma, \sigma') \in 2 - \mathrm{AT}(T)$.

(Uniqueness of Base Predicate)
$(\forall A, A' \in BP(T))(\exists N \models T, \mathbf{x} \in N)(N \models A(\mathbf{x}) \wedge A'(\mathbf{x})) \rightarrow \emptyset \Vdash (A(\mathbf{x}) \leftrightarrow A'(\mathbf{x}))$

Lets go through what each of these conditions says. First lets consider (Prediction). This is one of the most important axioms we have. Those of you who are familiar with [8], will recognize it as being very similar to

Generalized Saturation for Archetypes (Proposition 4.3.2.1 of [8]) which is probably the most important theorem of the paper (for a discussion of that specific Proposition see Section 5.2).

So what exactly is (Prediction) saying. To understand (Prediction) we need to understand the intended connection between the base predicates and the archetypes. Intuitively the archetypes are supposed to contain all important information about the tuple they describe. But, this in and of itself isn't particularly useful as we are not even requiring that the archetypes be formulas of $\mathcal{L}_{\infty,\omega}$. As such, we need some way to get a handle on what the archetypes are saying. This is where the base predicates come in.

What we would like to be able to say is that if we have an archetype $\sigma(\mathbf{x})$, and we know which base predicate $\mathbf{xy}$ satisfies, as well as which base predicates extensions of $\mathbf{xy}$ satisfy, then in fact we know what the archetype of $\mathbf{xy}$ is. This way all the information about an extension of a tuple would be contained in the the base predicates and the archetype of the original tuple. This would then allow us to completely describe our model by the archetypes of arity 1 which it realizes. This is because we know by (Homogeneity of Base Predicates) that all base predicates are extended in every consistent way. So, we could produce a back and forth argument guaranteeing that if two models satisfied these same archetypes of arity 1 they are $\mathcal{L}_{\infty,\omega}$ equivalent.

However, we can't quite do this. And in fact upon further thought, if we could do this it would lead to (at best) a theory where all models have the same spectrum. To see this suppose we start with an archetype which places it's sole tuple x at $-\infty$. Now this archetype will be realized in every model. Now suppose we have two models. One of which, M, has an element of color

α and one of which, N doesn't. Let a be such a tuple of color α in M. Then the archetype realized by xa is witnessed by some base predicate A. But then in N there is some b such that $N \models A(xb)$. Hence by assumption b has color α. $\Rightarrow\Leftarrow$.

After considering the previous argument, we realize that in addition to a base predicate witnessing the extension of our archetype we need something else which will force the archetype to be realized in the model we are looking at. That is what η_τ is for. η_τ is meant to contain the same colors as τ and so anytime τ is realized in a model, so is η_τ. This then allows us to pin down τ by comparing the colors of tuples in the domain to those in η_τ.

Now (Prediction up to a Slant Line) says essentially the same thing as (Prediction) except instead of requiring that we can extend an archetype to another, we only require that we can extend it up to a slant line. We hence require that this extension is witnessed by a base predicate extending the base predicates of both σ, σ' and $\eta_{t|sl}$. Now there are a couple of subtle points to notice about this axiom. The first of which is that this axiom implies that if $\sigma|sl = \sigma'|sl$ and $\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x})$ then there is an archetype $\tau'(\mathbf{x}, \mathbf{y})$ such that $\tau'(\mathbf{x}, \mathbf{y}) \Vdash \sigma'(\mathbf{x})$ and $\tau'|sl = \tau|sl$. This in and of itself is a VERY strong form of homogeneity (and will be crucial for determining the quantifier ranks of our models). It says not only if we have two different realizations that they must have exactly the same extensions (like (Prediction) says) but if we have two realizations of different archetypes which just happen to look the same then as far as they look the same their extensions must look the same as well.

The next subtle point to notice is that in some circumstances the archetype $\eta_{t|sl}$ and the base predicate $A_{\sigma|sl,\tau|sl}$ may be simply be the same as η_t and

$A_{\sigma,\tau}$. Specifically, this might be the case if $\sigma'|sl = \sigma$ and $\tau'|sl = \tau$ (with the obvious meanings).

On a similar note, it is worth understanding what happens in the case that $sl = \infty$. In this we find that the archetype τ' places all colors in exactly the same place as τ does and what is more that τ and τ' force exactly the same atomic formulas. The point of this clause though is that this does not necessarily mean that $\tau = \tau'$. And in fact if $\sigma \neq \sigma'$ then τ can't equal τ'. In particular it is possible that there is more information encoded in the archetypes we have chosen than just the color of their tuples and the atomic formulas they realized.

The reason for this, as we will see later, is that if models of the theory Θ which we are interested in, which have Spectrum $\{-\infty\} \cup \omega * \alpha + n$, come in two forms. What is more, these models will look fundamentally different above $\omega * \alpha$. But at the same time we won't be able to see this difference just by looking at the color of a tuple (which is above $\omega * \alpha$).

Hence, in addition to the the color archetype of its domain as well as which atomic formulas are satisfied by its domain, an archetype will have to keep track of which "type" of model it is allowed to be in. It will do this by using the "Extra Information" function (and in fact that was why we need the "Extra Information" function). Further, the way we treat this in regards to (Prediction up to a Slant Line) is we either require the slant line to be below the largest limit ordinal in the Spectrum, or we require the slant line to be ∞ and for both archetypes being compared to be "compatible" (i.e. both realizable in the same model)

The (Truth on Atomic Formulas) condition in Definition 3.3.0.13 says essentially the same thing as the (Truth on Atomic Formulas) condition in Definition 3.2.0.8. That is, if we know a tuple satisfies a base predicate then that information determines which atomic formulas that tuple satisfies.

(Amalgamation for Base Predicates) says essentially the same thing as (Amalgamation for Archetypes) in Definition 3.2.0.8. If we have two base predicates which agree on the intersection of their domain then it is possible to amalgamate them in such a way that all new tuples are given color $-\infty$.

What (Homogeneity of Base Predicates) says is that if it is consistent for a Base Predicate B to be extended to a Base Predicate A then for any tuple which satisfies B there are infinitely many extensions which realizes A.

(Extension of 1-Colors) is similar to (Extension of 0-Colors). The main difference is that this time instead of starting with an extension of the second element of a consistent pair of archetypes we start with an extension of the first element of the sequence. This makes things a little more complicated because it is not immediately clear what the choice for our extension of the second element of the sequence should be. What is more, it is not clear in this case (unlike in the case of (Extension of 0-Colors)) that this new consistent pair of archetypes should always be realized in any pair of modes in which the original sequence is realized (consider the case where τ forces a color larger than anything in the model in which τ' is realized in). (Extension of 1-Colors) is specifically to ensure that we can do this (and we will see that it is non-trivial that this can be done for the theory Θ from [8])

(Homogeneity of Consistent Pairs of Archetypes) is one of the main ways in which consistent pairs of archetypes are tied to consistent pair of base predicates. This says if we have a consistent pair of archetypes and we know that the consistent pair of base predicates which they force (say $(A, A'))$ can be extended to another consistent pair of base predicates (say $(B, B'))$, then we can extend our original consistent pair of archetypes to a consistent pair of archetypes over (B, B'), such that the extensions of our archetype are realized in the same same models as our original consistent pair of archetypes were realized in.

(Completeness of Extra Information) simply says a model realizes an archetype if and only if all the "extra information" about that archetype is realized in the model. This is a way of codifying the idea that the extra information simply describes types of tuples which are realized.

The purpose of (Completeness of Consistent Pairs of Base Predicate) is to ensure that the only information being used to determine whether or not a pair of archetypes is consistent is whether or not the pair satisfies (Consistency of Color) and if they are over a consistent pair of base predicates. This is very useful as, at the end of the day, it is only the base predicates which we actually can talk about in our theory. So this will then allow us also ensure that "enough" consistent pairs of archetypes exist.

The only condition left is (Uniqueness of Base Predicates). This just says that each tuple only realizes one Base Predicate up to equivalence.

Definition 3.3.0.14. Let T be such that it has a collection of archetypes $\text{AT}(T)$. If $\phi \in \text{AT}(T)$ and $A \in \text{BP}(T)$ such that $\phi(\mathbf{x}) \Vdash_T A(\mathbf{x})$ we say that $\underline{\phi \text{ is over } A}$

So, in particular, by (Uniqueness of Base Predicate) and (Truth of Atomic Formulas) every archetype is over exactly one basic predicate (up to equivalence).

3.4 Results

Lemma 3.4.0.15. *If* $\sigma, \tau \in AT(T)$ *both force* $A \in BP(T)$ *and* $\sigma(\boldsymbol{x}), \tau(\boldsymbol{y}) \Vdash ctype(\boldsymbol{x})|sl = ctype(\boldsymbol{y})|sl$ *then* $\sigma|sl = \tau|sl$ *for some slant line sl.*

Proof. This is by (Truth on Atomic Formulas) □

Lemma 3.4.0.16. *Let* $M \models T$, $\overline{a} \in M$, $\phi, \phi' \in AT(T)$. *If* $M \models \phi(\overline{a}) \wedge \phi'(\overline{a})$ *then* $\phi(\boldsymbol{x})|\infty = \phi'(\boldsymbol{x})|\infty$.

Proof. We know by (Truth on Color) that both $\phi(\mathbf{x})$ and $\phi'(\mathbf{x})$ must force ctype($\mathbf{x}$) to be the same. So all that is left is to make sure that they both force all predicates to be the same. But they must do this because $\phi(x_1, \ldots, x_n)$ forces $K(\langle x_i : i \in S\rangle)$ iff $M \models \phi(a_1, \ldots, a_n) \rightarrow K(\langle a_i : i \in S\rangle)$, and similarly for ϕ' (by (Truth on Atomic Formulas)) □

Definition 3.4.0.17. Let T have a collection of archetypes and $M \models T$. Define the $\text{ATYPE}_T(M) = \{\phi \in \text{AT}(T) : EI(\phi) \subseteq EI(M)\}$

If $M, N \models T$ and sl is a slant line, we say $\text{ATYPE}_T(M)|sl = \text{ATYPE}_T(N)|sl$ if

$$(\forall \phi \in \text{ATYPE}_T(M))(\exists \psi \in \text{ATYPE}_T(N))(\phi|sl = \psi|sl)$$

and

$$(\forall \psi \in \text{ATYPE}_T(N))(\exists \phi \in \text{ATYPE}_T(M))(\phi|sl = \psi|sl)$$

Lemma 3.4.0.18. *Let* T *have a collection of archetypes and* $M \models T$ *then* $(\forall \phi \in AT(T))\phi \in ATYPE(M) \leftrightarrow M \models (\exists \boldsymbol{x})(\phi(\boldsymbol{x}))$

Proof. This is immediate by (Completeness of Extra Information). □

Theorem 3.4.0.19. *Let T have a collection of archetypes. If $M, N \models T$ such that for each slant line $sl < \omega * \gamma$, $ATYPE(M)|sl = ATYPE(N)|sl$ and $\omega * \gamma \leq \min\{Spec(M) \cap ORD, Spec(N) \cap ORD\}$ then $M \equiv_{\omega*\gamma} N$*

Proof. In order to prove this result we need to define a sequence of partial isomorphisms from M to N of length at least $\omega * \gamma$.

Definition 3.4.0.20. Define $I^*{}_\eta(M, N) = I^*_\eta$ as follows:
$I^*{}_{\eta*\omega+n} = \{f : M \to N s.t. f$ is a bijection, $|\mathrm{dom}(f)| < \omega$, there exists a slant line $sl < (\eta + 1) * \omega$ such that if $M \models \sigma_f(\mathrm{dom}(f))$ and $N \models \tau_f(\mathrm{range}(f))$ then $\sigma_f|sl = \tau_f|sl$ and where $sl(|\mathrm{dom}(f)| + n) \geq \eta * \omega\}$

Let $f \in I_{\omega*\eta+n+1}$. Notice that f is a partial isomorphism because the range and domain of f satisfy archetypes which are equal up to some slant line, and hence f preserves all atomic formula of L_K

All that is left to show is that $\langle I_\zeta : \zeta < \omega * \beta\rangle$ satisfies the back and forth property. Let $\omega * \eta + n + 1 < \omega * \beta$ and let $a \in M$. We want to find a $b \in N$ such that $g(a) = b$, $f \subseteq g, g \in I_{\omega*\eta+n}$.

Let $\sigma' \in \mathrm{AT}(T)$ be such that $M \models \sigma'(\mathrm{dom}(f)a)$. Then we know by (Prediction up to a Slant Line) that there is an archetype $\eta_{t|sl}$ and a base predicate $A_{\sigma|sl,\sigma'|sl}$ such that

$$N \models A_{\sigma|sl,\sigma'|sl}(\mathbf{x}, b, \overline{c}, \overline{d}) \wedge \eta_{\tau|sl}(\overline{c}) \wedge \tau'(\mathbf{x}, b)$$

then $\tau'|sl = \tau|sl$.

But in particular we also know that $N \models (\exists \overline{c}, b, \overline{d})\eta_{\tau|sl}(\overline{c})) \wedge A_{\sigma|sl,\sigma'|sl}(\mathbf{x}, b, \overline{c}, \overline{d})$ (by the conditions on the "Extra Information" about $\eta_{\tau|sl}$ and (Homogeneity of Base Predicates)). So if we let b be as above then $g = f \cup (a, b) \in I_{\omega*\eta+n}$ (because $sl(|\mathrm{dom}(f)| + n + 1) = sl(|\mathrm{dom}(g)| + n)$.

We can then do the case where we are given a $b \in N$ and we need to find an $a \in M$ analogously and so we have proved that $\langle I_\zeta : \zeta < \omega * \gamma \rangle$ has the back and forth property. Hence that $M \equiv_{\omega * \gamma} N$. □

Theorem 3.4.0.21. *Let T have a collection of archetypes. If $M, N \models T$, such that $ATYPE(M)|\infty = ATYPE(N)|\infty$ then $M \equiv_\infty N$*

Proof. In order to prove this result we need to define a sequence of partial isomorphisms $I \subseteq I$ from M to N.

Definition 3.4.0.22. Define $I(M, N) = I = \{f : M \to N s.t. f$ is a bijection, $|\text{dom}(f)| < \omega$, if $M \models \sigma_f(\text{dom}(f))$ and $N \models \tau_f(\text{range}(f))$ then $\sigma_f|\infty = \tau_f|\infty\}$

Let $f \in I$. Notice that f is a partial isomorphism because the range and domain of f satisfy archetypes which are equal up to ∞ and hence f preserves all atomic formula of L_K.

All that is left to show is that $I \subseteq I$ satisfies the back and forth property. Let $a \in M$. We want to find a $b \in N$ such that $f \cup (a, b) \in I$.

Let $\sigma' \in \text{AT}(T)$ be such that $M \models \sigma'(\text{dom}(f)a)$. We then know that there is some $\tau'(\mathbf{x}, c) \in \text{ATYPE}(N)$ such that $\tau'|\infty = \sigma'|\infty$ (by the assumption of the theorem). So in particular if $\tau'(\mathbf{x}, c) \Vdash \tau^+(\mathbf{x})$ then $\tau^+|\infty = \sigma|\infty = \tau|\infty$. Hence by (Prediction up to a Slant Line) there must be some $\tau^*(\mathbf{x}, c)$ such that $\tau^*(\mathbf{x}, c) \Vdash \tau(\mathbf{x})$, $\tau^*|\infty = \tau'|\infty = \sigma'|\infty$ and $N \models (\exists b)\tau^*(\text{range}(f)b)$. So in particular if we let b be as above we have $f \cup (a, b) \in I$.

We can then do the case where we are given a $b \in N$ and need to find an $a \in M$ analogously and so we have proved that $I \subseteq I$ has the back and forth property. Hence that $M \equiv_\infty N$. □

The purpose of Theorem 3.4.0.19 and Theorem 3.4.0.21 is to show that if a theory T has a collection of archetypes, then in fact models of T are determined by the "types of archetypes" they realize.

Chapter 4

Gluing Theories with Collections of Archetypes

4.1 Definitions

We are now almost ready to begin the process of gluing models together. First though we will need a few more conditions on our theory other than having a collection of archetypes.

Definition 4.1.0.23. Let T_K be some sentence of $\mathcal{L}_{\infty,\omega}$ (with $T_K(\alpha) = T_K \cup (\forall \mathbf{x}) \neq \varphi^{\geq}_{\alpha}(x))$ such that

- $\models T_K \to T_P$
- $\{\text{Spec}(M)/-\infty : M \models T_K(\alpha)\}$
  - (1) is cofinal in α
  - (2) contains α

- There is a collection of archetypes ($\mathrm{AT}(T_K(\alpha))$)

We are now finally ready to present the theories we will be looking at. The theories will consist of two copies of T_K glued together. Specifically, one copy will have its spectrum fixed by the theory and the other copy will be under it (in an appropriate sense). First though we need our language.

Definition 4.1.0.24. Let

- $\mathcal{M} \models T_K$ have no tuples of color ∞
- $\mathrm{Spec}(\mathcal{M}) = \{-\infty\} \cup \alpha$.
- $L_Q = \{\langle c_i : i \in \mathcal{M}\rangle, Q(x)\}$
- $L_K(\mathcal{M}) = L_K^0 \cup L_K^1 \cup L_R^1 \cup L_Q$

Here the superscript is meant to distinguish different copies of the same language.

Definition 4.1.0.25. Let $T_K(\mathcal{M})$ be universal closure of the following $L_K(\mathcal{M})$ sentences:

Q:

- $Q(x) \leftrightarrow \bigvee_{a \in \mathcal{M}} x = c_a$
- $Q \models \phi(c_{a_1}, \ldots c_{a_n})$ in L_P^1 iff $\mathcal{M} \models \phi(a_1, \ldots a_n)$
- $Q(\mathbf{x}) \wedge \neg Q(\mathbf{y}) \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any predicate other than $R_{\leq}^1$ or P^1 and $|\mathbf{x}| > 0$

$\underline{L^1_K:}$

- $(\forall x)(\exists c)Q(c) \wedge R^1_{\leq}(x, c)$
- $(\forall c)(\exists x)\neg Q(x) \wedge R^1_{\leq}(x, c)$

Other Axioms:

- $\neg Q \models T^1_K$
- $\neg Q \models T^0_K$
- $\neg Q \models P^0(\mathbf{x}) \to P^1(\mathbf{x})$
- (Homogeneity) For all $(A, A_*), (B, B_*) \in 2-\mathrm{BP}$ such that $(A, A_*) \leq (B, B_*)$,

$$\neg Q \models [(\forall \mathbf{x})[A^0(\mathbf{x}) \wedge A^1_*(\mathbf{x})] \to (\exists^n \mathbf{y})(B^0(\mathbf{x}, \mathbf{y}) \wedge B^1_*(\mathbf{x}, \mathbf{y}))]$$

- (Completeness) $(\forall \mathbf{x})(\exists \mathbf{y}) \bigvee_{(A,A')\in 2-\mathrm{BP}}(A, A')(\mathbf{xy})$

The first thing to notice is that our theory $T_K(\mathcal{M})$ is in fact a sentence of $\mathcal{L}_{\infty,\omega}$. And, if $\mathcal{M}$ is countable it is a sentence of $\mathcal{L}_{\omega_1,\omega}$. As such $T_K(\mathcal{M})$ makes no explicit mention of the archetypes (which aren't required to be sentences in any particular logic).

Now the purpose of the Q axioms are to fix everything that can be said about any element which satisfies Q. In particular, we want the collection of elements which satisfy Q to be isomorphic to $\mathcal{M}$ in L^1_K and to have every element named. We further want nothing to be true in L^0_K of elements which satisfy Q. Finally, we want to be able to compare the 1-color (using $R^1_{\leq}$) of elements which satisfy Q with elements which satisfy $\neg Q$.

The purpose of the $\underline{L_K^1}$ axioms to guarantee that the spectrum the collection of elements which satisfy $\neg Q$ is the same as $\mathcal{M}$. It is worth mentioning explicitly that the only connection between elements satisfying $\neg Q$ and those satisfying Q is by the fact that in L_R^1 they must have the same spectrum. As such, if we were to restrict our models only to the part which satisfies $\neg Q$, we see that in fact all that matters concerning Q is the height of the tree $\mathcal{M}$, and nothing about its structure. But, in the theories we will be interested in and in the heights we will be interested in, there will only be a single tree of any height α.

As for the other axioms, the only ones which aren't self explanatory are (Homogeneity) and (Completeness). (Homogeneity) says that if we have a consistent pair of base predicates extending another consistent pair of base predicates, whenever the second is realized we can find an extension which realizes the first. This is very similar to (Homogeneity for Base Predicates) in Definition 3.3.0.13 except for consistent pair of base predicates instead of for single Base Predicates. Notice though that this axiom does not in fact make reference to archetypes.

(Completeness) on the other hand is there to guarantee that when ever we have a consistent pair of base predicates realized in the model, then it can be extended to a consistent one.

Definition 4.1.0.26. If $\sigma^0, \sigma^1 \in \mathrm{AT}(T_K(\alpha))$ and $M \models T_K(\mathcal{M})$. we say $M \models (\sigma^0, \sigma^1)(\mathbf{x})$ if $M|L_K^0 \models \sigma^0(\mathbf{x})$ and $M|L_K^1 \models \sigma^1(\mathbf{x})$.

4.1.1 Results

Lemma 4.1.1.1. *Let*

- (σ, σ') *be a consistent pair of archetypes*
- $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash \sigma(\boldsymbol{y})$
- $\tau'(\boldsymbol{y}, \boldsymbol{z}) \Vdash \sigma'(\boldsymbol{y})$
- $EI(\tau) \subseteq EI(M)$
- $EI(\tau') \subseteq EI(M')$

Then there is a consistent pair of archetypes (η, η') *such that*

- $\eta(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \tau(\boldsymbol{x}, \boldsymbol{y})$
- $\eta'(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \tau'(\boldsymbol{y}, \boldsymbol{z})$
- $EI(\eta) \subseteq EI(M)$
- $EI(\eta') \subseteq EI(M')$
- $(\eta, \eta') \leq (\sigma, \sigma')$

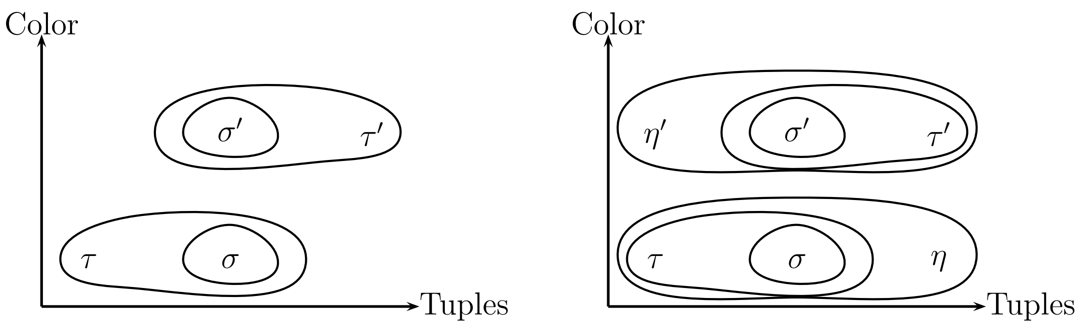
Proof. Let

- $(\sigma, \sigma')(\mathbf{x}) \Vdash (A, A')(\mathbf{x})$
- $\tau(\mathbf{x}, \mathbf{y}) \Vdash B(\mathbf{x}, \mathbf{y})$
- $\tau'(\mathbf{y}, \mathbf{z}) \Vdash B'(\mathbf{y}, \mathbf{z})$
- $A, A', B, B' \in BP(T)$

We know there is a base predicate $D(\mathbf{y}, \mathbf{z})$ such that any tuple with any element of $\mathbf{z}$ in it has color $-\infty$. Let $C(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the amalgamation of $D(\mathbf{y}, \mathbf{z})$

with $A(\mathbf{x}, \mathbf{y})$ guaranteed by (Amalgamation of Base Predicates). Let $\zeta(\mathbf{y}, \mathbf{z})$ be any archetype such that $\zeta(\mathbf{y}, \mathbf{z}) \Vdash \sigma(\mathbf{y}) \wedge D(\mathbf{y}, \mathbf{z})$ and $EI(\zeta) \subseteq EI(M)$ (we can find this by letting $\zeta(\mathbf{y}, \mathbf{z})$ be the trivial amalgamation of $\sigma(\mathbf{y})$ with $\upsilon(\mathbf{z})$ where υ puts everything at $-\infty$). Let $\eta(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be an amalgamation of $\zeta(\mathbf{y}, \mathbf{z})$ and $\tau(\mathbf{x}, \mathbf{y})$ realized in M (so $EI(\eta) \subseteq EI(M)$).

Now we know that $(\zeta, \tau')(\mathbf{y}, \mathbf{z})$ is a consistent pair of archetypes and $(\zeta, \tau')(\mathbf{y}, \mathbf{z}) \leq (\sigma, \sigma')(\mathbf{y})$ by the axiom (Extending 0-Colors) because ζ put all tuples not σ' at $-\infty$. In particular, by the axiom (Extending 1-Colors), there must be some $\eta'(\mathbf{x}, \mathbf{y}, \mathbf{z})$ such that $(\eta, \eta') \in 2-\mathrm{AT}(T)$, $\mathrm{EI}(\eta') \subseteq \mathrm{EI}(M')$, and $(\eta, \eta')(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq (\zeta, \tau')(\mathbf{y}, \mathbf{z})$. □



Lemma 4.1.1.2. *Let $N \models T_K(\mathcal{M})$. Let (σ_0, σ_1), $(\tau_0, \tau_1) \in 2-AT(T_K)$, $(\tau_0, \tau_1)(\boldsymbol{x}, \boldsymbol{y}) \leq (\sigma_0, \sigma_1)(\boldsymbol{x})$ and $EI(\tau_0) \subseteq EI(N|L_K^0), EI(\tau_1) \subseteq EI(N|L_K^1)$. Then $N \models (\forall \boldsymbol{x})(\sigma_0, \sigma_1)(\boldsymbol{x}) \rightarrow (\exists \boldsymbol{y})(\tau_0, \tau_1)(\boldsymbol{x}, \boldsymbol{y})$.*

Proof. We will prove this by two applications of (Prediction). First find $\eta_{\tau_0}(\mathbf{x}, \overline{a})$ as in (Prediction). Then we find $\eta_{\tau_1}(\overline{b})$. We know by Lemma 4.1.1.1 that there is a consistent pair of archetype such that $N \models (\eta_0, \eta_1)(\mathbf{x}, \overline{a}, \overline{b})$. $\eta_i(\overline{a}, \overline{b}) \Vdash \mathrm{Trivial}(\eta_{\tau_0}(\overline{a}), \sigma_i, \emptyset)$ (we know each of η_0, η_1 are realized in $N|L_K^0, N|L_K^1$ because each of $N|L_K^i \models T_K$ and $EI(\eta_{\tau_i}) \subseteq EI(N|L_K^i)$).

We also know that if $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \Vdash (B_0, B_1)$ and $(\sigma_0, \sigma_1)(\mathbf{x}) \Vdash (A_0, A_1)$

(where $A_0, A_1, B_0, B_1 \in \mathrm{BP}$) then (B_0, B_1) is a consistent pair of base predicates and $(B_0, B_1) \leq (A_0, A_1)$. In particular they must be realized by some (in fact infinitely many) $\mathbf{x}, \mathbf{y}$ in N. Now let $C_0(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{d}) \Vdash A_{\sigma_0,\tau_0}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ and $C_1(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{d}) \Vdash A_{\sigma_1,\tau_1}(\mathbf{x}, \mathbf{y}, \mathbf{z}')$ (where $A_{\sigma_0,\tau_0}, A_{\sigma_1,\tau_1}$ are from (Prediction)) and such that (C_0, C_1) is a consistent pair of base predicates (we get this the same way we got (η_0, η_1)). Simply extend first on $N|L^0_K$ and then extend (disjointly) on $N|L^1_K$ and look at the consistent pair of base predicates of the tuples you get. These are extendible to consistent pairs of base predicates by (Completeness).

We therefore know by the way these were constructed that

$$N \models (\sigma_0, \sigma_1)(\mathbf{x}) \rightarrow (\exists \mathbf{y}, \mathbf{z}, \mathbf{z}, \overline{a}, \overline{b}, \overline{d})(C_0, C_1)(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{d}) \wedge (\eta_0, \eta_1)(\mathbf{x}, \mathbf{y}, \overline{a}, \overline{b})$$

But, once again, by (Prediction) and how these consistent pairs of base predicates and consistent pairs of archetypes were created we know that we must also have $N \models (\sigma_0, \sigma_1)(\mathbf{x}) \rightarrow (\exists \mathbf{y})(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y})$. □

Now we are going to do something very similar but we will use (Prediction up to a Slant Line) instead of (Prediction).

Lemma 4.1.1.3. *Let $N \models T_K(\mathcal{M})$. Let sl_i ($i \in \{0, 1\}$) be slant lines with $sl_i < \omega * \gamma \leq Spec(N|L^i_K)$ or $sl_i = \infty$. Let (σ_0, σ_1), (σ'_0, σ'_1), $(\tau_0, \tau_1) \in 2 - AT(T_K(\alpha))$,*

- $sl_0 \leq sl_1$
- $(\tau_0, \tau_1)(\boldsymbol{x}, \boldsymbol{y}) \leq (\sigma_0, \sigma_1)(\boldsymbol{x})$
- $(\exists M \models T_K(\mathcal{M}))M \models (\exists \boldsymbol{x}, \boldsymbol{y})(\tau_0, \tau_1)(\boldsymbol{x}, \boldsymbol{y})$

- $\sigma_i|sl_i = \sigma'_i|sl_i$
- *If* $sl_i = \infty$ *then*
  - $(\exists N' \models T_K(\alpha) EI(\sigma_i) \cup EI(\sigma'_i) \subseteq N')$
  - $EI(\tau_i) \subseteq EI(N|L^i_K)$

Then there is a consistent pair of archetypes (τ'_0, τ'_1) *such that*

- $N \models (\forall \boldsymbol{x})(\sigma'_0, \sigma'_1)(\boldsymbol{x}) \to (\exists \boldsymbol{y})(\tau'_0, \tau'_1)(\boldsymbol{x}, \boldsymbol{y})$.
- $\tau'_i|sl_i = \tau_i|sl_i$

Proof. This proof will be almost identical to the proof of Lemma 4.1.1.2, except instead of using (Prediction) we will use (Prediction up to a Slant Line).

First we find $\eta_{\tau_0|sl_0}(\overline{a})$ as in (Prediction up to a Slant Line). Then we find $\eta_{\tau_1|sl_1}(\overline{b})$. We know that there is a consistent pair of archetype such that $N \models (\eta_0, \eta_1)(\mathbf{x}, \overline{a}, \overline{b})$ such that $\eta_0(\mathbf{x}, \overline{a}, \overline{b}) \Vdash \eta_{\tau_0|sl_0}(\overline{a}) \wedge \sigma_0(\mathbf{x})$ and $\eta_1(\mathbf{x}, \overline{a}, \overline{b}) \Vdash \eta_{\tau_1|sl_1}(\overline{b}) \wedge \sigma_1(x)$ (we know each of η_0, η_1 are realized in $N|L^0_K, N|L^1_K$ because each of $N|L^i_K \models T_K$ and $EI(\eta_{\tau_i|sl}) \subseteq EI(N|L^i_K))$.

We also know that if $(\tau_0, \tau_1)(\mathbf{x}, \mathbf{y}) \Vdash (B_0, B_1)$ and $(\sigma_0, \sigma_1)(\mathbf{x}) \Vdash (A_0, A_1)$ (where $A_0, A_1, B_0, B_1 \in$ BP) then (B_0, B_1) is a consistent pair of base predicates with $(B_0, B_1) \leq (A_0, A_1)$ (because (τ_0, τ_1) is realized in some model of $T_K(\mathcal{M})$). In particular, they must be realized by some (in fact infinitely many) $\mathbf{x}, \mathbf{y}$ in N. Now let $C_0(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{d}) \Vdash A_{\sigma_0|sl_0, \tau_0|sl_0}(\mathbf{x}, \mathbf{y}, \overline{a}, \mathbf{z})$ and $C_1(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{d}) \Vdash A_{\sigma_1|sl_1, \tau_1|sl_1}(\mathbf{x}, \mathbf{y}, \overline{b}, \mathbf{z}')$ (where $A_{\sigma_0|sl_0, \tau_0|sl_0}, A_{\sigma_1|sl_1, \tau_1|sl_1}$ are from (Prediction up to a Slant Line)) and such that (C_0, C_1) is a consistent pair (we get this in a similar way to how we got (η_0, η_1)).

Therefore by the way these were constructed

$$N \models (\sigma_0', \sigma_1')(\mathbf{x}) \rightarrow (\exists \mathbf{y}, \mathbf{z}, \mathbf{z}, \overline{a}, \overline{b}, \overline{d})(C_0, C_1)(\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{z}', \overline{a}, \overline{b}, \overline{d}) \wedge (\eta_0, \eta_1)(\mathbf{x}, \overline{a}, \overline{b})$$

But, once again, by (Prediction up to a Slant Line) and by how these consistent pairs of base predicates and consistent pairs of archetypes were created we know that we also have $N \models (\sigma_0', \sigma_1')(\mathbf{x}) \rightarrow (\exists \mathbf{y})(\tau_0', \tau_1')(\mathbf{x}, \mathbf{y})$, for some consistent pair of archetypes (τ_0', τ_1') where $\tau_i'|sl_i = \tau_i|sl_i$. □

Lemma 4.1.1.4. *Let* $(\sigma_i, \sigma_i'), i \in \{1, 2, 3\}$ *be consistent pair of archetype such that*

- $(\sigma_1, \sigma_1')(x_1, \ldots, x_n) \geq \sigma_3(\langle x_i : i \in S_1 \rangle)$
- $(\sigma_2, \sigma_2')(y_1, \ldots, y_m) \geq \sigma_3(\langle y_i : i \in S_2 \rangle)$
- $(\exists N \models T_K) EI(N) \supseteq \cup_{1 \leq i \leq 3} EI(\sigma_i)$
- $(\exists N' \models T_K) EI(N') \supseteq \cup_{1 \leq i \leq 3} EI(\sigma_i')$

where $|S_1| = |S_2|$*. Then there is a consistent* (σ_4, σ_4') *such that*

- $(\sigma_4, \sigma_4')(x_1, \ldots, x_{n+m-|S_1|}) \leq (\sigma_1, \sigma_1')(\langle x_i : i \in S_1^* \rangle)$
- $(\sigma_4, \sigma_4')(x_1, \ldots, x_{n+m-|S_1|}) \leq (\sigma_2, \sigma_2')(\langle x_i : i \in S_2^* \rangle)$
- $(\sigma_4, \sigma_4')(x_1, \ldots, x_{n+m-|S_1|}) \leq (\sigma_3, \sigma_3')(\langle x_i : i \in S_1^* \cap S_2^* \rangle)$

Proof. This is just a precise statement of (Amalgamation of Consistent Pairs of Archetypes). □

4.1.2 Construction of Models

In this section we will explicitly construct models of $T_K(\mathcal{M})$ such that our models look like $\mathcal{M}$ (when $|\mathcal{M}| = \omega$) in the language L^1_K and look like a countable model of our choosing in L^0_K. This will show that our theory $T_K(\mathcal{M})$ is consistent.

Theorem 4.1.2.1. *If $M^0 \models T_K$, and $|M^0| = |\mathcal{M}| = \omega$, then there is a model $M^* \models T_K(\mathcal{M})$ such that $M^*|L^0 = M^0$.*

Proof. We are going to construct our model M^* in the following manner. First we are going to create a bijection $f : M^0 \to \mathcal{M}$ such that if $M^0 \models \sigma(\overline{a})$ and $\mathcal{M} \models \sigma'(f[\overline{a}])$ then (σ, σ') can be extended to a is a consistent pair of archetypes (σ^*, σ^+) such that $M^0 \models \sigma^*(\overline{ac})$ and $\mathcal{M} \models \sigma'(f[\overline{ac}])$. Further we will do this is such a way that if (τ, τ') is a consistent pair of archetype such that

- $(\tau, \tau')(\mathbf{x}, \mathbf{y}, \mathbf{z}) \leq (\sigma^*, \sigma^+)(\mathbf{x}, \mathbf{y})$
- $EI(\tau) \subseteq (M^0)$
- $EI(\tau') \subseteq EI(\mathcal{M})$

then there is a $\overline{b} \in M^0$ such that $M^0 \models \tau(\overline{a}, \overline{c}, \overline{b})$ and $\mathcal{M} \models \tau'(f[\overline{a}, \overline{c}, \overline{b}])$.

We then will extend our initial model M^0 to a model M on $L^0_K \cup L^1_K \cup L^1_R$ so that $M \models (\sigma, \sigma')(\overline{a})$ iff $M^0 \models \sigma(\overline{a})$ and $\mathcal{M} \models \sigma'(f[\overline{a}])$. We then extend our model M to $M^* = M \cup \{c_m : m \in \mathcal{M}\}$ in such a way that M^* is a model in $L_K(\mathcal{M})$. We do this as follows:

- $M^* \models \neg Q(a) \leftrightarrow a \in M$

- $M^*|L^1 \models \phi^Q(c_{m_1}, \ldots c_{m_n})$ iff $\mathcal{M} \models \phi(m_1, \ldots m_n)$.
- $M^*|L^1 \models R_{\leq}(\overline{a}, \overline{b})$ iff $\|\overline{a}\|^1 \leq \|\overline{b}\|^1$.

It is immediate from this construction that all axioms of $T_K(\mathcal{M})$ are satisfied except possibly for (Homogeneity) ((Completeness) is trivially true as every tuple can be extended to be in some consistent pair of archetypes and hence must also satisfy a consistent pair of base predicates). To see that (Homogeneity) is satisfied assume we have a tuple $\overline{a} \in M^0$ such that

- $M^0 \models \sigma(\overline{a}) \wedge \tau(\overline{b})$ and $\mathcal{M} \models \sigma'(f[\overline{a}]) \wedge \tau'(f[\overline{b}])$.
- σ, τ are over a cell A
- σ', τ' are over a cell A'
- There is a consistent pair of archetypes $(\zeta, \zeta')(\mathbf{x}, \mathbf{y}) \Vdash (\tau, \tau')(\mathbf{x})$ such that (ζ, ζ') is over (C, C')
- $EI(\zeta) \subseteq EI(M^0)$
- $EI(\zeta') \subseteq EI(\mathcal{M})$

In order to prove homogeneity what we need to show is that there is an extension $(\eta, \eta')(\mathbf{x}, \mathbf{y}) \Vdash (\sigma, \sigma')(\mathbf{x})$ such that (η, η') is over (C, C') (because all such extensions are realized by our assumption on f). But this is exactly what the (Homogeneity on Consistent Pairs of Archetypes) says (and in fact why we have the axiom).

All that is left is to construct our bijection $f : M^0 \to \mathcal{M}$. We will do this by a method very similar to the creation of a term model. First let

$M^0 = \langle m_i : i \in \omega \rangle$ be an enumeration of M^0, $\mathcal{M} = \langle n_i : i \in \omega \rangle$ be an enumeration of $\mathcal{M}$, $\Sigma = \langle \sigma_i : i \in \omega \rangle$ be an enumeration of the archetypes realized in M^0 and $\Upsilon = \langle \tau_i : i \in \omega \rangle$ be an enumeration of the archetypes realized in $\mathcal{M}$.

Before we begin we will need some combinatorical definitions.

Definition 4.1.2.2. Let $w_3 : \omega \times \omega \times \omega \to \omega$ be a bijection with $w_3(i, j, k) \geq i$ for all i, j, k and let $w_2 : \omega \times \omega \to \omega$ be a bijection with $w_2(i, j) \geq i$ for all i, j.

Definition 4.1.2.3. Let $\langle (F^0_{i,j}, S_{i,j}) : j < \omega \rangle$ be a (countably redundant) enumeration of the archetypes such that $S_{i,j} \subseteq n$ and $\sigma_i(x_1, \ldots x_n) \Vdash F^0_{i,j}(\langle x_s : s \in S_{i,j})$.

Let $\langle (F^1_{i,j}, S_{i,j}) : j < \omega \rangle$ be a (countably redundant) enumeration of the archetypes such that $S_{i,j} \subseteq n$ and $\tau_i(y_1, \ldots y_n) \Vdash F^1_{i,j}(\langle y_s : s \in S_{i,j})$

Definition 4.1.2.4. Let $\langle G_i(\sigma, \tau) : i < \omega \rangle \subseteq \mathrm{AT}(T_K)$ be a (countably redundant) enumeration of the archetypes realized in $\mathcal{M}$ such that

- $G_i(\sigma, \tau)(\mathbf{x}, \mathbf{y}) \Vdash \tau(\mathbf{x})$
- $(\sigma, G_i(\sigma, \tau)) \in 2 - \mathrm{AT}(T_K(\alpha))$
- $(\sigma, G_i(\sigma, \tau)) \leq (\sigma|\mathrm{dom}(\tau), \tau)$

Intuitively this is the collection of extensions of τ in $\mathcal{M}$ which form a consistent archetype pair with σ.

Lemma 4.1.2.5. *If $(\sigma|dom(\tau), \tau)$ is consistent then this is non-empty.*

Proof. Immediate from the axiom (Extending 1-Color). □

Definition 4.1.2.6. Let $\langle H^0_{i,j,k} : k < \omega \rangle$, be the set of all archetypes $H^0_{i,j,k}(\mathbf{y})$ realized in M^0 extending $F^0_{i,j}$.

Let $\langle H^1_{i,j,k} : k < \omega \rangle$, be the set of all archetypes $H^1_{i,j,k}(\mathbf{y})$ realized in $\mathcal{M}$ extending $F^1_{i,j}$.

Stage -1:

Let $f(\emptyset) = \emptyset$. And let $(\zeta^0_0, \zeta^0_1) = (\emptyset, \emptyset)$. Let $\mathbf{x}_{-1} = \mathbf{y}_{-1} = \emptyset$

Let $m = w_2(n, l)$, and $n = w_3(i, j, k)$. $\zeta^0_i = \sigma_p$, $\zeta^1_i = \tau_q$

Stage 3m:

Let η be an amalgamation of ζ^0_i with $H^0_{p,j,k}$ around $F^0_{p,j}$ and let β be an amalgamation of η and ζ^0_{3m-1} around ζ^0_i (as it is included in the construction of ζ^0_{3m-1}).

Now let $\eta' = G_l(\eta, \zeta^1_i)$, so in particular (η, η') is a consistent pair of archetypes. Let β' be an amalgamation of η' and ζ^1_{3m-1} around ζ^1_i. We can now use Lemma 4.1.1.1 to find a consistent pair of archetypes $(\zeta^0_{3m}, \zeta^1_{3m})$ such that

- $\zeta^0_{3m}(\mathbf{x}, \overline{a}, \overline{b}, \mathbf{z}) \Vdash \beta(\mathbf{x}, \overline{a}, \overline{b})$
- $\zeta^1_{3m}(\mathbf{x}, \overline{a}, \overline{b}, \mathbf{z}) \Vdash \beta'(\overline{a}, \overline{b}, \mathbf{z})$
- $(\zeta^0_{3m}, \zeta^1_{3m})(\mathbf{x}, \overline{a}, \overline{b}, \mathbf{z}) \leq (\eta, \eta')(\overline{a}, \overline{b}) \leq (\zeta^0_{3m-}, \zeta^1_{3m-})(\overline{a})$
- $EI(\zeta^0_{3m}) \subseteq M^0$
- $EI(\zeta^1_{3m}) \subseteq \mathcal{M}$.

Now as ζ^0_{3m} extends ζ^0_{3m-1} we can find an extension $\mathbf{x}_{3m}$ of $\mathbf{x}_{3m-1}$ such that $M^0 \models \zeta^0_{3m}(\mathbf{x}_{3m})$. And similarly we can find an extension $\mathbf{y}_{3m}$ of $\mathbf{y}_{3m-1}$ such that $\mathcal{M} \models \zeta^1_{3m}(\mathbf{y}_{3m})$. So define $f[\mathbf{x}_{3m}] = \mathbf{y}_{3m}$.

Stage 3m+1:

Let r be the least such that $f(m_r)$ has not been defined yet. Let $\zeta^0_{3m+1} \in \mathrm{AT}(T_K(\alpha))$ be such that $M^0 \models \zeta^0_{3m+1}(\mathbf{x}_{3m}m_r\overline{c})$ with $\overline{c} \cap \mathbf{x}_{3m}m_r = \emptyset$. We know by (Extensions to 1-Color) that there is a ζ^1_{3m+1} such that $(\zeta^0_{3m+1}, \zeta^1_{3m+1})(\mathbf{x}, y, \mathbf{z}) \leq (\zeta^0_{3m}, \zeta^1_{3m})(\mathbf{x})$ and $(\zeta^0_{3m+1}, \zeta^1_{3m+1}) \in 2-\mathrm{AT}(T)$. Now finally let $b\overline{d}$ be such that $\mathcal{M} \models \zeta^1_{3m+1}(\mathbf{y}_{3m}b\overline{d})$ where $\mathbf{y}_{3m} \cap b\overline{d} = \emptyset$. Let $\mathbf{x}_{3m+1} = \mathbf{x}_{3m}m_r\overline{c}$, $y_{3m+1} = \mathbf{y}_{3m}b\overline{d}$ and $f[m_r\overline{c}] = b\overline{d}$.

Stage 3m+2:

Let r be the least such that n_r has not been defined yet. Let $\zeta^1_{3m+2} \in \mathrm{AT}(T_K)$ be such that $\mathcal{M} \models \zeta^1_{3m+2}(\mathbf{y}_{3m+1}n_r)$. We know by (Extensions to 0-Color) that there is a ζ^0_{3m+2} such that $(\zeta^0_{3m+2}, \zeta^1_{3m+2})(\mathbf{x}, y) \leq (\zeta^0_{3m+1}, \zeta^1_{3m+1})(\mathbf{x})$ and $(\zeta^0_{3m+2}, \zeta^1_{3m+2}) \in 2 - \mathrm{AT}(T)$. Now finally let a be such that $M^0 \models \zeta^0_{3m+1}(\mathbf{x}_{3m+1}a)$ where $a \notin \mathbf{x}_{3m+1}$. Let $\mathbf{x}_{3m+2} = \mathbf{x}_{3m+1}a$, $y_{3m+2} = \mathbf{y}_{3m+1}n_r$ and $f[a] = n_r$.

So the construction of f is done! Notice also that in this construction f is a bijection because it is injective and $\mathrm{dom}(f) = M^0$ and $\mathrm{range}(f) = \mathcal{M}$ (these last facts were taken care of in Steps 3m+1, 3m+2 respectively). Further, if $(\tau, \tau')(\mathbf{x}, \mathbf{y}) \Vdash (\sigma, \sigma')(\mathbf{x})$ and both are consistent pairs of archetype and $EI(\tau) \subseteq EI(M^0)$ and $EI(\tau') \subseteq EI(\mathcal{M})$, then for each $\overline{a} \in M^0$ such that $M^0 \models \sigma(\overline{a})$ and $\mathcal{M} \models \sigma'(f[\overline{a}])$ there is a $\overline{b}$ such that $M^0 \models \tau(\overline{b})$ and $\mathcal{M} \models \tau'(f[\overline{b}])$ (this was taken care of in Step 3m)

In particular we have finished the construction of the model. □

4.1.3 Quantifier Rank of $T_K(\mathcal{M})$

Theorem 4.1.3.1. *Let $M, N \models T_K(\mathcal{M})$ be such that $ATYPE(M|L^1_K) = ATYPE(N|L^1_K)$ and let $\omega * \gamma \leq \min\{Spec(M|L^0_K) \cap ORD, Spec(N|L^0_K) \cap ORD\}$. If for all slant lines $sl < \omega * \gamma$ $ATYPE(M|L^0_K)|sl = ATYPE(N|L^0_K)|sl$ then $M \equiv_{\omega * \gamma} N$*

Proof.

Definition 4.1.3.2. Define $I_\eta(M, N) = I_\eta$ as follows:
$I_{\omega*\eta+n} = \{f : M \to N$ such that

- f is a bijection, $|\mathrm{dom}(f)| < \omega$
- f preserves $Q, \langle c_m : m \in \mathcal{M}\rangle$
- If $q_f = a \in \mathrm{dom}(f), \neg Q(a)$, $M \models (\sigma^f_0, \sigma^f_1)(q_f)$ and $N \models (\tau^f_0, \tau^f_1)(f[q_f])$ then $\sigma^f_0|sl = \tau^f_0|sl$ and $\sigma^f_1|\infty = \tau^f_1|\infty$ (where sl is a slow slant line such that $sl < \omega * (\eta + 1)$, and $sl(|\mathrm{dom}(f)| + n) \geq \omega * \eta$)

Let $f \in I_{\omega*\eta+n+1}$, $\omega * \eta + n + 1 < \omega * \gamma$. The first thing to observe is that by the definition of $I_{\eta*\omega+n+1}$ f must preserve L^0_K, L^1_K and L_Q. The only atomic formulas we don't know immediately are preserved are those of L^1_R. But we do know that f preserves color on L^1_K (because $\sigma^f_1|\infty = \tau^f_1|\infty$) and hence f must preserve L^1_R (as $\mathcal{M}$ has no tuples of color ∞).

All that is left to show is that $\langle I_\zeta : \zeta < \beta\rangle$ satisfies the back and forth property. Choose $a \in M$. We want to find a $b \in N$ such that $g = f \cup (a, b)$ $g \in I_{\eta*\omega+n}$.

This breaks into two cases:

Case 1: $M \models Q(a)$

In this case we know that there is a $m \in \mathcal{M}$ such that $M \models c_m = a$. Let b be such that $N \models c_m = b$.

Case 2: $M \models \neg Q(a)$

Let $(\sigma_0, \sigma_1)(q_f a)$. We then know by Lemma 4.1.1.3 that there is a $(\tau_0, \tau_1) \in 2 - \mathrm{AT}(T_K)$ such that $\tau_0|sl = \sigma_0|sl$, $\tau_1|\infty = \sigma_1|\infty$ and $N \models (\exists b)(\tau_0, \tau_1)(f[q_f]b)$. Let b be such. We then have $g = f \cup (a, b) \in I_{\omega*\eta+n}$ because $sl(|\mathrm{dom}(g)| + n) = sl(|\mathrm{dom}(f)| + n + 1)$.

So $\langle I_\zeta : \zeta < \omega * \gamma \rangle$ witnesses (by Theorem 1.2.2.19) that $M \equiv_{\omega*\gamma} N$ $\square$

Notice that it is not obvious where we are using the assumption that $\mathrm{ATYPE}(M|L_K^0)|sl = \mathrm{ATYPE}(N|L_K^0)|sl$. This is because we really only need it to get the first step of the back and forth argument. In other words we only really need it to guaranteed that $I_{\omega*\eta+n}$ is non-empty and for every $a \in M$ there is a $b \in N$ (and vice versa) such that $(a, b) \in I_{\omega*\eta+n}$.

Also notice that there is an extra requirement for using Lemma 4.1.1.3 and that is that τ_1^f and σ_1^f be realized in some model together. But, we know that this must be the case because $M|L_K^1$ and $N|L_K^1$ have the same archetypes.

Now we want to prove an almost identical theorem except we want to show when the models satisfy the same archetypes in the language L_K^0 then they satisfy the same $\mathcal{L}_{\infty,\omega}$ sentences.

Theorem 4.1.3.3. *Let $M, N \models T_K(\mathcal{M})$ be such that $ATYPE(M|L_K^1) = ATYPE(N|L_K^1)$ and $ATYPE(M|L_K^0) = ATYPE(N|L_K^0)$ then $M \equiv_\infty N$*

Proof.

Definition 4.1.3.4. Define $I(M,N) = I$ as follows:
$I = \{f : M \to N$ such that

- f is a bijection, $|\mathrm{dom}(f)| < \omega$
- f preserves $Q, \langle c_m : m \in \mathcal{M}\rangle$
- If $q_f = a \in \mathrm{dom}(f), \neg Q(a)$, $M \models (\sigma_0^f, \sigma_1^f)(q_f)$ and $N \models (\tau_0^f, \tau_1^f)(f[q_f])$ then $\sigma_0^f|\infty = \tau_0^f|\infty$ and $\sigma_1^f|\infty = \tau_1^f|\infty$

Let $f \in I$. The first thing to observe is that by the definition of I we know that f must preserve L_K^0, L_K^1 and L_Q. The only atomic formulas we don't know immediately know from this are those of L_R^1. But we do know that f preserves color on L_K^1 (because $\sigma_1^f|\infty = \tau_1^f|\infty$) and hence f must preserve L_R^1 (as $\mathcal{M}$ has no tuples of color ∞).

All that is left to show is that $I \subseteq I$ satisfies the back and forth property. Choose $a \in M$. We want to find a $b \in N$ such that $g = f \cup (a,b)$ $g \in I$. This breaks into two cases:

Case 1: $M \models Q(a)$

In this case we know that there is a $m \in \mathcal{M}$ such that $M \models c_m = a$. Let b be such that $N \models c_m = b$.

Case 2: $M \models \neg Q(a)$

Let $(\sigma_0, \sigma_1)(q_f a)$. We then know by Lemma **??** that there is a $(\tau_0, \tau_1) \in$

$2-\mathrm{AT}(T_K(\alpha))$ such that $\tau_0|\infty = \sigma_0|\infty$, $\tau_1|\infty = \sigma_1|\infty$ and $N \models (\exists b)(\tau_0, \tau_1)(f[q_f]b)$. Let b be such. We then have $g = f \cup (a, b) \in I$ and so I witnesses (by Theorem 1.2.2.19) that $M \equiv_\infty N$ □

Notice that there is an extra requirement to use Lemma 4.1.1.3, and that is that τ_i^f and σ_i^f must be realized in some model together. But, we know that this must be the case because M and N have the same archetypes on both languages.

Corollary 4.1.3.5. *Let $M, N \models T_K(\mathcal{M})$ be such that $EI_{T_K}(M|L_K^0) = EI_{T_K}(N|L_K^0)$, $M|L_K^1 \cong N|L_K^1$ then $M \equiv_\infty N$.*

Proof. Immediate from Theorem 4.1.3.3 and the fact that if $EI_{T_K}(M|L_K^0) = EI_{T_K}(N|L_K^0)$ then $\mathrm{ATYPE}(M|L_K^0) = \mathrm{ATYPE}(N|L_K^0)$ □

Corollary 4.1.3.6. *Let $M, N \models T_K(\mathcal{M})$, $|M| = |N| = \omega$, $M|L_K^0 \cong N|L_K^0$, and $M|L_K^1 \cong N|L_K^1$. Then $M \cong N$.*

Proof. Immediate from Theorem 4.1.3.3 and Theorem 1.2.2.14 □

Corollary 4.1.3.7. *If $T_K(\alpha)$ has only κ many models then $T_K(\mathcal{M})$ has only κ many models up to $\mathcal{L}_{\infty,\omega}$ equivalence. In particular if $T_K(\alpha)$ has only countably many models all of which are countable, then $T_K(\mathcal{M})$ is scattered.*

Proof. Immediate from Corollary 4.1.3.6 □

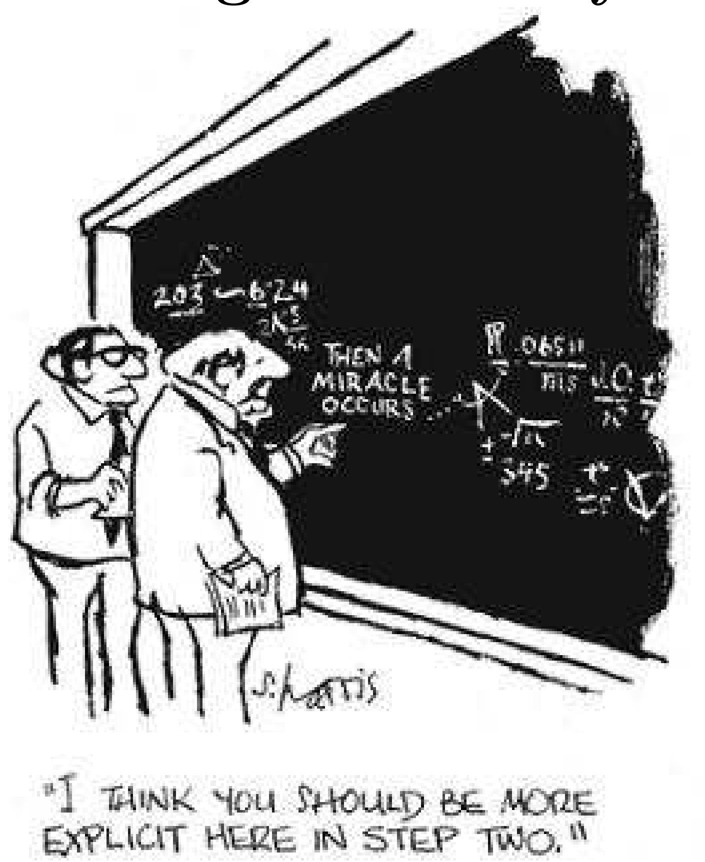
Corollary 4.1.3.8. *Let α be such that T_K has only ω many countable models. Further assume that the $\{qr(M) : M \models T_K\} \subseteq \omega * \alpha$ and is unbounded in $\omega * \alpha$.*

Let $\mathcal{M}$ be such that for all $\omega * \beta$ there is a $N \models T_K(\omega * \alpha)$ such that $ATYPE(N)|sl = ATYPE(\mathcal{M})|sl$ for all slant lines $sl < \omega * \beta$. Then $T_K(\mathcal{M})$ is a scattered sentence of $\mathcal{L}_{\omega_1,\omega}$ such that $\{qr(M) : M \models T_K(\mathcal{M})\} \subseteq \omega * \alpha$ and is unbounded in $\omega * \alpha$. Further, $qr(T_K(\mathcal{M})) = \max\{qr(T_K), \omega\}$.

Proof. Immediate from the previous results and the definition of $T_K(\mathcal{M})$. □

Chapter 5

Robin Knight's Theory Θ



[4]

Now that we have a way of constructing a scattered sentence with quantifier rank spectrum unbounded in $\omega * \alpha$ from a sentence which has a collection of archetypes all we need to do is come up with a sentence of $\mathcal{L}_{\infty,\omega}$ which has a collection of archetypes. However, this is easier said then done. The method we will use (and at the time of this writing the only known method) is to show that Θ from [8] has a collection of archetypes. For those readers who are already very familiar with [8] the proofs in this section should (hopefully) be pretty strait forward. However, for those who are not familiar with [8] this chapter will most likely be very difficult to follow. The only consolation I can make to the reader is that I have in fact gone through [8] line by line to ensure that everything followed from what came before. And, to the best of my ability as a proof checker the paper is sound (modulo trivial typos).

5.1 Introduction

Before we can prove that Θ has the properties we want, we will need two important new results concerning Θ. The first result is what we call "Generalized Saturation for Restricted Archetypes". This proof is a generalization the proof of what Knight calls "Generalized Saturation for Archetypes" (Proposition 4.3.2.1, in [8]). Generalized Saturation for Restricted Archetypes will be what allows us to get (Prediction up to a Slant Line). We will prove this in Section 5.2

The second new result we will allow us to define which pairs of archetype are consistent in our collection of archetypes. Specifically we will describe special pairs of cells and then describe when one pair of cells "extends" another. We then define the consistent pairs of archetypes to be those pairs of archetypes which are over one of these special pairs of cells and which also respects (Consistency of Color) from Definition 3.2.0.8. In Section 5.3 we will provide the necessary constructions to ensure that this can be done.

After we have all the background material we will show in Section 5.4 that Θ has a collection of archetypes and we will explicitly describe what it is.

After we show Θ has a collection of archetypes we will prove a strong bound on the quantifier ranks of the models of Θ which we are interested in (in Section 5.5). This will be the final step necessary to get a scattered theory with the desired quantifier rank spectrum.

We will end this Chapter by proving some other interesting results about Θ (Section 5.6).

It also important to note that much of the proof that Θ has the necessary properties is actually shown in [8]. What is more, the many of the cases which aren't shown in [8] rely on theorems which are very similar to theorems found in [8]. As such, there are some parts where we would like to say something like "We know this by methods identical to those in [8]". However, as [8] is not common knowledge (or even published at the time of writing this), in these cases we have reproduced the argument from [8] (with very minor modifications if necessary). To signify this, we put in parentheses after the number of the theorem a * and the number of the Theorem/Proposition/Lemma in [8] from which the argument originally appears.

As [8] only deals with countable models we will assume that all models of Θ will be countable. For the rest of this chapter a thorough understanding of [8] is assumed.

5.2 Generalized Saturation for Restricted Archetypes

5.2.1 Introduction

Before we start the proof of Generalized Saturation for Restricted Archetypes it is worthwhile to make sure we understand the proof of Generalized Saturation for Archetypes (Proposition 4.3.2.1 of [8]). Recall what Generalized Saturation says. It says that if we have an archetype $\sigma(\mathbf{x})$ and we have another archetype $\tau(\mathbf{x}, \mathbf{y})$ such that $\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x})$ and we have a model M such that $M \models \sigma(\overline{a})$ and τ doesn't place any tuples somewhere that is inconsistent with M (i.e. tuples which are above the spectrum of M or are inaccessible when M has no inaccessibles) then there is some $\overline{b}$ such that $M \models \tau(\overline{a}, \overline{b})$.

In fact the proposition says just a little bit more than this. Not only do we know that there is such a $\overline{b}$ but we can find an archetype extending σ (what is called υ in the proof of Proposition 4.3.2.1 of [8]) as well as a cell (what is called K_4) such that if $M \models \upsilon(\overline{a}, \mathbf{x}) \wedge K_4(\mathbf{x}, \mathbf{y}, \overline{a}, \overline{b})$ then $M \models \tau(\overline{a}, \overline{b})$. (The similarity to our (Prediction) from Definition 3.3.0.13 is no coincidence).

Lets go through the general idea of the proof. The main goal of this proof is to find the cell K_4. The way we want to get K_4 is to in some way translate down all the information about τ to being encoded in colors less than ω. That way they can be talked about in our cell.

However there is a problem. The problem is that cells by themselves can't say anything about what the actual color of tuples is. All cells can talk about is the relative position of tuples. For this reason, if we want to force a tuple in τ to be at a certain point in $o \in \Omega$ we had better have some other point "near" Ω so that we can compare them. This is the purpose of υ. υ is simply an extension of σ with some tuples in the right place to allow comparisons. In particular, notice that υ can easily be viewed as a trivial amalgamation of σ with an archetype υ^* which simply places tuples in approximately the same place as they are in τ (this isn't mentioned explicitly in the proof in [8] but is immediate form how υ is defined).

Now that we have a general idea of what is going on with Generalized Saturation for Archetypes we can begin to think about Generalized Saturation for Restricted Archetypes. Here we do almost exactly the same thing except this time we only talk about tuples up to a slant line. Beyond the slant line we aren't going to care what happens.

There is one subtle point to mention here and that is that it is okay for the slant line to be ∞. In this case (as long as σ and σ' can be realized in the same model (i.e one doesn't try to put a tuple somewhere which is inconsitent with where the other places a tuple)) the proof of Generalized Saturation for Archetypes in [8] gives us Generalized Saturation for Restricted Archetypes up to ∞.

5.2.2 Definitions

First, it will be useful to recall from [8] the definition of an Archetype.

Definition 5.2.2.1 (*3.2.1.1)**.** An archetype is a triple $E = \langle K, \mathcal{C}, \phi \rangle$ where

1. K is a cell. If $K = \langle \mathbf{x}_{i \in N}, \Phi \rangle$, say $\Phi(\mathbf{x}_{i \in N}) = P^N_{\mathcal{F}}$

2. $\phi = \bigcup_{1 \le i \le N} \phi * i$ where $\phi * i$ is a partial function whose domain includes $\{-\infty\}$, some i-sections in Ω and possibly ∞ and where

   (a) $\phi * i(-\infty)$ is an ambiguity tree $\langle T_i, F_i \rangle$ of characteristic arity i on $\mathbb{P}x_{i \in N}$

   (b) For all i, there exists $< T, F \rangle \in \mathcal{F}$ such that $\langle T_i, F_i \rangle \Leftarrow \langle T, F \rangle$

   (c) $\langle T_i, F_i \rangle$ is the projection of $\langle T_{i+1}, F_{i+1} \rangle$

   (d) If $\gamma \in dom\phi * i$ and γ' i its immediate predecessor in $dom\phi * i$ and $\phi * i(\gamma) \neq \top$ (see clause (e)) then

      i. If γ' is trivial, then $\phi * i(\gamma)$ is radically included in $\phi(\gamma')$

      ii. If γ' is non-trivial, then $\phi * i(\gamma)$ is a bough of $\phi(\gamma')$

      iii. If γ is non-trivial, then γ' is trivial, and $\phi * i(\gamma) = \phi * i(\gamma')$, and is rooted.

      iv. ∞ counts as being non-trivial for this purpose.

   (e) Top elements:

      i. The top element of γ_i of $dom\phi * i$ is one of

         A. ∞ or

         B. a non-trivial section; and $\phi * i(\gamma_i) = \langle \{n\}, \{\langle n, F_i(n) \rangle\} \rangle$ where n is atop node of T_i, or

         C. $\phi * i(\gamma_i) = \top, \gamma_i$ non-trivial [the idea is that in this case, all remaining i-tuples are suppressed to the top of γ_i]

         D. γ_i is a non-trivial section, $\phi * i(\gamma_i) \neq \top$ [and we think of all remaining i-tuples as being arity suppressed to the top element of γ.]

ii. Monotonicity: either γ_i is ∞, or it is strictly above γ_{i+1}, or $\gamma_{i+1} = -\infty$.

(f) Consistency: suppose $\gamma \in \text{dom}(\phi * i)$ and γ is non above γ_{i+1} Then the $i+1$-section γ' that γ belongs to is in $dom\phi * (i+1)$. Moreover, $\phi * (i+1)(\gamma')$ projects at γ to $\phi * i(\gamma)$ and the layout of $\phi * (i+1)(\gamma')$ correctly describes the position of γ

3. $\mathcal{C}$ is a precoloring such that if $\mathcal{C}(\mathbf{x}_{i\in X}) = a$, $|S| = i$, and a is the top element of γ, then $\gamma \in dom\phi * i$ and according to the layout of $\phi * i(\gamma), \mathbf{x}_{i\in S}$ is positioned in γ; or else $\phi * i(\gamma) = \top$.
Let $\mathcal{T}(E,n) = \phi * n(-\infty)$ and let $\mathcal{B}(E,n)$ be the branch of $\mathcal{T}(E,n)$ inferred from $\phi * n$

Definition 5.2.2.2. Let $\sigma = \langle K, \mathcal{C}, \phi\rangle$ be an archetype. We say σ is complete if $(\exists M)M \models$ "Robin Knight's Theory Θ" (from [8]) and $M \models \exists\mathbf{x}\sigma(\mathbf{x})$ as in Definition 4.2.2.5 of [8].

Definition 5.2.2.3. Let f be a slow slant line. If $\sigma = \langle K, \mathcal{C}, \phi\rangle$ is a regular archetype such that for all $n \leq |\text{dom}(\sigma)|$ $(\exists\gamma)\phi * n(\gamma) = \top \Rightarrow E\#n = f(n)$ then we say σ is an f-archetype.

Intuitively this means that for each arity either no suppression takes place (in the form of $\top$), or if it does take place, it happens uniformly at the slant line f.

Lemma 5.2.2.4 (*3.3.2.1)**.** *Let $\sigma = \langle K, \mathcal{C}, \phi\rangle$ be an archetype with precolors in $\{-\infty\} \cup \Omega \cup \{\infty\}$ (with domain$(\sigma) = \boldsymbol{x}_{(i\in R)}$) and let f be a slow slant line* *of rank greater than or equal to the arity of the largest tuple on σ with color above base(f). Then there is an archetype $\langle K, \mathcal{C}', \phi' \rangle$ with the same underlying cell-structure, such that $\mathcal{C}'^{\flat}(\boldsymbol{x}) \leq f(|\boldsymbol{x}|)$ and $\mathcal{C}^{\flat}(\boldsymbol{x}) < f(|\boldsymbol{x}|) \rightarrow \mathcal{C}'^{\flat}(\boldsymbol{x}) = \mathcal{C}^{\flat}(\boldsymbol{x})$.*

Proof. Let $f(1) = \lambda * \omega + n$(=base$(f)$+rank$(f)$). If $\mathcal{C}(\mathbf{x}) \geq \langle \lambda, n \rangle$, then let $\mathcal{C}'(\mathbf{x}) = \langle \lambda, n, 0, n-1, 0, \ldots 0, |\mathbf{x}| - 1 \rangle$. So, if $\mathcal{C}$ makes K true, then by arity suppression, so does $\mathcal{C}'$. We can now easily construct ϕ'. Namely, $\zeta \in$dom$\phi' * j$ iff $\zeta \in$dom$\phi * j$ and ζ is not above $\langle \lambda, n, \ldots, j-1 \rangle$; if $\zeta < \langle \lambda, n, \ldots j-1 \rangle$, then $\phi' * j(\zeta) = \phi * j(\zeta)$; if max$C$"$\mathbb{P}_j \mathbf{x}_{(i \in R)} \geq \langle \lambda, n, \ldots j-1 \rangle$, then $\phi' * j(\langle \lambda, n, \ldots j-1 \rangle) = \top$. □

Essentially what we are doing here is placing any tuple above the slant line f on it. Then, we are suppressing all information above the slant line to that slant line (if necessary). We need that the rank of the slant line is at least the arity of σ because, if it isn't than we might have n-tuples (for large enough n) which have to have their colors suppressed but can't be placed on the slant line.

Definition 5.2.2.5. Let $\sigma = \langle K, \mathcal{C}, \phi \rangle$ be an archetype. Let f be a slow slant line with rank(f) greater than the arity of the largest tuple with color above base(f) on σ. We define $\sigma | f$ to be the archetype defined in the previous lemma.

Notice that by the definition, for any regular τ, $\tau | f$ is an f-archetype for any slant line with rank greater than or equal to the arity of τ.

Definition 5.2.2.6. Let $\gamma = \omega * \lambda \in w_1$. Let $\varpi^{\sigma}_{\gamma} = \max\{\beta < \lambda : (\exists \mathbf{x})$ a tuple on σ such that $\mathcal{C}^{\flat}(\mathbf{x}) + |\mathbf{x}| > \omega * \beta\}$ and let $S^{\sigma}_{\gamma} = \max\{r : (\exists \mathbf{x})$ a tuple on σ such that $\mathcal{C}^\flat(\mathbf{x}) + |\mathbf{x}| = \omega * \beta + r\}$.

Let $f^{\tau,\sigma}_\gamma$ be a slant line with base$(f) \geq \max(\varpi^\tau_\gamma, \varpi^\sigma_\gamma)$ and rank$(f) > \max(S^\tau_\gamma, S^\sigma_\gamma)$.

We say $\tau|\gamma = \sigma|\gamma$ if $(\forall g > f^{\tau,\sigma}_\gamma)(g(1) < \gamma) \to (\tau|g = \sigma|g)$.
Similarly we say that $\tau|\gamma$ witnesses some fact ψ if $(\forall g > f_\tau)(g(1) < \gamma) \to (\tau|g = \sigma|g)$ witnesses ψ (where rank$(f_\tau) > S^\tau_\gamma$ and base$(f_\tau) \geq \varpi^\tau_\gamma$)

We will sometimes abuse notations and talk about the archetype $\tau|\gamma$. However, it will always be clear what we mean.

Lemma 5.2.2.7. *If $\tau|f^{\tau,\sigma}_{\gamma*\omega} = \sigma|f^{\tau,\sigma}_{\gamma*\omega}$ then $\tau|\gamma * \omega = \sigma|\gamma * \omega$.*

Proof. The idea is that in both τ and σ nothing happens in between $f^{\tau,\sigma}_{\gamma*\omega}$ and $\gamma * \omega$ and what is more every point which is on $f^{\tau,\sigma}_{\gamma*\omega}$ comes from a reduction of a point above $\gamma * \omega$. So, if $f^{\tau,\sigma}_{\gamma*\omega} < g$ and $g(1) < \gamma * \omega$ then both τ and σ will think there is a point on g iff they think there is one on $f^{\tau,\sigma}_{\gamma*\omega}$ in the same place. And, they will both think that below $f^{\tau,\sigma}_{\gamma*\omega}$ the exact same stuff happens as below g. (This is because of how we choose $f^{\tau,\sigma}_{\gamma*\omega}$). So, $\tau|g = \sigma|g$ and so $\tau|\gamma * \omega = \sigma|\gamma * \omega$. □

The idea is that saying two archetype are the same below a limit ordinal γ is the same as saying for all sufficiently large slant lines g below γ, their restriction to g is the same. But this is the same as saying that if you go above all interesting points on either archetype (below γ) and restrict the archetypes to such a slant line, then the restricted archetypes are the same. Also note that $\tau|\infty = \tau$.

Lemma 5.2.2.8. *If $\tau = \sigma|f$ with σ a regular complete archetype then τ is an f-archetype.*

Proof. We know that because σ is complete, it is the result of completely determining where, on the appropriate trees, it's subtuples are relative to some fixed realization ($\overline{a}$). This is done when we define what it means for σ to be realized (and hence complete) (Definition 4.2.2.4 and Definition 4.2.2.5 of Knights paper). As there can't be any unknown questions about where a fixed $\overline{a}$ is (as would be implied by a value of $\top$), we know that $\top$ isn't in the range of $\bigcup_i \phi_\sigma * i$. In particular, either range$(\phi_\tau * i) \subseteq f(i)$ and $\top \notin$ range$(\phi_\tau * i)$ or we are suppressed to $\top$ at f for arity i. But, because σ is complete and hence regular, this is exactly what is necessary to say τ is an f-archetype. □

5.2.3 Generalized Saturation for Restricted Archetypes

We now show a version of generalized saturation for complete f-archetypes. We do this by showing that if we have an archetype σ, if rank(f) is large enough, and if σ's restriction to f can be extended to an f-archetype τ then σ can be extended to a τ' such that $\tau = \tau'|f$. Intuitively what this is saying is that stuff higher up in color can't force stuff below to happen. But it is worth pointing out that stuff below can force stuff higher up to happen, which is the key to Knight's proof of Generalized Saturation for Complete Archetypes in [8].

Before we start it is important to mention again that this Proposition is a generalization of "Generalized Saturation for Archetypes" (Proposition 4.3.2.1 of [8]). And, as it is a generalization, after appropriate modifications of the definitions, the proof in [8] only requires small modifications (modulo typos). As such we have tried to follow the general layout of the proof in [8] in the hopes it will make understanding the proof easier.

Theorem 5.2.3.1 (Generalized Saturation for Restricted Archetypes)**.** *Suppose* $\sigma = \langle K^0, \mathcal{C}^0, \phi^0 \rangle$ *where* $K^0 = \langle x_{(i\in n^0)}, \Phi^0 \rangle$ *is a complete archetype, where there is a tuple* $x_{(i\in n^0)} \in \mathcal{M}$ *such that* $\mathcal{M} \models \sigma(x_{(i\in n^0)})$ *(that is* $x_{(i\in n^0)}$ *realizes* σ *in* $\mathcal{M}$*)*

Let f *be a slant line. Let* $\tau = \langle K^1, \mathcal{C}^1, \phi^1 \rangle$ *be an* f*-archetype where* $K^1 = \langle x_{(i\in n^1)}, \Phi^1 \rangle$*. If* $\tau \leq \sigma | f$*, then there exists* $\tau' \leq \sigma$ *such that* $\tau' | f = \tau$*. Further, if spec*$\mathcal{M} \cap \omega_1 = \omega * \zeta + n$ *and* $f(1) < \omega * \zeta$ *then*

$$\mathcal{M} \models \exists x_{(i\in[n^0,n^1))} \tau(x_{(i\in n^1)})$$

Also notice that if base$(f) \notin$ *Spec*$(\mathcal{M})$ *(or base*$(f) = \infty$*) then* τ *is a complete archetype, and this would just reduce to the statement of full Generalized Saturation.*

Before start the proof there is something which it is important to notice. For the archetype extension to be realized we require f to be below the highest limit ordinal which is a subset of the spectrum or ∞. The reason for this requirement is that in the case where $\mathcal{M}$ has inaccessibles we run into difficulty. The problem is that while we know that the points in τ which are on f are in fact above f, we don't have any control on how they are situated. So, if we allowed f to be above the largest limit, then the result of this process might put the tuples on f in a place they can't be in the model (because remember, this process is independent of the model we are working in so long as the model realizes $\tau|f$). For example, suppose $\mathrm{Spec}(\mathcal{M}) \cap \omega_1 = \omega * \gamma + m$, and $\mathrm{rank}(f) = n < m$, $\mathrm{base}(f) = \gamma * \omega$. Then, if τ is an f-archetype, extending the empty archetype, then the result of this process will be an archetype τ'. But τ', because it was created independently of any specific model, will have no idea where it should put elements of τ that were on f. Say, for example, it puts a 1-tuple at color $\omega * \gamma + n + 1$. Then, it could only hollow models in which it can be realized are those with spectrum $\omega * \gamma + n + 2$. (In fact, it wouldn't be able to be realized in any hallow model with $\text{spec}(\mathcal{M}) \cap \omega_1 \leq \gamma * \omega + \omega$ as all tuples with colors $\geq \gamma * \omega$ are inaccessible. But, that would take some effort to show and isn't needed.) Hence we need some sort of requirement to say which f-archetypes we are considering.

We will actually see later on that all models of Θ with $\text{Spec}(\mathcal{M}) \cap \omega_1 \geq \omega * \gamma$ "look the same" among tuples with color less than $\omega * \gamma$.

We are now ready to start the proof. Our method is to proceed in two stages. First, in Section 5.2.3.1, we construct "scaffolding" which contains information about the colors that we wish to manifest in τ'. This "scaffolding" is contained in an archetype υ with cell cell K_2.

Next, in Section 5.2.3.2, we recursively use witnessing tuples to transmit the information about υ down to the finite level. But, when we do this, we only worry about information below $\text{base}(f) + \omega$. We then obtain a "witnessing cell" of K_4 with K_1, such that whenever K_2 realizes υ, and K_1 realizes τ', $\tau'|f = \tau|f$.

5.2.3.1 The Scaffolding

Definition 5.2.3.2. An ordinal λ is bounded if it is in the spectrum and there are arbitrarily large tuples above it.

We then define a scaffolding archetype υ to contain σ and be based on

τ. We will do this by first defining υ^* which will contain all the information we need. We will then amalgamate υ^* with σ to get υ. This is very similar to how it was done in Knight's original proof of Generalized Saturation for Archetypes (although in his proof there were minor typos where he used cells when the author believes archetypes were intended).

Definition 5.2.3.3. υ^* contains some u^μ such that $\|u^\mu\| \in [\mu, \mu + \omega)$ whenever there is a tuple $\overline{a}$ on τ such that $\mathcal{C}^{1\flat}(\overline{a}) \in [\mu, \mu + \omega)$ and $\mu \leq \gamma$. Let A_μ be defined so that $\|u^\mu\| + |u^\mu| = \mu + A_\mu$.

Lemma 5.2.3.4. *This is possible, if $\mu <$ Spec$(\mathcal{M}) \cap \omega_1$.*

Proof. spec$\mathcal{M} \cap \omega_1$ is an ordinal. □

Definition 5.2.3.5. Let υ be the trivial amalgamation of υ^* and σ. Let $\upsilon = \langle K^2, \mathcal{C}^2, \phi^2 \rangle$

Lemma 5.2.3.6. *υ is an f-archetype*

Proof. Every tuple $\mathbf{x}$ which is added to υ can be chosen such that $\mathcal{C}^{2\flat}(\mathbf{x}) + |\mathbf{x}\| \leq f(|\mathbf{x}|)$ □

5.2.3.2 The witnessing cell

We define K_3, the witnessing cell.

Definition 5.2.3.7. Let $K_2^0 = K_2$, $\lambda_0 =$base$(f) + \omega$.

Now for $r > 0$, let K_2^r be the strong witness of K_2^{r-1} at λ_{r-1}, and let λ_r be the drop of the strong witnessing and let N_r be its breadth. Let p be least such that $\lambda_p = -\infty$.

Let υ^r be an archetype which is an extension of υ^{r-1} and realizes K_2^r.

Let $K_3 = K_2^p$.

Notice that the main difference between the above definition and the one which is used in Generalized Saturation for Archetypes is that we only require the strong witnesses describe colors up to base$(f) + \omega$. If we preserve all the information below base$(f)+\omega$ then everything below f is preserved. But, the point is that τ can't say anything above f and so by preserving everything below base$(f) + \omega$ we guarantee that whatever we get when restricted to f will get us τ back again.

Definition 5.2.3.8. Now, let $\upsilon^p = \langle\langle x_{(i \in n^0 \cup [n^1, n^3))}, \Phi^3\rangle, \mathcal{C}^3, \phi^3\rangle$. Let $\Phi^3(x_{(i\in n^0\cup[n^1,n^3))}) = P_{F^3}^{n^3-n^1+n^0}$ and let $\Phi^1(x_{(i\in n^1)}) = P_{F^1}^{n^1}$.

We have now constructed the witnessing cell.

5.2.3.3 Achieving Generalized Saturation

Definition 5.2.3.9. Let $\mathcal{F}^{*,4} = \mathcal{F}^1 \times \mathcal{F}^3$.

Definition 5.2.3.10. Let $\mathcal{F}^4$ be the subset of $\mathcal{F}^{*,4}$ defined as follows.
Let H_3^r, for $r \leq p$, be the N_r-history witnessed by $\mathbb{B}(\upsilon^r, N_r)$ as far as λ_r.
We insist that, in any element of $\mathcal{F}^4$:

1. If $\langle T, F\rangle \in \mathcal{F}^4$ has characteristic arity N_p, then the witnessing tuples declaring that H_3^p holds, also declare that the desired history H_1^p of $\mathcal{F}^1$ holds as far as $\lambda_p + N_p$: say such a branch is p-special.
2. Any p-special branch projects to an r-special branch for each r.

In these two conditions, the order is in terms of position in the ambiguity tree of appropriate characteristic arity.

These conditions are exactly what we need to guarantee that our cell K^4 forces $\mathbf{x}_{(i\in n^1)}$ to satisfy τ up to base$(f)+\omega$ when $\mathcal{M}\models \upsilon(\mathbf{x}_{(i\in n^0)\cup[n_1,n_2)})$, which will intern imply that if $\mathcal{M}\models \tau'(x_{(i\in n^1)})$ then $\tau'|f=\tau$.

Before we go on though, it is important to understand how K^4 and σ can force τ to be realized up to f. To do this we have to understand the relationship between, sensible trees, forests, witnessing tuples, the definition above and archetypes.

First, it is important to understand what a sensible tree is supposed to represent. We have a sensible tree on a tuple $\mathbf{x}$, that sensible tree is supposed to represent possible colors that the subtuples of $\mathbf{x}$ could have. But as we follow any single path through a sensible tree we are forced to make choices about what the colors of the tuples actually are. Now despite this generality, it isn't quite general enough, which is where forests come in. The idea behind a forest $\mathcal{F}$ is that it is just a collection of sensible trees such that if a tuple realizes $P_{\mathcal{F}}$ then it must be able to be placed on a sensible tree in $\mathcal{F}$.

Now for witnessing tuples. The idea behind the witnessing tuples is to be able to bring information about about tuples at a certain color down below that color. But, just knowing which sensible tree a tuple is on isn't enough to actually say what the color of that tuple is. So, the witnesses bring information about all possible colors that a tuple can have (depending on the path you follow). Also, when we are placing these witnessing tuples we are doing it in such a way that if we follow a path up a tree and we come a cross a witnessing tuple, it will be talking about the path we are on.

In order to get υ, what we are doing is we are placing some tuples in an expansion σ in such a way that it tells us some information about τ (and possibly our model). Then, we repeatedly take the witness transform until we get that all information below base$(f)+\omega$ can be found by looking at the restriction to colors less than ω (i.e. at cells) and comparing the results to the actual colors in υ. We now have a forest $\mathcal{F}^3$ which tells us everything that will happen all the way up to base$(f)+\omega$ for tuples in $\mathcal{F}^1$ (relative to υ). We then combine it with a copy of $\mathcal{F}^1$ to get $\mathcal{F}^{*,4}$. But, this isn't enough because while we know all the things that "could" happen, we don't know what does. So, the first thing we have to do is to make sure that we only consider the elements of $\mathcal{F}^{*,4}$ which always follow the same paths up both trees. In other words, we consider the subforest $\mathcal{F}^4$ of $\mathcal{F}^{*,4}$ consisting only of those trees where going up a path in $\mathcal{F}_3$ means you are going up the same path in $\mathcal{F}_1$. That is exactly what p-special means.

From now on the proof is almost identical to the proof of Generalized Saturation for Archetypes in Knight's paper.

Lemma 5.2.3.11 (*4.3.5.3)**.** *$\mathcal{F}^4$ is a forest and is $\leq \mathcal{F}^1, \mathcal{F}^3$.*

Proof. $\mathcal{F}^4 \leq \mathcal{F}^1$ is easy. Let $\mathcal{T}_1 \in \mathcal{F}^1$, let $\mathcal{T}^0$ be such that $\mathcal{T}_1^*|\mathbb{P}x_((i \in n^0)) \to \mathcal{T}_0$, where if s is the characteristic arity of $\mathcal{T}_1$, then $\mathcal{T}_1^*$ is the projection of $\mathcal{T}_1$ to arity $\min(n^0, s)$, and let $\mathcal{T}_3 \in \mathcal{F}^3$ be such that $\mathcal{T}_0 \Rightarrow \mathcal{T}_3|\mathbb{P}x_{(i\in n^0)}$.

Now the characteristic arity of $\mathcal{T}_1$ is $\leq n^1$, and we have assumed that the characteristic arity of $\mathcal{T}_3$ is $\leq n^0$.

Let $\mathcal{T}_4$ be an ambiguity tree formed from the weak ambiguity tree $\mathcal{T}_1 \times \mathcal{T}_3$ by harmonization. Then since $n^1 < N_p, \mathcal{T}_4 \in F^4$, and $\mathcal{T}_4|\mathbb{P}x_{(i\in n^1)} \Leftarrow \mathcal{T}_1$.

To prove $F^4 \leq F^3$, suppose $\mathcal{T}_3 \in F^3$. If the characteristic arity j of $\mathcal{T}_3$ is $< N_p$, then there is no difficulty. There exists $\mathcal{T}_0$ and $\mathcal{T}_1 \in F^1$ such that $\mathcal{T}_3^*|\mathbb{P}x_{(i\in n^0)} \to \mathcal{T}_0 \Rightarrow \mathcal{T}_1|\mathbb{P}x_{(i\in n^0)}$, where $\mathcal{T}_3^*$ is the projection to arity n^0 of $\mathcal{T}_3$.

Form $\mathcal{T}_3 \times \mathcal{T}_1$ accordingly and harmonize.

Suppose the characteristic arity of $\mathcal{T}_3$ is N_p. We construct a "transform" $\varpi\mathcal{T}_3$ such that $\varpi\mathcal{T}_3 \in F^4$, and $\mathcal{T}_4 \Rightarrow \varpi\mathcal{T}_3|\mathbb{P}x_{(i \in n^3 n^1 \cup n^0)}$.

Firstly, if $\mathcal{T}_3$ is not rooted, we may express it as a sum $+_{i=1}^{k}\mathcal{T}_{3,i}$; then let $\varpi\mathcal{T}_3 = +_{i=1}^{k}\varpi\mathcal{T}_{3,i}$.

So without loss of generality $\mathcal{T}_3$ is rooted, with root r, say. Let $\mathcal{T}_3 = \langle T_3, F_3 \rangle$.

We ask what the locus of $F^3(r)$ is: in particular, we ask whether it is at $\{-\infty\}$ or not. We argue that we may assume that it is not. For if it is at $\{-\infty\}$, then let $\check{\mathcal{T}}_3 = \langle \check{T}_3, \check{F}_3 \rangle$ be the deracination of $\mathcal{T}_3$; let $\varpi\check{\mathcal{T}}_3 = \langle \tilde{T}_3, \tilde{F}_3 \rangle$; letting $\{r_j : j < p\}$ be the set of roots of this, where $\tilde{F}_3(r_j) = \langle \mathbb{S}_j, H_j \rangle$, let $\overline{T}_3$ be the result of adding a root r to $\check{T}_3$, and let $\overline{F_3}$ be the extension of $\tilde{F}_3$ such that $\overline{F_3}(r) = \langle \bigcup_j \mathbb{S}_j, H \rangle$ where H is an appropriately chosen superficial history; let $\varphi\mathcal{T}_3$ be the harmonization of $\langle \overline{T_3}, \overline{F_3} \rangle$. It will be worth noting (once we have done the construction) that in the case where the root r of T_3 has locus above $\{-\infty\}$, that if $F_3(r) = \langle \mathbb{S}, H \rangle$ and $\mathbb{S}$ has just one element, then this property of $\langle T_3, F_3 \rangle$ is preserved by ϖ - so in the case we are examining at the moment, the harmonization of $\langle \overline{T_3}, \overline{F_3} \rangle$ will be filled.

So now let us assume that the locus of $F_3(r)$ is above $\{-\infty\}$. Let $F_3(r) = \langle \mathbb{S}, H \rangle$. We ask what H says about $\{-\infty\}$ and about the first N_p-section of Ω. We ask, in particular, whether it describes the interval $[-\infty, \langle 0, N_p - 1 \rangle]$ in the archetype υ^p.

If it does not, then we proceed as in the case where the characteristic arity of $\mathcal{T}_3$ is less than N_p.

If it does, we let H_4 be a common superficial history incorporating information about τ: specifically, H_4 just describe the condition of being p-special (see Definition 5.2.3.10): we describe how this is done. We begin by looking at H, and see what it tells us.

Consulting Definition 5.2.3.7, we note that H tells us that a passive witnessing tuple from K_2^{p-1} exists in $[-\infty, \langle 0, N_p - 1\rangle]$ at arity N_{p-1}, telling us about an initial portion of the history of the archetype υ^{p-2}; this initial portion includes a witnessing tuple from K_2^{p-2} which tells about an initial portion of the history of υ^{p-3}, and so on. Proceeding backwards in this way, we find that we have information about all of υ, namely a history where the locus of υ (by which is meant: the locus of the history corresponding to the hightest non-inaccessible node of the branch $\mathbb{B}(\upsilon, N)$ of $\langle T_2, F_2\rangle = \mathcal{T}(\upsilon, N)$) is some node x of T_2 which is either a maximal node of T_2 or has the property that all its successors are in the inaccessible region. (Even though τ has no inaccessibles, σ might and so there might be some inaccessibles in υ)

To gain an insight into what is happening here, we examine the movement from K_2^r to K_2^{r-1} in a little more detail. Consider for instance the first step, going from K_2^p to K_2^{p-1}. Reading the witnessing tuple from K_2^{p-1} does not tell us exactly what is happening in the next N_{p-1} section up in K_2^{p-1} – that is, in the first significant N_{p-1}-section not contained in the first N_p-section in K_2^p– for it does not specify exactly which N_{p-1}-tuples are located in it. However, it does specify precisely everything else about that N_{p-1}-section – including the portion of the history of K_2^{p-2} contained in it. So in the N_{p-1}-section, we have partial information about K_2^{p-1}, and exact information about K_2^{p-2}.

Proceeding in this way, we see that – on the assumption that no arities have been suppressed – we know precisely what is happening in the cell K_2 right the way up to base$(f) + \omega$.

Now, using arguments similar to those used prior to Lemma 4.3.5.3 in Knights paper, without loss of generality if the node x is not maximal in T_2, then it has a unique successor y: let $X = x$ if x is maximal, and let $X = y$ otherwise. Let H_4 be a superficial history restricting to the superficial history corresponding to the history of $F_2(X)$ on K_2, and to the history of τ on K_1.

Now define $\check{\mathcal{T}}_4$ to be the harmonization of the product $\langle\uparrow X, F_2| \uparrow X\rangle \times_{H_4} \mathcal{T}_1$, where $\mathcal{T}_1$ is $\mathcal{T}(\tau, P)$ where P is the characteristic arity of the archetype τ (recall that all f-types are regular)

It is now possible to define a tree $\mathcal{T}_4^*$ at the same characteristic arity such that

1. $\check{\mathcal{T}}_4$ is an up-closed subtree of $\mathcal{T}_4^*$,
2. The projections of $\mathcal{T}_4^*$ to the appropriate respective arities refine $\mathcal{T}_1$ and closely refine $\mathcal{T}_2$, with the map witnessing close refinement of $\mathcal{T}_2$ including the map which witnesses that $\check{\mathcal{T}}_4$ closely refines an up-closed subtree of $\mathcal{T}_2$.

We do this as follows. Once we have constructed $\check{T}_4$, the history $\hat{H}$ of its root yields a partial map θ taking the nodes from T_2 not above X, to initial segments of $\hat{H}$.

So, if $w \in \mathit{dom}\theta$, and w' is its immediate successor in domθ (if w' doesn't exist, then we are at X already), we form a product

$$\overline{\psi(w)} = \langle\uparrow \{w\}, F_2| \uparrow \{w\}\rangle \times_{\theta(w)} \mathcal{T}_1$$

where $\uparrow \{w\}$ is the upclosure of $\{w\}$ in T_2. Now define

$$\tilde{\psi}(w) = \overline{\psi}(w) \looparrowleft \tilde{\psi}(w').$$

(if $w = X$, just let $\tilde{\psi}(w) = \check{\mathcal{T}}_4$.)

Now if $\mathcal{T}_2$ is rooted and s is its root, let $T_4^* = \tilde{\psi}(s)$. If $\mathcal{T}_2$ is not rooted, let S be its set of roots, and let s be that root which is blow X. Then let

$$\mathcal{T}_4^* = (+_{s' \in S\{s\}} \langle \uparrow \{s'\}, F_2 | \uparrow \{s'\} \rangle \times \mathcal{T}_1) + \tilde{\psi}(s).$$

Now, noting that $\mathcal{T}_2 \Rightarrow T_4^* | \mathbb{P}x_{(i \in n^2 n^1 \cup n^0)}$, with this being witnessed by a function including the root of $\check{\mathcal{T}}_4$ in its domain, we may form the product $\mathcal{T}_3 \times \mathcal{T}_4^*$, and harmonize to obtain a tree $\mathcal{T}_4$. This is then p-special, because of the special role of the node X and the tree $\check{\mathcal{T}}_4$; so $\mathcal{T}_4 \in \mathcal{F}^4$. And also, $T_3 \Rightarrow \mathcal{T}_4 | \mathbb{P}x_{(i \in n^3 n^1 \cup n^0)}$, as required. We may define $\varpi \mathcal{T}_3 = \mathcal{T}_4$.

Finally, if the characteristic arity of $\mathcal{T}_3$ is above N_p, let T_3^* be its projection to characteristic arity N_p, and let $\varpi \mathcal{T}_3$ be the harmonization of $\mathcal{T}_3 \times \varpi \mathcal{T}_3^*$. □

Lemma 5.2.3.12 (*4.3.5.4)**.** *K_4 is realized as an extension of K_3, by some tuple $x_{(i \in n^4)}$ on $\mathcal{M}$.*

Proof. By the axiom of Generalized Saturation for Θ. □

Definition 5.2.3.13. Let $\tau^* = \langle K_4, \mathcal{C}_4, \phi_4 \rangle$ be a complete archetype such that

$$\mathcal{M} \models \tau^*(x_{(i \in n_4)})$$

and, let $\tau' = \tau^* | \mathbb{P}x_{(i \in n_1)}$.

Corollary 5.2.3.14. *$\tau' | f = \tau$.*

Proof. We have constructed $\mathcal{F}^4$ specifically to make this true. □

And so we are done!

Before we continue, we want to be clear why we didn't need to worry about inaccessible elements and the hollowness of our models as Knight did. What this process did was come up with a method for extending a given cell K^0 to a cell K^4 that would witness an initial portion of the domain $\mathbf{x}_{(i \in n_1)}$ would satisfy our τ up to f (assuming the appropriate other conditions on the rest of its domain). Now, we choose f specifically so we wouldn't have to encode anything above the largest limit ordinal. However, this does not mean that there are no elements in τ' which are in the inaccessible region. In fact there may even be some elements of τ' which must be in the inaccessible region (i.e. if there are elements of σ in the inaccessible region). However, what it does mean is the only time this construction will force new tuples to be placed in the inaccessible region is if there were already inaccessibles in σ (and so then the inaccessibles aren't a problem).

To be more precise all we know is that whenever K_0 is a cell of an archetype, we can find an archetype extension with K_4 as a cell. But, we don't know that this extension will force $\tau|f$ to be realized unless we know that the archetype over K_3 is in fact υ.

To end this section we will prove a couple of corollaries to Generalized Saturation for Restricted Archetypes.

Corollary 5.2.3.15. *Let* σ *be as in Theorem 5.2.3.1. Let* f_γ^τ *be the slant line with* $\varpi^\tau_{\omega*(\gamma+1)} > f^\tau_\gamma(1), f^\tau_\gamma(\max(S^\tau_{\gamma*\omega}, \textit{arity of } \tau) + 1) \geq \varpi^\tau_{\omega*\gamma}$. *Now, assume* $\tau|f^\tau_\gamma \leq \sigma|f^\tau_\gamma$, *then* $\exists \tau' \leq \sigma$ *such that* $\tau'|f^\tau_\gamma = \tau|f^\tau_\gamma$.

Proof. The idea is that if we know τ is an extension of σ below γ but above anything that happens in τ or σ then we can find an actual extension of σ which agrees with τ below anything that happens in τ below γ. $\square$

Corollary 5.2.3.16. *Let σ be as in Theorem 5.2.3.1. If $\tau|\gamma * \omega + \omega \leq \sigma|\gamma * \omega + \omega$, then $\exists \tau' \leq \sigma$ such that $\tau'|\gamma * \omega = \tau|\gamma * \omega$.*

Proof. If $\tau|\gamma * \omega + \omega \leq \sigma|\gamma * \omega + \omega$ then $\tau|f^{\tau}_{\gamma} \leq \sigma|f^{\tau}_{\gamma}$. But we then know by Corollary 5.2.3.15 that there is a $\exists \tau' \leq \sigma$ such that $\tau'|f^{\tau}_{\gamma} = \tau|f^{\tau}_{\gamma}$. But we then also have $\tau'|\gamma * \omega = \tau|\gamma * \omega$ because restriction is transitive and how we defined f^{τ}_{γ}. $\square$

One might hope that we could actually get $\tau|\omega * \gamma + \omega = \tau'|\omega * \gamma + \omega$ (because $\tau|\omega * \gamma + \omega \leq \sigma|\omega * \gamma + \omega$), but we can't in general. The reason is that for the construction we have to pick a point (below $\omega * \gamma + \omega$) which is above everything that happens in τ. Now we can pick that point as high as we want, but once we pick it we can't say anything about what is above it. So, while we faithfully preserve everything below the slant line ending at that point, what is above it is given free reign to do whatever it wants (so long as it is consistent with σ). So it is possible for some of the tuples which in τ are above $\omega * \gamma + \omega$ to be between $\omega\gamma$ and $\omega * \gamma + \omega$ in τ'.

In fact, not only do we know that this proof doesn't create a τ' such that $\tau|f(1) + \omega = \tau'|f(1) + \omega$ and $\tau' \leq \sigma$ but in the general this can't be done if $\gamma * \omega + \omega <$spec$M \cap \omega_1$. To see why this can't in general be done, assume it could in general be done. Let $I = \{g : g$ is finitely generated bijection from M to N s.t. if $\sigma(\text{dom}(g)), \tau(\text{range}(g))$ then $\tau|f(1)+\omega = \sigma|f(1)+\omega\}$. But, by our assumption and Generalized Saturation for Restricted Archetypes, if $g \in I$

and $m \in M$ then there is an $n \in N, g' \supseteq g$ with $g'(\text{dom}(g)^\frown m) = \text{range}(g)^\frown n$ and $g' \in I$ (and similarly, for each $n \in N$ there exists an $m \in M$ which makes this true). But then I is a set of partial isomorphisms (by construction) and hence $M \cong N$ (because both are countable). Except, we only assumed that $\text{Spec}(M) \cap \omega_1$, $\text{Spec}(N) \cap \omega_1$ were larger than $\gamma * \omega + \omega$ and not even that they were equal. $\Rightarrow\Leftarrow$

5.3 Consistent Pairs of Cells

The goal of this section is to define consistent pairs of cells and archetypes in such a way as will allow us to show the collection of archetypes satisfies (Homogeneity of Consistent Pairs of Archetypes) and (Extension of 1-Color). Suppose we have a consistent pair of cells (A, A') and we have a cell B which extends A. We then want to find a cell B' which extends A' such that (B, B') is a consistent pair of cells which extends (A, A').

The first questions we have to answer before we can show this is "What Exactly is a Consistent Pair of Cells?", and "What Does it Mean for A Consistent Pair of Cells to Extend Another?".

The central question which we will consider when we look at consistent pairs of cells is what is happening to the colors of the two cells. We are going to intuitively want (A, A') to be a consistent pair of cells if whenever there is a color C which is according to A there is also a color C' which is according to A' such that the color of a tuple according to C' is always at least as large as the color according to C.

While this is a good first approximation of what it means for (A, A') to be a consistent pair of cells, it turns out this isn't going to be quite enough. We will also need to understand the allowable ways in which a consistent pair of cells can be extended. Specifically we only want to allow extensions which allow all pairs of colors to have an extension. To be more precise, if (C, C') are precolors according to (A, A') and D is a precolor according to B whose restriction to dom(A) is C then if (B, B') is a consistent pair of cells there should be a precolor D' according to B' such that D' places every tuple at least as high as D and D' restricted to domain(C') is C'

For technical reasons this isn't quite what we want. We also want to be able to weaken the conditions on the two colors (C, C') so that we only require C to place nodes below C' i the node on C is below some fixed $o \in \Omega$ (and we don't worry about what is above them). Of course the precolors (D, D') will also only be "consistent" up to o.

As the actual cell structure are very complicated we will prove the main theorem of this section recursively in stages which mimic the definition in [8] of a cell. But first we will want a definition.

Definition 5.3.0.17. Let (S, S') be a "consistent" pairs of "structures" we are considering. Then we say that S is Below S' and S' is Above S.

Definition 5.3.0.18. If C, C' are precolors we say that (C, C') is a consistent pair up to $o \in \Omega$ if C, C' are on the same domain and to each tuple $\overline{a}$, $\min\{C(\overline{a}), o\} \leq \min\{C'(\overline{a}), o\}$. We say that (C, C') are consistent if they are consistent for all $o \in \Omega$.

5.3.1 Cells

Before we continue it is worth recalling the definition of a cell:

Definition 5.3.1.1 (*2.9.2.2)**.** A *cell* is a pair $K = \langle \mathbf{x}_{(i<n)}, \Phi \rangle$, where

1. $\mathbf{x}_{(i<n)}$ is a non-repeating tuple of variable letters

2. Φ is a relation with domain $\{\mathbf{x}_{(f(i):i<m)} : m \leq n, f \text{ injective}\}$, and range the set of predicates $P^m_{\mathcal{F}}$, and

3 For all m, for all one-to-one functions $f : m \to n$, $\langle \mathbf{x}_{(f(i):i<m)}, P^k_{\mathcal{F}} \rangle \in \Phi$ implies $k = m$

4 K satisfies LE (Local Embedding). That is if $f : k \to l$ and $g : l \to n$ are functions, and if $P^k_{\mathcal{F}'}$ and $P^l_{\mathcal{F}}$ are letters, where either f is not one-to-one or $\mathcal{F}'$ is not extensible to $\mathcal{F}$ along f, then

$$(\langle \mathbf{x}_{(g(i):i<l)}, P^k_{\mathcal{F}} l \rangle \in \Phi) \to \neg(\langle \mathbf{x}_{(g \circ f(i):i<k)}, P^k_{\mathcal{F}'} \rangle \in \Phi)$$

So Φ is a partial function.

If $\langle \mathbf{x}_{(i:i \in S)}, \Phi \rangle$ and $\langle \mathbf{x}_{(i:i \in S')}, \Phi' \rangle$ are cells, we say $\langle \mathbf{x}_{(i:i \in S)}, \Phi \rangle \geq \langle \mathbf{x}_{(i:i \in S')}, \Phi' \rangle$ iff $S' \subset S$, and $\Phi' | \mathbb{P}\mathbf{x}_{(i \in S)} = \Phi$

It is important to notice that a cell is "almost" completely determined by the forest which the domain of the cell realizes. The only other information are the subforests each of which has to closely refine ($\Rightarrow$) the large forest on the appropriate arity. In particular if we could show that our construction "respects" close refinement, then all that would be needed would be to show the construction for Forests.

To be precise lets say we have two "structures" S, S' and an extension of S to a structure of the same type T. We will construct a structure T' in terms of T and S'. Let us call the structure we construct $\mathrm{Const}(T, S')$. What we want to show is that if $S \Rightarrow R$, $S' \Rightarrow R'$ (where R' extends R) and $T \Rightarrow Q$ then $\mathrm{Const}(T, S') \Rightarrow \mathrm{Const}(Q, R')$.

Definition 5.3.1.2. If C, C' are cells we say that (C, C') is a consistent pair if C, C' are on the same domain for each tuple $\mathbf{x} \subseteq \mathrm{dom}(C)$ if C witnesses $P_F(\mathbf{x})$ and C' witnesses $P_{F'}(\mathbf{x})$ then (F, F') is a consistent pair of forests.

Definition 5.3.1.3. If (C, C'), (D, D') are consistent pairs of cells we say $(D, D')(\mathbf{x}, \mathbf{y})$ is a Consistent Extension of $(C, C')(\mathbf{x})$ ($(D, D')(\mathbf{x}, \mathbf{y}) \leq (C, C')(\mathbf{x})$)if for each tuple $\mathbf{x} \subseteq \mathrm{dom}(C)$

- (C, C') witnesses $(P_{F_x}, P_{F_x{}'})(\mathbf{x})$
- (D, D') witnesses $(P_{G_x}, P_{G_x{}'})(\mathbf{xy})$
- $(F_x, F'_x) \geq (G_x, G'_x)$

(see Section 5.3.2 for an explanation of $\leq$ on pairs of forests)

5.3.2 Forests

Now recall the definition of a forest.

Definition 5.3.2.1 (*2.8.1.1)**.** A forest on $\mathbb{P}\mathbf{x}_{(i \in N)}$ is a set $\mathcal{F}$ of reduced, filled ambiguity trees of all characteristic arities up to N on $\mathbb{P}\mathbf{x}_{(i \in N)}$, closed under

1. Projection: so if $\mathcal{T} \in \mathcal{F}$, then its projection is in $\mathcal{F}$, and

2. Disentangleing ambiguity trees into their component sensible trees: that is, if $\langle F, T \rangle$ is an ambiguity tree in $\mathcal{F}$, $x \in T$, and $F(x) = \langle S, H \rangle$, then for each $\mathcal{T} \in \mathcal{S}$, the ambiguity tree representation of $\mathcal{T}$ is in $\mathcal{F}$

3. Retrojection: so if $\mathcal{T} \in \mathcal{F}$ has characteristic arity $n < N$, its retrojection is the ambiguity tree representation of the sensible tree $\langle\langle \sqsubset, \Upsilon\rangle, \mathcal{T}, \phi\rangle$, where

   (a) $\Upsilon = \{-\infty\} \cup \Omega$

   (b) If $|\mathbf{x}| = n + 1$ then $\mathbf{x} \sqsubset -\infty$

   (c) $\mathrm{dom}(\phi) = \{\{-\infty\}\}$ and $\phi(\{-\infty\}) = \mathcal{T}$

Definition 5.3.2.2. Let F, F' be forests. We say F' is above F if for every precolor C according to an ambiguity tree $T \in F$ there is a precolor C' according to an ambiguity tree $T' \in F'$ such that (C, C') is a consistent pair (and T, T' have the same arity).

Definition 5.3.2.3. Let F, F', G, G' forests such that

- F' is above F
- G' is above G
- $G(\mathbf{x}, \mathbf{y}) \Vdash F(\mathbf{x})$
- $G'(\mathbf{x}, \mathbf{y}) \Vdash F'(\mathbf{x})$
- If $C \in S \in F, C' \in S \in F'$ and $D \in T \in G$ such that

– (C, C') is a consistent pair of precolors up to $o \in \Omega$

– S, S' have the same arity

– $D|dom(C) = C$

Then there is a $D' \in T' \in G'$ such that

– (D, D') is a consistent pair of precolors up to $o \in \Omega$

– T, T' have the same arity

– $D'|dom(C) = C'$

We then say that (G, G') is a Consistent Extension of (F, F') $((G, G') \leq (F, F'))$.

Definition 5.3.2.4. If arity(F) =arity$(F') = 1$ then we say (F, F') is a consistent pair of forests if F' is above F.

For arbitrary arity we say that (G, G') is a consistent pair of forests if $(\exists(\emptyset, \emptyset) = (F_0, F_0'), \ldots (F_n, F_n') = (G, G'))$ where (F_{i+1}, F_{i+1}') is a consistent extension of (F_i, F_i') and

Lets consider what exactly is going on here. The first thing to notice is that the forests don't usually say much about the colors of tuples. All the forests really talk about is the relative position of colors. This is the reason why just having F over F' isn't enough for (F, F') to be a consistent pair of forests.

Hence what we are really interested isn't whether there is some consistent pair of precolors (C, C') according to forests (F, F') which is consistent (as there usually will be). Rather what we care about is that once we have our precolors they have enough extensions. This is a common theme when dealing with forests and cells. The forests and cells can't tell you exactly what is going on, they can only tell you what is going on relative to something else. But if you already have a list of colors of Ω which you can use as reference then the forest can tell you exactly what is going on (if you remember this was the reason why we needed υ^* in the proof of Theorem 5.2.3.1).

Specifically in this case we want (G, G') to be a consistent extension of (F, F') if whenever we have a conistent pair of colors (C, C') according to (F, F') and we extend the bottom one to a precolor D in a way consistent with G then we can extend the top one to a precolor D' in a way consistent with G' so that we still have a consistent pair (D, D').

Remember our original goal. Given a consistent pair of forests (F, F') and an extension of F to a forest G we want to find some forest G' such that (G, G') is a consistent extension of (F, F'). In particular remember our forests are composed of ambiguity trees. Hence it is enough to show that given a "consistent pair of ambiguity trees" (S, S') relative to a consistent pair of precolors (C, C'), and an ambiguity T and precolor D as in Definition 5.3.2.3 that we can find the an ambiguity tree T' and a precolor D' such that (T, T') is a "consistent extension" of (S, S') relative to (D, D'). This is enough because we can then let our forest G' be the collection of all such ambiguity trees.

Of course we still have to define what a "consistent pair of ambiguity trees relative to a consistent pair of precolors" is. And so we will now start our recursive definition.

5.3.3 Augmented Unitary Trees

The first stage of the recursion occurs at the level of Augmented Unitary Trees. An augmented unitary tree is it is meant to provide the most basic framework for assigning possible colors to a subtuples. One way to think about an augmented unitary tree is an augmented unitary tree is to a precolor what a tree ordering is to a linear order.

Definition 5.3.3.1. An augmented unitary tree of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i\in N)}$ is a pair $\langle \sqsubset, \Upsilon \rangle$ such that

1. Υ is a tree of colors of characteristic arity n
2. $\sqsubset$ is a relation with domain some subset of $\mathbb{P}_{\leq n}\mathbf{x}_{(i\in N)}$ and range Υ;
3. Writing $\breve{\sqsubset}$ for $\sqsubset |\mathbb{P}_n\mathbf{x}_{(i\in N)}$, $\langle \breve{\sqsubset}, \Upsilon \rangle$ is a unitary tree of characteristic arity n;
4. For all S, $\{a; \mathbf{x}_{(i\in S)} \sqsubset a\}$ is an antichain (not necessarily maximal), and if $\mathbf{x}_{(i\in S)} \sqsubset a$, then $|a| = |S|$;
5. For all S and a such that $\mathbf{x}_{(i\in S)} \sqsubset a$, there exists T of arity n and $b \geq a$ such that $\mathbf{x}_{(i\in T)} \sqsubset b$ (so the tuples of arity n delimit the augmented unitary tree);
6. If $V \supset U$, and $\mathbf{x}_{(i\in U)} \sqsubset a$, then there is $b < a$ such that $\mathbf{x}_{(i\in V)} \sqsubset b$

Definition 5.3.3.2. Let U, U' be augmented unitary trees of arity n on the same domain and (C, C') be a consistent pair of colors up to o. Then we say that (U, U') is a Consistent Pair of Augmented Unitary Trees with Respect to

(C, C') (up to o) if there is a branch of the augmented unitary trees U, U' which places all $\leq n$ tuples in the same place as C, C' respectively.

Definition 5.3.3.3. We say an augmented unitary tree of arity n is extendible (relative to o) if every branch point has a predecessor and o is an $n+1$ successor.

Before we continue lets clarify what it means for an augmented unitary tree to be extendible. Because of how the augmented unitary trees of arity n are defined we know every branch point projects to a point $o \in \Omega$ which is an $n-1$ successor. The reason for this is that the intended interpretation of a tuple being placed at a point in Ω is that that tuples color plus its arity is at that point. Hence, if we are going to place an n-tuple somewhere it had better be at a place which is an $n-1$ successor (or the color function ceases to make sense).

While this definition of augmented unitary tree is fine, it makes it difficult, in general, to increase the arity of an augmented unitary tree. This is because, apriori, there is nothing to stop a branch point from being a $n-1$ successor in Ω but not an n successor and hence we will not be able to place any tuples of arity $n+1$ right after the branch point (which is not good). We say an augmented unitary tree is extendible if in fact we can shift the branches just a little (but keep the placement of all tuples) to allow us to place $n+1$ tuples after the branch.

Lets suppose we have a consistent pair of augmented unitary trees (U, U') of arity n with respect to (C, C') which are both extendible (say the domain of (U, U') is $\mathbf{x}$). Now lets further suppose that V is an $n+1$ arity augmented unitary tree (on domain $\mathbf{x}y$) such that when V is restricted to $\mathbf{x}$ and such that every branch point of V is moved to its predecessor then you get U. Further let D be some path through V which restricts to C on $\mathbf{x}$. We now want to find a V' such that V' restricted to $\mathbf{x}$, and with the modified branch points, gives U'. Further we want a path D' on V' which when restricted to $\mathbf{x}$ gives C', and such that (D, D') is a consistent pair.

There are two things we have to consider when trying to construct our augmented unitary tree V'. The first is that the collection of points where any $n+1$ tuple is placed is an anti chain with at most one placement not being immediately after a branch point. The second is that $n+1$ tuples delimit the tuples of arity $\leq n$ on the tree.

The way we are going to construct our tree is as follows. First lets look at the branch that $\mathcal{C}'$ follows. Suppose we have a tuple $\overline{a} \subseteq \mathbf{x}$ which is placed at a point which projects to $o' \in \Omega$. What we do will be determined by the nature of o'.

- If o' is on an $n+1$ section, then place $\overline{a}, y$ on the same section (so in particular this means that if $\overline{a}$ is immediately after a branch point, so is $\overline{a}y$)
- If o' is not on a $n+1$ section (so we can't place $\overline{a}y$ immediately before it) then we have thwo cases
  - Case 1: $o' \leq o$

    Place $\overline{a}y$ below $\overline{a}$ but above anything that happens on D below where $\overline{a}$ is placed (on the branch $\mathcal{C}'$ follows in U'). We can do this because we know that $\overline{a}$ is at least as high on C' as on C because we are below o.

- Case 2: $o' > o$

  Here we want to put $\overline{a}y$ at some point above o (because we only require that (D, D') will be consistent up to o). The problem is it is conceivable that o and o' are in the same n section and in this case we know we can't place $\overline{a}y$ above o (o is not an $n+1$ successor). This though is a problem because even if we place $\overline{a}y$ above everything else in D (which is below o) we still don't have (D, D') is consistent up to o as one precolor might place $\overline{a}y$ below o and the other might place it above o (if D places $\overline{a}$ above where D' places $\overline{a}$).

  However, in the case of extendible augmented unitary trees up to o we know this isn't a problem as o has to be on an $n+1$ section and so can't be in the same n-section as o' (and in fact this was why we had this requirement). Hence we can place $\overline{a}y$ at some $n+1$ section above o.

Finally place y along D' and above anything which happens anywhere. And place $\overline{a}y$ immediately after the branch points in all other branches.

So our constructed V' has all the properties we want and is almost an augmented unitary tree. All that is left is to ensure that there are $n+1$-tuples delimiting the tree. But that is easy. Simply add for each branch a new $n+1$ tuple $\mathbf{z}$ and place it beyond anything on that branch in V'. Similarly place all such $\mathbf{z}$ at $-\infty$ on V.

For notational consistency we will define $\mathrm{Const}([V, D], [U', C']) = [V', D']$

5.3.4 Sensible Trees of arity 1

As with [8], we will start our recursion with the case of sensible trees of arity 1.

Let S, S' be consistent extendible sensible trees of arity 1 on $\mathbf{x}$ relative to a consistent pair of precolors (C, C'). (recall that a sensible tree of arity 1 is the same thing as a unitary tree of arity 1). Now let T be a sensible tree of arity 2 whose restriction to $\mathbf{x}$ at arity 1 is S. Further let D be a precolor according to T (i.e. a path through the tree) whose restriction to S at arity 1 is C.

We know by the previous section that we can find an augmented unitary tree T' of arity 2 with the properties that we want. All that is left is to turn this augmented unitary tree into a sensible tree. To do this we need to assign to each point in range($\sqsubset$) an ambiguity tree of arity 1.

Recall from the definition of ambiguity tree that an ambiguity tree of arity 1 consists of a finite tree and a collection of ambiguity nodes of arity 1, which are placed on the nodes of the finite tree. Further notice that we require that the gist (see Definition 2.4.1.6 of [8]) of the root of the ambiguity tree is the same as the gist of the point on which it was placed (when then node is non-trivial).

But, this isn't hard to do. We simply create an ambiguity tree where the tree part is isomorphic (as a partial order) to the nodes on the augmented unitary tree T'. Then at the root of this ambiguity tree place an ambiguity node which contains only one sensible tree of arity 1 (i.e. the tree S'). We then follow the tree S' along. Suppose we come to a point N with a node on it. Then this node corresponds to a node N^* on our ambiguity tree. At this node place an ambiguity node with 1-history up to N^* and only one element which is the unitary tree S' pruned below N. So in particular the gist of the ambiguity node at this point will be the same as the gist of N.

Another and perhaps more intuitive way to think about the process is by considering only the roots of the ambiguity trees placed at each point in range($\sqsubset$). Here what we are doing is simply placing an ambiguity node at each point in range($\sqsubset$) in such a way that the history of the node is exactly the path through the sensible tree S' up until that point. Further, the only tree in the ambiguity node is the tree S'. Then, in order to find the ambiguity tree which is placed at each point in T' all we do is combine all the ambiguity nodes at or above the point in T' into an ambiguity tree.

In particular we know that there is a path D' along this sensible tree of arity 2 such that D' restricted to $\mathbf{x}$ at arity 1 is C' and further (D, D') is a consistent pair of colors (this was the whole point of Section 5.3.3).

Technically there is one more thing to worry about. What we have actually defined here isn't in fact a sensible tree because we need to place unrooted ambiguity trees at trivial 1-sections in between any two rooted ambiguity trees. But this isn't a problem as there is an obvious way to choose the ambiguity trees (just chop off the root of the first tree) and between any two non-trivial 1-sections there is a trivial one (Lemma 2.2.3.5 of [8]). So we can define $\mathrm{Const}([T, D], [S', C']) = [T', D']$

All that is left is to prove that if $S \Rightarrow S_*|\mathrm{dom}(S)$ and $S' \Rightarrow S'_*|\mathrm{dom}(S)$ are all unitary trees of arity 1 such that $(S, S'), (S_*, S'_*)$ are consistent pair and $T \Rightarrow T_*|\mathrm{dom}(T)$ are augmented unitary trees of arity 2 such that T extends S and T_* extends S_* then $\mathrm{Const}([T, D], [S', C']) \Rightarrow \mathrm{Const}([T_*, D], [S'_*, C']) =$

$[T'_*, D']$ (were C, C', D, D') are the appropriate colors. But this is trivially true because on sensible trees of arity 1, $X \Rightarrow Y$ iff $X = Y$.

While we are done with the construction (and the construction does in fact ensure what we want) we still want to make one more observation. The observation is that if we have S', S^* such that they agree on the 1-history which C', C^* respectively follow then in fact the corresponding T', T^* will produce paths D', D^* which agree on the 2-history. The reason is simply that everything that was done to create a new branch is completely determined by the old branch we are looking at (i.e. we never care about something which happens on the tree outside of the branch we are looking at).

5.3.5 Ambiguity Nodes

Recall the definition of an ambiguity node.

Definition 5.3.5.1 (*2.5.1.1)**.** An ambiguity node of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i \in N)}$ is a pair $\langle \mathcal{S}, H \rangle$ such that

1 H is an n-history of characteristic arity n, and

2 $\mathcal{S}$ is a non-empty set of order-representations of sensible trees of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i \in N)}$ for each of which H is an n-history

Our inductive assumption will be, given a consistent pair of extendible sensible trees (S, S') of arity n relative to a consistent pair of colors (C, C') and such that T is an extension of S and D is an extension of C then we can find an extension of T' with color D' such that

- D' is an extension of C'

- (T, T') is consistent relative to (D, D')
- T' restricted to domain of S' and arity n is S'

Definition 5.3.5.2. We say that a pair of ambiguity nodes (A, A') is Consistent Relative to a consistent pair of colors (C, C') if for every sensible tree S in A such that S follows C along the history of A then there is a sensible tree S' in A' such that S' follows C' along the history of A' (i.e (S, S') are consistent relative to (C, C')).

Definition 5.3.5.3. We say that a consistent pair of ambiguity nodes (B, B') of arty $n+1$ extends a consistent pair of ambiguity nodes of arity n (A, A') on $\mathbf{x}$ if

(1) (B, B') restricted to arity n and $\mathbf{x}$ equals (A, A')

(2) Whenever we have a consistent pair of sensible trees (S, S') relative to (C, C'), where $(S, S') \in (A, A')$ and S, S' follow the colors C, C' along the histories of A, A' and a $T \in B$ extending S, there is a $T' \in B'$ extending S' such that (T, T') is a consistent pair of sensible trees relative to (D, D') (where D, D' are the colors according to (T, T') along the histories of (B, B')).

This definition, while a little complicated, says exactly what we want it to. The idea behind an extension is that whenever we have a consistent pair of "structures" and we extend the bottom one, we can find some extension of the top one. This is what Definition 5.3.5.3 says.

Before we continue it is worth pointing out something explicitly. Notice that here we have only required the colors to be the same as far as the history of the ambiguity nodes goes. This is because beyond the history of A, A' the ambiguity nodes can't really say anything about the trees. In fact, the only reason why we insisted that we consider consistent pairs of precolors up to $o \in \Omega$ instead of just consistent pairs of precolors was so that we could handle this case.

Our goal is given (A, A') a consistent pair of ambiguity nodes relative to precolors (C, C'), and an ambiguity node B extending A to come up with an ambiguity node B' and color D' so that (B, B') is a consistent pair of ambiguity nodes relative to (D, D'). But this is easy by the inductive hypothesis. Let $B' = \{\mathrm{Const}([T, D], [S', C']) : (S, S')$ is a consistent pair of sensible trees relative to (C, C') (which follow the histories of the ambiguity nodes) up to the height of corresponding histories of the ambiguity nodes with $S' \in A'$ and $T \in B$ is an extension of $S \in A\}$

Now B' is an ambiguity node because whenever we have two sensible trees which follow the same branch (i.e as in the case of an ambiguity node) then this construction applied to both the trees returns the same branch. So we can let the history of B' be the branch common to all of the sensible trees. Further (B, B') is a consistent extension of (A, A') relative to (D, D') (the colors extending (C, C')) by the way it was constructed.

All that is left is to show if

- $A \Rightarrow A_*|\mathrm{dom}(A)$ and $A' \Rightarrow A'_*|\mathrm{dom}(A)$ are all ambiguity nodes of arity n
- $(A, A'), (A_*, A'_*)$ are consistent pairs of ambiguity nodes.

- $B \Rightarrow B_*|\text{dom}(B)$ are ambiguity nodes of arity $n+1$
- B extends A and B_* extends A_*

then $[B', D'] = \text{Const}([B, D], [A', C']) \Rightarrow \text{Const}([B_*, D], [A'_*, C'])|\text{dom}(B) = [B'_*, D']|\text{dom}(B)$ (were C, C', D, D') are the appropriate colors.

But recall what it means for $X \Rightarrow Y$ when X, Y are ambiguity nodes.

Definition 5.3.5.4 (*2.5.2.2 (2))**.** Suppose $\langle \mathcal{S}, \mathcal{H} \rangle$ and $\langle \mathcal{S}', \mathcal{H}' \rangle$ are ambiguity nodes of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i \in N)}$
Then we say $\langle \mathcal{S}, \mathcal{H} \rangle \Rightarrow \langle \mathcal{S}', \mathcal{H}' \rangle$ iff

(a) for all $\mathcal{T} \in \mathcal{S}$, there exists $\mathcal{T}' \in \mathcal{S}'$ such that $\mathcal{T} \Rightarrow \mathcal{T}'$; and

(b) for all $\mathcal{T} \in \mathcal{S}$, there exists $\mathcal{T}' \in \mathcal{S}'$ such that $\mathcal{T} \Rightarrow \mathcal{T}'$; and

(c) H and H' have the same superficial history.

In particular if we have a $T_* \in [B'_*, D']$ we need to find a $T \in [B', D']$ such that $T_* \Rightarrow T$. But this is easy as T_* must have come (via the construction) from a $S_* \in [A'_*, C']$ (i.e. $T_* = \text{Const}([T^+_*, D], [S_*, C'])$, with $T^+_* \in B_*$). And, by assumption there must be some $S \in [A', C']$ such that $S_* \Rightarrow S$ and a $T^+ \in B$ such that $T^+_* \Rightarrow T^+$. But, because we know that this construction preserves $\Rightarrow$ on sensible trees we then have that $\text{Const}([T^+, D], [S, C']) \Rightarrow T_*$.

We get the other direction (i.e. starting with $T \in [B'D']$) in exactly the same way. So, we are done with the ambiguity node case of the recursive definition.

5.3.6 Ambiguity Trees

First of all recall the definition of an ambiguity tree.

Definition 5.3.6.1. An ambiguity tree of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i\in N)}$ is a pair $\langle T, F\rangle$ such that

1. T is a finite tree
2. F is a function from T to the set of ambiguity nodes of characteristic arity n on $\mathbb{P}\mathbf{x}_{(i\in N)}$ and,
3. if x is an immediate successor of y in T then $F(x) \triangleright F(y)$

Definition 5.3.6.2. We say that a pair of ambiguity trees (A, A') is consistent relative to a consistent pair of colors (C, C') (up to a pair of nodes (N, N')) if A, A' below N, N' are isomorphic (as linear orders) and the ambiguity nodes (N, N') are consistent.

Definition 5.3.6.3. We say that a consistent pair of ambiguity trees (B, B') of arty $n + 1$ (up to nodes (N, N')) extends a consistent pair of ambiguity nodes of arity n (A, A') on $\mathbf{x}$ (where (A, A') are consistent relative to (C, C') and up to nodes (M, M')) if

(1) (B, B') restricted to arity n and $\mathbf{x}$ closely refines (A, A') ($B \Rightarrow A$ and $A' \Rightarrow B'$)

(2) (N, N') is an extension of (M, M') as consistent ambiguity nodes relative to (C, C')

Before we can understand what these definitions mean we need a sense of what exactly an ambiguity tree is. Recall that an ambiguity node is a collection of sensible trees all of which agree on a common path, up to a point (the history of the node). One way to think about an ambiguity node is as a collection of things which "look a like" up to a point. The idea being that we have chosen some "facts" about our universe (the history) and all of our sensible trees in the ambiguity node have to agree on the facts we have chosen. But, beyond those facts we don't know what the universe looks like.

To understand what an ambiguity tree looks like, lets consider the tree with out the sensible trees at each node. In this case an ambiguity tree is a finite list of possible paths. In other words, along each branch of the tree we can glue together the histories (i.e. facts we have chosen). Further, if we glue all these histories together then what we get, considered as a partial ordering, will look like the tree part of the ambiguity tree.

What we are doing is assuming we have two pairs of ambiguity trees which "look the same" along a branch. Then when we enlarge the lower one, we want to make sure that we can enlarge the upper one so that for a fixed "new" branch in the lower one there is a branch in the upper one which looks the same.

This is also why we only require that the extension be a close refinement of the tree it is extending. It is possible that the extension might add in some new nodes along the history and we want to allow for that.

Assume we have a consistent pair of ambiguity trees (A, A') of arity n on the same tuple $\mathbf{x}$ with n-histories H, H' up to nodes (M, M') which are a consistent pair of ambiguity nodes relative to (C, C'). Also assume we have B an ambiguity tree with node N, which is an extension of M and a color D which is an extension of C.

One of the most important things to realize about this construction is that we don't actually care about what happens outside of the branch below N. The first thing we are going to do is to create to create the branch ending at $N' = \mathrm{Const}([N, D], [M', C'])$.

We do this by first going along the branch of M' in A'. Suppose we have already applied the construction to all nodes below P'. What we want to do is look at the branch M is on and find the highest node P such that if C follows the history of P up to o and C' follows the history of P' up to o then (C, C') is a consistent pair up to $\min\{o, o'\}$. Now we want to find some ambiguity node Q in B and our color D which follows the history of Q and where D extends C

We then apply our construction to $[Q, D], [P', C']$. However, there is a small problem. The histories of P, P' might not be "the same size" (i.e. we only know that (C, C') is a consistent pair relative to $\min\{o, o'\}$). But this isn't that big a problem as we can simply ignore the extra information when we do our construction to get the node Q'. Then, because the construction is done independent of the length of the path through the tree, there will be an extension of our history in Q' which is according to all of the sensible trees in Q' (the only purpose of having a point below for comparisons was to ensure that in the result we still get a consistent pair of colors at least as far as we had the original colors consistent.)

The point to realize in this construction is that we really want is to just apply the construction to N, M' and then say that the lower nodes on the tree are just restrictions. The problem is that we have to ensure our resultant tree closely refines the tree we started with. In particular, for every node we started with, that node had better not "disappear" on the end result.

So all we have to worry about is what to do with the nodes of A' which aren't below M'. Well if a node P' is not below M' then what we want to do is temporarily ignore the history of P' above where it is inconsistent with that of M'. Call this history H'. Then we want to apply the construction with the node N_α extending M_P where M_P is the ambiguity node whose sole element is the sensible tree equal to the history (so (M_P, P) is a consistent pair of ambiguity nodes)

We then will get a node $\overline{Q'}$. We know that because the construction on sensible trees is independent of the length of the path we are choosing we will be able to "reconstruct" a history on Q' which extends the history of P'. The node Q' we will place in the tree B' will then be $\overline{Q'}$ with the reconstructed history. We Const$([B, N], [A', M'])$ as the tree just described.

All that is left is to show $\Rightarrow$ is preserved. Let $A \Rightarrow A_*|\text{dom}(A)$ and $A' \Rightarrow A'_*|\text{dom}(A)$ are all ambiguity trees of arity n, $(A, A'), (A_*, A'_*)$ are a consistent pairs relative to $(C, C'), (C_*, C'_*)$ (up to nodes $(M, M'), (M_*, M'_*)$). Also let $B \Rightarrow B_*|\text{dom}(B)$ be ambiguity trees of arity $n+1$ such that B extends A and B_* extends A_*. Then let $[B', N'] = \text{Const}([B, N], [A', M']) \Rightarrow \text{Const}([B_*, N_*], [A'_*, M'_*])|\text{dom}(B) = [B'_*, N'_*]|\text{dom}(B)$. But recall what it means for $X \Rightarrow Y$ when X, Y are ambiguity trees.

Definition 5.3.6.4 (*2.5.4.5)**.** Suppose $\langle T, F\rangle$ and $\langle T', F'\rangle$ are ambiguity trees or weak ambiguity trees on $\mathbb{P}\mathbf{x}_{(i\in S)}$, of characteristic arity n.

1. Then say $\langle T, F\rangle \rightarrow \langle T', F'\rangle$ iff there is a non-strictly order-preserving $\phi : T \rightarrow T'$ such that for all $a \in T$, $F(a) \Leftarrow F'(\phi(a))$.

   We say that $\langle T, F\rangle$ is a refinement of $\langle T', F'\rangle$

2. Say $\langle T, F\rangle \Leftarrow \langle T', F'\rangle$ iff there is a function ϕ witnessing $\langle T, F\rangle \to \langle T', F'\rangle$, and there is a one-to-one strictly order-preserving map $\psi : T' \to T$, with $\phi\circ\psi$ being the identity, such that for all $b \in T'$, $F(\psi(b)) \Leftarrow$. We say that $\langle T, F\rangle$ is a close refinement of $\langle T', F'\rangle$.

In particular all we need to show is that the tree part of $[B', N']$ can be injected into the tree part of $[B'_*, N'_*]$ by a function f such that $Q \Rightarrow f(Q)$ for all nodes in B'. But, the tree part of B' is the same as the tree part of A' and similarly the tree part of B'_* is the same as the tree part of A'_*. So there is an injective map f from the tree part of B' to the tree part of $B'_*|\text{dom}(B)$ (it is the same map which we know must exist because $A' \Rightarrow A'_*|\text{dom}(A)$). And because $\Rightarrow$ is preserved on ambiguity nodes this map witnesses that $B' \Rightarrow B'_*|\text{dom}(B)$. (Notice that just having an injection from $X \to Y$ is not enough to ensure $X \Rightarrow Y$. We also need a surjection going the other way (where the composition is the identity in the direction that makes sense). But, as our construction doesn't modify the tree part of an ambiguity tree we can just use the maps between $A', A'_*|dom(A)$).

5.3.7 Sensible Trees

We are almost done with the construction. All that is left is the case of sensible trees of arity > 1. Recall the definition of a sensible tree:

Definition 5.3.7.1 (*2.6.2.1)**.** If Υ is a tree of colors of characteristic arity, and ϕ is a finite partial function from the union of $\{\{-\infty\}\}$ with the set of $n-1$ sections to pairs $\langle T', F \restriction T'\rangle$, where $T' \subseteq T$, and $\langle T, F\rangle$ is an ambiguity tree of characteristic arity n-1, then ϕ is **orderly** iff

1. For all $\gamma \in$dom ϕ if $\phi(\gamma) = \langle T', F \upharpoonright T' \rangle$, then T' is up-closed in T;

2. $\{-\infty\} \in$dom ϕ and $\phi(\{-\infty\}) = \langle T, F \rangle$. [The history of the root T could say that nothing has happened yet]

3. If $\gamma \in$dom ϕ is non-trivial or equal to $\{-\infty\}$ then

   (a) $\phi(\gamma)$ is rooted

   (b) if γ' is an immediate successor of γ in domϕ, then $\phi(gamma')$ is a bough of $\phi(\gamma)$.

   (c) If γ is not $\{-\infty\}$ then there exists $\gamma' < \gamma$ with γ' trivial with $\phi(\gamma) = \phi(\gamma')$, or else non-trivial or equal to $\{-\infty\}$ with $\phi(\gamma)$ being the deracination of $\phi(\gamma')$.

4. If $\gamma \in$ dom ϕ is trivial and γ' is an immediate successor of γ in dom ϕ, then $\phi(\gamma')$ is radically included in $\phi(\gamma)$.

5. dom ϕ is as small as possible. Specifically, if γ and γ' are in dom ϕ are trivial and γ' is an immediate successor of γ in dom ϕ, then

   (a) If γ and γ' in are trivial, then $\phi(\gamma') \neq \phi(\gamma)$.

   (b) If γ' is trivial and γ is non-trivial or is equal to $\{-\infty\}$, then $\phi(\gamma')$ is not the deracination of $\phi(\gamma)$.

Definition 5.3.7.2 (*2.6.2.2)**.** A **clear tree** of characteristic arity n on $\mathbb{P}\mathbf{x}N$ is a triple $\langle\langle\sqsubseteq, \Upsilon >, \langle T, F\rangle, \phi\rangle$ such that

1. $\langle\sqsubseteq, \Upsilon\rangle$ is an augmented unitary tree of characteristic arity n on $\mathbb{P}\mathbf{x}N$

2. $\langle T, F \rangle$ is a reduced, filled ambiguity tree of characteristic arity n-1 on $\mathbb{P}\mathbf{x}N$

3. ϕ is an orderly partial function on the union of $\{\{-\infty\}\}$ with the set of n-1 sections on Υ such that $\phi(\{-\infty\}) = \langle T, F \rangle$.

Definition 5.3.7.3 (*2.6.2.3)**.** A clear tree $\langle \sqsubseteq, \Upsilon, \langle T, F \rangle, \phi \rangle$ on $\mathbb{P}\mathbf{x}N$ is **sensible** iff

1. whenever $\gamma \in \text{dom } \phi$ is non-trivial, the gist of γ in $\langle \sqsubseteq, \Upsilon \rangle$ is the same as that of the root of $\phi(\gamma)$, and

2. Every n-1 section meeting $\sqsubseteq$ "$\mathbb{P}_{<n}\mathbf{x}N$ belongs to dom ϕ.

Suppose we have a consistent pair of extendible sensible trees (S, S') of arity n on $\mathbf{x}$ and relative to two precolors (C, C') (up to $o \in \Omega$). Now let T be an extension of S to arity $n+1$ and let D be an extension of the precolor C on the tree T. We want to find an extension T' of S' and a precolor D' extending C' such that (T, T') is a consistent pair of sensible trees relative to (D, D').

The approach is going to be very similar to the case where $n = 1$. The first thing we are going to do is extend the augmented unitary tree S' to an augmented unitary tree T' as in Section 5.3.3. Then all that is left to do is to place the ambiguity trees.

The method we are going to use to place the ambiguity tree will be to first consider the roots of the trees. We will then place ambiguity nodes at each point in the sensible tree so that if one ambiguity node is placed at a point greater than another, then the history of the ambiguity node which was placed at the greater point will extend the history of the ambiguity node placed at the lower point.

Notice that (by the construction in Section 5.3.3) all new tuples which are not placed on the branch we are considering (i.e. the branch that D' follows) are placed immediately after a branching point. In particular if we have a point X in S' which is immediately after a branching point then there is some tuple which is placed there. Now on any node $Y \in \text{range}(\sqsubset)$ which is off the path we are considering in S' there is some ambiguity tree A' of arity $n-1$. We want to extend the root of this ambiguity tree to an ambiguity node of arity n by simply putting all new tuples in all histories and all sensible trees at the same point immediately after the branching point (we can do this because there is already a tuple there and so all we have to do is put all new tuples at the same place as the tuple that already is there).

However, the construction of ambiguity trees for nodes off the main branch isn't important. The only thing we care about is the branch that D' follows, and the only reason we put ambiguity trees on the nodes on the other branches is to ensure that our end result will be a sensible tree of the correct arity.

The most important thing we know about this main branch is that (D, D') follow the branch and what is more is (D, D') are a consistent pair of precolors up to o (which was as far as we assumed that (C, C') were consistent). So when we come across any node in T' (at a point X below o) which has a node in S' which was at the same place (modulo the shifting in branch points), find the largest node of T below that point (call it Y) and apply the construction to the root of the ambiguity tree at X with the root of the ambiguity tree at Y but only requiring that the colors (C, C') are consistent as far as the history of Y. If we look at the construction $\mathrm{Const}(X, Y)$ on ambiguity nodes we see that the construction did not in fact use anything about the histories of either node when it was modifying the sensible trees in X. The only reason we kept track of the history was to ensure that $(Y, \mathrm{Const}(X, Y))$ was a consistent pair. In particular, there is a unique history extending the history of $\mathrm{Const}(X, Y)$ to the point on which X was placed. So the ambiguity node we place at X is the ambiguity node gotten from $\mathrm{Const}(X, Y)$ by extending the history up to the point where X was.

It is worth pointing out explicitly for the purposes of the above construction we consider any point on T' which is above o to be on the branch we are considering. To be more precise if we look at a node X above o we apply the construction to X and N and then extend the history to go all the way up to X.

There is only one more case we have to consider. If we remember the construction of the branch, it was possible that we could have an m tuple $\overline{a}$ at a point which was not an $m+1$ successor. This case corresponded to when we take an extension and the color of the extension falls below a limit ordinal which the original tuple was above. In this case we wanted to put $\overline{a}x$ (where x was the new element) at some node which was above everything important that happened in D.

In this case, we want to put the extension of the ambiguity tree that was at $\overline{a}$ in S' at the $\overline{a}x$ in T' except we are going to add a single node below the root. This node will be identical to the root except that the history will only go as far as $\overline{a}x$ (and not as far as $\overline{a}$).

The reason why we do this is because we have to have some ambiguity tree to place at the $n+1$ section of $\overline{a}$ and we need the ambiguity tree to be a rooted subtree of that placed at $\overline{a}x$.

We have now placed an ambiguity node at every point in T' which has to have an ambiguity tree. The next step is to combine all of these ambiguity nodes into ambiguity trees. We do this by simply creating the ambiguity tree of all nodes above the one we are at (and including it). We then obviously get a rooted ambiguity tree (the histories of the ambiguity nodes were defined in such a way as to be consistent, and if we started with a subset of an ambiguity node, when we applied the construction we get a subset of the result). Further the ambiguity trees are obviously all rooted.

The next thing we need to do is the same as had to do in the case of $n = 1$. In between any of the non-trivial $n+1$ sections with rooted trees (which have branches immediately above them) we need to choose a trivial $n+1$ section and place at that $n+1$ section the lower tree with the root cut off. Finally we just need to observe that the gist of the any point is in fact the same as the gist of the root of the ambiguity tree (by construction).

We can now define $\mathrm{Const}([T, D], [S', C'])$ to be the constructed tree. All that is left to show is that the construction preserves $\Rightarrow$. But this is trivially true because $\Rightarrow$ on the augmented unitary trees implies they are the same and we know by assumption that this construction preserves $\Rightarrow$ on the ambiguity trees because $\Rightarrow$ is preserved in our construction on ambiguity node and ambiguity trees.

So we are done with the construction!

Lets just recap what this construction gets us.

Theorem 5.3.7.4. *Suppose we have a consistent pair of cells (C, C') and D is an extension of C. Then there is a consistent pair of cells (E, E') such that E is an extension of D and E' is an extension of C'*

Note we can't just find an extension of C' such that (D, D') is consistent because we might have had to add in some dummy variables. But, in the other direction we don't have to worry about that.

Theorem 5.3.7.5. *Suppose we have a consistent pair of cells (C, C') and D' is an extension of C'. Then there is a consistent pair of cells (D, D') such that D is an extension of C.*

Proof. Simply let D be the extension of C which places every new tuple at $-\infty$. This is obviously a consistent pair as (C, C') is. □

5.3.8 Archetypes

Now that we have proved Theorem 5.3.7.4 we will want something similar for archetypes.

Definition 5.3.8.1. We say that (σ, σ') is a Consistent Pair of Archetypes if

- $\sigma(x_1, \ldots, x_n), \sigma'(y_1, \ldots, y_n) \Vdash C(x_1, \ldots, x_n) \wedge C'(y_1, \ldots, y_n)$
- $\sigma(x_1, \ldots, x_n), \sigma'(y_1, \ldots, y_n) \Vdash \bigwedge_{S \subseteq n} \|\{x_i : i \in S\}\| \leq \|\{y_i : i \in S\}\|$

- (C, C') is a consistent pair of cells
- For some $M, N \models \Theta$, $M \models (\exists \mathbf{x})\sigma(\mathbf{x})$, $N \models (\exists \mathbf{x})\sigma'(\mathbf{x})$.

We say that (τ, τ') is a Consistent Extension of (σ, σ') ($(\tau, \tau') \leq (\sigma, \sigma')$) if

- (τ, τ') is a consistent pair of archetypes
- $\tau(\mathbf{x}, \mathbf{y}) \Vdash \sigma(\mathbf{x})$
- $\tau'(\mathbf{x}, \mathbf{y}) \Vdash \sigma'(\mathbf{x})$
- $\tau(\mathbf{x}, \mathbf{y}), \tau'(\mathbf{x}', \mathbf{y}') \Vdash B(\mathbf{x}, \mathbf{y}) \wedge B'(\mathbf{x}', \mathbf{y}')$
- $(B, B')|\mathbf{x} \geq (B, B')(\mathbf{x}, \mathbf{y})$

Note that the last condition in the definition of consistent pair of archetypes was to ensure that the archetypes are in fact complete (see Definition 5.2.2.2). Essentially this says that a pair of archetypes is consistent if they obey (Consistency of Color) (of Definition 3.3.0.13) and they are over a consistent pair of cells. Similarly a pair of archetypes (τ, τ') is an extension of (σ, σ') if (τ, τ') forces (σ, σ') to hold on their domain and further the consistent pair of cells that (τ, τ') is over is an extension of the consistent pair of cells that (σ, σ') are over.

Theorem 5.3.8.2. *Suppose we have*

- *A consistent pair of archetypes* (σ, σ')
- $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash \sigma(\boldsymbol{x}) \wedge B(\boldsymbol{x}, \boldsymbol{y})$ *for some cell* B
- *Suppose* (B, B') *is a consistent pair of cells*

- $(B, B')(\boldsymbol{x}, \boldsymbol{y}) \leq (A, A')(\boldsymbol{x})$
- $\sigma'(\boldsymbol{x}) \Vdash A'(\boldsymbol{x})$.

Then there is an archetype τ' such that

- $\tau'(\boldsymbol{x}, \boldsymbol{y}) \Vdash B'(\boldsymbol{x}, \boldsymbol{y}) \wedge \sigma(\boldsymbol{x})$
- (τ, τ') *is a consistent pair of archetypes*
- $(\tau, \tau') \leq (\sigma, \sigma')$

Before we begin the proof lets consider exactly what this is saying. It is saying that if we have a consistent pair of archetypes and we extend the cells which they are over then we can also extend the archetypes. What is more, we can also choose the archetype which is in the bottom half of the pair. It was to be able to prove this that we went through so much effort in the previous sections.

Proof. This construction is immediate from our definitions. We simply treat the archetype like several different sensible trees which only have one branch (one tree for each arity). We then apply the construction exactly as we did in the case of the sensible trees.

There are only a few things we have to check. First we have to check that when we apply the construction to the ambiguity trees the resultant ambiguity trees don't violate any of the conditions on being an archetype. Notice that our construction preserves projection (i.e. if $A|\text{dom}(A_*) = A_*$, $A'|\text{dom}(A'_*) = A'_*$, $B|\text{dom}(B_*) = B_*$ then $\text{Const}(B, A)|\text{dom}(B_*) = \text{Const}(B_*, A_*)$. (This is immediate from how we built our construction.))

The second observation we need to make is to ensure that each of our new ambiguity trees at $-\infty$, according to $\psi{*}i$, closely refines a tree in B'. But this is true because as we have shown our construction preserves $\Rightarrow$ and further, because of how we defined what it meant for a pair of forest to extend one another, we know that when we applied the construction to the ambiguity trees which $\phi' * i(-\infty)$ closely refines, we get something in our forest B'.

All that is left to show is that in fact we can also find a color $\mathcal{D}'$ which extends $\mathcal{C}'$ (the color of σ') and makes this constructed τ' into an archetype. But, we know that (B, B') is a consistent extension of (A, A') and this immediately implies such a $\mathcal{D}'$ exists by the definition of a consistent extension (and in fact why we defined a consistent extension that way instead of just as a consistent pair of archetypes which extended another). Further, we know that $(\tau, \tau') \leq (\sigma, \sigma')$ by the definition of what it means for a pair of archetypes to extend another.

There is only one more thing which we need to observe from this construction. This is that not only do we get a τ', but we can find a τ' so that every new color is less than $\sup\{\alpha+\omega, \beta+\omega : \tau(x_1, \ldots, x_m) \Vdash \|x_i\| = \alpha, i \in m, \sigma'(y_1, \ldots, y_n) \Vdash \|y_i\| = \beta, i \in n\}$. □

Notice that this theorem really is saying two things. First it is saying if you have a consistent pair of archetypes over a consistent pair of cells, then you can always extend that pair of archetypes to be over any consistent extension of cells. This is crucial if we want to find a way to glue to models together so that they satisfy a form of homogeneity. I.e. if we have a realization of some consistent pair of cells then we can find a realization of every consistent extension.

The second thing this theorem is saying is that if we have a consistent pair of archetypes and we have some archetype which extends the bottom one, then we can find some archetype which extends the top one and what is more the archetype which extends the top one doesn't force anything "new" (at least not beyond a new limit ordinal). This will be very important as it will allow us to find a consistent way of gluing two models together.

We would also like to have the other direction of Theorem 5.3.8.3 (i.e. given (B, B') extending (A, A'), (σ, σ') over (A, A') and τ' over B' extending σ' then we can find a τ over B extending σ such that (τ, τ') is a consistent pair of archetypes extending (σ, σ'). However, we run into a problem. That problem is we don't now that for every extension of σ' to a τ' we can find an archetype τ under τ' if we require τ to be over a prechosen B. So we will have to settle with being able to find some τ, an extension of σ, such that τ is over some B and such that (τ, τ') is a consistent pair of archetypes.

Theorem 5.3.8.3. *Suppose we have a consistent pair of archetypes* (σ, σ'). *Further suppose* $\tau'(\boldsymbol{x}, \boldsymbol{y}) \Vdash \sigma'(\boldsymbol{x}) \wedge B'(\boldsymbol{x}, \boldsymbol{y})$ *for some cell* B'. *Then there is a cell* B *such that* (B, B') *is a consistent extension and of* (A, A') *(where* $\sigma(\boldsymbol{x}) \Vdash A(\boldsymbol{x}), \sigma'(\boldsymbol{x}) \Vdash A'(\boldsymbol{x})$*) and such that there is a* $\tau(\boldsymbol{x}, \boldsymbol{y}) \Vdash B(\boldsymbol{x}, \boldsymbol{y})$ *and* (τ, τ') *is a consistent pair of archetypes.*

Proof. Just let $\tau(\mathbf{x}, \mathbf{y})$ be the archetype which puts every tuple not in $\mathbf{x}$ at $-\infty$. Then this trivially satisfies all the conditions needed. $\square$

There is just one more theorem we will need in from this section.

Theorem 5.3.8.4. *Suppose we have consistent pairs of archetypes* $(\sigma, \sigma'), (\tau, \tau')$ *(possibly with some overlap on their domain). Then there is a consistent pair* *of archetypes* (η, η') *such that*

- $\eta(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \sigma(\boldsymbol{x}, \boldsymbol{y}) \wedge \tau(\boldsymbol{y}, \boldsymbol{z}) \wedge A'(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}$

- $\eta'(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \sigma'(\boldsymbol{x}, \boldsymbol{y}) \wedge \tau'(\boldsymbol{y}, \boldsymbol{z}) \wedge A'(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{x}$

- $(A, A')(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$ *is a consistent pair of cells*

- $(\eta, \eta')(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \leq (\eta, \eta')|\boldsymbol{y}$

Proof. Let η, η' be the archetypes which put every new tuple at $-\infty$. Also note that this gives rise to an identical amalgamation of consistent pairs of cells. □

Finally, the last point to notice is that no where in this construction did we use the color ∞. So in fact if everything we have done doesn't mention ∞ then we can apply our construction such away that nothing new has precolor at ∞

5.4 A Collection of Archetypes for Θ

In this section we will finally show that Θ (with slight modifications) satisfies the conditions in Definition 3.3.0.13.

Definition 5.4.0.5. Let $L_{\Theta^*} = L_\Theta \cup \{C_i : i \text{ is a Cell in [8]}\}$ (there are only countably many cells by Lemma 2.10.1.1 of [8]).

Let $\Theta^* = \Theta \cup \{(\forall \mathbf{x})(C_i(\mathbf{x}) \leftrightarrow \mathbf{x} \text{ satisfies Cell } i) : i \text{ is a Cell in [8]}\}$. Note that Θ^* is still a 1st order theory because "$\mathbf{x}$ satisfies Cell i", i is a Cell is expressible in L_Θ. (see Definition 2.9.2.2 in [8]).

Lemma 5.4.0.6. *Θ^* is a conservative extension of Θ (i.e for each model of Θ there is a unique way to extend it to a model of Θ^*).*

Proof. Because each predicate in $L_{\Theta^*} - L_\Theta$ is equivalent to a formula of L_Θ. $\square$

Theorem 5.4.0.7. *Θ^* satisfies all the conditions in Definition 4.1.0.23 except having a collection of archetypes for Θ.*

Proof. First note that $T_P \subseteq \Theta^*$. Also observe that $\{\mathrm{Spec}(M)/-\infty : M \models \Theta\} = \mathrm{ORD}$ (i.e there are models without tuples of color ∞ of all ordinal height). $\square$

5.4.1 Weak Collection of Archetypes

Definition 5.4.1.1. Let $\mathrm{ATYPE}(\Theta^*) = \{$complete archetypes in [8]$\}$.
Define $2-\mathrm{ATYPE}(\Theta^*) = \langle\{(\sigma,\sigma') : (\sigma,\sigma')$ is a consistent pair of archetypes$\}, \leq\rangle$ (as in Definition 5.3.8.1)
Define $EI_{\Theta^*}(\phi(x_1,\cdots,x_n))$ as

> $\{\langle\|\{x_j : j \in S\}\|, |S|, \circ_S\rangle : S \subseteq n, \circ_S = i$ if $\{x_j : j \in S\}$ is an inaccessible and a otherwise$\}$

Then define $EI_{\Theta^*}(M) = \bigcup\{EI_{\Theta^*}(\phi) : M \models (\exists \mathbf{x})\phi(\mathbf{x})\}$
Finally define $\mathrm{BP}(\Theta^*) = \{C_i : i$ a cell as in [8]$\}$

So what exactly is this definition saying. Well the archetypes which we want in our "Collection of Archetypes" for Θ^* are, not surprisingly, just the archetypes of Θ as defined in [8]. Similarly, the consistent pairs of archetypes are just the consistent pairs of archetypes and the base predicates are just the predicates added to the language of L_{Θ^*} to talk about the cells.

What is going on with the "Extra Information Function" is a little bit less clear. The idea behind this function is we want it to include all "types" of tuples which the model/archetype force to exist. Well, as it turns out, what matters about a tuple in this context its color, it's arity, and whether or not it is inaccessible. And, this is exactly the information about a tuple that EI_{Θ^*} captures.

There is just one more subtle point worth mentioning explicitly. In the general definition of collection of archetypes (Definition 3.3.0.13) we defined the pairs of archetypes we were interested in and then defined the consistent pairs of base predicates in terms of them (as the consistent pairs of base predicates which were realized). However, because of the (Completeness of Consistent Pairs of Base Predicate) axiom, we know that we could have just as easily defined the pairs of cells we were interested in and then defined the archetypes in terms of those (as we did here)

Theorem 5.4.1.2. *$ATYPE(\Theta^*), 2-ATYPE(\Theta^*)$, is a weak collection of archetypes (see Definition 3.2.0.8) for Θ^*.*

Proof. (Truth on Atomic Formulas):

If $\sigma(\mathbf{x})$ is an archetype then σ determines the color of each subset of $\mathbf{x}$. In particular it determines whether or not a subset satisfies P (and hence has color). $\sigma(\mathbf{x})$ also determines the cell that $\mathbf{x}$ satisfies. So, by the definition of what it means for a tuple to realize a cell (Definition 2.9.2.3 of [8]) we know that the cell which a tuple satisfies determines exactly which of the forest predicates its subtuples satisfy. Hence, the cell of a tuple also determines which cells its subtuples satisfy (and which predicates in $L_{\Theta^*} - L_{\Theta}$).

(Truth on Color):

This is because any archetype determines the color of all subtuples of it's domain.

(Restriction of Arity for Archetypes):

If $\sigma(\mathbf{x})$ is an archetype then $\sigma(\mathbf{x})|\mathbb{P}\mathbf{y}$ (as defined in Definition 3.2.2.1 of [8]) is an archetype on $\mathbf{y}$ for any $\mathbf{y} \subseteq \mathbf{x}$.

(Restriction of Arity for Consistent Pairs of Archetypes):

This is because if $(B, B')(\mathbf{x}, \mathbf{y}, z) \leq (A, A')(\mathbf{x})$ (for consistent pairs of cells $(A, A'), (B, B')$) then we must also have $(B, B')|(\mathbf{x}, \mathbf{y}) \leq (A, A')(\mathbf{x})$. To see this assume we have a consistent pair of precolors (C, C') according to (A, A') and we have a precolor D extending C and according to $B|(\mathbf{x}, \mathbf{y})$ and we want to find a D' according to $B'|(\mathbf{x}, \mathbf{y})$ such that (D, D') is a consistent pair of precolors.

Well just extend D to D_* which is according to B and is an extension of D. Then we know we can find a D'_* according to B' which is an extension of C' because (B, B') is a consistent extension of (A, A'). We can then just let $D' = D'_*$ restricted to $(\mathbf{x}, \mathbf{y})$.

(Completeness for Archetypes):

This is immediate from how Archetypes are defined

(Amalgamation for Archetypes):

First note that all archetypes which are realized in a model are regular (i.e. they contain all the information about the tuples the describe). Then notice that in the amalgamation of regular archetypes in Lemma 3.2.3.2 of [8], any "new" tuple is placed at color $-\infty$.

(Amalgamation for Consistent Pairs of Archetypes):

This is exactly Theorem 5.3.8.4

(Consistency of Color):

This is immediate from the definition of $2 - \text{ATYPE}(\Theta^*)$.

(Consistency of $\leq$):

This is immediate from the definition of $2 - \text{ATYPE}(\Theta^*)$.

(Extension of 0-Colors):

This simply says that if we have a consistent pair of archetypes and we extend the second archetype, we can extend the first archetype trivially (i.e. putting everything new at $-\infty$) and we get a consistent pair of archetype extending the first. This is immediate from the definition of archetype pair and Theorem 5.3.8.3. □

5.4.2 Collection of Archetypes

And now we show that in fact we have a collection of archetypes.

Theorem 5.4.2.1. *$ATYPE(\Theta^*), 2 - ATYPE(\Theta^*), EI_{\Theta^*}, BP(\Theta^*)$ for a Collection of Archetypes for Θ^* (see Definition 3.2.0.8)*

Proof. We know by Theorem 5.4.1.2 that these are a weak collection of archetypes for Θ^*.

(Prediction):

This is exactly what Generalized Saturation for Archetypes (Proposition 4.3.2.1 of [8]) says. You just have to observe (as in the discussion of Section 5.2) that υ comes from a trivial amalgamation of an υ^* with σ.

(Prediction up to a Slant Line):

This breaks up into two cases.

Case 1: $sl = \infty$

In this case we just observe that in the proof of Generalized Saturation for Archetypes (Proposition 4.3.2.1 of [8]) if we amalgamate υ^* with an archetype σ' which is not σ but which does have $\sigma'|\infty = \sigma|\infty$ then in fact the archetype (τ') which is witnesses as an extension of σ by $A_{\sigma,\tau}$ satisfies $\tau'|\infty = \tau|\infty$. This is because τ' has all the colors of its tuples in the same place as τ does (because σ has all tuples in the same place as σ'). And also must satisfy the same cell as τ as the cell is simply the restriction of the cell $A_{\sigma,\tau}$ to the domain of τ.

Case 2: $sl \neq \infty$

This is exactly Generalized Saturation for Restricted Archetypes (Theorem 5.2.3.1) and in fact why we proved it.

(Truth on Atomic Formulas):

Similarly to the case of (Truth on Atomic Formulas) in Theorem 5.4.1.2 we have that the cell structure of a tuple determines the cells of all subtuples as well as whether or not a subtuple has color.

<u>(Amalgamation for Base Predicates):</u>

This is because given any two archetypes $\sigma(\mathbf{x},\mathbf{y}), \tau(\mathbf{y},\mathbf{z})$ which can be realized in the same model there is always an archetype $\eta(\mathbf{x},\mathbf{y},\mathbf{z})$ which amalgamates τ and σ around $\mathbf{x}$ and puts all new tuples at $-\infty$. Hence we can just look at the Basic Predicates which σ, τ, η are over.

<u>(Homogeneity for Base Predicates):</u>

Suppose A is a cell and $A(\mathbf{x},\mathbf{y}) \Vdash B(\mathbf{x})$. Let $\tau(\mathbf{x},\mathbf{y}) \Vdash A(\mathbf{x},\mathbf{y}) \wedge \sigma(\mathbf{x})$. We can then let τ_n be an amalgamation of n copies of τ around σ. (See Corollary 3.2.3.4 of [8]). Let C be the cell such that $\tau_n(\mathbf{x},\mathbf{y}) \Vdash C_n(\mathbf{x},\mathbf{y})$. Then we know $B(\mathbf{x}) \Vdash (\exists \mathbf{z}) C(\mathbf{x},\mathbf{z})$ (By the Generalized Saturation Axiom of Θ^*) and hence $B(\mathbf{x}) \Vdash (\exists \mathbf{y}_1, \cdots \mathbf{y}_n) \bigwedge_{i \in n} A(\mathbf{x},\mathbf{y}_i)$ and$(\mathbf{y}_i \neq \mathbf{y}_j)$ if $i \neq j$.

<u>(Extension of 1-Colors):</u>

This is immediate from Theorem 5.3.8.2 (and one of the purposes of Section 5.3)

<u>(Homogeneity of Consistent Pairs of Archetypes) :</u>

This is immediate from 5.3.8.2 (and one of the purposes of Section 5.3)

<u>(Completeness of Extra Information):</u>

This is immediate from how we define EI_{Θ^*}

(Completeness of Consistent Pairs of Base Predicate):

This is immediate from how we defined consistent pairs of archetypes.

(Uniqueness of Base Predicate):

This is because, by the Local Embedding Axiom of Θ^* and the definition of cell (Definition 2.9.2.2 of [8]), no tuple can satisfy two distinct cells. □

Definition 5.4.2.2. Let $M_{\omega*\alpha} \models \Theta^*$ be the unique model of Θ^* such that $\text{Spec}(M_{\omega*\alpha}) = \{-\infty\} \cup \omega * \alpha$ (Note it is the unique model by Proposition 4.3.6.1 of [8])

Let $\text{ATYPE}(\Theta^*(\omega * \alpha)) = \{$complete archetypes in [8] which are realized in $M_{\omega*\alpha}\}$.

Define $2 - \text{ATYPE}(\Theta^*(\omega * \alpha)) = 2 - \text{ATYPE}(\Theta^*) \cap \text{ATYPE}(\Theta^*(\omega * \alpha))$

Define $EI_{\Theta^*(\omega*\alpha)}(X) = EI_{\Theta^*}(X)$

Finally define $\text{BP}(\Theta^*(\omega * \alpha)) = \text{BP}(\Theta^*)$

Corollary 5.4.2.3. *Let $\Theta^*(\omega * \alpha) = \Theta^* \cup (\forall \boldsymbol{x} \|\boldsymbol{x}\| < \omega * \alpha)$. Then $\Theta^*(\omega * \alpha)$ has a collection of archetypes (which in fact are those in Definition 5.4.2.2)*

Proof. This is immediate from Definition 5.4.2.2, Theorem 5.4.2.1 and Theorem 5.4.1.2 □

5.5 Quantifier Rank Equivalence

In this section we will completely categorize when two models M, N of Θ satisfy the same formula's of quantifier rank less than or equal to $\omega*\gamma (M \equiv_{\omega*\gamma} N)$ (and even a little more). First though, we need a Lemma.

Lemma 5.5.0.4. *If $M, N \models \Theta, M \equiv_{\omega*\gamma+n+1} N$ and Spec$(M) \cap \omega_1 < \omega*\gamma+n$ then Spec$(M) =$ Spec(N).*

Proof. Lets assume Spec$(M) \cap \omega_1 = \omega * \beta + m < \omega * \gamma + n$

First lets show that Spec$(N) \cap \omega_1 =$ Spec$(M) \cap \omega_1$. Let $\varphi^{=}_{\alpha}(x)$ be the statement from Corollary 2.2.2.8. Then $M \models (\exists x)\varphi^{=}_{\alpha}(x)$ for all $\alpha \in$ Spec$(M)\cap$ORD and so Spec$(M)\cap$ORD $\subseteq$ Spec$(N)\cap$ORD as $\alpha \in$ Spec$(M)\cap$ ORD $\Rightarrow$ qr$(\varphi^{=}_{\alpha}) \leq \omega*\beta+m \leq \gamma*\omega+n$. But, if Spec$(M) = \omega*\beta+m$ then we also have $M \models (\forall x)\neg\varphi^{=}_{\omega*\beta+m}(x)$ which has quantifier rank $\leq \omega*\beta+m+2 \leq \omega*\gamma+n$. So, $N \models (\forall x)\neg\varphi^{=}_{\omega*\beta+m}(x)$ as well. So, because Spec$(N)\cap$ORD is an ordinal Spec$(N)\cap$ORD $\leq \omega*\beta+m$ and so Spec$(N)\cap$ORD $=$ Spec$(M)\cap$ORD.

All that is left is to show is that $\infty \in$ Spec$(M) \Leftrightarrow \infty \in$ Spec(N). But we know that $\infty \in$ Spec$(M) \Leftrightarrow M \models (\exists x)\varphi^{=}_{\omega*\beta+m}(x) \Leftrightarrow N \models (\exists x)\varphi^{=}_{\omega*\beta+m}(x) \Leftrightarrow \infty \in$ Spec(N). (The first and last equivalences are true because we know that $\beta*\omega+m$ is not in the spectrum, and so if we have an element whose color is at least $\beta*\omega+m$, then it must have color ∞.) □

Lemma 5.5.0.5. *If $M, N \models \Theta, M \equiv_{\omega*\gamma+n+1} N$ and Spec$(M) \cap$ ORD $= \omega*\beta+m+2 < \omega*\gamma+n+1$ then M is hollow iff N is hollow.*

Proof. First note that by Lemma 5.5.0.4 we know Spec$(M)\cap$ORD $=$ Spec$(N)\cap$ ORD $= \omega*\beta+m+2$. Then we know the M (or N) is not-hollow precisely when it models $[(\exists x)\varphi^{=}_{\omega*\beta+m+1}(x)] \wedge [(\exists y)\varphi^{=}_{\omega*\beta+m}(y)]$. But as this is a formula of quantifier rank $\omega*\beta+m+3 \leq \omega*\gamma+n+1$ so we know that M is hollow iff N is hollow. □

It is worth making explicit why this argument does not work if Spec$(M) = \gamma*\omega+1$. The reason is that in this case we can't distinguish between the hollow and non-hollow models simply by looking at the combinations of (tuple size, color of tuple) which are satisfied by the model.

Intuitively what is happening in the hollow case is that you have your normal model with colors less than $\omega * \beta$ and then you have a collection of elements all of which have been placed at one point above $\omega * \beta$ (and by one point we mean they are all in the same n section in Ω in the archetype, for appropriate n). What is more there is an infinite descending chain of these elements (this can be done because Ω looks like $\mathbb{Q}$). Because these points are all in the "same place", the result of looking at the colors of the model is that all these points are on the same slant line. This is what Knight means when he says he thinks the hollow models should be viewed as models whose limit is a model with ∞ in the Spectrum. In other words, they have an infinite descending chain of precolors, but when we go to colors, we can no longer tell that it is an infinite descending chain. That is the reason why we can't distinguish between the hollow and non-hollow cases when Spec$(M) = \gamma * \omega + 1$ just by looking at the (tuple size, color of tuple) pairs satisfied in M.

Lemma 5.5.0.6. *If $M, N \models \Theta$, $M \equiv_{\omega*\gamma+\omega} N$ and Spec$(M) \cap ORD = \omega*\gamma+1$ then M is hollow iff N is hollow.*

Proof. Let $\psi_n \leftrightarrow \bigvee_{k\in\omega}(\exists x_1, \cdots x_n)(\exists \mathbf{y})\Phi_k(\mathbf{y}^\wedge x_1{}^\wedge \cdots {}^\wedge x_n) \wedge \varphi^{=}_{\omega*\gamma}(x) \bigwedge_{i\in n} \varphi^{=}_{\omega*\gamma}(x_i)$ where $\{\Phi_k : k < m\} = \{$ Cells which say x_i is above x_{i+1} but have the same color$\}$.

(Note it is the authors belief that we don't actually need an infinite disjunction of cells. However, we do need an infinite conjunction of φ_n so removing the disjunction won't lower the quantifier rank)

The point is that if M thinks $x_1 \dots x_n$ is a descending sequence of elements and further M says all of $x_1, \dots, x_n$ have color $\omega * \gamma$, then that will be witnesses by some cell containing $x_1 \dots x_n$ as well as the formula $\varphi^{=}_{\omega*\gamma}$ (because it is witnessed by the archetype containing $x_1 \dots x_n$). So, $M \models \psi_n$ iff M thinks there is some descending sequence of n elements all of which have color $\omega * \gamma$ (i.e. inaccessible). Further, $\mathrm{qr}(\psi_n) = \omega * \gamma + \omega$

If we let $\psi' = \bigwedge_{n\in\omega} \psi_n$ then $\mathrm{qr}(\psi') = \gamma * \omega + \omega$ and $M \models \psi'$ iff M has an infinite descending chain of elements all with color $\gamma * \omega$. But this is exactly what it means for M to be hollow. So, M is hollow iff N is hallow because $M \equiv_{\gamma*\omega+\omega} N$ (and $\mathrm{Spec}(M) \cap \mathrm{ORD} = \mathrm{Spec}(N) \cap \mathrm{ORD}$ by Lemma 5.5.0.4). □

We are now ready for one of the main theorems of this section.

Theorem 5.5.0.7. *If $M, N \models \Theta, M \equiv_{\omega*\gamma+\omega} N$ and Spec$(M) \cap$ ORD $<$ $\omega * \gamma + \omega$ then $M \cong N$.*

Proof. By Lemmas 5.5.0.4, Lemma 5.5.0.5, and Lemma 5.5.0.6 we know M and N have the same spectrum and M is hollow iff N is hollow. So, by Proposition 4.3.6.1 of [8], $N \cong M$. □

Theorem 5.5.0.8. *If Spec$(M) \cap$ ORD $\geq \omega * \iota$, Spec$(N) \cap$ ORD $\geq \omega * \iota$ then $M \equiv_{\omega*\iota} N$*

Proof. Lets first define our sequence of partial isomorphisms.

Definition 5.5.0.9. Define $I_\zeta(M, N)(= I_\zeta)$ as follows:
$I_{\omega*\eta+n} = \{f : M \to N s.t. f$ is a bijection, $|\mathrm{dom}(f)| < \omega$, f preserves all atomic formula's in L_Θ and if $M \models \sigma(\mathrm{dom}(f))$ and $N \models \tau(\mathrm{range}(f))$ then

$\sigma|g = \tau|g$ where $g < \omega * \iota$ is some slow slant line with base$(g) \geq \eta * \omega$ and rank$(g) \geq |\text{dom}(f)| + n\}$.

Let $f \in I_{\omega*\eta+n+1}$, $\overline{a} = \text{dom}(f), \overline{b} = \text{range}(f)$, $M \models \sigma(\overline{a})$, $N \models \tau(\overline{b})$ and g be the slow slant line required to exist by $I_{\omega*\eta+n+1}$. Then $f \in I_{\eta*\omega+n}$ then f is a partial isomorphism by construction. So all that is left is to show that $\langle I_\zeta : \zeta < \omega * \iota \rangle$ has the back and forth property.

Let $a \in M$ and $M \models \sigma^*(\overline{a}^\wedge a)$. By the definition of $I_{\eta*\omega+n}$ we know that $\sigma|g = \tau|g$ and therefore $\sigma^*|g \leq \tau|g$. We also know that $\sigma^*|g$ is a g-archetype, because σ^* is a complete archetype. But then, by Generalized Saturation for Restricted Archetypes (Theorem 5.2.3.1) we can find an archetype τ^* realized in N such that $\tau^*|g = \sigma^*|g$.

Let b be the element corresponding to a in τ^* and let $h(a) = b$. Then $h \supseteq f$ and $h \in I_{\eta*\omega+n}$ because rank$(g) \geq |\text{dom}(f)| + n + 1 = |\text{dom}(h)| + n$.

We further know $I_\zeta \supseteq I_{\zeta'}$ if $\zeta' < \zeta$ by the transitivity of restrictions (i.e. if $\tau|g = \sigma|g$ and $g' \leq g$ then $\tau|g' = \sigma|g'$). Hence, $M \equiv_{\omega*\eta+n} N$ if $I_{\omega*\eta+n} \neq \emptyset$.

So, all we need to show is that $I_{\eta*\omega+n} \neq \emptyset$ for all $\eta * \omega + n < \iota * \omega$. Fix $\overline{a} \in M$ such that $M \models \sigma(\overline{a})$ with all colors of subtuples of $\overline{a}$ less than $\omega * \iota$. Then, by Generalized Saturation for Archetypes (Proposition 4.3.2.1 of [8]) we know that there must be some $\overline{b} \in N$ such that $N \models \sigma(\overline{b})$. Now take g to be any slow slant line above any color which occurs in σ. Hence if we let $f(\overline{a}) = \overline{b}$ then $f \in I_{\omega*\eta+n}$ for all $\omega * \eta + n < \omega * \iota$ (because $\sigma|g = \sigma|g$). □

Theorem 5.5.0.10. *If Spec$(M) \cap$ ORD $= \eta * \omega + n$ then $\omega * \eta <$ qr$(M) \leq \omega * \eta + \omega$.*

Proof. By Theorem 5.5.0.7 we know that $M \equiv_{\omega*\eta+\omega} N$ implies $M \cong N$, so

$\omega*\eta+\omega \geq \mathrm{qr}(M)$. But, by Theorem 5.5.0.8 we know that if $\mathrm{Spec}(N)\cap\mathrm{ORD} \geq \omega * \eta$ then $M \equiv_{\omega*\eta} N$ and hence $\omega * \eta < \mathrm{qr}(M)$. □

Theorem 5.5.0.11. *For each α there is a sentence $\Theta^*(\mathcal{M}_{\omega*\alpha})$ of $\mathcal{L}_{\omega_1,\omega}$ of quantifier rank $\leq \omega$ such that*

- $\Theta^*(\mathcal{M}_{\omega*\alpha})$ *is scattered.*
- $\{qr(M) : M \models \Theta^*(\mathcal{M}_{\omega*\alpha})\} \subseteq \omega * \alpha$
- $\{qr(M) : M \models \Theta^*(\mathcal{M}_{\omega*\alpha})\}$ *is unbounded in $\omega * \alpha$.*

Proof. By Theorem 5.5.0.8 and Theorem 5.4.2.3 we know that Θ^* satisfies the conditions on T_K in Corollary 4.1.3.8 □

5.6 Extensions of Θ

In this section, we will prove several results about models of Θ. Specifically we will look at what happens to a model if we remove a single tuple. We will find that when we do this we still have a model of Θ (under the appropriate definition)

5.6.1 Models of Θ Extending Tuples

Definition 5.6.1.1. Let $M \models \Theta$, let $\overline{a} \in M$ be a tuple. Let $M-\overline{a}$ be defined as follows:

- $M - \overline{a} \models P'(\overline{b}) \Leftrightarrow M \models P(\overline{a}^\wedge\overline{b})$
- $M - \overline{a} \models P'_{\mathcal{F}'}(\overline{b}) \Leftrightarrow (\exists F) M \models P_{\mathcal{F}}(\overline{a}^\wedge\overline{b})$ and $F' = F|\overline{b}$

Theorem 5.6.1.2. *$M - \overline{a} \models \Theta$ in the language $P', P'_{\mathcal{F}}$*

Proof. Lets go through the proof in detail.

Local Color(LC)

This is true in $M - \overline{a}$ because it is true in M.

Local Compactness(LK)If $\overline{b}$ is non-repeating, $\exists \overline{c}$ such that $M \models P_{\mathcal{F}}(\overline{a}^\wedge \overline{b}^\wedge \overline{c})$ by (LK). So, $M - \overline{a} \models P_{\mathcal{F}|\overline{b}^\wedge \overline{c}}(\overline{b}^\wedge \overline{c})$, so $M - \overline{a} \models$ (LK)

Generalized Saturation (GS)

Let A be a cell, and let B be a cell to which it can be extended. If $M - \overline{a} \models A(\mathbf{x})$ then $\exists$ a cell A' such that $M \models A'(\overline{a}^\wedge x), A'|\mathbf{x} = A$. This is by the definition of $M - \overline{a}$.

Now, as A can be extended to be B, there must exist some B' extending A' such that $B'|\mathbf{x}^\wedge \mathbf{y} = B$. (As cells are compatible and we can always find a cell containing any tuple). So $\exists y$ such that $M \models B'(\mathbf{x}^\wedge \mathbf{y}^\wedge \overline{a})$, and hence, $M - \overline{a} \models B(\mathbf{x}, \mathbf{y})$ therefore $M - \overline{a} \models$(GS)

Local Order (LO)

This says that if $P_{\mathcal{F}}(\mathbf{x})$ holds, then it is true as far as any finite slant line.

But, this doesn't in any way take into account (by itself) the global structure. So, if $P_{\mathcal{F}}(\mathbf{x}^\wedge \overline{a})$ is true as far as f (a slant line) in M then the same axiom with P' replacing P guarantees in $M - \overline{a}$ $P_{\mathcal{F}|\mathbf{x}}(\mathbf{x})$ is true as far as $f - |\overline{a}|$.

Local Extension(LE)

Let $\mathcal{F}'$ not be extensible to $\mathcal{F}$ along f.

We need to show that $M - \overline{a} \models P_{\mathcal{F}}(\mathbf{x}_i) \to \neg P_{\mathcal{F}'}(x_{f(i)})$

So, we need to show $M \models [(\exists \overline{\mathcal{F}}), \overline{\mathcal{F}}|\mathbf{x} = \mathcal{F}, P_{\overline{\mathcal{F}}}(\mathbf{x}^\wedge \overline{a})] \to [(\forall \overline{\mathcal{F}}'), \overline{\mathcal{F}}'|\mathbf{x} = \mathcal{F}' \to \neg P_{\overline{F'}}(\mathbf{x}^\wedge \overline{a})]$

Claim 5.6.1.3. *It suffices to show that if* $\mathcal{F}(\boldsymbol{x}^\wedge \overline{a}) \Rightarrow \mathcal{F}'(\boldsymbol{x}^\wedge \boldsymbol{y}^\wedge \overline{a})|\boldsymbol{x}^\wedge \overline{a}$ *than* $(\exists \mathcal{G}, \mathcal{G}')$ *s.t.* $\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \mathcal{F}(\boldsymbol{x}, \boldsymbol{y})$ *and* $\mathcal{G}'(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \mathcal{F}'(\boldsymbol{x}, \boldsymbol{y})$ *and* $\mathcal{G}, \mathcal{G}'$ *put all tuples (of arity* ≥ 2*) not in dom(*$\mathcal{F}$*) at* $-\infty$ *and* $\mathcal{F}(\boldsymbol{x}^\wedge \overline{a})|\boldsymbol{x} \Rightarrow \mathcal{F}'(\boldsymbol{x}^\wedge \boldsymbol{y}^\wedge \overline{a})|\boldsymbol{x}$*.*

Proof. Let $M \models P_{\mathcal{F}}(\mathbf{x}_i)$ so $\exists \overline{\mathcal{F}}$ s.t. $\overline{\mathcal{F}}|\mathbf{x} = \mathcal{F}, M \models P_{\overline{\mathcal{F}}}(\mathbf{x}_i{}^\wedge \overline{a})$.

Assume to get a contradiction $\exists \overline{\mathcal{F}}', \overline{\mathcal{F}}'|\mathbf{x}_{f(i)} = \mathcal{F}'$, $P_{\overline{\mathcal{F}}'}(\mathbf{x}_{f(i)}{}^\wedge \overline{a})$

So we know $\mathcal{F}'$ is not extensible to $\overline{\mathcal{F}}$ along $\overline{f}$ (where $\overline{f}(i) = i$ if $\mathbf{x}_i \in \overline{a}, \overline{f}(i) = f(i)$ otherwise) because $M \models P_{\mathcal{F}}(\mathbf{x}, \overline{a}) \to \neg P_{\overline{\mathcal{F}}'}(\mathbf{x}_{f(i)}{}^\wedge \overline{a})$ and $M \models \Theta$.

So, $\overline{f}^* \overline{\mathcal{F}}' \Rightarrow \overline{\mathcal{F}}|\mathbf{x}_{f(i)}{}^\wedge \overline{a}$, but by the claim this means $\overline{f}^* \overline{\mathcal{F}}'|\mathbf{x}_{f(i)} \Rightarrow (\overline{F}|\mathbf{x}_{f(i)}{}^\wedge \overline{a})|\mathbf{x}_{f(i)}$ which is the same as $f^* \mathcal{F}' \Rightarrow \mathcal{F}|\mathbf{x}_{f(i)}$ because of how we constructed $\overline{f}$. But, this means $\mathcal{F}'$ is extensible to $\mathcal{F}$ along f. $\Rightarrow\Leftarrow$. □

Claim 5.6.1.4. *If* $\mathcal{F}(\boldsymbol{x}^\wedge \overline{a}) \Rightarrow \mathcal{F}'(\boldsymbol{x}^\wedge \boldsymbol{y}^\wedge \overline{a})|\boldsymbol{x}^\wedge \overline{a}$ *then* $(\exists \mathcal{G}, \mathcal{G}')$ *s.t.* $\mathcal{G}(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}) \Vdash \mathcal{F}(\boldsymbol{x}, \boldsymbol{y})$ *and* $\mathcal{G}'(\boldsymbol{x}, \overline{a}, \boldsymbol{z}) \Vdash \mathcal{F}'(\boldsymbol{x}, \overline{a})$ *and*
mcG$, \mathcal{G}'$ *put all tuples (of arity* ≥ 2*) not in dom(*$\mathcal{F}$*) at* $-\infty$ *and* $\mathcal{F}|\boldsymbol{x} \Rightarrow \mathcal{F}'|\boldsymbol{x}$*.*

Proof. It suffices to show if $(\mathcal{F}(\mathbf{x}^\wedge \overline{a}) \Rightarrow \mathcal{F}'(\mathbf{x}^\wedge \overline{a}))$ then $\mathcal{F}|\mathbf{x} \Rightarrow \mathcal{F}'|\mathbf{x}$. To do this we need to break down the analysis of $\Rightarrow$.

<u>Forests:</u> $\mathcal{F} \Rightarrow \mathcal{F}' \forall T \in \mathcal{F} \exists T' \in \mathcal{F}'$ s.t. $T \Rightarrow T'$ and

$\forall T' \in \mathcal{F}' \exists T \in \mathcal{F}$ s.t. $T' \Rightarrow T$

Sensible Trees: $T \Rightarrow T'$ iff $A \Rightarrow A'$ where A, A' are the ambiguity trees and if we "fill in" the process of ϕ, ϕ' then $\phi(\gamma) \Rightarrow \phi'(\gamma)$ (and $\phi(\gamma)$ is rooted). Where by fill in we mean as we go up ϕ' we make sure that we have a point immediately above γ (say γ') such that $\phi(\gamma')$ is the deracination of $\phi(\gamma)$.

Ambiguity Nodes: $\langle S', H' \rangle \Rightarrow \langle S, H \rangle'$ iff $\forall T \in S \exists T' \in S'$ s.t. $T \Rightarrow T'$ and
$\forall T' \in S' \exists T \in S$ s.t. $T' \Rightarrow T$

And H, H' have the same superficial histories, which means at least as far as the histories are concerned $\langle S, H \rangle, \langle S', H' \rangle$ "place" tuples in the same place on the augmented unitary trees part of the sensible trees in S (but what they place might be different).

Ambiguity Trees: This is where the heart of $\Rightarrow$ is defined.

$\langle T', F' \rangle \Rightarrow \langle T, F \rangle'$ iff $\exists$ maps ϕ, ψ s.t. $\phi : T \to T', \psi : T' \to T, \phi \circ \psi = \text{id}$

ϕ is non-strictly order preserving.

ψ is order preserving

$\forall a \in T, F(a) \Leftarrow F(\phi(a))$

$\forall b \in T', F(\psi(b)) \Leftarrow F'(b)$

The idea is that T is somehow a "larger" tree than T' and every node of T is "larger" then the corresponding node in T'

But, at the same time T isn't too much larger than T' because the nodes which are in T and not in T' can be "projected" onto nodes in T' in such a way that the node in T is "larger" than the node is is "projected" to. $\square$

This is a complicated recursion but the key point is that it mainly takes place at the ambiguity tree stage.

Lets look at this recursion in the case of $\mathcal{F}(\mathbf{x}^\wedge\overline{a}) \Rightarrow \mathcal{F}'(\mathbf{x}^\wedge\overline{a})$ where $|\overline{a}| = n$.

Now lets try and "remove" the mention of $\overline{a}$ from the forest and see what happens.

First lets go all the way down the recursion and look at the base case where we only have 1-tuples on sensible trees (and forests).

Here $T \Rightarrow T'$ iff $T = T'$, so if we remove all mention of $\overline{a}$ from T and all mention of $\overline{a}$ from T' we still have $T|\mathbf{x} = T'|\mathbf{x}$ and so $T|\mathbf{x} \rightarrow T'|\mathbf{x}$. And similarly if T, T' are forests.

Now if we look at ambiguity nodes of arity 1 we see that $\langle S, H\rangle \Rightarrow \langle S', H'\rangle$ iff $\langle S', H'\rangle = \langle S, H\rangle$, and so if we remove all mention of $\overline{a}$ from $\langle S', H'\rangle, \langle S, H\rangle$, we still have $\langle S', H'\rangle|\mathbf{x} = \langle S, H\rangle|\mathbf{x}$, so $\langle S', H'\rangle|\mathbf{x} \Rightarrow \langle S, H\rangle|\mathbf{x}$

Now lets look at ambiguity trees of arity 1. Assume $\langle T, F\rangle \Rightarrow \langle T', F'\rangle$. This means we have an injection $\psi : T \rightarrow T'$ and a surjection $\phi : T' \Rightarrow T$ each preserving $\Rightarrow$. Now, if $F(\mathbf{y})$ talks about tuples in $\overline{z}$ and $F'(\overline{c}) \Leftarrow F(\mathbf{y})$, then $F'(\overline{c})$ talks about <u>exactly</u> the same tuples (although it may say different things). So, if we restrict our maps only to ambiguity nodes in T, T' which don't talk about $\overline{a}$ (i.e. $\langle T, F\rangle|\mathbf{x}, \langle T', F'\rangle|\mathbf{x}$) then our maps ψ, ϕ still witness $\langle T, F\rangle|\mathbf{x} \Rightarrow \langle T', F'\rangle|\mathbf{x}$.

Now we are finally ready to consider tuples of arbitrary size.

Assume for all tuples $\mathbf{x}'$ $|\mathbf{x}'| = n$, and $Z \Rightarrow Z'$ implies $Z|\mathbf{x}' \Rightarrow Z'|\mathbf{x}'$

where Z is either a sensible tree, ambiguity tree, forest, or ambiguity node on $\mathbf{x}'^\wedge\overline{a}'(|\overline{a}'| \leq |\overline{a}|$

Now if $W \Rightarrow W'$ is a sensible tree, forest, ambiguity node on $\mathbf{x}^\wedge\overline{a}'$ such that $|\mathbf{x}| = n+1, |\overline{a}'| \leq |\overline{a}|$ then $W|\mathbf{x} \Rightarrow W'|\mathbf{x}$ by the definition of $\Rightarrow$ for these structures.

Now let $W = \langle T, F\rangle, W' = \langle T', F'\rangle$ be ambiguity trees. First we extend W, W' to ambiguity trees V, V' by placing all new 1-tuples at $-\infty$ and placing a sequence of 1-tuples going up the tree with a new tuple added to the history at each step. Now if $W' \Rightarrow W$ and the maps $\phi : T \to T'$, $\psi : T' \rightarrowtail T$ witness this then they immediately extend to witness $V' \Rightarrow V$. Now remove all mention of $\overline{a}$ from the ambiguity trees V', V. Then the maps which witnessed $V' \Rightarrow V$ still witness that these restrictions closely refine each other. The only thing we had to worry about is that when we removed some tuples that two old ambiguity nodes which weren't the same get collapsed to the same thing and now ψ is no longer injective. But, this was exactly why we added extra dummy elements (and why we had to).

$\square$

Corollary 5.6.1.5. *If $\infty \notin Spec(M)$ and M is non-hollow, then $M - \overline{a} \cong M - \overline{b}$ iff $M \models \|\overline{a}\| = \|\overline{b}\|$.*

Proof. By construction of $M - \overline{a}$, $M - \overline{a} \models \|\mathbf{y}\| = \alpha \Leftrightarrow M \models \|\overline{a}^\wedge\mathbf{y}\| = \alpha$ and $M - \overline{a} \models \mathbf{y}$ is above $\mathbf{x} \Leftrightarrow M \models \overline{a}^\wedge\mathbf{y}$ is above $\overline{a}^\wedge\mathbf{x}$. This is because taking the restriction of cells preserves the order of points.

So in particular, if M models there are no inaccessibles then $M - \overline{a}$ models there are no inaccessibles as well.

$\square$

Chapter 6

Other Component Trees

Having a collection of archetypes is a very strong property and allows us to get very sharp results concerning the sentence which has the collection of archetypes. However, there are two main problems with the study of such sentences. The first draw back is that the only known examples of a sentence with a collections of archetypes only has such a collection for countable models. As such, the approach breaks down when we try to look at languages beyond $\mathcal{L}_{\omega_1,\omega}$. The second (more asthenic) draw back is that the only known examples are somewhat unwieldy to deal with. In this chapter we will consider which have models that look very similar to those in Chapter 3, but which do not in fact have a collection of archetypes.

In this chapter we will introduce two theories of trees which we believe can be fruitfully used as components in larger theories in a similar way as sentences with a collection of archetypes were in Chapter 4. In Section 6.2 we will add a small amount of information onto the basic tree structure T_P to get theories T_Λ, T_Ω. This information will allow us to tell when two tuples have the same tree structure extending them.

While being able to tell when two tuples have the same trees extending them is a very nice feature to have, unless we have some way of limiting trees we are considering we very quickly end up with 2^ω many countable models. In Section 6.3 we will introduce a theory T_Ω where we allow ourselves to compare colors and thereby ensure that our trees are homogeneous. We will also calculate the quantifier rank of most such models.

However, as we will see, when we use our ability to compare the colors of a tree to ensure homogeneity we loose much of the sharp relationship between quantifier rank and the spectrum of the model which sentences with a collection of archetypes have (Theorem 3.4.0.19). As such, we need some new machinery to study these components. The notation which will be used in this machinery as well as the background notation for T_Λ is introduced in Section 6.1

6.1 Notation

Definition 6.1.0.6. Define

- $\Xi(1) = w$
- $\Xi(\alpha + 1) = \omega * \Xi(\alpha)$
- $\Xi(\gamma) = \bigcup_{\beta<\gamma} \Xi(\beta)$ if γ a limit.

6.1.1 Ordinal Equivalence

Definition 6.1.1.1. We say that two ordinals are equivalent up to ζ ($\alpha \equiv_\zeta \beta$) iff $(\forall x < \zeta)[(\exists a)a + x = \alpha \leftrightarrow (\exists b)b + x = \beta]$. We also define $\natural(\alpha) = \{x : (\exists a)a + x = \alpha\}$

For notational convenience we will assume $-\infty \not\equiv_\zeta \beta$ for all $\beta, \zeta \in \text{ORD}$.

Lemma 6.1.1.2. *$\alpha \equiv_\zeta \beta$ iff $\natural(\alpha) \cap \zeta = \natural(\beta) \cap \zeta$*

Proof. Immediate from the definitions. □

Corollary 6.1.1.3. *For all $\alpha, \beta, \gamma, \zeta$*

- $\alpha \equiv_\zeta \alpha$
- $\alpha \equiv_\zeta \beta \leftrightarrow \beta \equiv_\zeta \alpha$
- $\alpha \equiv_\zeta \beta \wedge \beta \equiv_\zeta \gamma \rightarrow \alpha \equiv_\zeta \gamma$

Proof. Immediate from 6.1.1.2 the definitions. □

Lemma 6.1.1.4. *Let $\alpha \equiv_\zeta \beta$ and let $\gamma < \zeta$. Then $\alpha + \gamma \equiv_\zeta \beta + \gamma$*

Proof. Let $x < \zeta$ such that $(\exists a)a + x = \alpha + \gamma$. If $x \leq \gamma$ then let b be such that $b + x = \gamma$. Then $\beta + b + x = \beta + \gamma$. If $x > \gamma$ then let z be such that $a + z = \alpha$. So there must exist a b such that $b + z = \beta$ (because $z \leq x < \zeta$ and $\alpha \equiv_\zeta \gamma$). Hence $\beta + \gamma = b + z + \gamma = b + x$.

The other direction (starting with β) is identical. □

Lemma 6.1.1.5. *$(\forall \alpha, n)(\exists \beta \leq \Xi(n))(\forall x)\alpha \equiv_{\Xi(n)} x + \Xi(n) + \beta$.*

Proof. First observe that for all x, if $\rho \equiv_{\Xi(n)} \Xi(n)$ then $x + \rho + \beta \equiv_{\Xi(n)} \beta$. This is because for each $z < \Xi(n), y < x + \rho, y + z < x + \rho < x + \rho + \beta$. So it suffices to prove that for each α there is a $\beta < \Xi(n)$ such that $\alpha \equiv_{\Xi(n)} \beta$

Now let $Y = \{y : y < \alpha, (\exists z(y) < \Xi(n))y + z(y) = \alpha\}$. Now if Y is empty then $\alpha \equiv_{\Xi(n)} \Xi(n)$ and we are done.

So lets assume Y is not empty. Then Y has a least element y' with $z(y') < \Xi(n)$. Now notice that if $(\exists a < \Xi(n), b < y')$ such that $b + a = y'$ then we have $b+a+z(y') = \alpha$ and $a+z(y') < \Xi(n)$ contradicting minimality of y'. So, we must have $y' \equiv_{\Xi(n)} \tau(n)$.
Hence $\alpha = y' + z(y') \equiv_{\Xi(n)} z(y')$ and we are done. $\square$

The idea is that we are going to be looking at two maps into the ordinals and we want to say when they look the same up to a certain "distance".

Definition 6.1.1.6. Let $f, g : X \to \mathrm{ORD}$ be maps from a set X into the ordinals. Let $\alpha \in \mathrm{ORD}$. We say <u>f is the same as g up to distance α</u> ($f|_d\alpha = g|_d\alpha$) if

- $(\forall x, y \in X)f(x) \leq f(y)$ iff $g(x) \leq g(y)$
- $(\forall x \in X, y \in X)(\forall \beta < \alpha)f(x) + \beta = f(y)$ iff $g(x) + \beta = g(y)$
- $(\forall x \in X)f(x) \equiv_\alpha g(x)$.

The idea here is that two functions are the same if they "look the same up to distance α". In other words, if two elements get mapped to ordinals (by f) which are less than α apart, then we know that they must get mapped by g to ordinals which are the same distance apart. But, if two elements get mapped to ordinals which are greater than or equal to α apart, all we know is that they are too far apart to talk about.

In addition we require one other conditions. We individual elements "look the same up to α" from below. What this means is that if a point gets mapped to two different ordinals, then those ordinals must "look the same from below". So in particular, if there is an ordinal below $f(x)$ from which $f(x)$ is less than α away, then there must also be such an ordinal below $g(x)$.

Definition 6.1.1.7. Let $f, g : A \times X^{<\omega} \to \text{ORD}$ be maps from finite subsets of X indexed by A into an ordinal α. Let $L : \omega \to \text{ORD}$.

Let $\langle x_1, \ldots, x_n \rangle \in X^{<\omega}$. Let $f_k = f$ restricted to $A \times \langle x_i, \ldots, x_k \rangle^{<\omega}$, Let $g_k = g$ restricted to $A \times \langle x_i, \ldots, x_k \rangle^{<\omega}$.
For a set A we say $\underline{f \text{ is the same as } g \text{ relative to } L \text{ on } \langle x_1, \ldots, x_n \rangle, A}$
$f(A, \langle x_1, \ldots, x_n \rangle)|_d L = g(A, \langle x_1, \ldots, x_n \rangle)|_d L)$ if:

$f_k|_d L(k) = g_k|_d L(k)$ for all $k \leq n$.

There are a few points worth mentioning about this definition. First, notice that the order of the tuple really does matter when determining whether or not two functions are the same up to L. This is because we want to eventually consider when functions are the same up to a slant line for the purpose of back and forth arguments. So, when we add on a new element we will only require all the new tuples to be equivalent up to the slant line at the arity of the whole tuple. As we will see, the fact that we are only looking at new tuples up to the slant line at the arity of the whole tuple and not the arity of the tuple we are considering will be crucial to allow us to get our back and forth argument.

The second point worth noting explicitly is that the set A is there so that we can essentially have for each function, multiple ordinals on each tuple $\langle x_1, \ldots, x_n \rangle$.

6.2 T_Λ

6.2.1 Introduction

In this section we will define our theory T_Λ. The goal is to ensure as much saturation/homogeneity as we can without being able to actually say when one tuple has color greater than another.

6.2.2 Basic Theory

6.2.2.1 Definitions

Definition 6.2.2.1. Let $L_\Lambda = L_{S_=}$.

Definition 6.2.2.2. Let T_Λ be the universal closure of the following $L_{S_=}$ sentences:

- $T_{S_=}$
- (Homogeneity for Archetypes) For each $m \in \omega$

$$(\forall \mathbf{x}, \mathbf{y}, \overline{a})(\exists^m \overline{b}) E_a(\mathbf{x}, \mathbf{y}) \rightarrow E_a(\mathbf{x}\overline{a}, \mathbf{y}\overline{b})$$

The purpose of (Homogeneity for Archetypes) is to ensure that if two tuples look identical (with respect to $S_=$) then they can be extended in exactly the same ways. Something that is important to notice though is that there E_a is as in Definition 2.3.2.1 and not as in Definition 2.3.1.2(as $R_{\leq}$ isn't even in the language of T_Λ).

6.3 T_Ω

6.3.1 Introduction

In this section we will look at a theory with enough expressive power to guarantee that the models are homogeneous. We will use the tools of Section 2.3.1 to study these trees.

We are going to want two things from our trees. First we are going to want the γ-type of a tuple to be completely determined by what the ordinal colors of it's subtuples look like up to $\Xi(\gamma)$ (where Ξ is as in Definition 6.1.0.6). Second, we want there to be a huge amount of homogeneity. In other words we want (in some strong sense) if we are looking at some finite part of the model and it is consistent that it can be extended in a certain way, then we must be able to extend it that way.

The reason why we want this new language is because the properties previously mentioned are ones which can't be defined in the language $L_{S_=}$.

6.3.1.1 Color Infinity

It is worth mentioning again that our description above isn't completely accurate. There is one time when we won't worry if the γ-type of a tuple is determined by the color of the subtuples (and in general when it won't be). This is when some subtuple has color infinity. The reason why we won't worry about this case is that when we put together these trees (as they are just components) we will make sure we combine them in such a way as to guarantee that there are no models with ill-founded branches.

This lack of a color infinity makes things significantly easier. This is because we know, by Theorems 2.3.1.3 and Theorem 2.3.1.4 that if there are no tuples of color infinity then $R_{\leq}$ actually defines when one tuple has color less than the other. And as we shall see, this will allow us to guarantee that all the information about the models is contained in the colors of its tuples.

However, if there are tuples of color infinity in our model, then there is no easy way tell when two tuple of color infinity have the same γ-type (for arbitrary γ). And in fact, much of the effort in [8] is due to a need to force that the archetypes of tuples of color infinity do in fact determine everything about the tuples.

It is finally worth mentioning that whenever we assume that a model has no tuple of color infinity in a theorem (without actually proving that it must be so) it suffices for our purposes to only assume that $R_{\leq}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\| \leq \|\overline{b}\|$ for the model.

6.3.2 Basic Theory

6.3.2.1 Definitions

Definition 6.3.2.1. Let $L_{\Omega} = L_{R_{\leq}}$.

Definition 6.3.2.2. Let T_{Ω} be universal closure of the following L_{Ω} sentences:

- T_R

- $(\forall \overline{a})(\forall \mathbf{x})(\exists \mathbf{y}) R_{=}(\overline{a}\mathbf{x}, \overline{a}\mathbf{y}) \wedge |\mathbf{x}| + 1 = |\mathbf{y}|$

- (Saturation for Archetypes) For each $m \in \omega$

$$(\forall x_1, \dots x_n)(\forall \{y_S : S \subseteq n\})(\exists^n b) \bigwedge_{S \subseteq n} [R_{<}(y_S, \{x_i : i \in S\}) \rightarrow R_{=}(y_S, \{x_i : i \in S\} b)]$$

The first new axiom is there to ensure that the ordinal part of the spectrum of the model is a limit. This isn't strictly necessary, but it will make calculations later on a little bit cleaner.

The second new axiom is the more important one. This says essentially that if an extension is consistent with the colors of a tuple then that extension must be realized infinitely often. So in a strong sense, everything that can happen does.

Definition 6.3.2.3. Let $M, N \models T_{\Omega}$. Define

$$\text{ATYPE}(M) = \{\text{ctype}(\overline{a}) : \overline{a} \in M\}$$

We also say $\text{ATYPE}(M)|_d\gamma = \text{ATYPE}(N)|_d\gamma$ if

- $(\forall X \in \text{ATYPE}(M))(\exists Y \in \text{ATYPE}(N))[X|_d\gamma = Y|_d\gamma]$

- $(\forall Y \in \text{ATYPE}(N))(\exists X \in \text{ATYPE}(M))[X|_d\gamma = Y|_d\gamma]$

Consider color types as functions on finite tuples (see Definition 6.1.1.6)

6.3.2.2 Results

Theorem 6.3.2.4. *If $M \models T_{\Omega}$ and has no tuples of color ∞ then Spec$(M) \cap$ ORD is a limit.*

Proof. Assume $\mathrm{Spec}(M) \cap \mathrm{ORD} = \beta + 1$.

So $(\forall \mathbf{x} \in M)\|\mathbf{x}\| \leq \beta$, and there is an element of M with color β. So, there is an 2 tuple xx' such that $\|xx'\| = \beta$. But then $\|x\| \geq \beta + 1$. $\Rightarrow\Leftarrow$.

□

Theorem 6.3.2.5. *Let $M, N \models T_\Omega$ and have no tuples of color ∞. If $ATYPE(N)|_d\Xi(\gamma) = ATYPE(M)|_d\Xi(\gamma)$ and $(Spec(M) \cap ORD)|_d\Xi(\gamma) = (Spec(N) \cap ORD)|_d\Xi(\gamma)$, then $M \equiv_\gamma N$.*

Proof. First we define our sequence of partial isomorphism.

Definition 6.3.2.6. Define $I_\eta(M, N) = I_\eta$ as follows:
$I_\eta = \{f : M \to N s.t. f$ is a bijection, $|\mathrm{dom}(f)| < \omega$, f preserves all atomic formulas in L_Ω and $\mathrm{ctype}(\mathrm{dom}(f))|\Xi(\eta) = \mathrm{ctype}(\mathrm{range}(f))|\Xi(\eta)\}$

Let $f \in I_\eta, \eta \leq \gamma$. Notice that f is a partial isomorphism by definition. All that is left to show is that $\langle I_\zeta : \zeta \leq \gamma\rangle$ satisfies the back and forth property. Let $\eta + 1 \leq \gamma$ and let $a \in M$. We want to find a $b \in N$ such that $g(a) = b$, $f \subseteq g, g \in I_\eta$.

Let $A = \mathrm{ctype}(\mathrm{dom}(f)a)$. We will now define a color archetype B on $(\mathrm{range}(f), b)$.

The idea is that we want the structure of where tuples are placed in ORD to look the same (up to $\Xi(\eta)$) no matter if we are looking at $\mathrm{dom}(f)a$ or at $\mathrm{range}(f)b$. We will do this in two stages. In the first stage we are going to make sure that all the new tuples in $\mathrm{range}(f)b$ get placed in between the correct tuples of $\mathrm{range}(f)$.

In the second stage we will organize the new tuples in the right order, and with the correct distances and between the boundaries of the interval.

While doing this second stage we will fix the actual values.

First some notation though. Let $C = \langle(\overline{c}_i, \alpha_i) : \|\overline{c}_i\| = \alpha_i, i \leq j \rightarrow \alpha_i \leq \alpha_j$,

- $\overline{c}_i \subseteq \mathrm{dom}(f)$ or
- $\overline{c}_i = M$ and $\alpha_i = \mathrm{Spec}(M) \cap \mathrm{ORD}$ or
- $\overline{c}_i = -\infty$ and $\alpha_i = -\infty\rangle$

We will further abuse notation by saying $f(-\infty) = -\infty$, $f(M) = N$, and $\| - \infty\| = -\infty$, $\|M\| = \mathrm{Spec}(M) \cap \mathrm{ORD}, \|N\| = \mathrm{Spec}(N) \cap \mathrm{ORD}$ and $f(\alpha_i) = \|f[\overline{c}_i]\|$.

Stage 1: Let $\overline{d}$ be a subtuple of range(f). We define our interval $[\beta(\overline{d}b), \zeta(\overline{d}b)]$ as follows. Let $(\overline{c}_j, \alpha_j)$ be such that α_j is the greatest ordinal in C which is less than $\|f^{-1}(\overline{d})a\|$. Let $\beta(\overline{d}b) = \|f[\overline{c}_j]\|$. Let $(\overline{c}_k, \alpha_k)$ be such that α_k is the least ordinal in C which is greater than $\|f^{-1}(\overline{d})a\|$. Let $\zeta(\overline{d}b) = \|f[\overline{c}_k]\|$. Note that these always exist by our abuse of notation and the fact that C is finite.

Stage 2: First let $\langle\overline{e}_1 a, \ldots \overline{e}_n a\rangle$ be the subtuples of dom$(f)a$ such that $\beta(f(\overline{e}_i)b) = \beta$, $\zeta(f(\overline{e}_i)b) = \zeta$ and such that $i \leq j \rightarrow \|\overline{e}_i a\| \leq \|e_j a\|$. Further, by an abuse of notation, let $\|\overline{e}_0 a\| = \beta$ (even if there is no tuple containing a with color β). We will define the colors of $f(e_i)b$ inductively on i.

Assume we have defined $\|f(\overline{e}_j)b\|$ for all $j < i$.

We will break this into cases.

- Case 1: $\beta + \Xi(\eta + 1) \geq \zeta$.

  Let $\|f(\overline{e}_i)b\| = \beta + \rho$ where $\|\overline{e}_i a\| = f^{-1}(\beta) + \rho$.

- Case 2: $\|\overline{e}_{j-1}a\| + \Xi(\eta) \geq \|\overline{e}_j a\|$.
  Let $\|f(\overline{e}_i)b\| = \|f(\overline{e}_{j-1}a)\| + \rho$ where $\|\overline{e}_{j-1}a\| + \rho = \|\overline{e}_j a\|$.

- Case 3: $\|\overline{e}_j a\| + \Xi(\eta) \geq \zeta$.
  Let $\|f(\overline{e}_i)b\| = \mu$ for some μ such that $\mu + \rho = \zeta$ where $\|\overline{e}_j a\| + \rho = f^{-1}(\zeta)$.

- Case 4: Everything else.
  Let $\rho < \Xi(\eta)$ such that $\rho \equiv_{\Xi(\eta)} \|\overline{e}_i a\|$
  Let $\|f(\overline{e}_i)b\| = \|f(\overline{e}_{j-1}a)\| + \Xi(\eta) + \rho$

Before we continue it is worth explaining what is going on in each of the four cases. First notice that by the definition of our sequence of partial isomorphism we require that dom(f) looks like range(f) up to $\Xi(\eta + 1)$. Hence in Case 1 we are in an interval which, in the context both of dom(f) and range(f) is completely determined. In particular we know the size of the interval is exactly the same in both M and N. So, we can put our tuples from range(f)b in exactly the sample places in the interval as the corresponding tuples of dom(f)a were.

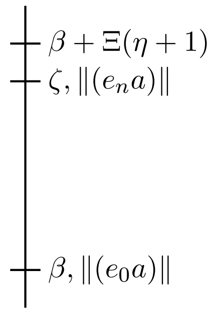
In Case 2 we know that in M we placed the tuple we are looking at less than $\Xi(\eta)$ from the previously placed tuple. So even if we are only considering the tuples up to $\Xi(\eta)$ we still know exactly the distance from the current tuple and the previous one. Hence that distance must be the same for the corresponding tuples in N.

Now here we also have to be a little bit careful for two reasons. The first reason is that we have to make sure that the ordinal on which our tuple in N is placed is "the same up to $\Xi(\eta)$" as the ordinal on which the corresponding tuple in M was placed. But this is true by Lemma 6.1.1.4

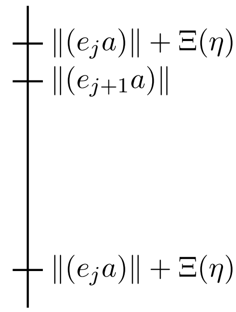
The second reason you need to be a little careful is that it is conceivable that at some point by placing successive copies of $\Xi(\eta)$ end to end that we might over take the end of the interval (which is the next highest tuple). However, in this case we don't have to worry about this happening as we are only placing a finite number of tuples and we know that the interval has to have length at least $\omega * \Xi(\eta) = \Xi(\eta + 1)$.

Case 3 is the case where something like the above happened. In Case 3 we find that in M we have placed a tuple so that we can now reach ζ in less than $\Xi(\eta)$. But we know that ζ and $f^{-1}(\zeta)$ are equivalent up to $\Xi(\eta + 1)$ (and hence up to $\Xi(\eta)$ by assumption) and so there must be some ordinal (possibly not unique) below ζ which corresponds to where the tuple in M was placed.

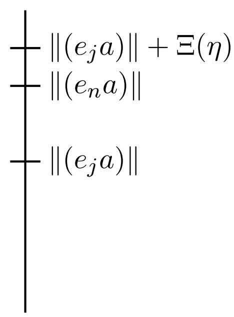
Now finally we are at Case 4. This is the case where everything is so far away from where we are placing our point in M that we can't tell the exact distance in either direction (up to $\Xi(\eta)$). So as long as we pick an ordinal that is at least $\Xi(\eta)$ from the ordinals on either end, and such that the ordinal is equivalent up to $\Xi(\eta)$ with the ordinal that the tuple in M is on then we are okay. But, we know we can always find such an ordinal because the distance between the previously placed point and the end of the interval is at least $\Xi(\eta + 1)$ and by Lemma 6.1.1.5.



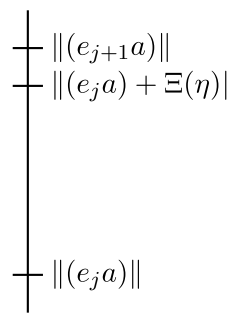
Case 1



Case 2



Case 3



Case 4

Now that we have defined the color archetype B that we want range$(f)b$ to satisfy, we need to show two things. First we need to show that B is a consistent extension of A (and hence realized) and then we need to show that if range$(f)b$ satisfies this color archetype then in fact $f \cup (a, b) \in I_\eta$.

To check consistency we only need to check that it is consistent with the spectrum of N and that if $\bar{a} \subsetneq \bar{b}$ then $\|f[\bar{a}]\| \leq \|f[\bar{b}]\|$. But we know by construction that $\|f[\bar{a}]\| \leq \|f[\bar{b}]\|$ iff $\|\bar{a}\| \leq \|\bar{b}\|$. And that is enough because we know (ctype(dom$(f)a$)) is consistent. To see that the colors are consistent with (i.e. less than) the spectrum of N observe that $(M, \mathrm{Spec}(M) \cap \mathrm{ORD}) \in C$ and so all colors in B are less than $\mathrm{Spec}(N) \cap \mathrm{ORD}$ as all tuples in M are less than $\mathrm{Spec}(N) \cap \mathrm{ORD}$.

So all that is left is to make sure that $f \cup (a, b)$ is in I_η. Well to do that we have to show $\text{ctype}(\text{dom}(f)a)|_d\Xi(\eta) = \text{ctype}(\text{range}(f)b)|_d\Xi(\eta)$. But each of the above 4 Cases in the definition of the color archetype were specifically designed to ensure this was the case. $\square$

Corollary 6.3.2.7. *Let A, A' be color archetypes such that $A|_d\Xi(\gamma+1) = A'|_d\Xi(\gamma+1)$. Then for all $B \leq A$ such that $|dom(B)| = |dom(A)| + 1$ there exists a B' extending A' with $|dom(B')| = |dom(A')| + 1$ and $B|_d\Xi(\gamma) = B'|_d\Xi(\gamma)$.*

Proof. This is immediate from the proof of Theorem 6.3.2.5. (And in fact this was the central idea needed in the proof) $\square$

Theorem 6.3.2.8. *Let $M, N \models T_\Omega$ and have no tuples of color ∞ Also let $Spec(M) \cap ORD = \Xi(\gamma_M)$ and $Spec(N) \cap ORD = \Xi(\gamma_N)$ and let $\zeta + 1 < \min\{\gamma_M, \gamma_N\} = \gamma'$. Then $M \equiv_\zeta N$*

Proof. By Theorem 6.3.2.5 It suffices to show that

(1) $\Xi(\gamma_M) \equiv_{\Xi(\zeta)} \Xi(\gamma_N)$

(2) $\text{ctype}(M)|_d\Xi(\zeta) = \text{ctype}(N)|_d\Xi(\zeta)$

To show (1) observe that if $\alpha \geq \zeta$ there is no $x < \Xi(\alpha)$ such that $x + \Xi(\zeta) > \Xi(\alpha)$. This is because if α is a limit then such an x would be less than some $\Xi(\zeta')$ and so $x + \Xi(\zeta) < \Xi(\zeta') + \Xi(\zeta) < \Xi(\zeta'+1) < \Xi(\alpha)$. And, if $\alpha = \gamma^* + 1$ then such an x would have to be less than $n * \Xi(\gamma^*)$ for some $n < \omega$ and hence $x + \Xi(\zeta) < (n+1) * \Xi(\gamma^*) < \Xi(\gamma')$.

To show (2) assume we have a color archetype A of M or N. We will then define a color archetype B realized in both N and M as follows. First order all the colors of subtuples A by their color. Suppose the n and $n+1$'s tuples are separated by α_n. If $\alpha_n < \Xi(\zeta)$ then add α_n to the color of the n tuple of B to get the color of the $n+1$'s tuple. If $\alpha_n \geq \Xi(\zeta)$ then add $\Xi(\zeta) + \alpha_n^*$ (where $\alpha_n^* \equiv_{\Xi(\zeta)} \alpha_n$ and $\alpha_n^* < \Xi(\zeta)$) to the color of the n tuple of B to get the color of the $n+1$'s tuple.

By construction we have that $A|_d\Xi(\zeta) \equiv B|_d\Xi(\zeta)$ and B is realized in both M and N (as all colors in B are less than the Spectrum of either M or N respectively. So in particular we have (because A was an arbitrary color archetype of either M or N) that $\text{ATYPE}(M)|_d\Xi(\zeta) = \text{ATYPE}(N)|_d\Xi(\zeta)$ and hence $M \equiv_\zeta N$. $\square$

Chapter 7

Almost Scattered Sentences

7.1 Introduction

In this chapter we will study quantifier rank spectrum for almost scattered sentences. In Section 7.2 we will give a procedure which will turn any scattered sentence into an almost scattered sentence which is not scattered but which has exactly the same quantifier rank spectrum.

In Section 7.3 we will consider another construction which will yield an almost scattered structure. The construction is very similar to that in Section **??** except we will not assume that the component structures have a collection of archetypes (see Definition 3.3.0.13). Further, we believe that, similar to the construction in Section **??** this construction will allow us to get a structure with the supremum of the quantifier rank spectrum what we wish. However, there is a problem. As the construction does not have a collection of archetypes we have had to "build into the theory" the idea of Homogeneity of Color Archetype pairs (see Definition 6.3.2.2). But, the obvious method poses a problem when it comes time to calculate a lower bound on the supremum of the quantifier rank spectrum. So, we will point out exactly what the problem is and conjecture that the structure still has the properties we want.

7.2 Almost Scattered Sentences which aren't Scattered

The way we are going to get an almost scattered sentence from a scattered one is by looking at κ many disjoint copies of our scattered sentence.

7.2.1 Axioms

First we need our almost scattered sentence which we will modify.

Definition 7.2.1.1. Let ϕ_{Sc} be an almost scattered sentence in the language L_{Sc}.

7.2.1.1 Language

Definition 7.2.1.2. Let $L_A(L_{Sc}, \kappa) = L_{Sc} \cup \{Q_i : i \in \kappa, \text{arity}(Q_i) = 1\}$.

7.2.1.2 Axioms

Definition 7.2.1.3. Let $T_A(\phi_{Sc}, \kappa)$ be universal closure of the following $L_A(L_{Sc}, \kappa)$ sentences:

- $\bigwedge_{i\in\kappa} Q_i(x)$

- $Q_i(x) \wedge Q_j(y) \rightarrow \neg U(\overline{a}x\overline{b}y\overline{c})$ if $i \neq j$ and U is any predicate in L_{Sc}.
- $(\exists x)Q_i(x)$ for each $i \in \kappa$
- $Q_i \models \phi_{Sc}$ for each $i \in \kappa$.

The idea is that we want our theory to consist of κ many disjoint copies of our sentence ϕ_{Sc}. We hence have the following obvious lemmas.

Lemma 7.2.1.4. $|L_A(L_{Sc}, \kappa)| = \max\{\kappa, |L_{Sc}|\}$, *and if* $\phi_{Sc} \in \mathcal{L}_{\beta,\omega}(L_{Sc})$ *then* $T_A(\phi_{Sc}, \kappa) \in \mathcal{L}_{\max\{\beta,\kappa\},\omega}(L_A(L_{Sc}, \kappa))$.

Proof. By definition of $L_A(L_{Sc}, \kappa)$ and $T_A(\phi_{Sc}, \kappa)$ □

Lemma 7.2.1.5. *Let* $M, N \models T_A(\phi_{Sc}, \kappa)$. *Then* $M \cong N$ *iff* $Q_i^M \cong Q_i^N$ *for all* $i \in \kappa$.

Proof. The right to left direction of the if and only if is trivial. Left to right is true because if $f_i : M|Q_i \rightarrow N|Q_i$ is a bijection which preserves predicates (i.e. an isomorphism) then $\bigcup_{i\in\kappa} f_i : M \rightarrow N$ is a bijection and also preserves all predicates because there is no interaction between elements which satisfy different Q_i's (i.e. all predicates with arguments from different Q-components are false). So $\bigcup_{i\in\kappa} f_i$ is an isomorphism. □

Lemma 7.2.1.6. *If* ϕ_{Sc} *has* β *many models of size* $\leq \alpha$ *then* $T_A(\phi_{Sc}, \kappa)$ *has* β^κ *many models of size* $\leq \max\{\kappa, \alpha\}$

Proof. Let $\langle M_i : i \in \beta\rangle$ be a list of all models of ϕ_{Sc} of size $\leq \alpha$. For each $f \in \beta^\kappa$ let $M_f \models T_A(\phi_{Sc}, \kappa)$ be the model where $M_f \models Q_i \cong M_{f(i)}$. Then every model of $T_A(\phi_{Sc}, \kappa)$ is of the form M_f for some $f \in \beta^\kappa$ and $M_f \cong M_g$ iff $f = g$ (by Lemma 7.2.1.5) □

Similarly we have the following theorem.

Theorem 7.2.1.7. *Let $M, N \models T_A(T_S, \kappa)$. Then $M \equiv_\gamma N$ iff $Q_i^M \equiv_\gamma Q_i^N$ for all i.*

Proof. First notice that the left to right direction of the iff is trivial. Now Let $\langle I_\zeta^i : \zeta \leq \gamma \rangle$ be a sequence of partial isomorphisms that witness that $Q_i^M \equiv_\gamma Q_i^N$. We then construct a sequence of partial isomorphisms, $\langle I_\zeta : \zeta \leq \gamma \rangle$, that witness $M \equiv_\gamma N$.

Define $I_\alpha = \{f : |\mathrm{dom}(f)| < w, f|(\mathrm{dom}(f) \cap Q_i^M) \in I_\alpha^i \text{ for each } i \in \kappa\}$. First of all notice that by construction if $f \in I_\alpha$ then f must preserve all predicates and hence is a partial isomorphism.

All that is left is to show that $\langle I_\zeta : \zeta \leq \gamma \rangle$ has the back and forth property. Let $f \in I_{\alpha+1}$ and let $a \in M$. We know that $M \models Q_i(a)$ for some $i \in \kappa$. We then also know that $f|(\mathrm{dom}(f) \cap Q_i^M) \in I_{\alpha+1}^i$ and so there must be some element $b \in Q_i^N$ such that $f|(\mathrm{dom}(f) \cap Q_i^M) \cup (a, b) \in I_\alpha^i$. But then by construction $f \cup (a, b) \in I_\alpha$. We can do the other direction (starting with a $b \in N$ and finding an $a \in M$) in exactly the same way and hence we know that $\langle I_\zeta : \zeta \leq \gamma \rangle$ has the back and forth property. So in particular we know $M \equiv_\gamma N$. □

Corollary 7.2.1.8. *If $M, N \models T_A(\phi_{Sc}, \kappa)$ and $\overline{a} \in M, \overline{b} \in N$ then $(M, \overline{a}) \equiv_\gamma (N, \overline{b})$ iff $(Q_i^M, \overline{a} \cap Q_i^M) \equiv_\gamma (Q_i^N, \overline{b} \cap Q_i^N)$ for all $i \in \kappa$.*

Proof. This is immediate from the construction of the sequence of partial isomorphisms $(\langle I_\zeta^i : \zeta \leq \gamma \rangle)$ from M to N in Theorem **??** and Theorem 1.2.2.19. □

Corollary 7.2.1.9. *For each α $M \models T_A(\phi_{Sc}, \kappa)$, if $p(\boldsymbol{x})$ is a type over $T_A(\phi_{Sc}, \kappa) \cup Th_\alpha(M)$ then let $(p)_i(\boldsymbol{x}) = \{\phi^{Q_i}(\boldsymbol{x}) : \phi \in L_{Sc}, qr(\phi) \leq \alpha, p \rightarrow \phi^{Q_i}(\boldsymbol{x})\}$. We then know each $(p)_i$ is a type over $\phi_{Sc} \cup Th_\alpha(M|Q_i)$. What is more, for each $\langle \phi(i) : i \in \kappa \rangle$ a sequence of types over ϕ_{Sc}, each with quantifier rank $\leq \alpha$, and such that all but finitely many $\phi(i)$ have no free variables, then there is a unique complete type p such that $qr(p) \leq \alpha$, p is a type over $T_A(\phi_{Sc}, \kappa)$ and $(p)_i = \phi(i)$ for all $i \in \kappa$.*

Proof. This is immediate from Corollary 7.2.1.8 says. □

Specifically what this Corollary says is that any α type is uniquely determined by it's components on each of the Q_i.
We therefore also have

Theorem 7.2.1.10. *Let $s(\phi_{Sc}) = \omega$. Then for each $\alpha < \omega_1$ there exists $\mathcal{S}_a(\alpha, T_A(\phi_{Sc}, \omega))$, a countable collection of formulas of $\mathcal{L}_{\omega_1,\omega}(L_A(L_{Sc}, \omega))$ such that*

(1) For all $M \models T_A(\phi_{Sc}, \omega)$ and $q \in \mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \omega))$ there is at least one $p \in \mathcal{S}_a(T_A(\phi_{Sc}, \omega), \alpha)$ such that $T_A(\phi_{Sc}, \omega) \models (\forall \boldsymbol{x})[p(\boldsymbol{x}) \wedge Th_\alpha(M) \wedge T_A(\phi_{Sc}, \omega)] \rightarrow q(\boldsymbol{x})$

(2) For all $M \models T_A(\phi_{Sc}, \omega)$ and $p \in \mathcal{S}_a(T_A(\phi_{Sc}, \omega), \alpha)$ there is at most one $q \in \mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \omega))$ such that $T_A(\phi_{Sc}, \omega) \models (\forall \boldsymbol{x})[p(\boldsymbol{x}) \wedge Th_\alpha(M) \wedge T_A(\phi_{Sc}, \omega)] \rightarrow q(\boldsymbol{x})$

Proof. Let $M \models T_A(\phi_{Sc}, \omega)$. Then we know by Corollary 7.2.1.9 that each $p \in \mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \omega))$, $\mathrm{qr}(p) \leq \alpha$ is uniquely determined by $(p)_i$ for all $i \in \omega$. But, we know that each $(p)_i$ is either uniquely determined by an element of $\mathcal{S}_a(\phi_{Sc}, \alpha)$ (for ϕ_{Sc}), or $(p)_i$ has no free variables. If $(p)_i$ has free variables call $(p)_i^*$ the element of $\mathcal{S}(Th_\alpha(\mathcal{M}), \phi_{Sc})$ which uniquely determines it. Otherwise let $(p)_i^*$ be $x = x$. We know that each element $p \in \mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \omega))$ is uniquely determined by the formula $Th_\alpha(M)^{Q_i} \cup T_A(\phi_{Sc}, \omega) \cup (\bigwedge_{i\in\omega} Q_i(\mathbf{x}) \wedge (p)_i^*)$ and further each such sequence $\langle (p)_i^* : i \in \kappa\rangle$ determines at most one element of $\mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \omega))$. So if we define $\mathcal{S}_a(T_A(\phi_{Sc}, \omega), \alpha) = \{\langle Th_\alpha(M)^{Q_i} \cup \bigwedge_{i\in\omega} Q_i(\mathbf{x}) \wedge (p)_i^* : i \in \omega\rangle :$ all but finitely many entries are empty$\}$ then $\mathcal{S}_a(T_A(\phi_{Sc}, \omega), \alpha)$ is countable and witnesses that $T_A(\phi_{Sc}, \omega)$ is almost scattered. □

Theorem 7.2.1.11. *Let $W, V \models ZFC, W \subseteq V[G]$ for some generic extension of V $\phi_{Sc} \in W \cap V$ and $W \models s(T_A(\phi_{Sc}, \kappa)) = \omega$. Then for each $(\alpha < \omega_1)^W$ there exists $\mathcal{S}_a(\alpha, T_A(\phi_{Sc}, \kappa))$, a collection of formulas of $[\mathcal{L}_{\omega_1,\omega}(L_A(L_{Sc}, \omega))]^W$, such that*

(1) For all $M \models T_A(\phi_{Sc}, \kappa)$ and $q \in \mathcal{S}(Th_\alpha(M))$ there is exactly one $p \in \mathcal{S}_a(T_A(\phi_{Sc}, \kappa), \alpha)$ such that $T_A(\phi_{Sc}, \kappa) \models (\forall \boldsymbol{x})[p(\boldsymbol{x}) \wedge Th_\alpha(M)] \rightarrow q(\boldsymbol{x})$

(2) For all $M \models T_A(\phi_{Sc}, \kappa)$ and $p \in \mathcal{S}_a(T_A(\phi_{Sc}, \kappa), \alpha)$ there is at most one $q \in \mathcal{S}(Th_\alpha(M) \cup T_A(\phi_{Sc}, \kappa))$ such that $T_A(\phi_{Sc}, \kappa) \models (\forall \boldsymbol{x})[p(\boldsymbol{x}) \wedge Th_\alpha(M) \wedge T_A(\phi_{Sc}, \kappa)] \rightarrow q(\boldsymbol{x})$

Proof. This is immediate from Theorem 7.2.1.10 and the fact that $(s(T_A(\phi_{Sc}, \kappa)) = \omega)^W$ implies $|\kappa|^W = \omega$. □

Theorem 7.2.1.12. *(a) $T_A(\phi_{Sc}, \kappa)$ is Almost Scattered and has at least countably many models of size $\leq s(T_A(\phi_{Sc}, \kappa))$ then $T_A(\phi_{Sc}, \kappa)$ is not Scattered.*

(b) Further the quantifier rank spectrum of $T_A(\phi_{Sc}, \kappa)$ =Limits under κ sequences of the quantifier rank spectrum of ϕ_{Sc}.

Proof. To see (a) notice by Lemma 7.2.1.6 $T_A(\phi_{Sc}, \kappa)$ has 2^ω many countable models in any model of ZFC where $s(T_A(\phi_{Sc}, \kappa)) = \omega$.

(b) is an immediate consequence of Theorem 7.2.1.7 □

7.3 Construction of Almost Scattered Models

In this section we will construct explicit sentences which are almost scattered. We will construct these sentences by gluing together two copies of the sentence T_Ω in a very similar way to how we glued together two copies of T_K in Section 4.1. The crucial difference will be that, because we don't have a collection of archetypes, we will have to get homogeneity of our components in a different way. We will get the homogeneity of the components by explicitly saying that we have it in the language.

7.3.1 Definitions

7.3.1.1 Language

Definition 7.3.1.1. Let $\mathcal{M} \models T_\Omega$ be such that no tuples have color ∞. Let $L_Q = \{\langle c_i : i \in \mathcal{M}\rangle, Q(x)\}$ where Q is a 1-ary predicate.
We then define the language $L_a(\mathcal{M}) = L^0_\Omega \cup L^1_\Omega \cup L_Q$
(Here L^0_Ω, L^1_Ω are two distinct copies of L_Ω)

7.3.1.2 Axioms

Definition 7.3.1.2. Let $T_a(\mathcal{M})$ be universal closure of the following $L_a(\mathcal{M})$ sentences:

Q:

- $Q(x) \leftrightarrow \bigvee_{a \in \mathcal{M}} x = c_a$
- $Q \models \phi(c_{a_1}, \dots c_{a_n})$ in L^2_Ω iff $\mathcal{M} \models \phi(a_1, \dots a_n)$
- $Q(\mathbf{x}) \wedge \neg Q(\mathbf{y}) \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any predicate other than $R^2_\leq$ and $|\mathbf{x}|, |\mathbf{y}| > 0$
- $Q(\mathbf{x}) \rightarrow \neg U'(\mathbf{x})$ where U' is any predicate other than $R^2_\leq, P$

L^2_Ω :

- $(\forall x)(\exists c) Q(c) \wedge R^2_=(x, c)$
- $(\forall c)(\exists x) \neg Q(x) \wedge R^2_=(x, c)$

Other Axioms:

- $\neg Q \models T^1_\Omega$
- $\neg Q \models T^2_\Omega$
- (Homogeneity) For each $m \in \omega$,

$$\neg Q \models (\forall \overline{a}, \overline{b}, c)(\exists^n d)(R^0_a(\overline{a}, \overline{b}) \wedge E^2_a(\overline{a}, \overline{b}) \rightarrow E^1_a(\overline{a}c, \overline{b}d_i) \wedge E^2_a(\overline{a}c, \overline{b}d_i))$$

The intent of the Q axioms are to fix everything that can be said about any element which satisfies Q. In particular, we want the collection of elements which satisfy Q to be isomorphic to $\mathcal{M}$ in L^2_Ω and to have every element named. We further want nothing to be true in L^2_Ω of elements which satisfy Q. And, finally, we want to be able to compare the 2-color (using $R^2_\leq$) of elements which satisfy Q with elements which satisfy $\neg Q$.

The intent of the L^2_Ω axioms are to guarantee the spectrum the collection of elements which satisfy $\neg Q$ is the same as $\mathcal{M}$ in L^1_Ω. Now it is worth mentioning explicitly that the only connection between elements satisfying $\neg Q$ and those satisfying Q is the fact that in L_Omega^1 they must have the same spectrum. As such, if we were to restrict our models only to the part which satisfies $\neg Q$, we would get the same restrictions for any tree structures placed on Q of the same height.

As for the other axioms, the only one which isn't self explanatory is (Homogeneity). This says that if we have a pair of color archetypes which are realized by two different tuples in the same model, then we have to be able to extend both tuples in exactly the same ways. An important point is that we don't require anything specific about the possible ways of extending a given pair of color archetypes is. We just require that the ways i which a pair of color archetypes can be extended depends only on the color archetypes.

Definition 7.3.1.3. If $M \models T_a(\mathcal{M}), \overline{a} \in M$ we say that $\langle \text{ctype}^0(\overline{a}), \text{ctype}^1(\overline{a})\rangle (=\text{ats}(\overline{a}))$ is the Color Archetype Sequence of $\overline{a}$.

If $M \models T_a(\mathcal{M})$ we let $\text{ATS}(M) = \{\text{ats}(\overline{a}) : \overline{a} \in M\}$

7.3.2 Properties of $T_a(\mathcal{M})$

In this section we will construct an upper bound on the quantifier rank spectrum of $T_a(\mathcal{M})$ and show that $T_a(\mathcal{M})$ is almost scattered.

Theorem 7.3.2.1. *Let $M, N \models T_a(\mathcal{M})$ be such that $ATS(M) = ATS(N)$. Then $M \equiv_\infty N$.*

Proof.

Definition 7.3.2.2. Let $I = \{f : M \to N, \mathrm{dom}(f) < \omega$ and if $q_f = \{a \in \mathrm{dom}(f) : \neg Q(a)\}$ then $\mathrm{ats}(q_f) = \mathrm{ats}(f[q_f])$ and f preserves atomic formula on $Q\}$.

We want to show then that $I \subseteq I$ is a sequence of partial isomorphisms. Notice if $f \in I$ then f preserves all atomic formula by construction. So all that is left is to show that $I \subseteq I$ has the back and forth property.

Let $f \in I$ and $a \in M$. We then need to find a $b \in N$ such that $f \cup (a, b) \in I$. We can break this into two cases.

Case 1: $M \models Q(a)$

In this case we know that there is a $m \in \mathcal{M}$ such that $M \models c_m = a$. Let b be such that $N \models c_m = b$.

Case 2: $M \models \neg Q(a)$

Let $M \models (\sigma_0, \sigma_1)(q_f a)$ and $N \models (\tau_0, \tau_1)(f[q_f])$ where $(\sigma_0, \sigma_1), (\tau_0, \tau_1)$ are color archetype sequences. In particular we have there is some $\bar{c}c \in N$ such that $N \models (\sigma_0, \sigma_1)(\bar{c}c)$ because $\mathrm{ATS}(N) = \mathrm{ATS}(M)$. But, we then also must have $(\tau_0, \tau_1)(\bar{c})$ by our conditions on when $f \in I$. So, by (Homogeneity) there must be a $b \in N$ such that $(\sigma_0, \sigma_1)(f[q_f]b)$. Hence $f \cup (a, b) \in I$ and we are done. $I \subseteq I$ has the back and forth property and I witnesses that $M \equiv_\infty N$. □

Corollary 7.3.2.3. *Let $M, N \models T_a(\mathcal{M})$ and $\bar{a} \in M, \bar{b} \in N$. Then $(M, \bar{a}) \equiv_\infty (N, \bar{b})$ iff $M \equiv_\infty N$ and $ats(\bar{a}) = ats(\bar{b})$.*

Proof. This is immediate from the construction of the sequence of partial isomorphisms ($I \subseteq I$) from M to N in Theorem 7.3.2.1 and Theorem 1.2.2.19. □

Theorem 7.3.2.4. *Let $|\mathcal{M}| = \omega$. Then $T_a(\mathcal{M})$ is almost scattered.*

Proof. Let $\mathcal{S}_a(T_a(\mathcal{M}), \alpha) = \{\text{ats}(\overline{a}) : a \in M, M \models T_a(\mathcal{M})\}$
Now it is clear from Corollary 7.3.2.3 that

(1) For all $M \models T_a(\mathcal{M})$ and $q \in \mathcal{S}(Th_\alpha(M) \cup T_a(\mathcal{M}))$ there is at least one $p \in \mathcal{S}_a(T_a(\mathcal{M}), \alpha)$ such that $T_a(\mathcal{M}) \models (\forall \mathbf{x})[p(\mathbf{x}) \wedge Th_\alpha(M)] \rightarrow q(\mathbf{x})$

(2) For all $M \models T_a(\mathcal{M})$ and $p \in \mathcal{S}_a(T_a(\mathcal{M}), \alpha)$ there is at most one $q \in \mathcal{S}(Th_\alpha(M))$ such that $T_a(\mathcal{M}) \models (\forall \mathbf{x})[p(\mathbf{x}) \wedge Th_\alpha(M)] \rightarrow q(\mathbf{x})$

□

7.3.3 Conjecture about $T_a(\mathcal{M})$

7.3.3.1 The Conjecture

Now that we have an upper bound on the quantifier rank of models of $T_a(\mathcal{M})$ we would like to come up with some conditions gives us a lower bound on the quantifier ranks of some models. However, here we run into a little bit of a problem as every attempt we have made to find such conditions has failed. So we will make a conjecture on the quantifier rank spectrum of $T_a(\mathcal{M})$ and talk about why a few of the obvious problems.

Conjecture 7.3.1. *If $\gamma = \Xi(\gamma)$ and Spec$(\mathcal{M}) = \{-\infty\} \cup \gamma$ then the supremum of the quantifier ranks of $T_a(\mathcal{M})$ is γ.*

7.3.3.2 The Problem

Lets consider how we might prove this. What we will want to do is to construct two models which are the same up to formulas of quantifier rank α for any particular $\alpha < \gamma$. This then leads to the question of what can be said by formulas of quantifier rank α. Remember from Section **??** that two tuples "look the same" up to formulas of quantifier rank α in T_Ω if there color archetypes place tuples on ordinals which are equivalent up to $\Xi(\alpha)$.

To be more precise, we know that the most important information we know about a tuple in a model of T_Ω is what the colors of its subtuples look like up to equivalence of the ordinals. In other words, as we increase the quantifier rank of our formula we are able to say not just what the colors of tuples are, but also "how far apart" tuples are.

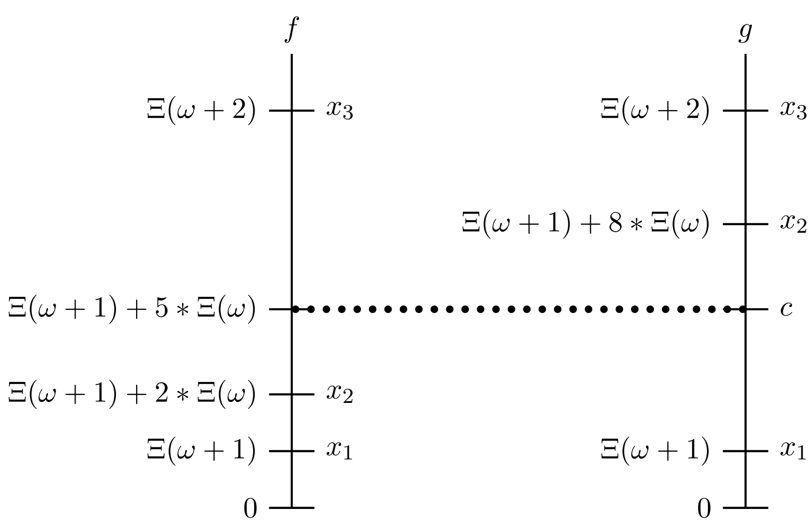
Now that we know what properties to consider, we would like to construct two models M, N of $T_a(\mathcal{M})$ such that $M \equiv_\alpha N$. In order to show this equivalence we must be able to create a sequence of partial isomorphisms from M to N of length α.

Recall how we proved the lower bound on the quantifier rank of a model of T_Ω (Theorem 6.3.2.5). The way we constructed the proof was we first noticed that if we had two finite sequences of ordinals which looked the same up to $\Xi(\gamma + 1)$ then we could add in a finite number of new ordinals to each and still maintain the equivalence up to $\Xi(\gamma)$ (including the order). We were then able to use this plus (Saturation) to get a back and forth property for a collection of partial isomorphism.

Remembering how we proved the lower bound on quantifier rank for models of T_Ω, we would hope that we would be able to do something similar in the case of $T_a(\mathcal{M})$. Specifically, we know that there is still only a finite number of ordinals to consider and we still have a version of (Homogeneity) so we would hope that we could do a similar argument to construct a sequence of partial isomorphisms with the back and forth property In fact, if we are only interested in models of $T_a(\mathcal{M})$ restricted to $L^0_\Omega \cup L^1_\Omega$ then we can do just that.

The problem is that in $T_a(\mathcal{M})$ we are introducing constants which make the back and forth argument fall apart. Specifically, given two finite sequences of ordinals which are $\Xi(\gamma)$ equivalent, if we are allowed to choose an arbitrary constant and look at the sequences with that constant added in, there is no reason to believe that they should still be $\Xi(\gamma)$ equivalent.

For example in the following two cases f and g are the same up to $\Xi(\omega)$ on $\{x_1, x_2, x_3\}$ but once we fix the constant value of $f(c) = g(c)$, f and g no longer even have the same ordering.



7.3.3.3 The First Approach

As we saw in Theorem 7.3.2.1, each model is determined by its collection of color archetype sequences which are realized. So the problem of constructing two different models which look the same up to quantifier rank α is the same as the problem of determining two collection of color archetype sequences which allow a back and forth argument to take place of length α.

The main difficulty in the back and forth argument is if we have two sequences of ordinals which "look the same" up to $\Xi(\gamma)$ and we add a constant value to each sequence then the sequences might not look the same any more. So one of the first solutions we would want to consider is to require all constants which could be added to be "above" any 1-color which is realized. That way we could never add a constant which could mess up the equivalence of our finite sequences of ordinals.

Attempt 7.3.3.1. *Say that if $M|L^1_\Omega \equiv_\gamma N|L^1_\Omega$ and in M, N, if (α', β') is a color sequence realized in M or N then either $\beta = -\infty$ or $\beta' > Spec(M)$ or $Spec(N)$*

To be more specific, what we are doing is construct two models with 1-spectrum α, β respectively. But we would require that any time we have a 1-tuple whose color is less than $\gamma >> \alpha, \beta$ then that tuples 1-color is $-\infty$. This way, we would hope that the back and forth argument that we would produce by only looking at L^1_Ω would carry over to $L_a(\mathcal{M})$.

The problem is that in this situation, even though we have required all actual constants to be above anything mentioned in the back and forth argument for L^1_Ω, there is no easy way to guarantee that we can't "transmit" information back down to the 1-color. Hence, we still have to deal with what is effectively a constant in the 1-color.

To see how this effective transmission of information can take place consider the model M such that for all $\overline{a} \in M$ such that $M \models (\|\overline{a}\|^1 \neq -\infty) \rightarrow (\|\overline{a}\|^1 + \alpha = \|\overline{a}\|^2)$ where $\alpha >> \text{Spec}(M) \cap \text{ORD}$. In this case we never can compare constants from the 2-colors with the sequences of 1-colors as the 2-colors are much to big. But, the constants still "force" the 1-colors to be a fixed value and hence adding a tuple with a given 2-color "forces" us to keep track of a fixed value among the 1-colors and this is just as bad as if we added a constant.

Chapter 8

Multiple Trees

8.1 Introduction

Now that we have defined out component trees in Chapter 6 in this chapter we will consider ways of "gluing" them together (in a very similar way to Section 4.1).

Specifically in Section 8.2 we will introduce some machinery which will allow us to study what happens when we glue together ω many copies of T_Ω. And, in Section 8.3 we will do just that. We will produce two different methods of constructing a model for such a theory. Next, in Section 8.4, we consider what happens to the theory when we place an internal bound on the spectrums.

Then in Section 8.6 we will consider something very similar but we will glue together copies of T_Λ instead of T_Ω.

8.2 Comparing Different Colors

8.2.1 Definitions

Our method for comparing color will be almost identical to 2.3.1.

Definition 8.2.1.1. Let $L^{\circ}_{R_{\leq},n} = \{R^{\circ,i}_{\leq,j,k} : j,k \leq n$ where $R^{\circ,i}_{\leq,j,k}$ is $a+b$ ary, $a,b \in \omega\}$.

For notational convenience we will treat $R^{\circ,a,b}_{\leq,i,j}$ as a predicate of two arguments (one a ary and one b ary). Further abusing notation (in a similar way as we did with P) we will consider $R^{\circ}_{\leq,i,j}$ as a two argument predicate on finite tuples (i.e. $R^{\circ}_{\leq,i,j}$ takes two finite tuples, one of size a one of size b as arguments).

Definition 8.2.1.2. Let L^n_R consist of $L^{\circ}_{R_{\leq},n}$ as well as n disjoint copies of L_R.

We will add a superscript to distinguish between different copies of the same relations. To further simplify notation when we want to distinguish between different copies of the language inside L^n_R we will omit the R (so L^1, L^2 are two distinct copies of L_R.)

Definition 8.2.1.3. Let T^n_R be universal closure of the following L^n_R sentences:

- $T^i_R, i \leq n$(i.e. T_R on the ith copy of L_Ω)
- $R^{\circ}_{\leq,i,j}(\mathbf{x},\mathbf{y}) \leftrightarrow [\neg P^i(\mathbf{x})] \vee [P^i(\mathbf{x}) \wedge P^j(\mathbf{y}) \wedge (\forall a)(\exists b) R^{\circ}_{\leq,i,j}(\mathbf{x}a, \mathbf{y}b)]$

8.2.2 Correctness

In this section we will show that if $R^{\circ}_{\leq,i,j}$ holds then $R^{\circ}_{\leq,i,j}$ accurately describes the relationship between the colors of its arguments. Further we will show that if our model has no tuple of color ∞ then we have $R^{\circ}_{\leq,i,j}(\overline{a},\overline{b})$ iff $\|\overline{a}\|^i \leq \|\overline{b}\|^j$

Theorem 8.2.2.1. *If $M \models T^n_R, \overline{a},\overline{b} \in M$ then $M \models R_{\leq}(\overline{a},\overline{b}) \rightarrow \|\overline{a}\| \leq \|\overline{b}\|$*

Proof. Assume $\|\overline{a}\|^i = -\infty$

$(\forall\overline{b})R^{\circ}_{\leq,i,j}(\overline{a},\overline{b})$ by the definition of $R_{\leq}$.

Assume if $\|\mathbf{x}\|^i < \alpha < \infty$ then $R^{\circ}_{\leq,i,j}(\mathbf{x},\mathbf{y}) \Rightarrow \|\mathbf{x}\|^i \leq \|\mathbf{y}\|^j$ and let $\|\overline{a}\|^i = \alpha$

Then $R^{\circ}_{\leq,i,j}(\overline{a},\overline{b}) \rightarrow [(\forall a')(\exists b')\|\overline{a}a'\| \leq \|\overline{b}b'\|]$ by the induction hypothesis. Therefore $R^{\circ}_{\leq,i,j}(\overline{a},\overline{b}) \rightarrow [\|\overline{a}\|^i = \sup\{\|\overline{a}a'\|^i + 1 : a' \in M\} \leq \sup\{\|\overline{b}b'\|^j + 1 : b' \in M\} = \|\overline{b}\|^j]$. So $R^{\circ}_{\leq,i,j}(\overline{a},\overline{b}) \rightarrow \|a\|^i \leq \|b\|^j$ and by induction this is true for any $\overline{a}$ such that $\|\overline{a}\|^i < \infty$.

Assume $\|\overline{a}\|^i = \infty$.

Then let $\overline{a}, a_0, a_1, \cdots$ be an infinite sequence such that $P^i(\overline{a}, a_0, \cdots, a_n)$ for all n (this exists by the definition of color ∞). Therefore there must exist a sequence $\overline{b}, b_0, b_1, \cdots$ such that $R^{\circ}_{\leq,i,j}(\overline{a}a_0\cdots a_n, \overline{b}b_0\cdots b_n)$ for all n. But then we have (by the definition of $R^{\circ}_{\leq,i,j}$), $P^j(\overline{b}, b_0, \cdots, b_n)$ for all n. Hence, $\|b\|^j = \infty$ by the definition of color ∞.

□

Theorem 8.2.2.2. *If $M \models T^n_R$, $\overline{a},\overline{b} \in M$ and $\|\overline{a}\|^i \leq \|\overline{b}\|^j < \infty$ then $M \models R^{\circ}_{\leq,i,j}(\overline{a},\overline{b})$.*

Proof. First notice that if $\|\overline{a}\|^i = -\infty$ then this is trivially true.

Now assume for all $\mathbf{x},\mathbf{y} \in M \models T^n_R$ if $\|\mathbf{x}\|^i < \alpha < \infty, \|\mathbf{x}\|^i \leq \|\mathbf{y}\|^j < \infty$ then

$R^{\circ}_{\leq,i,j}(\mathbf{x},\mathbf{y})$ and let $\|\overline{a}\|^i = \alpha$.

First off we know that $P^i(\overline{a}) \rightarrow P^j(\overline{b})$ by the definition of color. We also know by the definition of color that $(\forall a')(\exists b')$ such that $\|\overline{a}a'\|^i \leq \|\overline{b}b'\|^j$. Further, by the inductive hypothesis, we then have $R^{\circ}_{\leq,i,j}(\overline{a}a', \overline{b}b')$. But then by the definition of $R^{\circ}_{\leq,i,j}$ we then have $R^{\circ}_{\leq,i,j}(\overline{a}, \overline{b})$ and we are done.

So by induction we are done. □

8.3 T^n_Ω

8.3.1 Definitions

Definition 8.3.1.1. If $\langle C_i : i \leq n \rangle$ are color archetypes on the same domain $\mathbf{x}$ such that for all $\mathbf{y} \subseteq \mathbf{x}$ $\|\mathbf{y}\|^i \leq \|\mathbf{y}\|^j$ if $i \leq j$ (where $\|\cdot\|^i$ is that forced by C^i) then we say $\langle C_i : i \leq n \rangle$ is an n-color archetype sequence (we will leave out the n when the context is clear.)

If $M \models T^n_\Omega$ and $M \models \text{ctype}^i(\overline{a})$ for all $i \leq n$ then we say that $(\text{ctype}^1, \cdots, \text{ctype}^n)$ $(= \text{ats}(\overline{a}))$ is the Color Archetype Sequence of $\overline{a}$.

Definition 8.3.1.2. If $C_1, \ldots, C_n$ are color archetype sequences we say $C_1 \ldots C_n \geq D$ (for an color archetype sequence D) if domain$(C_i) = \mathbf{x}_i$, domain$(D) = \mathbf{x}_1 \ldots \mathbf{x}_n\mathbf{y}$ and $D \rightarrow C_i$ for each i. In other words every thing which is true about C_i is forced to be true by D.

(Notice that we are requiring that $\mathbf{x}$ be an initial segment. This is only for notational convenience and has no bearing on the intuitive meaning).

We are now ready to define our theory.

Definition 8.3.1.3. Let $L^n_\Omega = L^n_{R_\leq}$

Definition 8.3.1.4. Let T^n_Ω be universal closure of the following L^n_Ω:

- $P^i(\mathbf{x}) \to P^j(\mathbf{x})$ if $i < j \leq n$
- T^n_R
- (Homogeneity for Archetypes Sequences) For each $m \in \omega$

$$\bigwedge_{i \leq n} E^i_a(\mathbf{x}, \mathbf{y}) \to (\forall \bar{b})(\exists^m \bar{a}) \bigwedge_{i \leq m} E^i_a(\mathbf{x}\bar{a}, \mathbf{y}\bar{b})$$

- (Amalgamation of Archetype Sequences) For each $m \in \omega$

$$(\forall \mathbf{x}, \mathbf{y})(\exists^m \bar{a}, \bar{b}) \bigwedge_{i \leq n} E^i_a(\mathbf{x}, \bar{a}) \wedge E^i_a(\mathbf{y}, \bar{b}) \wedge \bigwedge_{\emptyset \neq S \subseteq \bar{a}, \emptyset \neq S' \subseteq \bar{b}, i \leq n} \neg P^i(SS')$$

- <u>If i are successor ordinals:</u>
  - $T^i_\Omega, i \leq n$(i.e. T_R on the ith copy of L_R)
  - (Saturation for Finite Sequences of Archetypes) For each $m \in \omega$, for each $\langle i_k : k \in p < \omega, s < t \to i_s \leq i_t \rangle \subseteq n -$ (limit point(n)) $(\forall x_1, \ldots x_n)(\forall \{y^k_S : S \subseteq n\})(\exists^m b) \bigwedge_{S \subseteq n} [\bigwedge_{k \leq k' \in p} R^{i_k}_<(y^k_S, \{x_i : i \in S\}) \wedge R^\circ_{k,k',\leq}(y^k_S, y^{k'}_S)] \to [\bigwedge_{k \in p} R^{i_k}_=(y^k_S, \{x_i : i \in S\} b)]$
- <u>If i is a limit ordinal:</u>
  - $P^i(\mathbf{x}) \leftrightarrow \bigvee_{j<i} P^j(\mathbf{x})$

The idea is we want to take n different models of T_Ω and put them all on the same set (here we only require $n \in$ ORD whereas we will look at the case of $n = \omega$). However, we don't want to do this arbitrarily for two reasons.

First of all, an arbitrary combination of models of T_Ω is so messy that we have no nice way to talk about it. Second, because an arbitrary combination of models is so messy that even given just two fixed models of size κ there are 2^κ many ways to combine them (and hence 2^κ many types as well as 2^κ many models).

The need for the axiom of (Amalgamation of Archetype Sequences) is a little less obvious. The reason we need it is that as we will see shortly, a model of T_Ω^n (which doesn't have a tuple of color ∞) is completely determined (in $\mathcal{L}_{\infty,\omega}$) by the archetype sequences it realize. So what this axiom says is that if we have two archetype sequences we can combine them in such a way that the color of any overlap is $-\infty$. This will allow us eventually to construct a large number of non-isomorphic models all $\mathcal{L}_{\infty,\omega}$ equivalent to a given model.

(Saturation for Finite Sequences of Archetypes) says essentially that given any color archetype sequence which is realized and a finite number of successor colors, if we have some consistent extensions of the color archetype of our tuple on those finite colors, then in fact that is realized somewhere.

(Homogeneity for Archetypes) says that given two different tuples which realize the same archetype, they can be extended in exactly the same ways.

Why we have a separation into the case of limit ordinals and non-limit ordinals, is not clear at first glance. The reason is that because the color of a limit is the limit of colors below it we have no reason to believe it should satisfy T_Ω (although it should satisfy T_Λ.

8.3.2 General Results

8.3.2.1 Introduction

Now that we have several ways of constructing models of T^n_Ω, we want to look at submodels of T^n_Ω. To be specific, we will show in this section that if we choose a single color (say the ith) and a single limit ordinal (say $\omega * \gamma$) and we only look at those elements whose i-color is less than $\omega * \gamma$ then in fact we still have a model of T^n_Ω.

8.3.2.2 Submodels

Theorem 8.3.2.1. *Let $M \models T^n_\Omega$, let $N^i_{\omega*\gamma}(M) = N^i_{\omega*\gamma} := \{a \in M : M \models \|a\|^i < \omega * \gamma\}$. Then $N^i_{\omega*\gamma} \models T^n_\Omega$ and $(\forall \overline{a} \in N^i_{\omega*\gamma})(\|a\|^j)^{N^i_{\omega*\gamma}} = (\|\overline{a}\|^j)^M$*

Proof. First off, it is obvious that $N^i_{\omega*\gamma} \models T^j_P$ for all $j \leq n$ because all axioms of T_P are universal.

Claim 8.3.2.2. *If $\overline{a} \in N^i_{\omega*\gamma}$ and let x be in M such that $(\forall \emptyset \neq \overline{b} \subseteq \overline{a})(\|\overline{b}x\|^j)^M < \omega * \gamma$. Then there exists an element $y \in N^i_{\omega*\gamma}$ such that $(\forall \emptyset \neq \overline{b} \subseteq \overline{a})(\|\overline{b}x\|^j)^M = (\|\overline{b}y\|^j)^M$*

Proof. The only condition that y must satisfy is that $(\|y\|^i)^M < \omega * \gamma$. We now need to break into two cases

- Case 1 $(i \geq j)$: In this case we can find by (Saturation for Finite Sequences of Archetypes) a y such that in M
  - $\|y\|^j < \min\{\omega * \gamma, \text{Spec}(M)^j\}$
  - $\|y\|^j > \|\overline{b}x\|^j$

- $\|y\|^j = \|y\|^i$
- $\|\bar{b}x\|^j = \|\bar{b}y\|^j = \|\bar{b}y\|^i$

- Case 2 ($i < j$): In this case we can find by (Saturation for Finite Sequences of Archetypes) a y such that in M
  - $\|y\|^j = \|\bar{b}\|^j$
  - $\|y\|^i = -\infty$
  - $\|\bar{b}x\|^j = \|\bar{b}y\|^j$

Hence $y \in N^i_{\omega*\gamma}$ and we are done. □

Claim 8.3.2.3. $(\forall \overline{a} \in N^i_{\omega*\gamma})(\|a\|^j)^{N^i_{\omega*\gamma}} = (\|\overline{a}\|^j)^M$

Proof. We have $(\|\overline{a}\|^j)^{N^i_{\omega*\gamma}} = \sup\{\|\overline{a}x\|^j + 1 : x \in N^i_{\omega*\gamma}\} \leq \sup\{\|\overline{a}x\|^j + 1 : x \in M\}$ because $N^i_{\omega*\gamma} \subseteq M$.

To see the other direction notice that if $(\|\overline{a}\|^j)^M \leq 0$ then the claim is true. So, assume it is true if $(\|\mathbf{x}\|^i)^M < \alpha$. Now let $(\|\overline{a}\|^j)^M = \alpha$. But then we know by Claim 8.3.2.2 that $(\forall x \in M)(\exists y \in N^i_{\omega*\gamma})$ such that $\alpha > (\|\overline{a}x\|^j)^M = (\|\overline{a}y\|^j)^M$. In particular we have $(\|\overline{a}x\|^j)^M = (\|\overline{a}y\|^j)^{N^i_{\omega*\gamma}}$ and so $(\|\overline{a}\|^j)^{N^i_{\omega*\gamma}} = \sup\{\|\overline{a}x\|^j + 1 : x \in N^i_{\omega*\gamma}\} \geq \sup\{\|\overline{a}x\|^j + 1 : x \in M\}$ and we are done. □

Corollary 8.3.2.4. *If $\overline{a} \in N^i_{\omega*\gamma}$ then $(ats(\overline{a}))^M = (ats(\overline{a}))^{N^i_{\omega*\gamma}}$*

Proof. This is because the only information in an archetype sequence is the colors of the subtuples of it's domain and these are indepnedent of which of $N^i_{\omega*\gamma}$ or M you are in by Claim 8.3.2.3 □

Corollary 8.3.2.5. *If C is an archetype sequence realized in $N^i_{\omega*\gamma}$ and $\boldsymbol{x}$ realizes C in M then $\boldsymbol{x} \in N^i_{\omega*\gamma}$*

Proof. This is because if C is realized in $N^i_{\omega*\gamma}$ it forces the i color of all 1-tuples to have color $< \omega * \gamma$ and hence any realization is in $N^i_{\omega*\gamma}$. □

To see that $N^i_{\omega*\gamma} \models T_\Omega$ on each L^n_Ω observe that we have

$$(\forall \overline{a})(\forall \mathbf{x})(\exists \mathbf{y}) R_=(\overline{a}\mathbf{x}, \overline{a}\mathbf{y}) \wedge |\mathbf{x}| + 1 = |\mathbf{y}|$$

because we know that the spectrum in $N^i_{\omega*\gamma}$ are limit's (and the color in of a tuple in $N^i_{\omega*\gamma}$ is the same as the color in M).

So we are now ready to show that $N^i_{\omega*\gamma} \models T^n_\Omega$. First notice that $N^i_{\omega*\gamma} \models P^i(\mathbf{x}) \rightarrow P^j(\mathbf{x})$ if $i < j$ because it is a universal statement. Also notice that it satisfies (Saturation for Finite Sequences of Archetypes) because M does and Corollary 8.3.2.5 (and hence $N^i_{\omega*\gamma}$ satisfies (Saturation for Archetypes) on every L_{i+1}). Seeing that it satisfies (Amalgamation of Archetype Sequences) is easy. Let C, D be a pair of archetype sequences realized in $N^i_{\omega*\gamma}$. Then in M there are realizations which witness (Amalgamation of Archetype Sequences). Hence, by Corollary 8.3.2.5 these realizations are in $N^i_{\omega*\gamma}$. Similarly, to see $N^i_{\omega*\gamma}$ satisfies (Homogeneity of Archetype Sequences) let $\overline{a}, \overline{b} \in N^i_{\omega*\gamma}$ be two tuples which have which have the same archetype sequence. Then if $\overline{a}$ can be extended to an archetype sequence C which is realized in $N^i_{\omega*\gamma}$ then there is an extension of $\overline{b}$ to C in M. But by Corollary 8.3.2.5 this extension must be in $N^i_{\omega*\gamma}$ and so we have $N^i_{\omega*\gamma}$ satisfies (Homogeneity of Archetype Sequences) and hence $N^i_{\omega*\gamma} \models T^n_\Omega$.

□

8.3.3 Quantifier Rank

Definition 8.3.3.1. Define $ATS(M) = \{\text{ats}(\overline{a}) : \overline{a} \in M\}$.

Theorem 8.3.3.2. *Let $M, N \models T^n_\Omega$ and be such that there are no tuples of color ∞ for any color on either model. If $ATS(M) = ATS(N)$ then $M \equiv_\infty N$ and for all $\overline{a} \in M, \overline{b} \in N * (\overline{a}) = *(\overline{b}) \to \overline{a} \equiv_\infty \overline{b}$.*

Proof. Let $M, N \models T^n_\Omega$ and be such that there are no tuples of color ∞ for any color on either model and $ATS(M) = ATS(N)$.
Let $I = \{f : M \to N, |\text{dom}(f)| < \omega \text{ ats}(\text{dom}(f)) = \text{ats(range(f))}\}$

Claim 8.3.3.3. *$\langle I, I\rangle$ is a partial isomorphism sequence.*

Proof. Notice that I is a collection of partial isomorphisms from M to N so it suffices to show that $I \subseteq I$ has the back and forth property.

Let $f \in I$. Now choose $a \in M$. We know that ats(dom$(f)a$) is realized in N by some sequence $\mathbf{x}c$. But we then know that ats$(\mathbf{x})$ = ats(range(f)). So, by (Homogeneity for Archetype Sequences) we know one of two things happens.

Case (1) $\mathbf{x}$ = range(f): In this case we can let $g = f \cup (a, c)$

Case (2) $\mathbf{x} \neq$ range(f): Then we know that ats$(\mathbf{x})$ = ats(range(f)) and so there is an element b such that ats(range$(f)b$) = ats$(\mathbf{x}c)$ = ats(dom$(f)a$). And in this case let $g = f \cup (a, b)$.

In either case we know that $g \supseteq f$ and by construction $g \in I$.

We can then do the other direction (i.e. starting with a an element of N and coming up with an element of M) in exactly the same way. □

We now have by Theorem 1.2.2.19 that $M \equiv_\infty N$ and that if $\overline{a} \in M, \overline{b} \in N$ and $\text{ats}(\overline{a}) = \text{ats}(\overline{b})$ then $(\exists f \in I) f(\overline{a}) = \overline{b}$ and hence $\overline{a} \equiv_\infty \overline{b}$. □

Theorem 8.3.3.4. *If $M \models T^n_\Omega$ and M has no tuples of color infinity at any arity then $qr(M) \leq sup\{Spec(M)^i + m : i \leq n, m \in \omega\}$*

Proof. Let D be an archetype sequence and let $\varphi^=_D(\mathbf{x})$ say that $\mathbf{x}$ satisfies D, using the formula's $\varphi^=_\alpha$ from Corollary 2.2.2.8. Now let $\varphi^M \leftrightarrow \bigwedge_{\overline{a}\in M}(\exists \mathbf{x})\varphi^=_{\text{ats}(\overline{a})}(\mathbf{x}) \wedge (\forall \mathbf{x}) \bigvee_{\overline{a}\in M} \varphi^=_{\text{ats}(\overline{a})}(\mathbf{x})$

We now need to notice two things. First of all we find that if $N \models \varphi^M$ then every color of every tuple of N is less than or equal to $\sup\{\text{Spec}(M)^i : i \leq n\}$. This is because if $N \models (\varphi^=_\alpha)^i(\overline{a})$ then we know that in L^i the color of $\overline{a} \in N$ is exactly α. Hence, if $N \models \varphi^M$ then N has no tuples of color infinity at any arity.

The second observation we need to make is that if $N \models \varphi^M$ then $ATS(N) = ATS(M)$ (in fact φ^M was defined specifically for this purpose). So, we then have by Theorem 8.3.3.2 that if $N \models \varphi^M$ then $N \equiv_\infty M$.

Hence $\text{qr}(M) \leq \text{qr}(\varphi^M) = \sup\{\text{Spec}(M)^i + m : i \leq n, m \in \omega\}$ □

The most important idea behind Theorem 8.3.3.2 and Theorem 8.3.3.4 is that the $\mathcal{L}_{\infty,\omega}$ types of a tuple is determined by its color archetype sequence (if we are in models where no tuples have color infinity).

8.3.4 First Construction of Models

While the theories T^n_Ω are very nice, we still need to show that they are consistent. In this section we will construct a collection of models of T^n_Ω such that there are 2^κ many of size κ. We will do this by first choosing a collection of archetype sequences for 1-tuples which are consistent (i.e a collection of sequences of non-decreasing ordinals). We will then construct model such that the only archetype sequences of 1-tuples which are realized are the ones we have chosen. But, we will maintain the necessary homogeneity of our models by forcing that for i-tuples with $i > 1$, every possible extension that can happen (consistent with our choice of 1-tuples) does.

Definition 8.3.4.1. If $X = \langle \alpha_1, \cdots, \alpha_n \rangle$ then define $X(i) = \alpha_i$ for $i \leq n$
Let $S \subseteq \{\langle \alpha_1, \cdots, \alpha_n \rangle : \alpha_i \leq \alpha_j \text{ if } i \leq j\} \subseteq (\omega * \zeta_1 \cup \{-\infty\}) \times \cdots \times (\omega * \zeta_n \cup \{-\infty\})$
If:

(1) For all $i \leq n$ there is a sequence $\omega * \eta_\gamma$ cofinal in ζ_i such that $\exists X_\gamma \in S$ where $X(j) = -\infty$ if $j < i$ and $X_\gamma(j) = \eta_\gamma$ if $j \geq i$

(2) There is a sequence $\langle X^\gamma : \gamma \leq \zeta_n \rangle \subseteq S$ where $X^\gamma(i) \leq X^{\gamma'}(i)$ if $\gamma \leq \gamma'$ and $\langle X^\gamma(i) : \gamma \leq \zeta_n \rangle$ is cofinal in ζ_i.

(3) If $\{\alpha_{i_k} : k \in p < \omega\}$ is such that

- $(\forall k \in p) i_k \in p - (\text{limit point}(n))$
- $\alpha_{i_k} < \omega * \zeta_{i_k}$
- $(\forall k, k' \in p) k < k' \rightarrow i_k \leq i_{k'} \rightarrow \alpha_{i_k} \leq \alpha_{i_{k'}}$
- Then there is a $\langle a_i : i \leq n \rangle$ such that $a_{i_k} = \alpha_k$

(4) If $\langle \zeta_1, \cdots, \zeta_n \rangle \in S$ and $\omega * \gamma \leq n$ then $\zeta_{\omega * \gamma} = \sup\{\zeta_i : i < \omega * \gamma\}$

then we say <u>S is a full subset of $\langle \zeta_1, \cdots, \zeta_n \rangle$</u>

Now these conditions may appear on first reading like they came out of thin air. But, what we intend for S to be (if S is full) is the collection of archetype sequences of 1-tuples in our model (with $\omega * \zeta_i$ =Spectrum in L^i). As we will see these conditions turn out to be exactly the conditions we need to make our argument work. So lets take a closer look at what they are saying.

The first condition will end up saying that given a copy of the language, say L^i, then there is a sequence of elements whose i-colors are confinal in the possible color of L^i but where the only relevant information concerning the tuple is it's i-color (i.e. if $j < i$ then all j-colors are $-\infty$ and if $j > i$ all j-colors are the same as the i-colors).

The second condition says that there is a sequence of elements whose colors are cofinal in all the colors simultaneously.

The final two conditions are relatively strait forward. The third condition is there to guarantee that (Saturation for Finite Sequences of Archetypes) holds and similarly the fourth condition is there to ensure that the condition on limit colors is preserved.

We can now begin our construction.

Definition 8.3.4.2.

Stage 1:

For each $c \in S$ let $K_c = \{k_i : k_i$ are elements, $i \in \omega$ such that $\langle \|k_i\|^1, \cdots, \|k_i\|^n \rangle = c\}$

Let $M^{(1,1)} = \bigcup_{c \in S} K_c$

Let $M^{(j,1)} = \{\mathbf{x} \subseteq M^{(1,1)} : |\mathbf{x}| = j$ and $\|\mathbf{x}\|^i = -\infty$ if $i > 1\}$

Notice that $|M^{(1,1)}| = \omega * |S|^n = |S|^n$

Stage i+1:

For each $\mathbf{x} \in M^{(i,i)}$, each archetype sequence D of arity $i+1$ such that $D(\mathbf{x}y) \to \text{ats}(\mathbf{x}) \leq D$, and $\text{ats}(b) \in S$, let $B^{\mathbf{x},D} = \{b_m^{\mathbf{x},D} : m \in \omega$ and each $b_m^{\mathbf{x},D}$ is new$\}$.

Then, for each $b \in B^{\mathbf{x},D}$ let $\text{ats}(\mathbf{x}b) = D$.

Let $M^{(1,i+1)} = M^{(1,i)} \cup \bigcup_{\mathbf{x},D} B^{\mathbf{x},D}$. So all that is left is to assign colors to the j-tuples. Let $M^{(j,i+1)} = \{\mathbf{x} \subseteq M^{(1,i+1)} : |\mathbf{x}| = j\}$. Now if $\mathbf{y} \in M^{(j,i+1)}$ we have three cases:

1) $\underline{\mathbf{y} \in M^{(j,i)}}$: Then just define the color of $\mathbf{y}$ in $M^{(j,i+1)}$ be the same as it was in $M^{(j,i)}$.

2) $\underline{\mathbf{y} = \mathbf{z}b_i^{\mathbf{x},D}, \mathbf{z} \subseteq \mathbf{x}}$: Then have the color of $\mathbf{y}$ be that determined by D.

3) Otherwise: $\|\mathbf{y}\|^r = -\infty$ for all $r \in n$.

Notice that this procedure puts a color on all $i+1$-tuples of $M^{(1,i+1)}$ in a unique and consistent way (this is important as it means we can amalgamate tuples in a unique and consistent way).

Also notice that $|M^{(j,i+1)}| = \omega * |M^{(i,i)}| * \sup\{x^r : x \in S\}|^n = |\sup\{x^r : x \in S\}|^n$

Let $M^{(i,\omega)} = \bigcup_{j \in \omega} M^{(i,j)}$

Let $M(S) = M^{(1,\omega)}$ along with the structure that for each $\mathbf{x} \in M(S), P^r(\mathbf{x})$ iff $\mathbf{x} \in M^{(|\mathbf{x}|,\omega)}$ and $\text{ats}(\mathbf{x}) \to P^r(\mathbf{x})$. Further we require $M(S) \models R^r_{\leq}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\|^r \leq \|\overline{b}\|^r$ and $M(S) \models R^{\circ}_{i,j,\leq}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\|^i \leq \|\overline{b}\|^j$.

Notice that $|M(S)| = |\sup\{x^r : x \in S\}|$

Okay, so now lets consider what it is we are doing here. The idea is that by stage i we have have an approximation of the model we want, which works just so long as we don't look at any tuples of arity $i+1$ or greater. At stage $i+1$ we add single elements which are meant to guarantee that all i tuples already defined can be extended in every way possible. This will then allow us show that the axioms on i tuples are all satisfied. But, we have the problem of making sure that these new elements don't interact with each other or with the other old tuples. The way we ensure this is to say that the color of any new tuple is $-\infty$ unless the tuple consists of an old tuple and a single element specifically added to extend it.

Because at stage i we will have shown all the axioms work on less than i tuples which are already defined, and because we have a unique way to amalgamate tuples to get tuples of higher arity, we will then be able to show that in the limit (i.e. at "stage ω") we will have determined the color of all finite tuples of our model and all the axioms will be satisfied.

Claim 8.3.4.3. *Let $\boldsymbol{x} \in M(S)$. Let $\gamma_r(\boldsymbol{x}) = \|\boldsymbol{x}\|^r$ as determined by the archetype assigned to it in the construction (i.e. in $M^{(|\boldsymbol{x}|,\omega)}$). Let $M(S) \models \gamma_r^*(\boldsymbol{x}) = \|\boldsymbol{x}\|^r$. Then $\gamma_r(\boldsymbol{x}) = \gamma_r^*(\boldsymbol{x})$.*

Proof. First notice that we ensured by our construction that every finite tuple of $M(S)$ is assigned an archetype in the construction (just let i be some stage greater than the point when all elements of $\mathbf{x}$ appear for the first time in the construction.)

Base Case:

Assume $\gamma_r(\mathbf{x}) = 0$ or $-\infty$.

We the know that for any y such that $\mathbf{x}y \in M^{(|\mathbf{x}y|,\omega)}$, $\gamma_r(\mathbf{x}y) = -\infty$ as this fact is witnessed by the archetype of $\mathbf{x}$. But, then we know that for all y $M(S) \models \neg P_r(\mathbf{x}y)$ (by definition of our color structure on $M(S)$ and so $\gamma_r^*(\mathbf{x}y) = -\infty$

But we also know by construction that $\gamma_r^*(\mathbf{x}) \geq 0$ (i.e. $M(S) \models P^r(\mathbf{x})$) iff $\gamma_r(\mathbf{x}) \geq 0$ (i.e. ats($\mathbf{x}$) in the construction witnesses that $P^r(\mathbf{x})$ holds). So, we know that $\gamma_r^*(\mathbf{x}) = \gamma_r(\mathbf{x})$.

Inductive Case:

Assume that if $\mathbf{z} \in M(S)$ and $\gamma_r(\mathbf{z}) < \alpha$ then $\gamma_r(\mathbf{z}) = \gamma_r^*(\mathbf{z})$. Let $\gamma_r(\mathbf{x}) = \alpha$.

So we have $\gamma_r^*(\mathbf{x}) = \sup\{\gamma_r^*(\mathbf{x}y) + 1 : y \in M^*\} = \sup\{\gamma_r(\mathbf{x}y) + 1 : \mathbf{x}y \in M(S)\} = \gamma_r(\mathbf{x})$

Now we have the last equality because by construction any consistent extension of $\mathbf{x}$ is realized and so there must be a $y_\beta \in M(S)$ such that $\gamma_r(\mathbf{x}y_\beta) = \beta$ for all $\beta < \gamma_r(\mathbf{x})$ (we have to be a little careful to make sure that fact that we don't have all 1-tuples of colors won't mess us up. Specifically we have to make sure that we can find y_β such that $\|y_\beta\|^r > \beta$. But this was exactly why we had our condition (2) in our definition of full S).

So by induction and the fact that $\gamma_r(\mathbf{x})$ is never is ∞ the claim is proved. □

Claim 8.3.4.4. *$M(S)$ is a model of T^n_Ω.*

Proof. $M(S)$ obviously models $T_{R_\leq}$ on each of the n copies of $L_{R_\leq}$ as we define the relation $R_\leq$ on $M(S)$ to make this so. Also $M(S) \models P^i(\mathbf{x}) \rightarrow P^j(\mathbf{x})$ if $i \leq j$ because all tuples are realized as part of an archetype sequence.

All that is left to check is (Saturation for Finite Sequences of Archetypes), (Homogeneity of Archetype Sequences), the (Amalgamation of Archetype Sequences) and the condition on limit colors (this is because (Saturation for Finite Sequences of Archetype) implies (Saturation on Archetypes) in each language).

Well (Saturation for Finite Sequences Archetypes) is easy. Lets say we have a j-tuple $\mathbf{x} \in M(S)$ and we want to show it can be extended to a color archetype sequence A such that $A(i_k) = B_k$ for a finite ordered sequence $\langle i_k : k \in p < \omega \rangle$.

First notice it suffices to consider extending by a single element (as doing this multiple times gets us any extension). Lets say $\mathbf{x}$ comes into the construction at stage i. In particular then, $\mathbf{x}$ is a subtuple of some i-tuple $\mathbf{y} \in M^{(i,i)}$. We know then that there must be an extension of ats($\mathbf{y}$) to an archetype sequence $D(\mathbf{y}, a)$ such that D implies

(1) If $\mathbf{z} \subseteq \mathbf{x}$, $i_k \leq i < i_{k+1}$ then $\|\mathbf{z}a\|^i = \|\mathbf{z}a\|^{i_k}$

(2) ctype$(\mathbf{x}, a) = B_k$

(3) The color of any other subtuple of $\mathbf{y}$ which contains a must have color $-\infty$.

The reason such an extension must exist is because the only three conditions which must be preserved by an extension of an archetype sequence are

(1) All subtuples must have strictly greater color than the tuple they are contained in

(2) If $i > j$ then the ith color must be at least as great as the jth.

(3) All archetype sequences of 1-tuples must be contained in S

and the conditions on S being full were designed to make this work.

As for (Amalgamation of Archetype Sequences), that is easily seen to hold as well. Say we have a pair of archetypes C, D realized in the model. Then, by construction they are realized by disjoint tuples $\mathbf{x}, \mathbf{y}$ respectively, for the first time at stage $i (= \max\{|\mathbf{x}|, |\mathbf{y}|\})$. But then by construction, at stage $i+1$ we see that $\|\mathbf{xy}\|^r = -\infty$ for each $r \leq n$, $\emptyset \neq \mathbf{x}' \subseteq \mathbf{x}$, $\emptyset \neq \mathbf{y}' \subseteq \mathbf{y}$

(Homogeneity for Archetype Sequences) trivially holds because if we have two elements $\mathbf{x}, \mathbf{y}$ which realize the same archetype sequence then they must be realized as subtuples of some $\overline{a}, \overline{b}$ respectively. Now let $C = \text{ctype}(\mathbf{x}, \overline{c})$. We can then amalgamate $\text{ctype}(\overline{b})$ and C around $\mathbf{y}$ (because we know $M(S)$ satisfies (Amalgamation for Archetype Sequences)) and get an archetype D which must be an extension of $\text{ctype}(\overline{b})$ consistent with the Spectra and with S. Hence D must be realized as an extension of $\overline{b}$ and that will give us the extension of $\mathbf{y}$ we need to witness (Homogeneity for Archetype Sequences)

The limit condition on colors is satisfied by virtue of the fact that all archetype sequences preserve the condition on limit colors and that the coordinate of a limit ordinal $\alpha \leq n$ in an element X of a full S is the limit of the coordinates of X which are less than α.

□

Lemma 8.3.4.5. $\{\langle \alpha_i : i \leq n \rangle : (\exists a \in N)|a| = 1, (\forall i \leq n)\|a\|^i = \alpha_i\} = S$

Proof. This is immediate from the construction. □

Lemma 8.3.4.6. *If $\boldsymbol{x} \in M(S)$ with archetype sequence C, and $D \leq C$ such that the archetype sequence of each individual element in the domain of D is in S, then D is realized as an extension of $\boldsymbol{x}$.*

Proof. This is just an iterated use of the construction at stage i+1 (iterated the number of times necessary to get the domain to have the same size as the domain of D) □

8.3.5 Second Construction of Models

8.3.5.1 Introduction

In Section 8.3.4 we started with a collection of consistent archetype sequences for 1-tuples and showed we could construct a model which realized them. In this section we will go through a similar construction but this time we will start with a model N of T^n_Ω and we will construct a collection of new models of T^n_Ω each with the same archetype sequences. This will allow us to show that each model of T^n_Ω with an uncountable spectrum (and without a tuple of color ∞) has a large number of models which are $\mathcal{L}_{\infty,\omega}$ equivalent to it.

We know by Theorem 8.3.3.2 any two models with the same archetype sequences are $\mathcal{L}_{\infty,\omega}$ equivalent. So, what we are going to do is construct a collection of uncountable models which all have the same archetypes sequences but which have a different number of realizations of the archetypes. To be specific, we know that each archetype sequence must be realized infinitely often, but we have no way of forcing what the actual number of these realizations are.

What we will do in this section is take a model of T^n_Ω and modify the sizes of the infinite tuples to get new models which are $\mathcal{L}_{\infty,\omega}$ to our original but not isomorphic to it. When we limit ourselves to the case when $n = 1$, this will allow us (among other things) to get a ω-categorical $\mathcal{L}_{\infty,\omega}$ complete sentence of $\mathcal{L}_{\omega_1,\omega}$ which has κ^ω many models of size $\aleph_\kappa$.

8.3.5.2 The Construction

Definition 8.3.5.1. Let $N \models T^n_\Omega$ and let $\kappa(N) = \sup\{\|\overline{a}\|^i : i \leq n, \overline{a} \in N\}$. Let $\mathcal{D} = \mathcal{D}(N)\{\langle D, m\rangle : (\exists \mathbf{x} \in N)|\mathbf{x}| = m \wedge \text{ats}(\mathbf{x}) = D\rangle\}$. Let $f : \mathcal{D} \to$ ORDand define $\mathcal{D}(f) = \{D : (\exists n)\langle D, n\rangle \in \text{dom}(f)\}$

The idea behind the construction is that for every tuple in N and every possible extension of that tuple in N there are infinitely many elements all of which realize that extension. So we want to ensure that for each extension of an archetype sequence D which is realized in N there are $\aleph_{f(D)}$ many extensions realized in our model.

Stage 1:

For each $\langle D, 1\rangle \in \mathcal{D}$ let $K_D = \{k_i : k_i$ are elements $i \in \aleph_{f(D)}$ such that $\text{ats}(k_i) = D\}$

Let $M^{(1,1)} = \bigcup_{\langle D,1\rangle \in \mathcal{D}} K_D$

Let $M^{(j,1)} = \{\mathbf{x} \subseteq M^{(1,1)} : |\mathbf{x}| = j$ and $\|\mathbf{x}\|^r = -\infty$ if $i > 1\}$

Notice that $|M^{(1,1)}| = \sup\{\kappa, \aleph_f(D) : \langle D, 1\rangle \in \mathcal{D}\}$

Stage i+1:

For each $\mathbf{x} \in M^{(i,i)}$ and each archetype sequence D of arity $i+1$ such that $D(\mathbf{x}y) \to \text{ats}(\mathbf{x}) \leq D$ and $\langle D, i+1\rangle \in \mathcal{D}$ let $B^{\mathbf{x},D} = \{b^{\mathbf{x},D}_m : m \in \aleph_f(D)$ and each $b^{\mathbf{x},D}_m$ is new$\}$.

Then, for each $b \in B^{\mathbf{x},D}$ let $\text{ats}(\mathbf{x}b) = D$.

Now let $M^{(1,i+1)} = M^{(1,i)} \cup \bigcup_{\mathbf{x} \in M^{(i,i)}, \langle D,i+1\rangle \in \mathcal{D}} B^{\mathbf{x},D}$. So all that is left is to assign colors to the j-tuples. Let $M^{(j,i+1)} = \{\mathbf{x} \subseteq M^{(1,i+1)} : |\mathbf{x}| = j\}$. Now if

$\mathbf{y} \in M^{(j,i+1)}$ we have three cases:

1) $\underline{\mathbf{y} \in M^{(j,i)}}$: Then just have the color of $\mathbf{y}$ in $M^{(j,i+1)}$ be the same as it was in $M^{(j,i)}$.

2) $\underline{\mathbf{y} = \mathbf{z}b_i^{\mathbf{x},D}, \mathbf{z} \subseteq \mathbf{x}}$: Then have the color of $\mathbf{y}$ be that determined by D.

3) Otherwise: $\|\mathbf{y}\|^r = -\infty$ for $r \in n$.

Notice that this procedure is almost identical to the one in Section 8.3.4. In particular it puts a color on all $i+1$-tuples of $M^{(1,i+1)}$ in a unique and consistent way (this is important as it means we can amalgamate tuples in a unique and consistent way). However, there is one point which we will need to check for later. That is that any archetype realized in $M^{(i,j)}$ is also realized in N. But, this is the case because all archetypes of tuples of $M^{(i,j)}$ consist of collections of archetypes of N which have all colors $-\infty$ on their over lap. And, because $N \models$ (Amalgamation of Archetype Sequences) we know that all such archetypes are in fact realized in N. Also notice that $|M^{(j,i+1)}| = \sup\{\kappa, f(D) : \langle D, j\rangle \in \mathcal{D}, j \leq i+1\}$

Let $M^{(i,\omega)} = \bigcup_{j \in \omega} M^{(i,j)}$. Let $M(f) = M^{(1,\omega)}$ along with the structure that for each $\mathbf{x} \in M(f), P^r(\mathbf{x})$ iff $\mathbf{x} \in M^{(|\mathbf{x}|,\omega)}$ and $\text{ats}(\mathbf{x}) \rightarrow P^r(\mathbf{x})$. In addition we add $M(f) \models R^r_{\leq}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\|^r \leq \|\overline{b}\|^r$ and $M(f) \models R^{\circ}_{\leq,i,j}(\overline{a}, \overline{b}) \leftrightarrow \|\overline{a}\|^i \leq \|\overline{b}\|^j$. Notice that $|M(f)| = \sup\{\kappa, f(D) : \langle D, j\rangle \in \mathcal{D}, j \leq \omega\}$

This construction mimics the one in the Section 8.3.4 very closely. There are only two main differences. First, when we look to see which extensions of a tuple we want, instead of forcing everything that could happen does, we only force extensions realized in N to be realized (and we always know there is a valid extension in this strategy because $N \models T^n_\Omega$). And second, instead of adding only ω many extensions of a given tuple with a given archetype sequence we allow ourselves to choose how many we want to add.

Claim 8.3.5.2. *Let $\boldsymbol{x} \in M(f)$. Let $\gamma_r(\boldsymbol{x}) = \|\boldsymbol{x}\|^r$ as determined by the archetype assigned to it in the construction (i.e. $M^{(|\boldsymbol{x}|,\omega)}$). Let $M(f) \models \gamma_r^*(\boldsymbol{x}) = \|\boldsymbol{x}\|^r$. Then $\gamma_r(\boldsymbol{x}) = \gamma_r^*(\boldsymbol{x})$.*

Proof. First notice that we ensured by our construction that the archetype of every finite tuple of $M(f)$ is determined in our construction (just let i be some stage greater than the point when all elements of $\mathbf{x}$ appear for the first time in the construction to find $\|\mathbf{x}\|^i$.)

Base Case:

Assume $\gamma_r(\mathbf{x}) = 0$ or $-\infty$.

We the know that for any y such that $\mathbf{x}y \in M^{(|\mathbf{x}y|,\omega)}, \gamma_r(\mathbf{x}y) = -\infty$ as this fact is realized by the archetype of $\mathbf{x}$. But, then we know that for all $y, \gamma_r^*(\mathbf{x}y) = -\infty$ by how we define color on $M(f)$.

But we also know by construction that $\gamma_r^*(\mathbf{x}) \geq 0$ (i.e. $M(f) \models P^r(\mathbf{x})$) iff $\gamma_r(\mathbf{x}) \geq 0$ (i.e. $\text{ats}(\mathbf{x})$ in the construction witnesses that $P^r(\mathbf{x})$). So, we know that $\gamma_r^*(\mathbf{x}) = \gamma_r(\mathbf{x})$.

Inductive Case:

Assume that if $\mathbf{z} \in M(f)$ and $\gamma_r(\mathbf{z}) < \alpha$ then $\gamma_r(\mathbf{z}) = \gamma_r^*(\mathbf{z})$. Let $\gamma_r(\mathbf{x}) = \alpha$.

So we have $\gamma_r^*(\mathbf{x})$ =sup$\{\gamma_r^*(\mathbf{x}y) + 1 : y \in M(f)\} = \sup\{\gamma_r(\mathbf{x}y) + 1 : \mathbf{x}y \in M(f)\} = \gamma_r(\mathbf{x})$

Now we have the last equality because by construction if ats($\mathbf{x}$) $= D$ and $C \leq D$ and C is realized as an extension of D in N then there is some extension of $\mathbf{x}$ which realizes C. So, because this last equality holds in N it must also hold in $M(f)$.

So by induction the claim is proved. □

Claim 8.3.5.3. *$M(f)$ is a model of T_Ω^n.*

Proof. So $M(f)$ obviously models T_R on each of the n copies of L_R.
Also $M(f) \models P^i(\mathbf{x}) \rightarrow P^j(\mathbf{x})$ if $i \leq j$ because all tuples are realized as part of an archetype sequence.

All that is left to check is (Saturation for Finite Sequences of Archetypes), (Homogeneity of Archetype Sequences), the (Amalgamation of Archetype Sequences) and the condition on limit colors (this is because (Saturation for Finite Sequences of Archetype) implies (Saturation on Archetypes) in each language).

Well (Saturation for Finite Sequences Archetypes) is easy. Lets say we have a j-tuple $\mathbf{x} \in M(S)$ and we want to show it can be extended to a color archetype sequence A such that $A(i_k) = B_k$ for a finite ordered sequence $\langle i_k : k \in p < \omega \rangle$.

First notice it suffices to consider extending by a single element (as doing this multiple times gets us any extension). Lets say $\mathbf{x}$ comes into the construction at stage i. In particular then, $\mathbf{x}$ is a subtuple of some i-tuple $\mathbf{y} \in M^{(i,i)}$. We know then that there must be an extension of ats($\mathbf{y}$) to an archetype sequence $D(\mathbf{y}, a)$ such that D implies

(1) If $\mathbf{z} \subseteq \mathbf{x}$, $i_k \leq i < i_{k+1}$ then $\|\mathbf{z}a\|^i = \|\mathbf{z}a\|^{i_k}$

(2) $\mathrm{ctype}(\mathbf{x}, a) = B_k$

(3) Any color of other subtuple of $\mathbf{y}$ which contains a must have color $-\infty$.

and D is realized in N (this is because N satisfies (Saturation for Finite Sequences of Archetypes)). Hence this extension must also be realized by the construction.

As for (Amalgamation of Archetype Sequences), that is easily seen to hold as well. Say we have a pair of archetypes C, D realized in the model. Then, they are realized in N and hence there trivial amalgamation is realized in N (because $N \models$(Amalgamation of Archetype Sequences)). In particular, by the construction, this means that the trivial amalgamation is realized in $M(f)$.

(Homogeneity for Archetype Sequences) trivially holds because if we have two elements $\mathbf{x}, \mathbf{y}$ which realize the same archetype sequence D then they both can be extended to a color archetype sequence E iff the color archetype sequence D can be extended in N to E.

The limit condition on colors is satisfied by virtue of the fact that all archetype sequences realized in N satisfy the condition.

□

Theorem 8.3.5.4. $\mathcal{D}(f) = ATS(M(f))$

Proof. The construction is designed to make this true. □

Corollary 8.3.5.5. *If $\mathcal{D}(f) = \mathcal{D}(g)$ then $M(f) \equiv_\infty M(g)$.*

Proof. This is a direct consequence of Theorem 8.3.5.4 □

Theorem 8.3.5.6. *For each $N \models T^n_\Omega$ (with no tuple of N has color ∞) and for each $\omega_\lambda \geq \kappa(N)$ there are at least $\lambda^{\kappa(N)}$ many distinct models of size $\aleph_\lambda$ which are all $\mathcal{L}_{\infty,\omega}$ to N.*

Proof. This is because $|\mathcal{D}(N)| = \kappa(N)$ and so there are λ^κ many functions from $\mathcal{D} \to \lambda$ □

8.4 $T^n_\Omega(\mathcal{M})$

8.4.1 Introduction

Now that we have defined our theories T^n_Ω we will want to place an upper bound the models in a similar way to our theories $T_a(\mathcal{M})$ and $T_K(\mathcal{M})$

8.4.2 Definitions

8.4.2.1 Language

Definition 8.4.2.1. Let $\mathcal{M} \models T_\Omega$ be such that no tuples have any color ∞. Let $L_Q = \{\langle c_i : i \in \mathcal{M}\rangle, Q(x)\}$ where Q is a 1-ary predicate.
We then define the language $L^\omega_\Omega(\mathcal{M}) = L^\omega_\Omega \cup \cup L_Q$.

8.4.2.2 Axioms

Definition 8.4.2.2. Let $T^n_\Omega(\mathcal{M})$ be universal closure of the following $L^n_*(\mathcal{M})$ sentences:

<u>Q:</u>

- $Q(x) \leftrightarrow \bigvee_{a \in \mathcal{M}} x = c_a$
- $Q \models \phi(c_{a_1}, \cdots c_{a_n})$ in L^2_Ω iff $\mathcal{M} \models \phi(a_1, \cdots a_n)$
- $Q(\mathbf{x}) \wedge \neg Q(\mathbf{y}) \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any predicate other than $R^\omega_\leq$ and $|\mathbf{x}|, |\mathbf{y}| > 0$
  $Q(\mathbf{x}) \rightarrow \neg U'(\mathbf{x})$ where U' is any predicate other than $R^\omega_\leq, P$

<u>L^ω :</u>

- T_R on L^ω
- $(\forall x)(\exists c) Q(c) \wedge R^\omega_=(x, c)$
- $(\forall c)(\exists x) \neg Q(x) \wedge R^\omega_=(x, c)$

<u>Other Axioms:</u>

- $\neg Q \models T^\omega_*$

Now the intent of the <u>Q</u> axioms is that we want to fix everything that can be said about any element which satisfies Q. In particular, we want the collection of elements which satisfy Q to have each element named and to have a spectrum which is the same as $\mathcal{M}$ in L^ω. We further want nothing else to be true in L^ω of elements which satisfy Q. Finally, we want to be able to compare the ω-color (using $R^\omega_\leq$) of elements which satisfy Q with elements which satisfy $\neg Q$.

The intent of the <u>L^ω</u> axioms is to ensure that ωth spectrum of a model is the same as the spectrum of $\mathcal{M}$. One point worth mentioning explicitly is that even if for all $i \in \omega$ our model looks like a model of T_Ω on L^i_* we still don't know that on L^ω it will look like a model of T_Ω. And, in general our models will not have the required saturation in the L^ωth language to be models of T_Ω. This is because in L^ω the color of a tuple has to be the limit of the color on all other languages. It is for this reason that we need a copy of T_R and not just T_S to bound the ωth color.

8.4.3 Theorems

Theorem 8.4.3.1. *Let $M, N \models T^\omega_\Omega(\mathcal{M})$. If $ATS(M) = ATS(N)$ then $M \equiv_\infty N$.*

This proof is almost identical to Theorem 8.3.3.2.

Proof. Let $M, N \models T^\omega_\Omega(\mathcal{M})$ with $ATS(M) = ATS(N)$.

Definition 8.4.3.2. Let $I = \{f : M \to N, \mathrm{dom}(f) < \omega$ and if $q_f = a \in \mathrm{dom}(f), \neg Q(a)$ then $\mathrm{ats}(q_f) = \mathrm{ats}(f[q_f])$ and f preserves atomic formula on $Q\}$.

We want to show then that $I \subseteq I$ is a sequence of partial isomorphisms. So if $f \in I$ then f preserves all atomic formula by construction. So all that is left is to show that $I \subseteq I$ has the back and forth property.

Let $f \in I$ and $a \in M$. We then need to find a $b \in N$ such that $f \cup (a, b) \in I$. We can break this into two cases.

Case 1: $M \models Q(a)$

In this case we know that there is a $m \in \mathcal{M}$ such that $M \models c_m = a$. Let b be such that $N \models c_m = b$.

Case 1: $M \models \neg Q(a)$

Let $M \models \sigma(q_f a)$ and $N \models \tau(f[q_f])$ where σ, tau are color archetype sequences. In particular we have by assumption that there is some $\bar{c}c \in N$ such that $N \models \sigma(\bar{c}c)$ because $\mathrm{ATS}(N) = \mathrm{ATS}(M)$. But, we then also must have $\tau(\bar{c})$ by our conditions on when $f \in I$. So, by (Homogeneity for Archetypes Sequences) (see Definition 8.3.1.4) there must be a $b \in N$ such that $\sigma(f[q_f]b)$. Further we know that $f \cup (a,b) \in I$ and we are done. $I \subseteq I$ has the back and forth property, and hence I witnesses that $M \equiv_\infty N$. □

Corollary 8.4.3.3. *Let $M, N \models T^\omega_\Omega(\mathcal{M})$ and $\bar{a} \in M, \bar{b} \in N$. Then $(M, \bar{a}) \equiv_\infty (N, \bar{b})$ iff $M \equiv_\infty N$ and $ats(\bar{a}) = ats(\bar{b})$.*

Proof. This is immediate from the construction of the sequence of partial isomorphisms ($I \subseteq I$) from M to N in Theorem 8.4.3.1 □

Theorem 8.4.3.4. *If $M \models T^n_\Omega(\mathcal{M})$ and M has no tuples of color ∞ at any color then $qr(M) \leq sup\{Spec(M)^i + m : i \leq n, m \in \omega\}$*

Proof. Let D be an archetype sequence and let $\varphi^=_D(\mathbf{x})$ say that $\mathbf{x}$ satisfies D, using the formula's $\varphi^=_\alpha$ from Corollary 2.2.2.8. Let $\varphi^M \leftrightarrow \bigwedge_{\bar{a} \in M} (\exists \mathbf{x}) \varphi^=_{\mathrm{ats}(\bar{a})}(\mathbf{x}) \wedge (\forall \mathbf{x}) \bigvee_{\bar{a} \in M} \varphi^=_{\mathrm{ats}(\bar{a})}(\mathbf{x})$

Notice if $N \models \varphi^M$ then $ATS(N) = ATS(M)$ (in fact φ^M was defined specifically for this purpose). We then have by Theorem 8.4.3.1 that if $N \models \varphi^M$ then $N \equiv_\infty M$. We then have that $\mathrm{qr}(M) \leq \mathrm{qr}(\varphi^M) = \sup\{\mathrm{Spec}(M)^i + m : i \leq n, m \in \omega\}$ □

The most important idea behind Theorem 8.4.3.1 and Theorem 8.4.3.4 is that the $\mathcal{L}_{\infty,\omega}$ types of a tuple is determined by it's archetype sequence.

8.4.4 Conjectures

Conjecture 8.4.1. *If*

- $Spec(\mathcal{M}) = \{-\infty\} \cup \alpha$
- $\Xi(\alpha) = \alpha$
- $M \models T^{\omega}_{\Omega}(\mathcal{M})$

then $qr(M) = \alpha$

So what is this conjecture saying? This conjecture says that if the spectrum of $\mathcal{M}$ is "nice enough" (i.e. a fixed point of Ξ) then all models of $T^{\omega}_{\Omega}(\mathcal{M})$ have exactly the same quantifier rank. First off notice that we have by Theorem 8.4.3.4 that the quantifier rank of any model of $T^{\omega}_{\Omega}(\mathcal{M})$ is at most α. So what we need to show is that given a model $M \models T^{\omega}_{\Omega}$ and an ordinal $\beta < \alpha$ we can find another model $N_\beta \models T^{\omega}_{\Omega}$ which "looks like" M up to β.

Now there is an obvious candidate for such a model N_β. Let i be some color such that $\mathrm{Spec}^i(M) \cap \mathrm{ORD} > \beta$ (we know this must exists as $\mathrm{Spec}^\omega(M) = \mathrm{Spec}(\mathcal{M}) = \alpha \cup \{-\infty\}$). Now let $\alpha > \lambda > \beta$ (we know such must exists as $\Xi(\alpha) = \alpha$). Now let $N_\beta = N^i_{\Xi(\lambda)}(M)$ from Theorem 8.3.2.1.

The reason why $N^i_{\Xi(\lambda)}(M)$ is a candidate to witness that $\mathrm{qr}(M)$ is at least β is that in each component $j \neq i$ $M|L^j_\Omega \cong N|L^j_\Omega$ and $M|L^i_\Omega \equiv_\beta N|L^i_\Omega$ by Theorem 6.3.2.5. However, there are two points we have to worry about. The first (and less troublesome) is the fact that given any tuple we now have to deal with countably many colors. So when we extend a tuple by an element we have to make sure that the new sequences of colors "look the same" up the the appropriate ordinal. As such we will need an analogous theorem to Theorem 6.3.2.5 to deal with infinite sequences of ordinals. But, that shouldn't be to difficult and the techniques used in the proof of Theorem 6.3.2.5 should generalized immediately (although you may need to look at slightly larger ordinal to guarantee the back and forth argument works). And it is the authors belief that for "most" $\lambda > \beta$ $N^i_{\Xi(\lambda)}(M) \equiv_\beta M$ in L^ω_Ω (notice this is not the same as being equivalent up to β in $L^\omega_\Omega(\mathcal{M})$)

The second problem we have to deal with in a proof of Conjecture 8.4.1 is that unlike in the case of T^ω_Ω we have a method for completely determining the ω-color of an arbitrary tuple. As it turns out this is a big problem (just as it was in the case of $T_a(\mathcal{M})$ in Section 7.3). The reason is that if we can nail down exactly any particular color than given any two sequences of ordinals which "look the same" we can distinguish them by choosing a color relative to which the order isn't preserved

Because of this second reason a new method of proof other than the one described above will be needed to prove Conjecture 8.4.1. It is because of that we have the next section.

8.5 T^n_Λ

8.5.1 Definitions

Definition 8.5.1.1. Let $L^n_\Lambda = n$ copies of L_Λ

Definition 8.5.1.2. Let T^n_Λ be the universal closure of the following L^n_Λ sentences:

- $P^i(\mathbf{x}) \rightarrow P^j(\mathbf{x})$ if $i < j \leq n$

- (Homogeneity for Archetypes Sequences) For each $m \in \omega$

$$\bigwedge_{i \leq n} E^i_a(\mathbf{x}, \mathbf{y}) \rightarrow (\forall \bar{b})(\exists^m \bar{a}) \bigwedge_{i \leq m} E^i_a(\mathbf{x}\bar{a}_i, \mathbf{y}\bar{b})$$

- (Amalgamation of Archetype Sequences) For each $m \in \omega$

$$(\forall \mathbf{x}, \mathbf{y})(\exists \bar{a}, \bar{b}) \bigwedge_{i \leq n} E^i_a(\mathbf{x}, \bar{a}) \wedge E^i_a(\mathbf{y}, \bar{b}) \wedge \bigwedge_{\emptyset \neq S \subseteq \bar{a}, \emptyset \neq S' \subseteq \bar{b}, i \leq n} \neg P^i(SS')$$

- <u>If i is a successor ordinals:</u>
  - $T^i_\Lambda, i \leq n$ (i.e. T_S on the ith copy of L_S)

- <u>If i is a limit ordinal:</u>
  - $P^i(\mathbf{x}) \leftrightarrow \bigvee_{j<i} P^j(\mathbf{x})$

The idea behind this theory is that we want to take n different models of T_Λ and put them all on the same set just as in Section 8.4. The main difference between this example and that of Section 8.4 is that here we are not able to compare different colors in different languages (or for that matter even comparing colors in the same language). As we will see this is actually useful as it means our proof won't fall through because we could access the constant value.

8.6 $T^n_\Lambda(\mathcal{M})$

8.6.1 Introduction

Now that we have defined our theories T_Λ we want to place an upper bound the models in a similar way to our theories $T_a(\mathcal{M})$, $T_K(\mathcal{M})$, and $T^n_\Omega(\mathcal{M})$.

8.6.2 Definitions

8.6.2.1 Language

Definition 8.6.2.1. Let $\mathcal{M} \models T_\Lambda$ be such that no tuples have any color ∞.
Let $L_Q = \{\langle c_i : i \in \mathcal{M}\rangle, Q(x)\}$ where Q is a 1-ary predicate.
We then define the language $L^\omega_\Lambda(\mathcal{M}) = L^\omega_R \cup \cup L^n_S \cup L_Q$

8.6.2.2 Axioms

Definition 8.6.2.2. Let $T^n_\Lambda(\mathcal{M})$ be universal closure of the following $L^n_*(\mathcal{M})$ sentences:

Q:

- $Q(x) \leftrightarrow \bigvee_{a\in\mathcal{M}} x = c_a$

- $Q \models \phi(c_{a_1}, \cdots c_{a_n})$ in L^2_Λ iff $\mathcal{M} \models \phi(a_1, \cdots a_n)$

- $Q(\mathbf{x}) \wedge \neg Q(\mathbf{y}) \rightarrow \neg U(\mathbf{x}, \mathbf{y})$ where U is any predicate other than $R^\omega_\leq$ and $|\mathbf{x}|, |\mathbf{y}| > 0$

$Q(\mathbf{x}) \to \neg U'(\mathbf{x})$ where U' is any predicate other than $R^{\omega}_{\leq}, P$

L^{ω} :

- T_R on L^{ω}
- $(\forall x)(\exists c)Q(c) \wedge R^{\omega}_{=}(x,c)$
- $(\forall c)(\exists x)\neg Q(x) \wedge R^{\omega}_{=}(x,c)$

Other Axioms:

- $\neg Q \models T^{n}_{\Lambda}$ if n is not a limit.
- $P^{i}(\mathbf{x}) \to P^{j}(\mathbf{x})$ if $i \leq j$

Now the intent of the Q axioms is that we want to fix everything that can be said about any element which satisfies Q. In particular, we want the collection of elements which satisfy Q to have each element named and to have a spectrum which is the same as $\mathcal{M}$ in L^{ω}. We further want nothing else to be true in L^{ω} of elements which satisfy Q. And, finally, we want to be able to compare the ω-color (using $R^{\omega}_{\leq}$) of elements which satisfy Q with elements which satisfy $\neg Q$.

8.6.3 Conjectures

Conjecture 8.6.1. *If*

- $Spec(\mathcal{M}) = \{-\infty\} \cup \alpha$
- $\Xi(\alpha) = \alpha$

- $M \models T^{\omega}_{\Lambda}(\mathcal{M})$

then $qr(M) = \alpha$

This conjecture is very similar to Conjecture 8.4.1, except it deals with $T^{\omega}_{\Lambda}(\mathcal{M})$ instead of $T^{\omega}_{\Omega}(\mathcal{M})$. Because it deals with different theories it presents a different set of obstacles than does Conjecture 8.4.1. To understand the differences we first really have to understand the differences between T_Ω and T_Λ.

The most important difference between T_Ω and T_Λ is that in T_Λ we are not able to say when one color is greater than another. This is important because it means unlike in the case of T^{ω}_{Ω} we don't have to worry about having colors named. Even if we know that some tuple has a fixed ω-color that doesn't tell us anything about any other colors of any other tuples in any other language in the model.

The downside to T_Λ though is that we no long have a nice description of the $\mathcal{L}_{\infty,\omega}$ type of an tuple. In T_Ω we know that the $\mathcal{L}_{\infty,\omega}$ type of a tuple is determined completely by its colors. And what is more we even have a way to describe when two $\mathcal{L}_{\infty,\omega}$ types in T_Ω are the same up to some quantifier rank (see Theorem 6.3.2.5). In T_Λ however we don't have this easy description. It is possible to have several (in fact infinitely many) different $\mathcal{L}_{\infty,\omega}$ types all with the same color (consider a tuple with color ω such that it only has extensions of color in S for some S unbounded in ω).

It is this reason why T_Λ doesn't work as a component when trying to build almost scattered sentences. But, in the case where we don't care about the number of models or the number of types, but only about the quantifier rank, then $\mathcal{T}_\Lambda$ is a very good sentence to use as a component.

It is the belief of the author that once a method has been developed for describing the $\mathcal{L}_{\infty,\omega}$ types over T_Λ then Conjecture 8.6.1 should follow from the techniques developed in this paper.

One last point which we should make is that the theory T_P was designed for use with T_Ω and for theories with collections of archetypes. It is the authors believe that T_P is not the theory best suited to deal with T_Λ. It is the authors belief that a better suited theory of trees would be one where each element can only extend at most one finite sequence, and further where we deal with finite sequences of elements instead of finite sets (i.e. unordered sequences) of elements.

One such theory which the author believes would be better suited for the study of T_Λ is the following:

Definition 8.6.3.1. Let $L_{P_*} = \{P^n_* : P^n_* \text{ is an n-ary predicate}\}$.

Definition 8.6.3.2. Let T_{P_*} be universal closure of the following L_{P_*} sentences:

- $P^n_*(\mathbf{x}, \mathbf{y}) \wedge P^n_*(\mathbf{x}', \mathbf{y}) \rightarrow \mathbf{x}' = \mathbf{x}$ for all tuples $\mathbf{x}, \mathbf{x}', \mathbf{y}$
- $P^{n+1}_*(x_0, \cdots, x_n) \rightarrow P^n_*(x_1, \cdots, x_n)$

Here the intended interpretation is that P_* is a tree under the order $\langle x_i : i \in n\rangle < \langle x_i : i \in m\rangle (m > n)$. I.e. in the order of extension of finite sequences.

If we replace T_P with T_{P_*} in the definition of T_S, T_Λ and T^ω_Λ then all the proofs of Section 8.6 should go through unchanged. And, the author believes that in this context it will be easier to study the $\mathcal{L}_{\infty,\omega}$ types of T_Λ and hence easier to study Conjecture 8.6.1

Appendix A

Vaught Tree

In Part A the most useful measure of the complexity of the collection of models of a theory was the theories quantifier rank spectrum. However, in the case that our theories are sufficiently well behaved, there is another method which highlights the relationships between the model and the theories they satisfy. This method is the Vaught tree. It is from the Vaught tree that we get many of our intuitions for what well behaved theories should look like. In this section we explain what the Vaught tree is, explain the relationship between the Vaught rank of a model and it's quantifier rank, and define in terms of both the Vaught tree what it means for a theory to be Scattered or Weakly Scattered (and show that they have the same meaning as the Definition 1.2.3).

A.1 Vaught Tree

A.1.1 Definition

Definition A.1.1.1. Let L_0 be a countable language A_0 a countable fragment of $\mathcal{L}_{\infty,\omega}(L_0)$ and T_0 a complete theory in A_0. Let $M \models T_0$ be a model. We define the theory of M at level α as follows

<u>Limit Ordinals</u>

- $L_{\omega*\alpha}(M) = \bigcup_{\zeta<\omega*\alpha} L_\zeta(M)$.
- $A_{\omega*\alpha}(M) = \bigcup_{\zeta<\omega*\alpha} A_\zeta(M)$.
- $T_{\omega*\alpha}(M) = \bigcup_{\zeta<\omega*\alpha} T_\zeta(M)$.

<u>Successor Ordinals</u>

- $L_{\zeta+1}(M) = L_\zeta(M) \cup \{P_p : p$ a complete type in $A_\zeta(M)$ over $T_\zeta(M)\}$
- $A_{\zeta+1}(M)$ to be the smallest fragment containing $A_\zeta(M)$ and $\{P_p(\mathbf{x}) \leftrightarrow \bigwedge_{\varphi\in p} \varphi(\mathbf{x})\}$
- $T_{\zeta+1}(M)$ =Theory of M in $A_{\zeta+1}(M)$.

We define the <u>Vaught Tree</u> of a sentence $T \in \mathcal{L}_{\infty,\omega}$ to be

$$\bigcup_{M\models T,\alpha\in \mathrm{ORD}} \mathrm{Theory}_{A_\alpha(M)}(M)$$

ordered by inclusion (where $A_0 = Frag(T)$, the smallest fragment containing T.

Intuitively we start with a theory in $\mathcal{L}_{\infty,\omega}$ which is in a fragment A. Then for each model M we add a node at Level 1 which corresponds to the theory of M in the language with names for all types over A.

To find out what happens at level 2 we repeat this procedure but this time looking at each individual node at level 1. We then continue this process forever and that is the Vaught tree.

The reason why this is useful is we are building up all the models of our theory 1 piece at a time. As such we get further up the Vaught tree we have better and better approximations for the model we are trying to describe. This is especially useful because we know, by Theorem 1.2.2.14, that for all models there is a Scott sentence, so for any model this "growing" of the model piece by piece must end.

Theorem A.1.1.2. *If T is a sentence of $\mathcal{L}_{\omega_1,\omega}(L)$ with $M \models T$ a countable model. Then there is an ordinal α such that the theory of M in $A_\alpha(M)$ has no non-principle types.*

Proof. See [3] Section 2. □

Definition A.1.1.3. We say Scott Rank(M) (sr(M)) = least α such that $T_\alpha(M)$ is atomic (i.e. has no non-principle types).

A.1.2 Theorems

Theorem A.1.2.1. *Each formula in $\varphi \in A_\gamma(M)$ is equivalent to a formula in $\varphi' \in \mathcal{L}_{|\kappa|,\omega}$ (where $\kappa = \max\{qr(\phi) : \phi \in A_\gamma(M)\}$) and such that $qr(\varphi') \leq \max\{qr(\phi) : \phi \in A_0\} + \omega * \gamma$.*

Proof. Assume this is true for all formulas in $A_\gamma(M)$.

Let $\varphi \in A_{\gamma+1}(M) - A_\gamma(M)$.

- If P_p is atomic and not in $A_\gamma(M)$, let $P_p' = [\bigwedge_{\varphi\in p}\varphi(\mathbf{x})]$
- If $\varphi = \neg\psi$ then let $\varphi' = \neg\psi'$.
- If $\varphi = \bigwedge_{i\in I}\psi_i$ let $\varphi' = \bigwedge_{i\in I}\psi_i'$
- If $\varphi = (\exists x)\psi$ then $\varphi' = (\exists x)\psi'$.

If $\varphi \in A_\gamma(M)$ let $\varphi' = \varphi$.

So in particular as $A_{\gamma+1}(M)$ contains only formulas which are finite conjunctions/disjunctions of finitely quantified formulas of $A_\gamma \cup L_{\gamma+1}$ and we know that $\mathrm{qr}(\varphi') \leq \max\{\mathrm{qr}(\phi) : \phi \in A_\gamma(M)\} \cup \mathrm{qr}(P_p') + \omega \leq \max\{\mathrm{qr}(\phi) : \phi \in A_\gamma + \omega\}$

Hence we are done by induction. □

Corollary A.1.2.2. *For all countable $M \models T$, $qr(M) \leq \omega * sr(M)$*

Proof. This is an immediate consequence of Theorem A.1.2.1. □

Theorem A.1.2.3. *For all countable $M \models T$ $sr(M) \leq qr(M)$*

Proof. Because the α-characteristic of $s \in M$ (see [**?**] Chapter VII §6.1) is definable in $A_\alpha(M)$ and because $M \models \sigma_N^\alpha \leftrightarrow M \equiv_\alpha N$ (see [**?**] Chapter VII §5, §6) we know that $N \models T_{\mathrm{qr}(M)}(M)$ implies that $M \cong N$ if N, M are countable.

Assume to get a contradiciton that $\mathrm{sr}(M) > \mathrm{qr}(M)$.

So this means that $T_{\mathrm{qr}(M)}(M)$ must have a non-principle type p over $A_{\mathrm{qr}(M)}(M)$. But then this type must be realized by a countable model N and omitted by a countable model N'.

$\Rightarrow\Leftarrow$ Any two countable models of $T_{\mathrm{qr}(M)}(M)$ must be isomorphic to M.

Hence $\mathrm{sr}(M) \leq \mathrm{qr}(M)$ □

A.2 Well Behaved Sentences

Now that we have seen what the Vaught tree we will can begin to understand the motivation behind the definition of scattered and weakly scattered.

A.2.1 Weakly Scattered

Definition A.2.1.1. Let A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and let T be a theory in A. T is Weakly Scattered if for all countable fragments $A' \supseteq A$ and finitely consistent and ω-complete $T' \subseteq T$ in A', T has only countably many types over T' in A'.

Intuitively weakly scattered theories are nice because not only can we build the Vaught tree as in the previous section, but we find that for each $T_\alpha(M)$ we also have a model M_α which is atomic over $T_\alpha(M)$. In this way not only can we approximate a model by the nodes of the Vaught tree, but we can approximate a model with other models M_α at those nodes.

A.2.2 Scattered

Definition A.2.2.1. Let A be a countable fragment of $\mathcal{L}_{\omega_1,\omega}$ and let T be a theory in A. T is Scattered if it is weakly scattered and for all countable fragments $A' \supseteq A$ the set $\{T' \subseteq A' : T'$ is finitely consistent and ω complete$\}$ is countable.

Intuitively a scattered sentence not only has an atomic model at each node of the Vaught tree (because it is weakly scattered) but there are only ω many nodes at any particular level of the tree.

A.2.3 Theorems

A.2.3.1 Consistency

Theorem A.2.3.1. *The definition of weakly scattered in Section A.2.1 and in Section 1.2.3 are the same.*

Proof. First of all it is obvious that if a sentence is weakly scattered in the sense of Section A.2.1 then it is weakly scattered in the sense of Section 1.2.3 by Theorem A.1.2.2 and Theorem A.1.2.3

To see the other direction notice that any countable fragment A is contained in $Th_\alpha(M)$ for some countable α and some M. □

Theorem A.2.3.2. *The definition of scattered in Section A.2.2 and in Section 1.2.3 are the same.*

Proof. First of all it is obvious that if a sentence is scattered in the sense of Section A.2.2 then it is scattered in the sense of Section 1.2.3 by Theorem A.1.2.2 and Theorem A.1.2.3

To see the other direction notice that any countable fragment A is contained in $Th_\alpha(M)$ for some countable α and M. So any fragment A containing a scattered (in the sense of Definition 1.2.3.7) T must have only countably many complete theories extending T over A. Further by Corollary 1.2.3.9 any scattered sentence in the sense of Section 1.2.3 is weakly scattered in the sense of Section 1.2.3 and hence (by Theorem A.2.3.1) is also weakly scattered in the sense of Section A.2.1. □

A.2.3.2 Miscellaneous Theorems

Theorem A.2.3.3. *If T is scattered and $\beta < \alpha < \omega_1$ and $L(\alpha, T)$ is Σ_1 admissible then $\langle\{T_\beta(M) : M \models T\}, \subseteq\rangle \in L(\alpha, T)$.*

Proof. See [3] Proposition 4.4. □

What this says is that the Vaught tree of a scattered theory is completely determined by the constructible universe over that theory. In particular it doesn't matter which set theoretic universe you are in, the Vaught tree over a scattered theory T will be the same.

Theorem A.2.3.4. *If T is a sentence of $\mathcal{L}_{\omega_1,\omega}$ which has less than 2^ω many countable models in some generic extension of the universe, then T is scattered.*

Proof. See [3] □

Corollary A.2.3.5. *Let T be a sentence of $\mathcal{L}_{\omega_1,\omega}$. Then in any model of set theory containing T, T has either $\leq \omega, \omega_1$ or 2^ω countable models.*

Proof. Immediate from Theorem A.2.3.4 and the definition of scattered □

It is because of this result that scattered sentences were originally studied.

Part B

Strong Separation Theorem for Projections of Sheaves

Chapter 9

Introduction

9.1 Summary

9.1.1 Goal

In Part B we will prove a generalization of the Suslin-Kleene separation theory for analytic sets. The theorem is called the Suslin-Kleene separation theorem because it was realized that two separation theorems, one due to Suslin and one due to Kleene, were really two instances of the same theorem (See Theorem 9.2.6.5 and Appendix B). In particular, Suslin's Separation Theorem says.

Theorem 9.1.1.1 (Suslin Separation Theorem)**.** *If A, B are disjoint projections (onto ω^ω) of closed sets in $\omega^\omega \times \kappa^\omega$ then there is a $\kappa + 1$-Borel set C such that $C \supseteq A$ and $C \cap B = \emptyset$.*

Proof. See [10] Chapter 2E Theorem 2E.1 □

As we will see, there is a strong relationship between closed sets in X^ω and sheaves on a particular topological space. It is this relationship that we will use to replace closed subsets of $\omega^\omega \times \kappa^\omega$ with sheaves. We then get that if A and B are "disjoint" subsheafs of $W \times K$ there is a X-Borelian set with "separates" them.

9.1.2 Approach

In Section 9.2 we will provide the necessary background material so that the reader can understand the relationship between sheaves and trees and hence follow the proof of our Separation Theorem.

In Chapter 10 we will introduce and discuss "Partial Grothendieck Topologies". These are a weakening of the usual axioms of a Grothendieck Topology which will allow us to prove our result in a more general context. In this chapter we will also prove many of the background results concerning partial Grothendieck topologies which we will need in our proof our separation theorem.

Finally in Chapter 11 and Chapter 12 we will prove our Separation Theorem. We will prove this in two ways. First, in Chapter 11, we will provide a proof in the case that our partial site is actually a topological space. The hope is that by seeing this case first the reader will be able to better follow the argument. Then, in Chapter 12 we will provide the full proof of the theorem.

Finally, in Appendix B we will provide a general discussion of the Suslin-Kleene Separation Theorem.

9.2 Background

9.2.1 Definition by Transfinite Induction

One of the most useful methods for defining complicated infinite objects is by transfinite induction (or transfinite recursion). Transfinite induction provides a way to assign an object to every node in a well-founded tree, in terms of the objects which extend the node in the tree. To understand how it works we first need to know what a well founded tree is.

Definition 9.2.1.1. Let X be a set. A Pretree on X is a subset of $X^{\leq\omega}$ closed under initial segments. That is $Y \subseteq X^{\leq\omega}$ is a pretree if

(P) $(\forall\langle y_i : i < n\rangle \in Y \cap X^n, n \leq \omega)(\forall m < n)(\langle y_i : i < m\rangle \in Y \cap X^m)$.

If $\langle x_i : i \in n\rangle \in X^{\leq\omega}$ and $m < n$ we define $\langle x_i : i \in n\rangle|m = \langle x_i : i \in m\rangle$

Definition 9.2.1.2. Let X be a set. A Tree on X is a pretree on X which is closed under "gluing together". That is $Y \subseteq X^{\leq\omega}$ is a tree if

(P) Y is a pretree

(S) $((\forall n \in \omega)\langle y_i : i < n\rangle \in Y \cap X^n) \to \langle y_i : i < \omega\rangle \in Y \cap X^\omega$

We say that a tree Y on X is Well-Founded if $Y \cap X^\omega \neq \emptyset$. We say a tree Y is Ill-Founded otherwise.

We will see later that the condition (P) is exactly what it means to say that Y is a subpresheaf of a particular sheaf $\overline{X}$ (for an appropriate topology). Similarly we will see that the condition (S) is exactly what it means for such a subpresheaf to be a sheaf (for the appropriate topology).

Definition 9.2.1.3. Let Y be a well-founded tree on X. Let F :Powerset$(X^{\leq\omega}\times A) \to A$. We say that $g : X^{\leq\omega} \to A$ is Defined by Transfinite Induction on Y using F if

$$(\forall b \in X^{\leq\omega}) g(b) = F(\{\langle c, g(c)\rangle : (\exists d \in X^{\leq\omega}) c = b^\wedge d \wedge c \in Y\})$$

Theorem 9.2.1.4. *If Y is a well-founded tree on X and F :Powerset$(X^{\leq\omega}\times A) \to A$, then there is a unique $g : X^{\leq\omega} \to A$ definable by transfinite induction on Y using F.*

Proof. This is a standard result. See [10] Chapter 2D. □

Lets consider what is going on here. Suppose we have some function F :Powerset$(X^{\leq\omega} \times A) \to A$ and some well founded tree Y and we want to define g by transfinite induction on Y using F. How should we do this? Well the procedure we want to use to calculate $g(\mathbf{x})$ for $\mathbf{x} \in X^{\leq\omega}$ is as follows

(1) Find the set $Y_{\mathbf{x}} = \{\mathbf{x}^\wedge\overline{c} : \overline{c} \in X^{\leq\omega} \wedge \mathbf{x}^\wedge\overline{c} \in Y\}$ of nodes which are extensions of $\mathbf{x}$ in the tree Y.

(2) Find the value of g on all nodes of $Y_{\mathbf{x}}$

(3) Construct the set $Y^A_{\mathbf{x}} = \{(\mathbf{y}, g(\mathbf{y})) : \mathbf{y} \in Y_{\mathbf{x}}\}$

(4) Set $g(\mathbf{x}) = F(Y^A_{\mathbf{x}})$

Now on first examining this procedure step (2) looks a little suspicious. It looks like (2) implies that in order to calculate $g(\mathbf{x})$ we have to calculate $g(\mathbf{y})$ for all $\mathbf{y}$ extending $\mathbf{x}$. But to calculate $g(\mathbf{y})$ we then have to calculate $g(\mathbf{z})$ for all $\mathbf{z}$ extending each $\mathbf{y}$. But to calculate $g(\mathbf{z})\ldots.$

At first glance it seems like because of (2) this procedure just doesn't make sense. But here is where well-foundedness comes to the rescue. Suppose this procedure produced a point $\mathbf{x}$ where g was undefined. We then know that $\mathbf{x} \in Y$ (because if $\mathbf{x} \notin Y$ then $g(\mathbf{x}) = F(\emptyset)$ and hence is defined). But, the only way in which $g(\mathbf{x})$ could be undefined is if there is some extension $\mathbf{x}^\wedge\mathbf{x}_1 \in Y$ on which g is undefined. But the only way $g(\mathbf{x}^\wedge\mathbf{x}_1)$ is undefined is if there is some extension $\mathbf{x}^\wedge\mathbf{x}_1{}^\wedge\mathbf{x}_2 \in Y$ on which g is undefined .... In this way we produce an infinite sequences $(\mathbf{x}, \mathbf{x}^\wedge\mathbf{x}_1, \mathbf{x}^\wedge\mathbf{x}_1{}^\wedge\mathbf{x}_2, \cdots) \in Y \cap X^\omega$. So, by condition (S) in Definition 9.2.1.2 we have there must be an infinite path $\mathbf{x}^\wedge\mathbf{x}_1{}^\wedge\mathbf{x}_2{}^\wedge \cdots \in Y$. But, as Y is well founded this can't happen. So our original assumption (that g was undefined somewhere) must be false.

9.2.2 Baire Space

Now that we understand what a tree is we can start to look at the space of all illfounded branches on a set X. Specifically we will look at spaces of the form X^ω. As it turns out such function spaces are very well behaved.

Definition 9.2.2.1. Let $X^\omega \supseteq O_f = \{x \in X^\omega : (\forall i \in \text{dom}(f))x(i) = f(i)\}$. Define $\mathcal{O} = \{\bigcup_{i\in\omega} O_{f_i} : f_i \in X^{<\omega}\}$ as the open sets for the topology on X^ω.

If $|X| = \omega$ we say X^ω is the Baire Space. If $|X| = 2$ we say X^ω is the Cantor Space

Theorem 9.2.2.2. *X^ω is complete meterizable. Further if $|X| \leq \omega$ X^ω is separable and if $|X| < \omega$ it X^ω is compact.*

Proof. This is a standard result. See [10] or [6] □

There are many other very nice properties of X^ω and we would refer the interested reader to [**?**] or [10].

9.2.3 Trees

One of the many nice properties of topological spaces of the form X^ω is that their topology has a succinct description in terms of trees. But, before we begin to discuss this we will need some definitions.

Definition 9.2.3.1. Let Y be a pretree on X. Define $[Y] = \{\langle y_i : i \in \omega\rangle \subseteq Y\} = Y \cap X^\omega$. Similarly if $Y \subseteq X^\omega$ define $\langle Y\rangle = \{\mathbf{x}|n \text{ s.t. } n \in \omega \text{ and } x \in Y\}$.

So $[Y]$ is the collection of infinite paths through the pretree Y. Similarly given a set $Y \subseteq X^\omega$ $\langle Y\rangle$ is the collection of initial segments of elements of Y.

Definition 9.2.3.2. Let Y be a tree on X. We say that Y is <u>Pruned</u> if $(\forall n < \omega)(\forall y \in Y \cap X^n)(\exists \mathbf{y} \in [Y])\mathbf{y}|\text{dom}(y) = y$.

In other words, we say a tree is pruned if every element of the tree belongs to an infinite path through the tree.

Pruned Tree Unpruned Tree

We are now ready to see the connection between the topology on X^ω

Theorem 9.2.3.3. *For all trees T on X, $[T] \subseteq X^\omega$ is a closed set. Further, for every set $U \subseteq X^\omega$, $[\langle U \rangle] = \overline{U}$ (the closure of U in X^ω).*

Proof. This is a standard result. See [10] Chapter 2C, Theorem 2.C1. □

So we find that the closed sets of X^ω correspond in a natural way to the pruned trees on X.

9.2.4 Sheaves on a Topological Space

Before we continue our study of function spaces of the form X^ω lets first review a little bit about the actual objects we will be studying, sheaves. Recall the definition of a presheaf on a topological space.

9.2.4.1 Presheaves

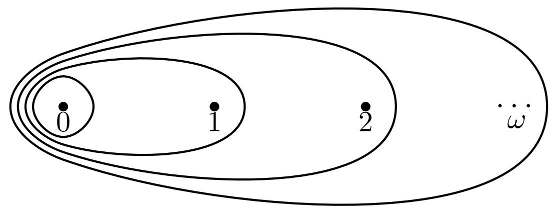
Definition 9.2.4.1. Let $(T, \mathcal{O})$ be a topological space. Let $\overline{T}$ be the category of open sets on X with inclusion maps as morphisms. A Presheaf on T is a functor $SET^{\overline{T}^{op}}$.

So, a presheaf on a topological space X is a function which assigns to each open set U a set $P(U)$. In addition, for all open $U \subseteq V$ the presheaf gives us a map $P(i_{V,U}) : P(V) \to P(U)$ in such a way that the maps commute. I.e. if $U \subseteq V \subseteq W$ then $P(i_{W,U}) = P(i_{W,V}) \circ P(i_{V,U})$.

To better understand the definition of presheaf, lets consider what presheaves look like on a very simple topological space.

Definition 9.2.4.2. Let $O_n = \{i \in \omega : i \leq n\}$. Let $\mathcal{O}_\omega = \{O_{\cdot}n \leq \omega\}$. Define $\overline{\omega} = (\omega, \mathcal{O}_w)$.

Now what does this topological space look like? Well the only open sets are those which are initial segments of ω or all of ω. (Notice that this topological space is closed under arbitrary unions and intersections as they correspond to sup, inf respectively).



So what does a presheaf P on $\overline{\omega}$ look like? Well we know that for each $n \leq \omega$ $P(n)$ is a set of elements. What is more, if $m < n$ then there is a map $P(i_{n,m}) : P(n) \to P(m)$ such that $P(i_{n,m}) \circ P(i_{k,n}) = P(i_{k,m})$.

Now this looks an awful lot like the condition (P) from Definition 9.2.1.2 of a tree. To make this even clearer, lets consider a special type of presheaf $\overline{X}$ where $\overline{X}(n) = X^n$ and let $\overline{X}(i_{n,m}) : \overline{X}(n) \to \overline{X}(m)$ simply ignore everything after the first m elements of the sequence. Now lets consider a presheaf $Y \rightarrowtail \overline{X}$.

The presheaf condition simply says that whey you ignore the tail end of an element of $Y(n)$ (beyond m) you get an element of $Y(m)$. In particular if we let $Y = \bigcup_{n \leq \omega} Y(n) \subseteq X^{\leq \omega}$ then the presheaf condition says if

$$[(\forall \langle y_i : i < n\rangle \in Y \cap X^n, n \leq \omega)] \to (\forall m < n)(\langle y_i : i < m\rangle \in Y \cap X^{\leq m}).$$

But this is exactly what condition (P) says in Definition 9.2.1.2.

9.2.4.2 Sheaves

Now that we understand the relationship between presheaves and pretrees in $X^{\leq\omega}$ lets consider what the sheaves look like. Recall the definition of a sheaf on a topological space.

Definition 9.2.4.3. Let $(T, \mathcal{O})$ be a topological space. Let P be a presheaf on T. We say that $\{x_i : i \in I\}$ is a compatible set of elements for P if $x_i \in P(U_i)$ and $P(i_{U_i, U_i \cap U_j})(x_i) = P(i_{U_j, U_i \cap U_j})(x_j)$.

We say that P is a Sheaf on T if for all $\{x_i : i \in I\}$, a compatible set of elements for P, there is a unique $x \in P(U)$ (where $U = \bigcup_{i \in I} \mathrm{dom}(x_i)$) such that $P(i_{U,U_i})(x) = x_i$.

We say that a presheaf is a sheaf if any time we have a collection of elements which pairwise "agree on their intersection", we can glue them together uniquely.

Lets consider what this means for presheaves on our topological space $\overline{\omega}$. The first thing we have to consider is the ways in which is it possible to "cover" a set. In other words, when is it possible to find $\langle U_i : i \in I\rangle$ such that $U = \bigcup_{i \in I} U_i$ for a U of our choosing. Well, if U is a finite open set in our topology then the only way $U = \bigcup_{i \in I} U_i$ is if $U = U_i$ for some i (this is because union corresponds to taking sup). Hence any time we have a compatible family of elements on U_i's, where $\bigcup_{i \in I} U_i = U = n$ for some $n < \omega$, the "gluing together" process simply returns the element already on n (we know one such element must exist because $n = U_i$ for some i)

All that is left is to consider is the case when $U = \omega$. Once again, if

$\omega = U_i$ for some i then the gluing just returns the element of the compatible family whose domain is ω. So lets assume $\omega \neq U_i$ for all $i \in I$. In this case we know that the U_i are finite and cofinal in ω. In particular we know that we can generate uniquely a consistent sequence $\langle x_i : i \in \omega \rangle$ such that $P(i_{n,m})(x_n) = x_m$ and $x_n \in P(n)$ (because the x_i are a compatible collection). Hence we know that any such sequence uniquely determines an element of $P(\omega)$ (and trivially any element of $P(\omega)$ determines such a sequence).

In particular if we consider our presheaf $Y(n) \subseteq X^{\leq n}$ then the sheaf condition corresponds to saying

$$(\forall n \in \omega)\langle y_i : i < n \rangle \in Y \cap X^n (= Y(n)) \rightarrow \langle y_i : i < \omega \rangle \in Y \cap X^\omega (= Y(\omega))$$

But this is exactly the condition (S) in Definition 9.2.1.2.

So in fact we can consider trees on X as nothing more than subsheaves on $\overline{\omega}$ of $\overline{X}$. It is also worth mentioning explicitly the following theorem

Theorem 9.2.4.4. *Let $\bar{\cdot}$ be the closure operator induced by the Grothendieck topology on T. Then a presheaf P is a sheaf if and only if $\overline{P} = P$.*

Proof. See [9] Chapter V □

Hence the closed presheaves on $\overline{\omega}$ which are subsheaves of $\overline{X}$ are essentially the same thing as the closed subsets of X^ω. It is this relationship which will inspire our generalization of the Suslin-Kleene separation theorem.

9.2.5 Partial Topologies

Before we continue we will need to understand the exact nature of what it means to be a topological space. Recall the definition of a topology.

Definition 9.2.5.1. Let X be a set. A Topology on X is a collection of Open Sets $\mathcal{O} \subseteq \text{Powerset}(X)$ such that

(1) $X \in \mathcal{O}$

(2) If $\{U_i : i \in n\} \subseteq \mathcal{O}$ is a finite collection of open sets $\bigcap_{i\in n} U_i \in \mathcal{O}$

(3) If $\{U_i : i \in I\} \subseteq \mathcal{O}$ is an arbitrary collection of open sets $\bigcup_{i\in I} U_i \in \mathcal{O}$

We further say $\mathcal{B} \subseteq \text{Powerset}(X)$ is a Basis for $(X, \mathcal{O})$ if

- $\mathcal{B} \subseteq \mathcal{O}$
- $(\forall U \in \mathcal{O})(\exists \{B_i : i \in I\} \subseteq \mathcal{B})\ U = \bigcup_{i\in I} B_i$

Given a set X and a basis $\mathcal{B}$ for a topological space we can ask what the topology on X generated by $\mathcal{B}$ will look like? (as an example to keep in mind recall Definition 9.2.2.1 where the basis is $\mathcal{B}_\omega = \{\bigcup_{i\in n} O_{f_i} : f_i : \omega \to X$ is a partial function with $|\text{dom}(f)| < \omega\})$

There is one phrase in the definition of topology though which we want to be careful of. This the phrase "an arbitrary collection". Given nothing more than the information above we have no way of knowing what exactly an arbitrary collection of open sets looks like. In fact, in order to determine what an arbitrary collection of subsets looks like we need to know what exactly the Powerset(Powerset(X)) is. And, to determine this we need to know what model of set theory we are working in.

As it turns out the universe of set theory we are working in, makes a great deal of difference to the topology generated by a given basis. To see this consider the topological space $\mathbb{R}$.

Definition 9.2.5.2. Let $\mathbb{R}$ be the real numbers and let $\mathcal{B}_{\mathbb{R}} = \{(a,b) : a,b \in \mathbb{Q} \cup \{-\infty, \infty\}\}$. We let the topology on $\mathbb{R}$ be that generated by arbitrary unions and finite intersection of elements of $\mathcal{B}_{\mathbb{R}}$.

We then have the following two theorems which are examples of how dependent the topology of the reals is on the background universe.

Theorem 9.2.5.3. *In L (the constructible universe) there is a Δ^1_2 well ordering of the reals and hence a Σ^1_2 non-Lebesgue measurable set of reals.*

Proof. This is a standard result of set theory. □

Theorem 9.2.5.4 (Solovay)**.** *Suppose that κ is an inaccessible cardinal and G is $Col(\omega, \kappa)$-generic. Then $V[G]$ has an inner model satisfying:*

(a) Every set of reals is Lebesgue measurable.

(b) Every set of reals has the Baire property

(c) Every set of reals has the perect set property.

(d) The Principle of Dependent Choices (DC)

Proof. See [5] Chapter 11 for a discussion. □

The reason the background universe makes such a big difference on the topology (even when the basis is the same) is that the background universe determines what subsets of the basis exists, and hence determine which open sets exists.

Despite the fact that the background universe makes such a difference on the actual topology, there is still something which is absolute between the universes (at least in the case of $\mathbb{R}$) ... the basis. No matter what model of set theory we are in (assuming it agrees with the real universe on ω), the rational numbers are the same. In particular this means that, in some sense, the basis for the topology on the real numbers is the same.

What we would like to do is to consider the pair $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ and look at what topologies are generated in various universes and compare them. The problem that we run into when trying to do this is that while the rationals (and hence the information in $\mathcal{B}_{\mathbb{R}}$) is the same no matter what model of set theory you are in, $\mathbb{R}$ itself is not. So, the structure $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ is not absolute.

To see this notice that if $\mathbb{R}^V \neq \mathbb{R}^W$, $V \subseteq W$, then

$$[\text{the topological space generated by } (\mathbb{R}^V, \mathcal{B}_{\mathbb{R}})]^W$$

is not the same thing as

$$[\text{the topological space generated by } (\mathbb{R}^W, \mathcal{B}_{\mathbb{R}})]^W = \mathbb{R}^W$$

So the only information which is absolute is the basis. We therefore want a way to express this fact. Specifically we want some piece of information which is absolute between models of set theory and such that we can recover our topology from it once we know what universe we are in and the exact set we are working on.

Definition 9.2.5.5. Let C be a lattice. We then say that C, considered as a category, is a Partial Topology. Let X be a set. Then a realization of a partial topology C on X is a faithful functor $C \rightarrow$Powerset(X) which preserves finite inf and finite sup.

The idea is that the lattice C contains all the information about the basis which is absolute. Our partial topology then assigns to each element of the lattice a unique "Basic Open Set" (i.e. an element of a basis for a topology on X) in such a way that finite unions and intersections are preserved.

As we will see in Section 10.2.2, our definition of a partial Grothendieck topology will be the natural generalization of a partial topology.

9.2.6 Admissible Sets

One of the most interesting features of the Suslin-Kleene separation theorem is that it historically was discovered in two very different forms. The Suslin Separation Theorem (Theorem 9.1.1.1) as well as the Kleene Separation theorem

Theorem 9.2.6.1 (Kleene Separation Theorem)**.** *If X, Y are disjoint Σ^1_1 sets of reals then there is a hyperarithmatic set Z such that $Z \supseteq X$ and $Z \cap Y = \emptyset$*

Proof. See [10] Chapter 7B □

Other than the superficial similarity in form, there is no obvious reason why the Suslin Separation Theorem and the Kleene Separation Theorem should be related, let alone should be instances of the same theorem. After all the Suslin Separation Theorem is talking about topological spaces and sets in the Borel and Analytic hierarchies whereas the Kleene separation theorem is talking about sets in the computability hierarchy. The connection between the Suslin Separation Theorem and the Kleene Separation theorem comes by considering very carefully which universe you are proving the theorem in.

9.2.6.1 Definitions

An admissible set can be thought as a small version of the universe of sets. In an admissible set we don't usually have anything near the full power of ZFC, but we do have enough power to do many of the actual calculations on sets that we want to do.

Definition 9.2.6.2. Let $L(\in) = \{\in, R\}$ (where R is a 1-ary predicate and $\in$ is a 2-ary predicate). The theory KPU is the universal closure of the following axioms in $L(\in)$

Extensionality: $(\forall x)(x \in a \leftrightarrow x \in b) \rightarrow a = b$

Foundation: $(\exists x)\varphi(x) \rightarrow (\exists x)[\varphi(x) \wedge \forall y \in x \neg\varphi(y)]$ for all $\varphi \in L(\in)$ in which y does not occur free.

Pair: $(\exists a)(x \in a \wedge y \in a)$

Union: $(\exists b)(\forall y \in a)(\forall x \in y)(x \in b)$

Δ_0 Separation: $(\exists b)(\forall x)(x \in b \leftrightarrow x \in a \wedge \varphi(x))$ for all Δ_0 formulas in which b does not occur free.

Δ_0 Collection: $(\forall x \in a)(\exists y)\varphi(x, y) \rightarrow (\exists b)(\forall x \in a)(\exists y \in b)\varphi(x, y)$ for all Δ_0 formulas in which b does not occur free.

We say a structure (A, ϵ) is an Admissible Structure if $(A, \epsilon) \models KPU$. We say a set A is an Admissible Set if A is transitive and $(A, \in)$ is an Admissible Structure.

Admissible sets have been studied in great detail over the years. For more information see [2] or [7].

Definition 9.2.6.3. Let $(A, \in)$ be an admissible set. If $b \in A$ we say that b is A-Finite. If $B \subseteq A$ is Δ_1 definable over A in $L(\in)$ we say that B is A-Recursive. If $B \subseteq A$ is Σ_1 definable over A in $L(\in)$ we say that B is A-Recursively Enumerable.

9.2.6.2 The Relationship between Suslin's and Kleene's Separation Theorems

Before we compare Suslin's and Kleene's separation theorems we need a definition.

Definition 9.2.6.4. Let $W(X, Y)$ =Lattice Generated by $\langle (X \times Y)^{<\omega}, \leq \rangle$ with $s \leq t$ if s is an initial segment of t. Then define $\text{Code}(\alpha, X, Y)$ as follows.

- $\text{Code}(0, X, Y) = \text{obj}(W(X, Y))$
- $\text{Code}(\omega * \alpha, X, Y) = \bigcup_{\lambda < \omega * \alpha} \text{Code}(\lambda, X, Y)$
- $\text{Code}(\beta + 1, X, Y)$ $=\text{Code}(\beta, X, Y) \cup \{(\neg, a) : a \in \text{Code}(\beta, X, Y)\} \cup \{(\cup, a) : a \subseteq \text{Code}(\beta, X, Y), |a| = \omega\}$

As we have already seen (see Section 9.2.5) the actual Borel sets in ω^ω depend heavily on actual value of ω^ω (and hence the set theoretic universe we are working in). However, the actual informal contained in the Borel sets (as with the closed sets) is in some sense more absolute than the sets themselves. This is what $\text{Code}(\alpha, X, Y)$ is meant to define. Intuitively we want to think of an element of $\text{Code}(\alpha, X, Y)$ as procedure for constructing a Borel sets of rank

α which is independent of the actual set values of $(\omega^\omega)^V$ (where V is our set theoretic universe.)

Now we can see how both the Suslin Separation Theorem and the Kleene Separation theorem are really special cases of the following

Theorem 9.2.6.5. *Let A be an admissible set containing $W(X,Y)$. Then there is an A-recursively enumerable function such that*

- $B : Code(1,X,Y) \cap A \times Code(1,X,Y) \cap A \to Code(\omega_1,X,Y) \cap A$
- *The domain of B consists of those codes for closed sets whose projections to X are disjoint.*
- *$B(a,b)$ is the code for a Borel set separating the projection of the set coded by a from the projection of the set coded by b.*

Proof. This is immediate from the classical proofs (See [10] Chapter 7B) □

We can now see how both the Kleene Separation Theorem and the Suslin Separation Theorem are special cases of Theorem 9.2.6.5. In the case of the Suslin Theorem we are letting the admissible set be our model V of ZFC. In this case we know that any such Borel Set must actually be in V. In the case of the Kleene Separation theorem we are working in the admissible set $L(\omega_1^{CK})$ and so we know that the code for the Borel set must be in $L(\omega_1^{CK})$ and hence must be constructible in a "computable way".

The proof of this theorem is then exactly the proof of either the Suslin Separation Theorem or the proof of the Kleene Separation Theorem except relativized to the appropriate set.

In fact this presentation of the Suslin-Kleene Separation theorem makes it obvious why, by the following theorem, the Kleene-Separation theorem relativizes.

Theorem 9.2.6.6. *$L(\omega_1^T)$ is admissible for all $T \subseteq \omega$. Further if $L(\alpha)$ is admissible and $\alpha > \omega$ then there is some T such that $\alpha = \omega_1^T$*

Proof. See [11] □

9.2.6.3 Admissible Categories

In a similar way to how the Suslin-Kleene separation theorem can be proved in an admissible set our proof will also be able to be done in an admissible set. As such we need to be able to define what it means for a category and for a presheaf to be on an admissible set.

Definition 9.2.6.7. Let A be an admissible set. A category $\mathcal{C}$ is an A -Admissible Category if

- $\text{obj}(\mathcal{C}), \text{morph}(\mathcal{C})$ are A-recursively enumerable sets
- $\text{dom}, \text{range} : \text{morph}(\mathcal{C}) \to \text{obj}(\mathcal{C})$ are A-recursively enumerable functions.
- Identity: $\text{obj}(\mathcal{C}) \to \text{morph}(\mathcal{C})$ is A-recursively enumerable
- $\circ : \text{morph}(\mathcal{C}) \times \text{morph}(\mathcal{C}) \to \text{morph}(\mathcal{C})$ is an A-recursively enumerable

Intuitively a category is A-admissible if it is a subset of A and everything we are interested in is A-recursively enumerable.

Definition 9.2.6.8. Let A be an admissible set, $\mathcal{C}$ an A -admissible category. $(\mathcal{C}, J)$ is an A-admissible partial presite if J is A recursively enumerable (and so in particular $J(C)$ is A-finite for all $C \in \text{obj}(\mathcal{C})$.

Definition 9.2.6.9. Let A be an admissible set, $\mathcal{C}$ an A-admissible category. A presheaf P is A-admissible if

- IN$\subseteq A \times \text{obj}(\mathcal{C})$ is A-recursively enumerable where IN$= \{(x, C) : x \in P(C)\}$

- EXTENSION: $A \times \text{morph}(\mathcal{C}) \to A$ is A-recursively enumerable where EXTENSION$(x, f) = y$ if $x \in \text{cod}(f)$ then $y = \{z \in \text{dom}(f) : z|f = x\}$ and if $x \notin \text{cod}(f)$ then EXTENSION(x, f) is undefined

Intuitively a presheaf is A-admissible if it is a subset of A and it is possible to calculate in an A-recursively enumerable way what the preimage of an element under a function is. In particular this means that given any element $x \in P(C)$ and any $f : C \to D$ we have $\{y : y|f = x\}$ is A-finite.

We will see that when we are working in an A-admissible category with A-admissible presheaves that in fact our methods of induction give us a function which is A-recursively enumerable.

Chapter 10

Partial Sites and Induction

10.1 Introduction

In this chapter we will not only introduce the idea of a partial Grothendieck topology but also our method of generalized induction. Partial Grothendieck topologies are intended to contain just enough information to allow our proof to go through. Further, any partial Grothendieck topology will generate a full Grothendieck topology in a way very similar to how a normal topology on a set can be generated from the basis of the topology.

In Section 10.2 we will give all the necessary definitions of a partial Grothendieck topology and discuss their relationships with full Grothendieck topologies and with admissible sets.

In Section 10.3 we will then prove some basic results concerning partial Grothendieck topologies which we will use in the proof of our separation theorem.

In Section 10.4 we will then introduce our generalization of induction to sheaves on a partial site and discuss how it relates to admissible sets.

10.2 Definitions

10.2.1 Grothendieck Topology

First we need to recall the definition of a Grothendieck topology.

Definition 10.2.1.1. Let $\mathcal{C}$ be a category. We say that a Sieve on an object $C \in \text{obj}(\mathcal{C})$ is a subfunctor of $Hom(-, C)$. In other words S is a sieve on C if $S \subseteq Hom(-, C)$ and $(\forall g \in \text{morph}(\mathcal{C}))(\forall f \in S)\text{cod}(g) = \text{dom}(f) \Rightarrow f \circ g \in S$ If S is a sieve on C and $h : D \to C$ then $h^*(S) = \{g : \text{cod}(g) = D \wedge h \circ g \in S\}$ is a sieve on D.

A sieve on C is a set of morphisms with codomain C which acts like a right ideal in algebra under $\circ$. If we let our category T be a topological space then a sieve S on C is just a collection of open subsets such that if $U \in S$ and $V \subseteq U$ then $V \in S$.

Definition 10.2.1.2. A Grothendieck Topology on a category $\mathcal{C}$ is a function J which assigns to each $C \in \text{obj}(\mathcal{C})$ a collection of sieves on C in such a way that

(i) (Identity) The maximal sieve $t_C = \{f : cod(f) = C\} \in J(C)$

(ii) (Stability Axiom) If $S \in J(C)$ and $h : D \to C$ then $h^*(S) \in J(D)$

(iii) (Transitivity Axiom) If $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C \in S$ then $R \in J(C)$.

We say that $(\mathcal{C}, J)$ is a Site.

To better see the connection with topologies (and partial topologies) it is often useful to consider a Grothendieck pretopology.

Definition 10.2.1.3. A Grothendieck Pretopology on a category $\mathcal{C}$ with pullbacks is a function K which assigns to each $C \in \text{obj}(C)$ a collection of Covering Families of morphisms with codomain C in such a way that

(i') (Isomorphism) If $f : C' \to C$ is an isomorphism then $\{f\} \in K(C)$

(ii') (Stability Under Base Change) If $\{g : C_\alpha \to C : \alpha \in I\} \in K(C)$ and $h : D \to C$ then $\{f_\alpha : C_\alpha \times_C D \to D\} \in K(D)$

(iii') (Stability Under Composition) If $\{f_\alpha : C_\alpha \to C : \alpha \in I\} \in K(C)$ and $\{g_{\alpha,\beta} : D_{\alpha,\beta} \to C_\alpha : \beta \in I_\alpha\} \in K(C_\alpha)$ then $\{f_\alpha \circ g_{\alpha,\beta} : D_{\alpha,\beta} \to C_\alpha \to C : \alpha \in I, \beta \in I_\alpha\} \in K(C)$

We say $(\mathcal{C}, K)$ is a Presite.

Theorem 10.2.1.4. *Let K be a Grothendieck pretopology on a category $\mathcal{C}$ with pullbacks. If J is defined as*

$$S \in J(C) \Leftrightarrow (\exists R \in K(C))R \subseteq S$$

then J is a Grothendieck topology on $\mathcal{C}$. Further, every Grothendieck topology arises in this way from a Grothendieck pretopology (although sometimes the same Grothendieck topology can arise from different Grothendieck pretopologies).

Proof. See [9] Chapter III §2 □

Theorem 10.2.1.5. *Let $\langle S_i : i \in I\rangle$ be sieves on $C \in obj(\mathcal{C})$. Then $\bigcap_{i\in I} S_i$ is a sieve.*

Proof. Let $f : D \to C \in \bigcap_{i\in I} S_i$ and $g : E \to D \in \text{morph}(\mathcal{C})$. Then we know $f \circ g \in \bigcap_{i\in I} S_i$ because each S_i is a sieve. □

Definition 10.2.1.6. Let $F \subseteq Hom(-, C)$. We define the Sieve generated by F to be $\bigcap\{S : S$ is a sieve and $F \subseteq S\}$

As is often useful when considering Grothendieck topologies, lets look at what these axioms are saying in the case of standard topological spaces. Intuitively a covering family $\{U_i \to U : i \in I\}$ for an open set U is meant to represent a collection of open subsets of U such that $U = \bigcup_{i\in I} U_i$.

Under this interpretation we see that the (Isomorphism) condition in Definition 10.2.1.3 says that for any set, the set itself is a cover. (Stability Under Base Change) on the other hand says that if we have a covering family $\{U_i \subseteq U : i \in I\}$ of U and $V \subseteq U$ then $\{U_i \cap V \subseteq U \cap V : i \in I\}$ is a covering family for V. If we consider a covering family (loosely) as a generalized open set then this looks like the condition on a topology which says that open sets are closed under intersection.

The condition of (Stability Under Composition) is the most interesting of the three conditions though (for our purposes). This says that if we have a cover $\{U_\alpha \subseteq U : \alpha \in I\}$ of U and for each α we have covers $\{V_{\alpha,\beta} \subseteq U_\alpha : \beta \in I_\alpha\}$ then in fact $\{V_{\alpha,\beta} \subseteq U : \alpha \in I, \beta \in I_\alpha\}$ is a cover of U. Once again, considering a covering family as (loosely) a generalized open set, this condition is saying that the union of generalized open sets is a generalized open set.

10.2.2 Partial Grothendieck Topology

As with classical topology hidden in the statement that an arbitrary union of (generalized) opens sets is a (generalized) open set lurks implicitly the existance of a fixed universe of set theory. Further, as in the case of classical topologies, if we were to change the background universe we would change the sets of morphisms which are forced to be covering families.

Further, in the same way as the partial topologies of Section 9.2.5 were meant to allow us to talk about the part of a topological space which was "independent" from the model of set theory we are working in, the following idea of a partial Grothendieck topology is meant to allow us to talk about that part which is independent of the model of set theory we are in. We will do this in a very similar way to how we defined partial topology. We will simply restrict the unions of open sets which we we require to be open to be finite unions (hence absolute in all models of set theory which agree on ω)

Definition 10.2.2.1. A Partial Grothendieck topology on a category $\mathcal{C}$ is a function J which assigns to each $C \in \mathrm{obj}(\mathcal{C})$ a collection of sieves on C in such a way that

(i) (Identity) The maximal sieve $t_C = \{f : cod(f) = C\} \in J(C)$

(ii) (Stability Axiom) If $S \in J(C)$ and $h : D \to C$ then $h^*(S) \in J(D)$

(iii) (Finite Transitivity Axiom) If $S \in J(C)$ is a sieve such that every element factors through one of $\{f_i : i \in n < \omega\}$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C \in S$ then $R \in J(C)$.

We say that $(\mathcal{C}, J)$ is a Partial Site.

Definition 10.2.2.2. A Partial Grothendieck Pretopology on a category $\mathcal{C}$ is a function K which assigns to each $C \in \text{obj}(\mathcal{C})$ a collection of Covering Families of morphisms with codomain C in such a way that

(i') (Isomorphism) If $f : C' \to C$ is an isomorphism then $\{f\} \in K(C)$

(ii') (Stability Under Base Change) If $\{f_\beta : C_\beta \to C : \beta \in I\} \in K(C)$ and $h : D \to C$ then there is a cover $\{g_\alpha : D_\alpha \to D : \alpha \in I'\} \in K(D)$ such that each $h \circ g_\alpha$ factors through some f_β.

(iii') (Finite Stability Under Composition) If $\{f_\alpha : C_\alpha \to C : \alpha \in n\} \in K(C)$, $n \in \omega$ and $\{g_{\alpha,\beta} : D_{\alpha,\beta} \to C_\alpha : \beta \in I_\alpha\} \in K(C_\alpha)$ then $\{f_\alpha \circ g_{\alpha,\beta} : D_{\alpha,\beta} \to C_\alpha \to C : \alpha \in I, \beta \in I_\alpha\} \in K(C)$

We say that $(\mathcal{C}, J)$ is a Partial Presite.

Theorem 10.2.2.3. *Let K be a partial Grothendieck pretopology on $\mathcal{C}$. If J is defined as*

$$S \in J(C) \Leftrightarrow (\exists R \in K(C))R \subseteq S$$

then J is a partial Grothendieck topology on $\mathcal{C}$. Further, every partial Grothendieck topology arises in this way from a partial Grothendieck pretopology (although sometimes the same partial Grothendieck topology can arise from different partial Grothendieck pretopologies).

Proof. This is exactly the same proof as Theorem 10.2.1.4 (See [9] Chapter III §2) □

This theorem says that the relationship between partial Grothendieck pretopologies and partial Grothendieck topologies is the same as the relationship between Grothendieck pretopologies and Grothendieck topologies.

This is something we would want to be true of any reasonable generalization of these concepts.

There is one difference in the definition of a partial Grothendieck topology and a Grothendieck topology worth mentioning explicitly. Our condition (Stability Under Base Change) for partial Grothendieck pretopologies is not quite the same as the condition we gave for Grothendieck topologies. This is because in the theorems we will be proving we do not want to assume that our categories have pullbacks. As such we have had to explicitly assume there are maps which "look like" pullbacks for the purpose of the axiom. This does not in any way change the relationships we are studying (it just allows us to look at a wider collection of categories). Further if we were to replace the (Stability Under Base Change) condition in the definition of Grothendieck pretopology with the (Stability Under Base Change) condition in the definition of partial Grothendieck pretopology all the previously mentioned results would still go through.

10.2.3 Sheaves

Now that we know what a partial site is we need to define what a sheaf on such a site is. First though we need some notation.

Definition 10.2.3.1. Let $\mathcal{C}$ be a category. We say that Q is a presheaf on $\mathcal{C}$ if $Q \in SET^{\mathcal{C}^{op}}$.

Definition 10.2.3.2. Let Q be a presheaf on the category $\mathcal{C}$. If $x \in Q(U)$ we say $\text{dom}(x) = U$. If $x \in Q(U), y \in Q(V), f : U \to V$ we say $y|f = x$ if $x = Q(f)(y)$.

Notice that we can assume that $Q(U) \cap Q(V) = \emptyset$ for $U \neq V$ (and hence dom$(\cdot)$ is well defined). This is because we can construct a sheaf Q^* where $Q^*(U) = \{(x, U) : x \in Q(U)\}$ with the obvious values on morphisms. And, Q^* is "essentially the same" as Q (for our purposes) (i.e. they are isomorphic, ect.)

We will now return to the definition of sheaf. The most important point to notice is that the definition of a sheaf on a full Grothendieck topology only refers to the covering sieves which exists and nothing about their relationships. Hence the definition should generalize to any collection of sieves.

Definition 10.2.3.3. Let $\mathcal{C}$ be a category and let $S(C)$ be a collection of sieves on $C \in \text{obj}(\mathcal{C})$. Let P be a presheaf on $\mathcal{C}$ ($P \in SET^{\mathcal{C}^{op}}$). We say P is a sheaf for S if for each $\alpha \in S(C)$ $P(C)$ is an equalizer in the following diagram

$$P(C) \xrightarrow{\;e\;} \prod_{f \,\in\, \alpha} P(\text{dom}(f)) \underset{a}{\overset{p}{\rightrightarrows}} \prod_{\substack{f,\,g \\ f \,\in\, \alpha \\ \text{dom}(f) \,=\, \text{cod}(g)}} P(\text{dom}(g))$$

where

- $e(x) = \langle P(f)(x) : f \in \alpha \rangle$
- $p(\langle x_f : f \in \alpha \rangle) = \langle x_{f \circ g} : \text{cod}(g) = \text{dom}(f), f \in \alpha \rangle$
- $a(\langle x_f : f \in \alpha \rangle) = \langle P(g)(x_f) : \text{cod}(g) = \text{dom}(f), f \in \alpha \rangle$

Lets go through exactly what this definition is saying. Suppose $\alpha \in S(C)$ and we have a collection of $\langle x_f : f \in \alpha \rangle$ such that

- $x_f \in P(\text{dom}(f))$

- For all $g \in \text{morph}(C)$, $f \in \alpha$, $x_f|g = x_{f \circ g}$

(we call such a collection a Compatible collection for a covering sieve α). Then in fact there is a unique element $x \in P(C)$ such that $P(f)(x) = x_f$.

Lets consider the case when our category comes from a topological space. In this case we know that a sieve α on an open set U is just a cover of U where $V' \subseteq V \in \alpha \rightarrow V' \in \alpha$. In this case a compatible collection is a set $\{x_V : V \in \alpha\}$ such that if $V, W \in \alpha$ then $x_V|V \cap W = x_W|V \cap W$. Hence a presheaf P is a sheaf if every compatible collection of elements which agree on the intersection of their domain comes from a unique element. I.e. can be "glued" together in a unique way.

10.3 Basic Results

In this section we will prove some basic results concerning partial sites which will useful later on.

10.3.1 Topology

Theorem 10.3.1.1. *Let $\mathcal{T}$ be a partial topology. Then $(\mathcal{C}, T)$ is a partial presite where $\{U_i \rightarrow U : i \in I\} \in T(U)$ if sup$\{U_i : i \in I\} = U$*

Proof. This is immediate from the definition of partial topology. □

10.3.2 Separation Lemmas

As with the usual proof of the Suslin-Kleene theorem our proof will rely on the following lemma (and it's generalization to the case of presheaves).

Lemma 10.3.2.1. *Let* $A = \bigcup_{i\in I} A_i$*,* $B = \bigcup_{j\in J} B_j$*, where* $A, B \subseteq X$ *are sets. If* $C_{i,j}$ *separates* A_i *from* B_j *then* $\bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}$ *separates* A *from* B*.*

Proof. Notice that for each i, j, $A_i \subseteq C_{i,j}$ and so $A_i \subseteq \bigcap_{j\in J} C_{i,j}$ and hence $A \subseteq \bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}$.

On the other hand, for each i, j $B_j \subseteq X - C_{i,j}$ and so $B \subseteq \bigcup_{j\in J} X - C_{i,j}$ for each i. So $B \subseteq \bigcap_{i\in I}\bigcup_{j\in J} X - C_{i,j}$ and so $B \cap \bigcup_{i\in I}\bigcap_{j\in J} C_{i,j} = \emptyset$ □

The version for presheaves will follow almost immediately from Lemma 10.3.2.1 by virtue of the fact that intersections and unions on presheaves are calculated componentwise. But first we have to define what it means for one presheaf to separate another two.

Definition 10.3.2.2. Let $\mathcal{C}$ be a category and let $U \in \mathrm{obj}(\mathcal{C})$. Let P, Q be presheaves on $\mathcal{C}$ such that

$$P(U) \cap Q(U) = \emptyset$$

We say a presheaf R on $\mathcal{C}$ separates P from Q up to U-elements if $P \rightarrowtail R$ and $Q(U) \cap R(U) = \emptyset$.

Lemma 10.3.2.3. *Let* $A = \bigcup_{i\in I} A_i$*,* $B = \bigcup_{j\in J} B_j$*, where* A, B, A_i, B_j *are presheaves on a category* $\mathcal{C}$*. If* C_i, j *separates* A_i *from* B_j *up to* U*-elements then* $\bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}$ *separates* A *from* B *up to* U*-elements.*

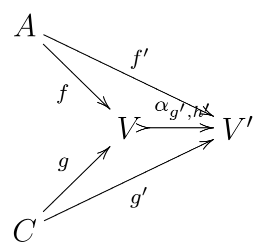
Proof. First off notice that in the category of presheaves limits/colimits are computed point wise (see [9]). So, unions and intersections are also calculated pointwise.

Now notice that $A(U) \subseteq [\bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}](U)$ and $[B \cap \bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}](U) = \emptyset$ by Lemma **??**. □

One more thing worth mentioning explicitly is that this separation theorem is using $\bigcup/\bigcap$ in the context of presheaves and not sheaves. Hence even if all $C_{i,j}$ are sheaves we will not in general have $\bigcup_{i\in I}\bigcap_{j\in J} C_{i,j}$ is a sheaf (at least not in the context we are using it).

10.3.3 Qusai-Supremums

Definition 10.3.3.1. Let $(\mathcal{C}, J)$ be a partial presite. We say that $(\mathcal{C}, J)$ has quasi-supreumums if for all $A, B \in \text{obj}(\mathcal{C})$ there is $f : A \to V, g : B \to V$ such that $\{f, g\} \in J(V)$ and if $f' : A \to V', g' : B \to V'$ with $\{f', g'\} \in J(V')$ then there is a (not necessarily unique) monic $\alpha_{f',g'} : V \rightarrowtail V'$ such that



commutes.

One of the nice features of a topological space when dealing with sheaves is that for any two open sets U, V there is a set $U \cup V$ such that $\{U, V\}$ is a cover of $U \cup V$ and for any W where $U \subseteq W$ and $V \subseteq W$ we know $U \cup V \subseteq W$. In particular if P is a sheaf and $x \in P(U), y \in P(V)$ and we want to know if there is an extension of both x and y in P (i.e. a W, $z \in P(W)$ such that $z|U = x$ and $z|V = y$) then we only have to look at one set ($U \cup V$).

Quasi-supreumums are supposed to be a generalization of this idea (and trivially any partial presite on a partial order with finite supremums also has quasi-supremums).

Specifically we have the following.

Definition 10.3.3.2. Let $(\mathcal{C}, J)$ be a partial presite with quasi-supremums. Let Q be a sheaf on $(\mathcal{C}, J)$ and let $f \in Q(W)$. Define

$$\begin{aligned} Q_f(U) = &\{s \in Q(U) : (\exists g : \mathrm{dom}(f) \to V, h : U \to V)(\exists a \in Q(V)) \\ &\{g, h\} \in J(V), a|h = s \text{ and } a|g = f\} \end{aligned}$$

Note that here $Q_\emptyset = Q$

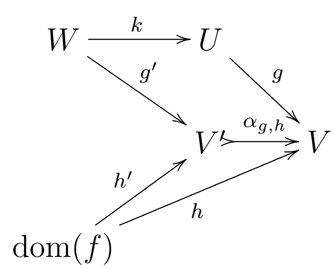
Theorem 10.3.3.3. *Let $(\mathcal{C}, J)$ be a partial presite with quasi-supremums. If Q is a sheaf on $(\mathcal{C}, J)$ then so is Q_f, and $Q_f \rightarrowtail Q$.*

Proof.

Claim 10.3.3.4. *Q_f is a presheaf on $\mathcal{C}$.*

Proof. Let $x \in Q_f(U)$ and let $g : U \to V$, $h : \mathrm{dom}(f) \to V$ be such that $\{g, h\} \in J(V)$ and there is $a \in Q(V)$ such that $x = a|g$ and $f = a|h$.

Let $k : W \to U$ and $g' : W \to V'$, $h : \mathrm{dom}(f) \to V'$ be such that $\{g', h'\} \in J(V')$ and $\alpha_{g,h} : V' \to V$ and the following diagram commutes



So we know $x|k = (a|g \circ k) = a|(\alpha_{g,h} \circ g') = (a|\alpha_{g,h})|g'$. But we also have $f = a|h = a|(\alpha_{g,h} \circ h') = (a|\alpha_{g,h})|h'$. In particular we then have that $a|\alpha_{g,h}$ witnesses that $x|k \in Q_f(W)$. So Q_f is a presheaf. $\square$

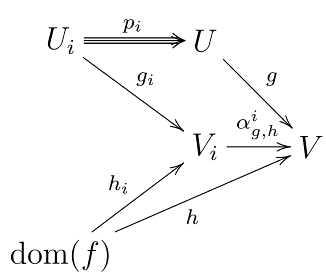
Claim 10.3.3.5. *$Q_f \rightarrowtail Q$.*

Proof. This is immediate from the definition and the fact that for presheaves $A \rightarrowtail B$ iff $A(U) \subseteq B(U)$ for all U (see [9]). □

Claim 10.3.3.6. *Q_f is a sheaf on $(\mathcal{C}, J)$.*

Proof. Let $\langle p_i : U_i \to U \text{ s.t. } i \in \kappa \rangle = S$ be a covering sieve of U and let $x_{p_i} \in Q_f(U_i)$ such that $x_{p_i}|k = x_{p_i \circ k}$ if $\mathrm{cod}(k) = \mathrm{dom}(p_i)$. (i.e. $\langle x_{p_i} : i \in \kappa \rangle$ is a compatible sequence of elements for the covering sieve S). We know that there is a unique element $y \in Q(U)$ such that $y|p_i = x_{p_i}$ because Q is a sheaf.

Let $g : U \to V$, $h : \mathrm{dom}(f) \to V$ be such that $\{g, h\} \in J(V)$. Further let $g_i : U_i \to V_i$, $h_i : \mathrm{dom}(f) \to V_i$ be such that $\{g_i, h_i\} \in J(V_i)$ and such that $(\forall i \in \kappa)(\exists a_i \in Q(V_i)) a_i|g_i = x_{p_i} \wedge a_i|h_i = f$. We know that these must exist by the definition of Q_f. In particular we have the following diagram.



Let $S' = Hom(-, \mathrm{dom}(f))$ be the trivial covering sieve of $\mathrm{dom}(f)$. We therefore have (by (Finite Stability Under Composition) of Definition 10.2.2.2) that $T = g(S) \cup h(S') = \{g \circ m : m \in S\} \cup \{h \circ m' : m' \in S'\}$ is a covering sieve of V.

Next let $R = \{q \in Hom(-, V) : q \text{ factors through } \alpha^i_{g,h} \circ h_i \text{ or } \alpha^i_{g,h} \circ g_i \text{ for some } i \in \kappa\} = \{q \in Hom(-, V) : q \text{ factors through } h \text{ or through } g \circ p_i \text{ for some } i \in \kappa\}$.

Claim 10.3.3.7. *$\langle y|p_i : i \in \kappa\rangle \cup \langle f|m : m \in S'\rangle$ is a compatible sequence of elements for the covering sieve T.*

Proof. All we need to show is that if $g \circ p_i = h \circ m$ then $f|m = x_{p_i}$ (because $y|p_i = x_{p_i}$). But $\alpha^i_{g,h} \circ g_i = g \circ p_i$ and $h \circ m = \alpha^i_{g,h} \circ h_i \circ m$. So in particular, because $\alpha^i_{g,h}$ is monic, we know $g_i = h_i \circ m$. Therefore $y|p_i = x_{p_i} = a|g_i = a|(h_i \circ m) = (a|h_i)|m = f|m$ □

So we know that $\langle y|p_i : i \in \kappa\rangle \cup \langle f|m : m \in S'\rangle$ come from a single element $z \in Q(V)$ such that $z|g = y$ and $z|h = f$. Hence z witnesses that $y \in Q_f(U)$.

Hence Q_f is a sheaf. □

□

As we will see later these particular sheaves (of the form Q_f for a sheaf Q) play an important role in our construction. It is because of this fact that our proof only works for partial sites with quasi-supremums.

Theorem 10.3.3.8. *Let $\mathcal{T}$ be a partial order with pairwise supremums. Then $\mathcal{T}$ has quasi-supremums.*

Proof. This is immediate because supremums are obviously quasi-supreumum.

□

Corollary 10.3.3.9. *Let $\mathcal{T}$ be a partial topology. Then $\mathcal{T}$ has quasi-supremums.*

Proof. This is immediate by the definition of a partial topology and Theorem 10.3.3.8. □

Theorem 10.3.3.10. *Let Q, P be sheaves on a partial site $(\mathcal{C}, K)$ with $Q \rightarrowtail P$. If $f \notin Q(dom(f))$ then $Q_f(V) = \emptyset$ for all $V \in obj(\mathcal{C})$.*

Proof. Assume there exists $z \in Q_f(\text{dom}(z))$.

Then $(\exists g : \text{dom}(z) \to V, h : \text{dom}(f) \to V)(\exists a \in Q(V))a|g = z \wedge a|h = f$.

But then $f \in Q(\text{dom}(f))$ as Q is a presheaf $\Rightarrow\Leftarrow$.

□

10.4 Induction

10.4.1 Motivation

Before we continue it is worth talking about how we will generalize the proof of the traditional Suslin-Kleene theorem. In the classical proof we start with two pruned trees A, B on $\omega^\omega \times \kappa^\omega$ and we want to separate the projections onto ω^ω of $[A]$ and $[B]$ when the projections don't intersect. In the classical proof (as in [10] Chapter 2E) we do this by finding a specific tree which is definable from the we construct a tree J in terms of the trees A and B in such a way that J is well-founded if the projections of $[A]$ and $[B]$ don't intersect. Then, once we have this tree we define by transfinite induction on J the Borel set we want to separate the projections.

We would like to do something similar for our separation theorem. However, there is one obvious problem that we will have to deal with first. In general our objects will be sheaves and not trees. As such the classical method of getting a well-founded tree from trees whose projections don't intersect just doesn't work. And, without a wellfounded tree we can't use transfinite induction to construct the Borel presheaves. So, the first thing we will have to do is to find a way to generalize the method of transfinite induction so that it allows us to deal with sheaves instead of trees.

Recall from Section 9.2.1 the way definition by transfinite induction works. We start with some tree W which has no infinite path through it. Then for each node in our tree we have some method for assigning a value to that node based on the values assigned to the nodes which extend it. We then know that if this procedure does not assign a value to a given node then we can find some extension of that node where a value wasn't assigned. We can then repeat this procedure and get an infinite path through the tree. But we know (by assumption) that there are no infinite paths through the tree. Hence our procedure must have assigned a value to each node.

Now recall from Section 9.2.4 our analogy between sheaves and trees. Specifically we found that trees turn out to be just (particular) sheaves on the topological space $\overline{\omega}$. Under this analogy a well founded tree T is just a sheaf such that $T(\omega) = \emptyset$. Now lets go through the definition of transfinite induction again but this time from the vantage point of sheaves on $\overline{\omega}$.

Suppose we have a sheaf T on $\overline{\omega}$ such that $T(\omega) = \emptyset$. We want to assign some value to each element of $T(n)$ for each $n \in \text{obj}(\overline{\omega})$. Further, given any $x \in T(n)$ we want to the value of x to be based on the values assigned to the elements of $\{y \in T(m) : y|n = x\}$. Now suppose there is a point $x_0 \in T(n_0)$ that this procedure doesn't assign a value to. Then we know that for some $n_1 \in \overline{\omega}$ (with $n_0 \subseteq n_1$) there must be a $y \in T(n_1)$ such that $y|n_1 = x_0$ and this procedure doesn't assign a value to y. Lets pick one such and call it x_1. Then, in a similar manner, we can find an $x_2 \in T(n_2)$ where $x_2|n_1 = x_1$.

So, we have produced a collection of elements $\mathbf{x} = \langle x_i : i \in T(n_i)\rangle$ where $x_i|n_i \cap n_j = x_j|n_i \cap n_j$. But we also know that $\{n_i : i \in \omega\}$ must be a cover of ω (in the site $\overline{\omega}$). So, $\mathbf{x}$ is a compatible collection of elements for a cover of ω. Hence, because T is a sheaf, there must be a unique element $x^* \in T(\omega)$ such that $x^*|n_i = x_i$. But we assumed that $T(\omega) = \emptyset$. $\Rightarrow\Leftarrow$

So our original assumption that there was an element that this procedure didn't assign a value to must be false.

10.4.2 Induction for Sheaves on a Topological Space

Notice that in the case of $\overline{\omega}$ the key idea which allows the induction it to work is we are able to construct a compatible sequence of elements and hence we know that there must be an amalgamation. In the case of a general sheaf on a topological space we are going to try and do something similar. The procedure will go as follows

10.4.2.1 General Definition

Let $(T, \mathcal{O})$ be a topological space and let $\langle U_i : i \in \kappa \rangle$ be a disjoint cover of T. Now let B, D be sheaves on T such that $B(T) = \emptyset$ and $B \rightarrowtail D$. Suppose we want to assign a value $I(x)$ to each element $x \in D(U)$ with $U \subseteq T$ open. Suppose further that there are functions G, F_i such that G assigns a value to every $x \in D(\text{dom}(x)) - B(\text{dom}(x))$ and F_i assigns a value to x assuming that I has been defined on all extensions of x in $B(\text{dom}(x) \cup U_i)$. Further suppose we want to define I so that $I(x) = G(x)$ if $x \in D(\text{dom}(x)) - B(\text{dom}(x))$, and $I(x) = F_i(x)$ for some $i \in \kappa$ otherwise.

To do this we will define I in stages.

- $I_\emptyset(x) = G(x)$ if $x \in D(\text{dom}(x)) - B(\text{dom}(x))$ and undefined otherwise.
- $I_{\omega * \alpha} = \bigcup_{\gamma < \omega * \alpha} I_\gamma$

- $I_{\beta+1}(x) =$
  - $I_\beta(x)$ if $I_\beta(x)$ is defined.
  - If i is least such that I_β is defined everywhere on $\{y \in B(\text{dom}(x) \cup U_i) : y|\text{dom}(x) = x\}$ then $I_{\beta+1}(x) = F_i(x)$.
  - $I_{\beta+1}(x)$ is undefined if no such i exists

We then define $I(x) = I_\alpha(x)$ if $(\exists\alpha)I_\alpha(x)$ is defined and we say $I(x)$ is undefined otherwise.

10.4.2.2 I is Defined Everywhere

Claim 10.4.2.1. *$I(x)$ is defined on all $x \in D(dom(x))$.*

Proof. Assume $I(x)$ is not defined for some $x \in D(\text{dom}(x))$.

The first thing to notice is that $x \in B(\text{dom}(x))$ because if $x \in D(\text{dom}(x)) - B(\text{dom}(x))$ then $I(x) = G(x)$. Next notice that we also have for each $i \in \kappa$ there is some $y_i \in B(\text{dom}(x) \cup U_i)$ such that $y_i|\text{dom}(x) = x$ and $I(y_i)$ is undefined.

But we know that $(U_i \cup \text{dom}(x)) \cap (U_j \cup \text{dom}(x)) = \text{dom}(x)$ (if $i \neq j$) and $\bigcup_{i\in\kappa} \text{dom}(y_i) = \bigcup_{i\in\kappa}(U_i \cup \text{dom}(x)) = T$, because $\langle U_i : i \in \kappa\rangle$ is a disjoint cover of T. Hence $y_i|\text{dom}(y_i) \cap \text{dom}(y_j) = y_i|\text{dom}(x) = x = y_j|\text{dom}(x) = y_j|\text{dom}(y_i) \cap \text{dom}(y_j)$. So $\langle y_i : i \in \kappa\rangle$ is a compatible sequence of elements for the cover $\langle U_i \cup \text{dom}(x) : i \in \kappa\rangle$ of T. In particular we know that there must be a unique $y \in B(T)$ such that $y|\text{dom}(y_i) = y_i$ because B is a sheaf. But $B(T) = \emptyset$. $\Rightarrow\Leftarrow$

So I is defined on all $x \in D(\text{dom}(x))$. □

Lets go through what is going on here. Just like in classical induction we want to calculate the value we assign to an element in terms of the elements which extend it. But, unlike when we are working with trees, in a sheaf on a topological space there are several different directions in which we can extend any given element. So, we have to decide which ones we care about.

The way that this induction is set up, if it fails at a point we know it must also fail at an extension of that point in each direction we are considering.

In other words if x isn't assigned a value, but it is possible to assign a value to x once values have been assigned for all extensions of x at $U \cup \mathrm{dom}(x)$, then we know there must be an extension of x at $U \cup \mathrm{dom}(x)$ which isn't assigned a value either.

However, while we know there is some such point we have no control over what that point is (other than it extends x). This is why this induction only works if we consider disjoint covers. Specifically what we have done is allow the value assigned to x to be defined in terms of the values of x's extensions on a whole bunch of different open sets. In this way, if x isn't assigned a value, for each open set we consider we will get a single element extending x. But, we have chosen our open sets so that ANY collection of elements extending x, whose domains are our open sets, is a compatible collection.

10.4.2.3 Admissible Sets

Before we continue it is worth talking about what happens when we apply this procedure to A-admissible sheaves. Specifically we have the following theorem.

Theorem 10.4.2.2. *Let A be an admissible set. Let $\mathcal{C}$ be an A-admissible* *category. If B, D are as in Section 10.4.2.1 and are A-admissible presheaves. Further let $F(x, i) = F_i(x)$, $G(x)$ be A-recursively enumerable functions and the cover $\{U_i : i \in I\}$ be A-recursively enumberable. Then $I(\alpha, x) = I_\alpha(x)$ is A-recursively enumerable.*

Proof. This is immediate by transfinite induction on an admissible set (see [2] or [12]). □

Theorem 10.4.2.3. *Assuming the same condition as Theorem 10.4.2.2 $I(ORD \cap A, x)$ is defined everywhere.*

Proof. This is because $I(\alpha, x)$ can be defined by a Σ_1 formula over A (because it is A-recursively enumerable) and $\{x : I(\alpha, x)\} \subseteq \{x : I(\beta, x)\}$ if $\alpha \subseteq \beta$. So the monotone relation corresponding to it yields a fixed point which is $\leq ORD \cap A$ (by [2] Chapter VI Corollary 2.8). But we know that $\bigcup_{\alpha \in ORD}\{x : I(\alpha, x)\}$ is defined everywhere in V and so if β is a fixed point of the monotone relation then $\bigcup_{\alpha \in ORD}\{x : I(\alpha, x)\} = \bigcup_{\alpha \leq \beta}\{x : I(\alpha, x)\}$. □

10.4.3 Disjoint Collection

Notice that in order for our construction in Section 10.4.2 to work we needed a disjoint cover of our topological space T. We needed this because given any x we wanted to find a collection of extensions of x whose domains didn't overlap (outside of dom(x)) and hence we know could be amalgamated (no matter what they were). However, in the case of a general partial site it isn't clear what should take the place of a disjoint cover (which is why we have the following definition)

Definition 10.4.3.1. Let $\mathcal{C}$ be a category and let $C \in \text{obj}(\mathcal{C})$. A Cone over C is a subset of

$$\bigcup_{D \in \text{obj}(\mathcal{C})} Hom(-, C)$$

A Co-Cone over C is a subset of

$$\bigcup_{D \in \text{obj}(\mathcal{C})} Hom(C, -)$$

Definition 10.4.3.2. Let $\mathcal{C}$ be a category. Let P be a presheaf on $\mathcal{C}$. We say $\mathcal{M} = \langle m_i : D_i \to D \text{ s.t. } i \in I \rangle$ are disjoint for P relative to $\mathcal{F} = \langle f_i : F \to D_i$ s.t. $i \in I \rangle$ (a co-cone over F) if $(\forall \mathcal{G} = \langle g_i : G \to D_i \text{ s.t. } i \in I \rangle)$ a co-cone over $\mathcal{M}$ either:

(1) $(\forall \langle x_i \in P(D_i) : i \in I \rangle)(\forall n, m \in I)(x_n | g_n) = (x_m | g_m)$.

(2) There is a map $\mathcal{G} \to \mathcal{F}$ (not necessarily unique).

(3) $|I| = 1$ (e.g. a singleton set is always a set of disjoint sets)

Lets consider what is going on here. We are trying to generalize the idea of a disjoint cover. Or more specifically, starting with an object F we want to find a way to extend elements of $P(F)$ so that any collection of extensions are compatible for a cover of D. But we want to be careful on how we do this. Specifically, given an $x \in P(F)$ we want to find some condition so that any sequence $\langle y_i \in P(D_i) \rangle$ where each $y_i | m_i = x$ is a compatible sequence.

In order to make this happen we have to make sure that for any $G \in \text{obj}(\mathcal{C})$ and maps from G to each of the D_i which commute with the maps from D_i to D, the restrictions of any combination of our elements give us the same thing (and hence they are compatible).

There are three possible ways we let this happen. First, in (1), we just strait out require that any restrictions are equal for all elements of $P(D_i)$. This is a very stringent and non-constructive condition. So in addition we also allow the case where the co-cone factors through our object F. In this case we know that when checking for compatibility we first go through F. But we will have by assumption that the compatible sequences all extend the same element in F and so they must all go to the same thing in G as well

Finally the last case we allow is when the covering D_i's is a single element (because then we don't have to worry about elements being restricted to incompatible things)

10.4.4 General Induction Argument

Now that we have discussed how we will generalize induction for sheaves on a topological space, and we have defined our generalization of disjoint maps for an arbitrary category, we can given an abstract example of our generalization of the induction argument for sheaves on a partial site.

10.4.4.1 General Definition

Definition 10.4.4.1. Let $(\mathcal{C}, K)$ be a partial presite and let $C \in \text{obj}(\mathcal{C})$. Let B, D be sheaves on $(\mathcal{C}, K)$ such that

- $B(C) = \emptyset$
- $B \rightarrowtail D$

Let $\overline{C} = \{E \in \text{obj}(\mathcal{C}) : (\exists f \in \text{morph}(\mathcal{C}))f : E \to C\}$. For each $U \in \overline{C}$ and each $x \in B(U)$ define $\mathcal{M}_x = \{m_i^x : U \to M_i^x \text{ s.t. } i \in \kappa_x\}$, $\mathcal{N}_x = \{n_i^x : M_i^x \to Q_x \text{ s.t. } i \in \kappa_x\}$ where

(1) $\mathcal{N}_x \in K(Q_x)$

(2) $(\exists q)q : C \to Q_x$

(3) $\mathcal{N}_x$ is disjoint relative to B and $\mathcal{M}_x$

Finally let $G : \bigcup_{E\in\overline{C}}(D(E) - B(E)) \to Z$ and F_i be a function such that if $F_i(x, y) = z$ then

- y is a function
- $\text{dom}(y) = \{\alpha \in B(M_i^x) : \alpha|m_i^x = x\}$
- $\text{range}(y) = Z$
- $z \in Z$

We see that in fact this is exactly the definition we want to make our induction go through. So we have the following.

Definition 10.4.4.2. Let

- $I_\emptyset(x) = G(x)$ if $x \in D(\text{dom}(x)) - B(\text{dom}(x))$ and undefined otherwise.
- $I_{\omega*\lambda} = \bigcup_{\gamma<\omega*\lambda} I_\gamma$
- $I_{\beta+1}(x) =$
  - $I_\beta(x)$ if $I_\beta(x)$ is defined.

- If i is least such that I_β is defined everywhere on $\{\alpha \in B(M_i^x) : \alpha|m_i^x = x\} = X_i$ then let $I_{\beta+1}(x) = F_i(x, I_\beta|X_i)$
- $I_{\beta+1}(x)$ is undefined if no such i exists

We then define $I(x) = I_\lambda(x)$ if $(\exists\lambda)I_\lambda(x)$ is defined and we say $I(x)$ is undefined otherwise.

10.4.4.2 I is Defined Everywhere

Claim 10.4.4.3. *$I(x)$ is defined on all $x \in D(dom(x))$.*

Proof. Assume $I(x)$ is not defined for some $x \in D(\text{dom}(x))$.

The first thing to notice is that $x \in B(\text{dom}(x))$ because if $x \in D(\text{dom}(x)) - B(\text{dom}(x))$ then $I(x) = G(x)$. Next notice that we then also have for each $i \in \kappa$ there is some $y_i \in B(M_i^x)$ such that $y_i|m_i^x = x$ and $I(y)$ is undefined.

But we know that $\mathcal{N}_x$ is pairwise disjoint relative to B and $\mathcal{M}_x$ and so in particular this means that $\langle y_i : i \in \kappa\rangle$ generates a compatible collection of elements on Q_x relative to the covering sieve generated by n_i^x. Hence we know there is a unique $y \in B(Q_x)$ such that $y|n_i^x = y_i$. But we also know that there is a map $q : C \to Q_x$ and so in particular this means that $y|q \in B(C)$. $\Rightarrow\Leftarrow$ (by assumption $B(C) = \emptyset$)

So I is defined on all $x \in D(\text{dom}(x))$. $\square$

The point to notice is that our conditions were chosen precisely to give us enough machinery for our induction argument on sheave on a topological space to generalize to sheaves on a category.

10.4.4.3 Admissible-Sets

Just as in the case of a topological space we get that this construction is A-recursively enumerable if all of its component are.

Theorem 10.4.4.4. *Let A be an admissible set and let $\mathcal{C}$ be an A-admissible category. Following the same notation as in Section 10.4.4.1 if*

- *B, D are A-admissible presheaves*
- *$\mathcal{M}(x, i) = m_i^x$ is A-recursively enumerable*
- *$\mathcal{N}(x, i) = n_i^x$ is A-recursively enumerable*
- *$q(x)$ is A-recursively enumerable where $q(x) : C \to Q_x$*
- *$F(x, y, i)$, $G(x)$ are A-recursively enumerable.*

then $I(\alpha, x) = I_\alpha(x)$ is A-recursively enumerable.

Proof. This is immediate by transfinite induction on an admissible set (see [2] or [12]). □

Theorem 10.4.4.5. *Assuming the same conditions as Theorem 10.4.4.4 then $I(ORD \cap A, x)$ is defined everywhere.*

Proof. This is because $I(\alpha, x)$ can be defined by a Σ_1 formula over A (because it is A-recursively enumerable) and $\{x : I(\alpha, x)\} \subseteq \{x : I(\beta, x)\}$ if $\alpha \subseteq \beta$. So the monotone relation corresponding to it yields a fixed point which is $\leq ORD \cap A$ (by [2] Chapter VI Corollary 2.8). But we know that $\bigcup_{\alpha \in ORD}\{x : I(\alpha, x)\}$ is defined everywhere in V and so if β is a fixed point of the monotone relation then $\bigcup_{\alpha \in ORD}\{x : I(\alpha, x)\} = \bigcup_{\alpha \leq \beta}\{x : I(\alpha, x)\}$. □

Chapter 11

Separation Theorem for Sheaves on a Topological Spaces

In this chapter we will prove our separation theorem in the context of sheaves on a topological space. Even though this result is a special case of Theorem 12.1.1.2 we still believe that it is useful to first consider our separation theorem in the context of sheaves on a topological space before looking at sheaves on a partial site.

All the ideas in Theorem 12.1.1.2 are present in this proof. But, because we know that there is at most a single map between any two objects (i.e. the inclusion map), the presentation is much cleaner and hence it will (hopefully) be easier to see the important elements of the proof.

11.1 Notation, Terminology and Basic Results

Definition 11.1.0.6. Let L be a topological space. Define $O(L) = \{U \subseteq L : U \text{ is open}\}$. If P is a presheave on $O(L)$, $f : U \to V$ and $x \in P(V)$ then we define the restriction of x to U $(x|U)$ is $P(f)(x)$ (i.e. $x|f$).

Definition 11.1.0.7. If A is a sheaf on the topological space L, let $PreSh(A) =$ subobjects of A in the category of presheaves on the topological space L and $Sh(A) =$ subobjects of A in the category of sheaves on the topological space L.

Definition 11.1.0.8. Let Q be a sheaf and let $f \in Q$. We define $Q_f(U) = \{s \in Q(U) : (\exists a \in Q(\text{dom}(s) \cup \text{dom}(f)))\ a|\text{dom}(s) = s \text{ and } a|\text{dom}(f) = f\}$. Note that here $Q_\emptyset = Q$

Theorem 11.1.0.9. *The above notation is consistent with Definition 10.3.3.2*

Proof. This is because $\text{dom}(s) \cup \text{dom}(f)$ is the supreumum of $\text{dom}(s)$ and $\text{dom}(f)$ and $\{\text{dom}(f), \text{dom}(s)\}$ is a cover for $\text{dom}(s) \cup \text{dom}(f)$. □

Lemma 11.1.0.10. *Let A be a sheaf and $Q \in PreSh(A)$. If $f \notin Q(dom(f))$ then $Q_f(V) = \emptyset$ for all $V \in O(L)$.*

Proof. This is immediate from the definition of Q_f and by virtue of the fact that for all $V \in O(L)$ and all $x \in Q(V)$ $x|\text{dom}(f) \neq f$ (otherwise $f \in Q(\text{dom}(f))$ because Q is a presheaf.) □

We will fix sheaves A and X. These will correspond to ω^ω and κ^ω in the case of Suslin's Separation Theorem.

Definition 11.1.0.11. Let $Q \rightarrowtail A \times X$. Define $p[Q](U)$ to be

$$\{a : (\exists x \in X(U))(a, x) \in Q(U)\}$$

Lemma 11.1.0.12. *Let $Q \in PreSh(A \times X)$. Then $p[Q] \in PreSh(A)$.*

Proof. Let $a \in p[Q](U)$, $V \subseteq U$. We then know there is an $x \in X(U)$ such that $(a, x) \in Q(U)$. So in particular we know $(a, x)|V = (a|V, x|V) \in Q(V)$ because Q is a presheaf. Hence $a|V \in p[Q](V)$. So, $p[Q]$ is a presheaf. $\square$

It is worth mentioning explicitly that even when Q is a sheaf we have no reason to believe that $p[Q]$ will be a sheaf. This is because if we have a compatible collection of elements $\langle a_i : i \in \kappa \rangle$ where $a_i \in p[Q](U_i)$, we know that each one comes from an $(a_i, x_i) \in Q(U_i)$. But, there is no reason (apriori) why this should mean $\langle x_i : i \in \kappa \rangle$ is a compatible collection.

Definition 11.1.0.13. The κ-Borelian presheaves on $PreSh(A)$ is the smallest collection of presheaves closed under κ-Unions and κ-Intersections (in $PreSh(A)$) and containing $Sh(A)$

This is the analog of being κ-Borelian in the descriptive set theory case. Notice though that this is not analogous to being κ-Borel. Specifically the collection of κ-Borel sets is the smallest collection closed under κ-Union, Complementation, and containing the Open Sets. Now while in the context of ω^ω, ω-Borelian sets are exactly those which are ω-Borel (See [6] Chapter I §3 Proposition I.3.7) in the context of sheaves we no longer even have that $\neg\neg Q = Q$ and so these concepts are very different.

Definition 11.1.0.14. Let $f \in \Pi_{i \in I} A_i(U)$. Define $(f)_k \in A_k(U)$ the projection onto the kth component.

Similarly, if $f_i \in A_i(U)$ then define $\langle f_i : i \in I\rangle \in \Pi_{i\in I} A_i(U)$ to be the product.

Definition 11.1.0.15. Let $U \subseteq T$ be an open set. We say a presheaf P on T is U-Complete if for all $x \in P(V)$ with $V \subseteq U$ there is a (not necessarily unique) $x_U \in P(U)$ such that $x_U|V = x$.

Intuitively the U-Complete presheaves correspond to the pruned trees (i.e. if X is a set $Y \rightarrowtail \overline{X}$, then Y is ω-complete if and only if Y considered as a pretree is pruned.)

11.2 The Separation Theorem

11.2.1 The Theorem

Theorem 11.2.1.1. *Let A, X be sheaves. $T, S \rightarrowtail A \times X$ be U-complete sheaves such that $p[T] \cap p[S](U) = \emptyset$. Then there is an W-Borelian element of $PreSh(A)$ which separates $p[T]$ and $p[S]$ in $PreSh(A)$ up to U-elements where*

$$W = \max_{\{x\in\bigcup_{V\subseteq U} T(V)\cup S(V)\}}\{$$

$$\min_{\{\langle U_i : i\in\kappa\rangle:\ \textit{disjoint cover of } U\}}\{$$

$$\max_{i\in\kappa} |\{y \in T(U_i \cup dom(x)) \cup S(U_i \cup dom(x)) : y|dom(x) = x\}|\}\}$$

(We will talk about where exactly this bound comes from after we have gone through the proof)

Proof. For simplicity of notation, if $f \in A \times X \times X(U)$ we are going to define $\tau(f) = \langle (f)_0, (f)_1 \rangle$, and $\sigma(f) = \langle (f)_0, (f)_2 \rangle$.
Let J be defined as

$$f \in J(U) \Leftrightarrow \tau(f) \in T(U) \wedge \sigma(f) \in S(U)$$

Claim 11.2.1.2. *$J \in PreSh(A \times X \times X)$.*

Proof. Let $f \in J(U)$ and $V \subseteq U$. So $\tau(f) \in T(U)$ and $\sigma(f) \in S(U)$. Therefore $\tau(f)|V \in T(V)$ and $\sigma(f)|V \in S(V)$. But, $\tau(f)|V = \langle (f)_0|V, (f)_1|V \rangle$ and $\sigma(f)|V = \langle (f)_0|V, (f)_2|V \rangle$. So, $\langle (f)_0|V, (f)_1|V, (f)_2|V \rangle = f|V \in J(V)$. Hence J is a subsheaf of $A \times X \times X$. □

Claim 11.2.1.3. *$J \in Sh(A \times X \times X)$.*

Proof. Let V be an open set in L and let $\{V_i : i \in I\}$ be a cover of V. Let $\{f_i : i \in I\}$ be a set of elements such that $f_i \in J(V_i)$ and $f_i|V_i \cap V_j = f_j|V_j \cap V_i$. We know $\{\sigma(f_i) : i \in I\}$ and $\{\tau(f_i) : i \in I\}$ are also compatible collections because $\{f_i : i \in I\}$ is a compatible collection. So, as $S, T \in Sh(A \times X)$ there are unique $g_S \in S(V), g_T \in T(V)$ such that $g_S|V_i = \sigma(f_i)$ and $g_T|V_i = \tau(f_i)$. But then $(g_S|V_i)_0 = (g_T|V_i)_0$ so $(g_S)_0 = (g_T)_0$ because A is a sheaf. Hence there must exist a unique g such that $(g)_0 = (g_S)_0 = (g_T)_0$, $(g)_1 = (g_T)_1$, and $(g)_2 = (g_S)_2$. Therefore $g \in J(V)$ and so J is a sheaf. □

Claim 11.2.1.4. *$J(U) = \emptyset$*

Proof. If $f \in J(U)$ then $\tau(f) \in T(U)$ and hence $(f)_0 \in p[T](U)$. But we then also have $\sigma(f) \in S(U)$ and so $(f)_0 \in p[S](U)$. But then $(f)_0 \in p[T] \cap p[S](U)$. $\Rightarrow\Leftarrow$ (we assumed $p[T] \cap p[S](U) = \emptyset$). □

Our goal will be to find, for each open $V \subseteq L$ and each $f \in \bigcup_{V \in O(L)} A \times X \times X(V)$, a W-Borelian presheaf C_f which separates $p[T_f]$ from $p[S_f]$ up to U-elements. Then we can just define our presheaf to be $C_\emptyset$ and we are done (as $T_\emptyset = T, S_\emptyset = S$ where $\emptyset$ is considered the sole element of any sheaf evaluated at $\emptyset$)

We are going to do this in the following way. For each α we are going to define a partial function $I_\alpha : \bigcup_{V \in O(L)} A \times X \times X(V) \rightarrow W-$Borelian presheaves on $PreSh(A)$. We will define this in such a way that if $\beta > \alpha, f \in \text{dom}(I_\alpha)$ then $I_\beta(f) = I_\alpha(f)$, and $I_\alpha(f)$ separates $p[T_f] \cap p[S_f]$ up to U-elements. And, I_α in no way uses α in defining any of it's values.

Under these conditions we know that at some ordinal this function stabilizes to a function I. It will be $I(\emptyset)$ which will give us our W-Borelian set (see Section 10.4).

Lemma 11.2.1.5. *$p[T_{\tau(f)}]$ separates $p[T_{\tau(f)}]$ from $p[S_{\sigma(f)}]$ up to U-elements.*

Proof. This is because $p[S_{\sigma(f)}] \cap p[T_{\tau(f)}] \rightarrowtail p[S] \cap p[T]$ and $p[S] \cap p[T]$ has no U-elements so $p[S_{\sigma(f)}] \cap p[T_{\tau(f)}]$ must not either. □

11.2.1.1 Definition of I_α

Define I_α as follows:

Base Case:

<u>α is a limit:</u>

$I_\alpha = \bigcup_{\beta < \alpha} I_\beta$.

α is not a limit:

If $\exists\beta < \alpha$ such that $I_\beta(f)$ is defined, let $I_\alpha(f) = I_\beta(f)$.

If $f \notin J$ then:

If $\tau(f) \notin T$

Then $T_{\tau(f)}(V) = \emptyset$ for all $V \in O(L)$ (by Lemma 11.1.0.10). Hence $p[T_{\tau(f)}](V) = \emptyset$ for all $V \in O(L)$ as well. So, $p[T_{\tau(f)}]$ is a sheaf and hence W-Borelian.

But by Lemma 11.2.1.5 we also have $p[T_{\tau(f)}]$ separates $p[S_{\sigma(f)}] \cap p[T_{\tau(f)}]$ up to U elements.

So we can let $I_\alpha(f) = p[T_{\tau(f)}]$.

Otherwise $\sigma(f) \notin S$ in which case

Then $S_{\sigma(f)}(V) = \emptyset$ for all $V \in O(L)$ (by Lemma 11.1.0.10). Hence $p[S_{\sigma(f)}](V) = \emptyset$ for all $V \in O(L)$ as well.

So we can let $I_\alpha(f) = A$.

Cover Condition:

Before we continue with the definition of I_α it is important to notice something. For each $f \in A \times X \times X(W)$, $U \supseteq W \supsetneq V$ we have

$$T_{\tau(f)}(W) = \bigcup_{g \in A\times X\times X(V), g|dom(f)=f} T_{\tau(g)}(W)$$

$$S_{\sigma(f)}(W) = \bigcup_{g \in A\times X\times X(V), g|dom(f)=f} S_{\sigma(g)}(W)$$

because both T and S are U-complete sheaves. As such we also have

$$p[T_{\tau(f)}](W) = \bigcup_{g \in A\times X\times X(V), g|\mathrm{dom}(f)=f} p[T_{\tau(g)}](W)$$

$$p[S_{\sigma(f)}](W) = \bigcup_{g \in A\times X\times X(V), g|\mathrm{dom}(f)=f} p[S_{\sigma(g)}](W)$$

Definition 11.2.1.6. For the rest of the definition of I_α, we are going to fix $\{U_\mu : \mu \in \kappa\}$ a disjoint cover of U

If $I_\alpha(f)$ is undefined but $\exists \mu \in \kappa$ such that for each $g \in A \times X \times X(\mathrm{dom}(f) \cup U_\mu)$ with $g|\mathrm{dom}(f) = f$, $I_\alpha(g)$ is defined then let U_μ be such and define $I_{\alpha+1}(f)$ as follows.

First notice by the above that it suffices to construct W-Borelian $D_{t,\zeta,s,\eta}$ (for each $\langle t, \zeta\rangle, \langle s, \eta\rangle \in A \times X(U_\mu \cup \mathrm{dom}(f))$ such that $\langle t, \zeta\rangle|\mathrm{dom}(f) = \tau(f)$, and $\langle s, \eta\rangle|\mathrm{dom}(f) = \sigma(f)$) such that $D_{t,\zeta,s,\eta}$ separates $p[T_{\langle t,\zeta\rangle}]$ from $p[S_{\langle s,\eta\rangle}]$ up to U-elements.

This is because because we can then let

$$I_{\alpha+1}(f) = \bigcup_{t,\zeta}\bigcap_{s,\eta} D_{t,\zeta,s,\eta}$$

and by the Lemma 10.3.2.3 $I_{\alpha+1}(f)$ separates $p[T_{\tau(f)}]$ from $p[S_{\sigma(f)}]$ up to U-elements, and is W-Borelian because each $D_{t,\zeta,s,\eta}$ is.

We will break the construction of $D_{t,\zeta,s,\eta}$ into cases:

<u>Case (1) $t = s$:</u>

Then $\langle s, \zeta, \eta\rangle \in A \times X \times X(U_\mu \cup \mathrm{dom}(f))$ and $\langle s, \zeta, \eta\rangle|\mathrm{dom}(f) = f$. So, by assumption $I_\alpha(\langle s, \zeta, \eta\rangle)$ separates $p[T_{\tau(\langle s,\zeta,\eta\rangle)}](= p[T_{\langle t,\zeta\rangle}])$ from $p[S_{\sigma(<s,\zeta,\eta\rangle)}](= p[S_{\langle s,\eta\rangle}])$ up to U-elements and we can let $D_{t,\zeta,s,\eta} = I_\alpha(\langle s, \zeta, \eta\rangle)$

<u>Case (2) $t \neq s$:</u>

Notice that $p[T_{\langle t,\zeta\rangle}] \rightarrowtail A_t$.

Assume (to get a contradiction) that $\exists a \in (A_t \cap p[S_{\langle s,\eta\rangle}])(U)$.

So, $(\exists x)\langle a, x\rangle \in S_{\langle s,\eta\rangle}(U)$. Hence $\langle a, x\rangle|(\mathrm{dom}(\langle s, \eta\rangle)) = \langle a, x\rangle|(U_\mu \cup \mathrm{dom}(f)) = \langle s, \eta\rangle$ because $U_\mu \cup \mathrm{dom}(f) \subseteq U$ (and the definition of $S_{\langle s,\eta\rangle}$). In particular, $a|(U_\mu \cup \mathrm{dom}(f)) = s$.

We also have $a|\mathrm{dom}(t) = t$ because $\mathrm{dom}(t) \subseteq U$ and $a \in A_t(U)$. But, because $\mathrm{dom}(t) = U_\mu \cup \mathrm{dom}(f)$ we therefore have $s = t \Rightarrow\Leftarrow$.

So $(A_t \cap p[S_{\langle s,\eta\rangle}])(U) = \emptyset$ and we can let $D_{t,\zeta,s,\eta} = A_t$ as A_t is a sheaf and separates $p[T_{\tau(\langle s,\zeta,\eta\rangle)}]$ from $p[S_{\sigma(\langle s,\zeta,\eta\rangle)}]$ up to U-elements.

Finally, if $I_\alpha(f)$ is undefined and $\forall \mu \in \kappa\ \exists g \in A \times X \times X(\mathrm{dom}(f) \cup U_\mu)$, such that $g|\mathrm{dom}(f) = f$, and $I_\alpha(g)$ is undefined then let $I_{\alpha+1}(f)$ be undefined. Now define $I = \bigcup_{\alpha\in\zeta} I_\alpha$ where $I_\zeta = I_{\zeta+1}$.

Claim 11.2.1.7. *$I(f)$ is defined for each $f \in \bigcup_{W\in O(L), W\subseteq U}(A \times X \times X)(W)$.*

Proof. Let $UD(I) = \bigcup_{W\in O(L), W\subseteq U}\{f \in (A \times X \times X)(U) : I(f) \text{ is undefined}\}$ Assume there exists $f \in UD(I)$

By the definition of I (and because $f \in UD(I)$) we know that for each $\mu \in \kappa$ there is some g such that $\mathrm{dom}(g) = U_\mu \cup \mathrm{dom}(f)$, $g|\mathrm{dom}(f) = f$ and $g \in UD(I)$. Let f_μ be one such g.

By assumption, $U_\mu \cap U_\beta = \emptyset$ if $\mu \neq \beta$. And so $\mathrm{dom}(f_\mu) \cap \mathrm{dom}(f_\beta) = \mathrm{dom}(f)$ (if $\mu \neq \beta$) and hence $f_\mu|(\mathrm{dom}(f_\mu) \cap \mathrm{dom}(f_\beta)) = f_\beta|\mathrm{dom}(f_\mu) \cap \mathrm{dom}(f_\beta) = f$ (because f_β, f_μ both extend f).

(Note it is to get this fact that we need $\{U_\mu : \mu \in \kappa\}$ is a disjoint cover of U and not just a cover.)

So, $\{f_\mu : \mu \in \kappa\}$ is a compatible collection of elements of J (as I is defined for any element of $A \times X \times X(V)$ not in $J(V)$ for all open $V \subseteq U$). Hence, as J is a sheaf, there must be an element of $J(\bigcup_{\mu\in\kappa} \mathrm{dom}(f_\mu))$ from which all of these come. But $\bigcup_{\mu\in\kappa} \mathrm{dom}(f_\mu) = U$ and $J(U) = \emptyset$ by assumption. $\Rightarrow\Leftarrow$

So $UD(I) = \emptyset$. □

Hence $I(\emptyset)$ is defined, separates $p[T]$ from $p[S]$ up to U-elements and is W-Borelian. □

11.2.2 Corollaries

Lemma 11.2.2.1. *If $\{A_i : i \in I\}$ are sheaves, so is $\bigcap_{i\in I} A_i$.*

Proof. Limits are preserved by the inclusion functor $Sheaves(L) \to Presheaves(L)$. (See [9]) □

Lemma 11.2.2.2. $p[\bigcup_{i\in I} A_i] = \bigcup_{i\in I} p[A_i]$.

Proof. This is true because it is true point wise (i.e. point wise there is a witness to an element being $p[\bigcup_{i\in I} A_i]$ iff there is a witness to it being in $p[A_i]$ for some $i \in I$) □

Corollary 11.2.2.3. *If* $\{S_{i,j} : i \in I, j \in J\}$, $\{T_{i,j} : i \in I', j \in J'\}$ *are sheaves such that* $p[\bigcup_{i\in I}\bigcap_{j\in J} S_{i,j}] \cap p[\bigcup_{i\in I'}\bigcap_{j\in J'} T_{i,j}](U) = \emptyset$ *then* $p[\bigcup_{i\in I'}\bigcap_{j\in J'} T_{i,j}]$ *can be separated from* $p[\bigcup_{i\in I}\bigcap_{j\in J} S_{i,j}]$ *by a* $\max\{W, I, I'\}$*-Borelian presheaf.*

Proof. First note that because the intersection of sheaves is still a sheaf it suffices to consider the case $p[\bigcup_{i\in I} S_i] \cap p[\bigcup_{i\in I'} T_i](U) = \emptyset$.

But, by the definition of projection we have $p[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} p[S_i]$. We also have $p[S_j] \cap p[T_i]$ has no U-elements and so $p[S_j]$ and $p[T_i]$ are separated up to U-elements by W-Borelian presheaves $C_{i,j}$. Hence, by Lemma **??** $p[\bigcup_{i\in I'} T_i], p[\bigcup_{i\in I} S_i]$ are separated up to U-elements by $\bigcup_{i\in I}\bigcap_{j\in I'} C_{i,j}$ which is a $\max\{W, I, I'\}$-Borelian presheaf. □

11.2.3 Examples

Now that we have proved the theorem it will be worthwhile to go through a couple of examples and calculate the exactly how complicated our Borelian presheaves are.

11.2.3.1 $\overline{\omega}$

So as a first example lets look at our topological space ω and see what this generalization of the Suslin-Kleene Separation theorem tells us about what Borelian presheaves can separate the analog of analytic sets.

Lets start by considering the sheaves $A = \overline{\omega^\omega}$ and $X = \overline{\kappa^\omega}$. Recall the definition of W in Theorem 11.2.1.1. In order to calculate W for this space lets first consider what disjoint covers of $\overline{\omega}$ look like. Well, as the topology on $\overline{\omega}$ has the property that for any two open sets U, V either $U \subseteq V$ or $V \subseteq U$

we know the the only disjoint open cover of any open set is itself. In other words, the only disjoint open cover of ω is $\{\omega\}$.

In particular we know that $W = \max_{x \in \overline{\omega^\omega} \times \overline{\kappa^\omega}} \{|\{y \in T(\omega) \cup S(\omega)\}|\}$. But, in all but a few cases this will in fact be $|\omega^\omega|$.

However, we already knew that for any space X^ω and any disjoint sets there is a $|X^\omega|$-Borel subset separating them (See Theorem B.1.2.3). So in this particular case we don't find out anything new.

11.2.3.2 κ

Instead of looking at $\overline{\omega}$ lets consider the topological space κ with the discrete topology.

Definition 11.2.3.1. For all $U \subseteq \kappa$ let $\overline{\lambda^\kappa}(U) = \{f : U \to \lambda\}$. If $V \subseteq U$ and $f \in \overline{\lambda^\kappa}(U)$ let $f|V(i) = f(i)$ for all $i \in V$.

Lemma 11.2.3.2. $\overline{\lambda^\kappa}$ *is a sheaf*

Proof. $\overline{\lambda^\kappa}$ is a presheaf by definition of restriction. Similarly if $\{x_i : i \in I\}$ is a compatible collection where x_i and x_j agree on $\mathrm{dom}(x_i) \cap \mathrm{dom}(x_j)$ and we can define $x(n) = x_j(n)$ for all $i \in \bigcup_{i \in I} \mathrm{dom}(x_i)$ and $n \in \mathrm{dom}(x_j)$, and so $\overline{\lambda^\kappa}$ is a sheaf. □

Now lets consider the sheaves $A = \overline{\lambda^\kappa}, X = \overline{\lambda^\kappa}$. In order to find the minimum bound on Borelian presheaf, we are going to want to find a cover which minimizes the number of extensions of each $x \in \overline{\lambda^\kappa} \times \overline{\lambda^\kappa}(U)$ for all U. Now notice the larger we can make the cover (i.e. the more elements of the cover) the less extensions there will be to any particular element of the cover. So we are looking for a maximal disjoint cover.

Now there is an obvious maximal disjoint open cover which is $\{\{i\} : i \in \kappa\}$. So lets consider what bound we get with this cover. Well given any $x \in \overline{\lambda^\kappa} \times \overline{\lambda^\kappa}(U)$ and for all $\{i\}$ in our cover we need to consider the number of $y \in \overline{\lambda^\kappa} \times \overline{\lambda^\kappa}(U \cup \{i\})$ such that $y|U = x$. But, as any such y is just a function the number of such y is just the $|\{f : (U \cup \{i\} - U) \to \lambda\}| = \lambda$.

Further, because this is true for all x and this cover is obviously a maximal one we know that in this context $W = \lambda$. And more to the point we know that in fact W is (in general) far less than the trivial upper bound we knew before of λ^κ (see Theorem B.1.2.3)

Chapter 12

Separation Theorem for Sheaves on a Site

In this chapter we will finally prove our separation theorem for sheaves and presheaves on a partial site.

Definition 12.0.3.3. Let $\mathcal{C}$ be a category. We say that a presheaf P on $\mathcal{C}$ is U-complete (for $U \in \mathrm{obj}(\mathcal{C})$) if

$$(\forall g : V \to U)(\forall x \in P(V))(\exists y \in P(U))y|g = x$$

U-complete presheaves are the generalization of pruned trees to the context of sheaves on an arbitrary partail site.

Definition 12.0.3.4. If A is a sheaf on the partial site $(\mathcal{C}, L)$, let $PreSh(A) = \{$ subobjects of A in the category of presheaves on $\mathcal{C}\}$ and $Sh(A) = \{$ subobjects of A in the category of sheaves on $(\mathcal{C}, L)\}$.

Definition 12.0.3.5. Let $Q \rightarrowtail A \times X$. Define $p[Q](U)$ to be

$$\{a : (\exists x \in X(U))(a, x) \in Q(U)\}$$

Lemma 12.0.3.6. *Let $Q \in PreSh(A \times X)$. Then $p[Q] \in PreSh(A)$.*

Proof. Let $a \in p[Q](U)$, $f : V \to U$. We then know there is an $x \in X(U)$ such that $(a, x) \in Q(U)$. So in particular we know $(a, x)|f = (a|f, x|f) \in Q(V)$ because Q is a presheaf. Hence $a|f \in p[Q](V)$ and $p[Q]$ is a presheaf. □

Definition 12.0.3.7. The κ-Borelian presheaves on $PreSh(A)$ is the smallest collection of presheaves closed under κ-Unions and κ-Intersections (in $PreSh(A)$) and containing $Sh(A)$.

12.1 Basic Results

12.1.1 The Separation Theorem

First a little notation

Definition 12.1.1.1. Let $f \in \Pi_{i \in I} A_i(U)$. Define $(f)_k \in A_k(U)$ the projection onto the kth component. Similarly, if $f_i \in A_i(U)$ then define $\langle f_i : i \in I\rangle \in \Pi_{i \in I} A_i(U)$ to be an element of product.

Theorem 12.1.1.2. *Let A, X be sheaves on $(\mathcal{C}, J)$ where $(\mathcal{C}, J)$ is a partial site with quasi-supremums and an initial object 0. Let $T, S \rightarrowtail A \times X$ such that $p[T] \cap p[S](U) = \emptyset$. Then there is a W-Borelian element of $PreSh(A)$*

which separates $p[T]$ and $p[S]$ in $PreSh(A)$ up to U-elements where

$$W = \max_{\{x \in \bigcup_{V \subseteq U} T(V) \cup S(V)\}} \{$$

$$\min_{\{\langle \mathcal{M}_i^x : i \in \kappa \rangle :\ \textit{disjoint cover with the properties in the proof}\}} \{$$

$$\max_{i \in \kappa} |\{y \in T(M_i^x) \cup S(M)_i^x : y|m_i^x(x) = x\}|\}\}$$

Proof. For simplicity of notation, if $x \in A \times X \times X(U)$ we are going to define $\tau(x) = \langle (x)_0, (x)_1 \rangle$, and $\sigma(x) = \langle (x)_0, (x)_2 \rangle$.

Let J be defined as

$$x \in J(V) \Leftrightarrow \tau(x) \in T(V) \wedge \sigma(x) \in S(V)$$

Claim 12.1.1.3. *$J \in PreSh(A \times X \times X)$.*

Proof. Let $x \in J(Y)$ and $f : V \to Y$. So $\tau(x) \in T(Y)$ and $\sigma(x) \in S(Y)$. Therefore $\tau(x)|f \in T(V)$ and $\sigma(x)|f \in S(V)$. But $T(f)(\langle a, x \rangle) = A \times X(f)(\langle a, x \rangle)$, $S(f)(\langle a, x \rangle) = A \times X(f)(\langle a, x \rangle)$ and $A \times X(f)(\langle a, x \rangle) = \langle A(f)(a), X(f)(x) \rangle$ because T, S are subobjects (i.e. subfuctors) of the sheaf $A \times X$. So, $\tau(x)|f = \langle (x)_0|f, (x)_1|f \rangle \in T(V)$ and $\sigma(x)|f = \langle (x)_0|f, (x)_2|f \rangle \in S(V)$. Hence, $\langle (x)_0|f, (x)_1|f, (x)_2|f \rangle = x|f \in J(V)$ and J is a presheaf of $A \times X \times X$. □

Claim 12.1.1.4. *$J \in Sh(A \times X \times X)$.*

Proof. Let $V \in \text{obj}(\mathcal{C})$ and let $\{f_i : V_i \to V \text{ s.t. } i \in I\}$ be a covering family of V. Let $\{x_i : i \in I\}$ be a set of compatible elements such that $x_i \in J(V_i)$. Now we know because $S, T \in Sh(A \times X)$ there are unique $g_S \in S(V), g_T \in T(V)$ such that $g_S|f_i = \sigma(x_i)$ and $g_T|f_i = \tau(x_i)$. But by construction $(g_S|f_i)_0 = (g_T|f_i)_0$ so there must exist some g such that $(g)_0 = (g_S)_0 = (g_T)_0$, $(g)_1 = (g_T)_1$, and $(g)_2 = (g_S)_1$ (because A is a sheaf). Therefore $g \in J(V)$, and $g|f_i = x_i$. So J is a sheaf. $\square$

Claim 12.1.1.5. $J(U) = \emptyset$.

Proof. If $z \in J(U)$ then $\tau(z) \in T(U)$ and hence $(x)_0 \in p[T](U)$ and similarly $\sigma(x) \in S(U)$ and so $(x)_0 \in p[S](U)$. In particular if $J(U) \neq \emptyset$ then $p[T] \cap p[S](U) \neq \emptyset$. But we have assumed for this theorem that $p[T] \cap p[S](U) = \emptyset$. $\Rightarrow\Leftarrow$ $\square$

Now our goal is to find for each $U \in \text{obj}(\mathcal{C})$ and each $x \in \bigcup_{U \in \text{obj}(\mathcal{C})} A \times X \times X(U)$ a W-Borelian set C_x which separates $p[T_x]$ from $p[S_x]$ up to U-elements. We can then just define our W-Borelian presheaf to be $C_\emptyset$ and we are done (as $T_\emptyset = T, S_\emptyset = S$ where $\emptyset$ is considered the sole element of any sheaf evaluated at 0)

We are going to do this in the following way. We are going to define a partial function $I_\alpha : \bigcup_{U \in \text{obj}(\mathcal{C})} A \times X \times X(U) \to W-$Borelian presheaves on $PreSh(A)$. We will define this in such a way that if $\beta > \alpha, x \in \text{dom}(I_\alpha)$ then $I_\beta(x) = I_\alpha(x)$, $I_\alpha(x)$ separates $p[T_x] \cap p[S_x]$ up to U-elements, and I_α in no way uses α in defining any of it's values.

Under these conditions we know that at some ordinal these functions stabilizes to a function I. And, it will be $I(\emptyset)$ which will give us our W-Borelian presheaf separating $p[T]$ from $p[S]$ up to U-elements.

Before we continue we need one lemma.

Lemma 12.1.1.6. *$p[T_{\tau(x)}]$ separates $p[T_{\tau(x)}]$ from $p[S_{\sigma(x)}]$ up to U-elements.*

Proof. This is because $p[S_{\sigma(x)}] \cap p[T_{\tau(x)}] \rightarrowtail p[S] \cap p[T]$ and $p[S] \cap p[T](U) = \emptyset$ so $p[S_{\sigma(x)}] \cap p[T_{\tau(x)}](U) = \emptyset$ also. □

12.1.1.1 Definition of I_α

Define I_α as follows:

Base Case:

α is a limit:

$I_\alpha = \bigcup_{\beta<\alpha} I_\beta$.

α is not a limit:

If $\exists \beta < \alpha$ such that $I_\beta(f)$ is defined, let $I_\alpha(f) = I_\beta(f)$.

If $f \notin J$ then:

If $\tau(f) \notin T$

Then $T_{\tau(f)}(V) = \emptyset$ for all $V \in \text{obj}(C)$ by Lemma 10.3.3.10. So, $p[T_{\tau(f)}]$ is a sheaf and hence W-Borelian.

But by Lemma 11.2.1.5 we also have $p[T_{\tau(f)}]$ separates $p[S_{\sigma(f)}]$ from $p[T_{\tau(f)}]$ up to U elements.

So we can let $I_\alpha(f) = p[T_{\tau(f)}]$.

Otherwise $\sigma(f) \notin S$ in which case

Then $S_{\sigma(f)}(V) = \emptyset$ for all $V \in \text{obj}(C)$ (by Lemma 10.3.3.10). So we can let $I_\alpha(f) = A$.

Cover Condition:

Before we continue we need to define the covers of our space. For each $x \in J(V)$ define $\mathcal{M}_x = \{m_i^x : V \to M_i^x \text{ s.t. } i \in \kappa_x\}$, $\mathcal{N}_x = \{n_i^x : M_i^x \to Q_x$ s.t. $i \in \kappa_x\}$ where

(1) $\mathcal{N}_x$ is a cover of Q_x in the site

(2) There is a map $q_x : U \to Q_x$

(3) If $r, r' : M_i^x \to V, t : U \to V$ and $\{r, t\}$, $\{r', t\}$ are covering families of V then $V \cong U$ and $r = r'$.

(4) $\mathcal{N}_x$ is disjoint relative to J and $\mathcal{M}_x$

(5) For each $i \in I$

$$T_{\tau(x)}(V) = \bigcup_{y|m_i^x = x} T_{\tau(y)}(M_i^x)$$

$$S_{\sigma(x)}(V) = \bigcup_{y|m_i^x = x} S_{\sigma(y)}(M_i^x)$$

Before we continue it is worth discussing what these conditions say. So the point of (1) and (4) is to get us a disjoint cover of Q_x. Now intuitively we think we would want the cover to be a cover of U but as it turns out we don't actually need this. The reason is that we are going to use our disjoint cover to get an element of $J(U)$. But, if we know there is a map from $U \to Q_x$ (as in (3)) then it suffices to know that there is an element of $J(Q_x)$.

Item (3) is a little less clear. It is a technical requirement that we will need to ensure that the presheaf we want actually separates the presheaves we want it to. Intuitively though (3) is meant to be the generalization of the fact that in a topological case we only consider $V \subseteq U$. What it says is that when ever we have U and another element which cover a third, then that third element must be isomorphic to U. Further it implies that there is a unique map from M_i^x to U (because $\{id_U\}$ is a covering family for U). So in fact, we see that U "looks like" a terminal object.

Finally (5) allows us to get

$$p[T_{\tau(x)}](V) = \bigcup_{y|m_i^x = x} p[T_{\tau(y)}](M_i^x)$$

$$p[S_{\sigma(f)}](V) = \bigcup_{y|m_i^x = x} p[S_{\sigma(y)}](M_i^x)$$

And it is worth noticing that (5) isn't immediate from the previous axioms. But what it says essentially is that the maps $J(m_i^x)$ are surjective.

Now if $I_\alpha(x)$ is undefined but $\exists \mu \in \kappa_x$ such that for each $y \in A \times X \times X(M_\mu^x)$, such that $y|m_i^x x$, $I_\alpha(x)$ is defined then let μ be the least such and define $I_{\alpha+1}(x)$ as follows.

Notice by the above that it suffices to construct W-Borelian $D_{t,\zeta,s,\eta}$ (for each $\langle t, \zeta\rangle, \langle s, \eta\rangle \in A \times X(M_\mu^x)$ where $\langle t, \zeta\rangle|m_\mu^x = \tau(x)$, and $\langle s, \eta\rangle|m_\mu^x = \sigma(x)$) such that $D_{t,\zeta,s,\eta}$ separates $p[T_{\langle t,\zeta\rangle}]$ from $p[S_{\langle s,\eta\rangle}]$ up to U-elements.

This is because because we can then let

$$I_{\alpha+1}(x) = \bigcup_{t,\zeta} \bigcap_{s,\eta} D_{t,\zeta,s,\eta}$$

And by the Lemma 10.3.2.3 $I_{\alpha+1}(x)$ separates $p[T_{\tau(x)}]$ from $p[S_{\sigma(x)}]$ up to U-elements, and is W-Borelian because each $D_{t,\zeta,s,\eta}$ is.

We will break the construction of $D_{t,\zeta,s,\eta}$ into cases:

Case (1) $t = s$:

Then $\langle s, \zeta, \eta\rangle \in A \times X \times X(M^x_\mu)$ and $\langle s, \zeta, \eta\rangle | m^x_\mu = x$. So, by assumption $I_\alpha(\langle s, \zeta, \eta\rangle)$ separates $p[T_{\tau(\langle s,\zeta,\eta\rangle)}](= p[T_{\langle t,\zeta\rangle}])$ from $p[S_{\sigma(\langle s,\zeta,\eta\rangle)}](= p[S_{\langle s,\eta\rangle}])$ up to U-elements and we can let $D_{t,\zeta,s,\eta} = I_\alpha(\langle s, \zeta, \eta\rangle)$

Case (2) $t \neq s$:

Notice that $p[T_{\langle t,\zeta\rangle}] \rightarrowtail A_t$.

Assume (to get a contradiction) that $\exists a \in (A_t \cap p[S_{\langle s,\eta\rangle}])(U)$.

So, $(\exists x)\langle a, x\rangle \in S_{\langle s,\eta\rangle}(U)$. But then we know there is a morphism $h : M^x_\mu \to U$ and $\langle a, x\rangle | h = \langle s, \eta\rangle$ (by condition (3) on the covers and the fact that quasi-supremums exists). But, also by condition (3) on the covers (and in fact why we had it) h is the unique such. Hence we must have $a|h = t$ (as $a \in A_t$.) $\Rightarrow\Leftarrow$. We are assuming $s \neq t$.

So $(A_t \cap p[S_{\langle s,\eta\rangle}])(U) = \emptyset$ and we can let $D_{t,\zeta,s,\eta} = A_t$ as A_t is a sheaf and separates $p[T_{\tau(\langle s,\zeta,\eta\rangle)}]$ from $p[S_{\sigma(\langle s,\zeta,\eta\rangle)}]$ up to U-elements.

Finally, if $I_\alpha(x)$ is undefined and $\forall \mu \in \kappa$, $\exists y \in A \times X \times X(M^x_\mu)$, such that $y|m^x_i = x$ and $I_\alpha(y)$ is undefined then let $I_{\alpha+1}(x)$ be undefined as well. Now define $I = \bigcup_{\alpha\in\zeta} I_\alpha$ where $I_\zeta = I_{\zeta+1}$.

Claim 12.1.1.7. *$I(x)$ is defined for each $x \in \bigcup_{U\in obj(C)}(A \times X \times X)(U)$.*

Proof. Let $UD(I) = \bigcup_{U\in ob(C)}\{x \in (A \times X \times X)(U) : I(x) \text{ is undefined}\}$

Assume there exists $x \in UD(I)$

By the definition of I (and because $x \in UD(I)$) we know that there is some $y|m^x_\mu = x$ such that $\mathrm{dom}(y) = M^x_\mu$ and $y \in UD(I)$. Let x_μ be one such y.

Now by assumption, $\{x_\mu : \mu \in \kappa_x\}$ is a compatible compatible set of elements for J over the covering family $\mathcal{N}_x$ (because I is defined for all elements which aren't in J and so we must have $x_\mu \in J(M^x_\mu)$ for all μ). Hence the they must all come from a single element in $J(Q_x)$. (This was why we defined disjointness the way we did. So that it wouldn't matter the choices of x_μ just so long as they were in the images of the right objects). But if $J(Q_x) \neq \emptyset$ then we must have $J(U) \neq \emptyset$ as there is a map from $U \to Q_x$ by assumption. $\Rightarrow\Leftarrow$

So $UD(I) = \emptyset$. □

Hence $I(\emptyset)$ is defined, separates $p[T]$ from $p[S]$ up to U-elements and is W-Borelian. □

It is worth mentioning explicitly that if all the presheaves are A-finite than we can define a "code" which represents the presheaf which separates $p[T]$ from $p[S]$ up to U-elements in an identical way to Definition 9.2.6.4 and Theorem 9.2.6.5 and the codes will also be A-Recursively Enumerable.

12.1.2 Corollaries

Lemma 12.1.2.1. *If $\{A_i : i \in I\}$ are sheaves, so is $\bigcap_{i\in I} A_i$.*

Proof. Limits are preserved by the inclusion fuctor

$$Sheaves(\mathcal{C}, L) \to Presheaves(\mathcal{C}, L)$$

□

Lemma 12.1.2.2. $p[\bigcup_{i\in I} A_i] = \bigcup_{i\in I} p[A_i]$.

Proof. This is true because it is true point wise (i.e. pointwise there is a witness to an element being $p[\bigcup_{i\in I} A_i]$ iff there is a witness to it being in $p[A_i]$ for some i. □

Corollary 12.1.2.3. *If* $\{S_{i,j} : i \in I, j \in J\}$, $\{T_{i,j} : i \in I', j \in J'\}$ *are sheaves such that* $p[\bigcup_{i\in I} \bigcap_{j\in J} S_{i,j}] \cap p[\bigcup_{i\in I'} \bigcap_{j\in J'} T_{i,j}]$ *has no* U*-elements then* $p[\bigcup_{i\in I'} \bigcap_{j\in J'} T_{i,j}]$ *can be separated from* $p[\bigcup_{i\in I} \bigcap_{j\in J} S_{i,j}]$ *by a* $\max\{W, I, I'\}$*-Borelian presheaf.*

Proof. First note that because the intersection of sheaves is still a sheaf it suffices to consider the case when $p[\bigcup_{i\in I} S_i] \cap p[\bigcup_{i\in I'} T_i]$ has no U-elements.

But, by the definition of projection, we have $p[\bigcup_{i\in I} S_i] = \bigcup_{i\in I} p[S_i]$. So we also have $p[S_j] \cap p[T_i](U) = \emptyset$ and so $p[S_j]$ and $p[T_i]$ are separated up to U-elements by W-Borelian presheaves $C_{i,j}$. Hence, by Lemma 10.3.2.3, $p[\bigcup_{i\in I'} T_i], p[\bigcup_{i\in I} S_i]$ are separated up to U-elements by $\bigcup_{i\in I} \bigcap_{j\in I'} C_{i,j}$ which is a $\max\{W, I, I'\}$-Borelian presheaf. □

Appendix B

Suslin-Kleene Separation Theorem

B.1 Suslin Theorem

B.1.1 Definition

Definition B.1.1.1. Let T be a set. $X \subseteq P(T)$ is a κ algebra on T if whenever $\langle U_i : i \leq \alpha \in \kappa \rangle \subseteq X$ then $\bigcup_{i\in\omega} U_i \in X$ and $\bigcap_{i\in\omega} U_i \in X$

Definition B.1.1.2. Let T be a topological space. The κ-Borelian sets on T is the smallest κ algebra containing the open sets. The κ-Borel sets on T is the smallest κ algebra containing both the open and closed sets.

Definition B.1.1.3. Let T be a topological space. We say a set $A \subseteq T$ is κ-Suslin if there is a closed set $X \subseteq T \times \kappa^\omega$ such that the projection of X to T is A ($p[X] = A$).

Definition B.1.1.4. We say a set is $\mathbf{\Sigma^1_1}$ if it is ω-Suslin. We say a set is $\mathbf{\Pi^1_1}$ if it is the complement of a $\mathbf{\Sigma^1_1}$ set. We say a set is $\mathbf{\Delta^1_1}$ if it is both $\mathbf{\Sigma^1_1}$ and $\mathbf{\Pi^1_1}$.

B.1.2 Results

Theorem B.1.2.1 (Suslin Separation Theorem)**.** *If A, B are disjoint projections (onto ω^ω) of closed sets in $\omega^\omega \times \kappa^\omega$ then there is a $\kappa+1$-Borel set C such that $C \supseteq A$ and $C \cap B = \emptyset$.*

Proof. See [10] Chapter 2E Theorem 2E.1 □

Corollary B.1.2.2. *A subset of ω^ω is $\omega+1$-Borel iff and only if it is $\mathbf{\Delta^1_1}$*

Proof. See [10] for the implication from left to right. To see the direction right to left let A be $\mathbf{\Delta^1_1}$. Then we know by Suslin's Separation Theorem that A and $\neg A$ can be separated by a $\omega+1$-Borel set B (because they are both $\mathbf{\Sigma^1_1}$). So we must have $B = A$. □

This is a particularly important consequence of the Suslin Separation theorem as $\omega+1$-Borel sets are very common and very useful in mathematics. This corollary gives us a concrete way to describe the entire class of $\omega+1$-Borel at once (as opposed to just saying they are the smallest set closed under certain operations) which is very useful.

Theorem B.1.2.3. *Let T be a T_0 topological space with $A \subseteq T$. Then A is $|T|+1$-Borelian.*

Proof. Let $O(x, y)$ be open in T such that $x \in O(x, y)$ and $y \notin O(x, y)$ (that we can always do this is exactly what it means for the space to be

T_0). Let $B_x = \bigcap_{y \in T} O(x,y)$. Then B_x is $|T|+1$-Borelian and $B_x = x$. So $A = \bigcup_{x \in A} B_x$ is $|T|+1$-Borelian □

This shows that just being able to be separated two sets by a κ-Borelian set for some κ isn't apriori a very strong property for a set to have. Rather what is important is the bound we can put on κ.

B.2 Kleene Separation Theorem

B.2.1 Definitions

B.2.1.1 Recursion Theory

Definition B.2.1.1. The class of Kleene Schemes relative to g (for $g : \omega \to \omega$) is the smallest class of functions which contains the following 3 groups of "trivial functions", g and which is closed under the following 3 schemes. If f is a function defined by Kleene Schemes relative to g, then we say f is recursive in g

Trivial Functions:

Here, $\{x_1, \cdots x_k, n\} \subseteq \omega$ and c is a constant.

Successor: $S(n) = n + 1$

Constant c on k arguments: $C^k_c(x_1, \cdots x_k) = c$

Projection of k arguments onto the i^{th}: $P^k_i(x_1, \cdots x_k) = x_i$

Schemes:

Here,$x \in \omega^i, n, m \in \omega$ and the range of all functions is ω

Composition: Given $g_1, g_2, \cdots, g_j : \omega^i \to \omega$ and $h : \omega^j \to \omega^k$ then define $f : \omega) \to \omega^k$:
$f(x) = h(g_1(x), g_2(x), \cdots, g_j(x))$.

Recursion: Given $h(n, m, x)$ and $g(x)$ then define:
$f(0, x) = g(x)$
$f(n+1, x) = h(f(n, x), n, x)$.

Minimization: Given $g(n, x)$ such that $(\sharp)\forall x \in \omega^i \exists n_x \in \omega$ such that $g(n_x, x) = 0$, then define $f(x) = \mu n[g(n, x) = 0]$ (i.e. f(x) is the least n such that $g(n, x) = 0$. If we do not assume $(\sharp)$, then the functions are not necessarily defined everywhere, and so we say f is partial recursive.)

If $g = id$ and f is recursive in g then we say f is recursive.

B.2.1.2 Basic Sets

Definition B.2.1.2. A basic space is a pair $(X, (N(X, s))_{s\in\omega})$ with a recursive function $R : \omega^3 \to \omega$ such that X is second countable, $(N(X, s))_{s\in\omega}$ is an enumeration (possibly with repetitions) of a countable basis for the topology of X and

$$N(X, m) \cap N(X, n) \Leftrightarrow \bigcup_p N(X, R(n, m, p)),$$

where R is called the witness function.

Definition B.2.1.3. Let X be a basic space. We say a set $S \subseteq X$ is semirecursive in g if there is a function f recursive in g such that

$$S = \bigcup_{n\in\omega} N(X, g(n)).$$

Definition B.2.1.4. Let X be a basic space. We say a set $S \subseteq X$ is recursive if both S and $X - S$ are semirecursive.

Intuitively the basic spaces are meant to be generalizations of ω. They are designed to give us enough control over the topology to so that we can generalize the idea of recursive and recursive enumerable sets (which is case correspond to recursive and semirecursive sets).

B.2.1.3 Hyperarithmatic Sets

Definition B.2.1.5. We define the set of Borel Codes for $\boldsymbol{\Sigma^0_\varphi}$ to be denoted BC_φ, by induction on φ as follows.

$$BC_0 := \{\alpha : \alpha(0) = 0\}$$

$$BC_\varphi := \{\alpha : \alpha(0) = 1 \wedge (\forall n)[\{\alpha^\star\}(n) \downarrow \wedge \{\alpha^\star\}(n) \in \bigcup_{\xi<\varphi} BC_\xi]\}$$

and finally

$$BC := \cup_\varphi BC_{\varphi<\omega_1}.$$

Definition B.2.1.6. Let X be a basic space. We define the functions $\pi c^X_\varphi : BC_\varphi \rightarrow \boldsymbol{\Sigma^0_\varphi}$ inductively over φ as follows:

$$\pi c^X_0(\alpha) := N(X, \alpha(1))$$

$$\pi c^X_\varphi(\alpha) := \cup_n (X - \pi c^X_{\beta(n)}(\{\alpha^\star\}(n)))$$

Where $\beta(n) = \mu\varphi[\{\alpha^\star\}(n) \in BC_\varphi]$.Finally

$$\pi c^X := \cup_\varphi \pi c^X_\varphi.$$

We say $\pi c^X(\alpha)$ is the set with Borel code α.

Definition B.2.1.7. Let X be a basic space. We say a subset A is hyperarithmatic in g if there is a Borel code α recursive in g such that $\pi c_{\varphi}^{X}(\alpha) = A$ for some φ. If A is hyperarithmatic in id then we say A is hyperarithmatic

Intuitively the hyperarithmatic sets are those Borel sets in which there is a computable way to build up the Borel set from the basic open sets.

Definition B.2.1.8. Let X be a basic space. We say a subset A is $\Sigma_1^1(g)$ if there is a Borel code α

- α is recursive in g
- $\pi c_{\varphi}^{X\times\omega^{\omega}}(\alpha) = C$
- C is closed in $X \times \omega$
- A is the projection of C onto X.

We say A is Σ_1^1 if it $\Sigma_1^1(id)$

The Σ_1^1 sets are the recursive analog of the ω-Suslin sets.

B.2.2 Results

Theorem B.2.2.1. *Let $s : \omega \to \omega^{<\omega}$ be a bijection. Then ω^{ω} is a basic space with $N(\omega^{\omega}, n) = \{x \in \omega^{\omega} : x|dom(s) = s\}$ and*

$$R(n, m) = n \text{ if } s(m) \subseteq s(n)$$

$$R(n, m) = m \text{ if } s(n) \subseteq s(m)$$

$$R(n, m) = s^{-1}(\emptyset) \text{ otherwise}$$

Proof. Immediate from the definition of basic space. □

Theorem B.2.2.2 (Kleene Separation Theorem)**.** *If X, Y are disjoint Σ^1_1 sets of reals then there is a hyperarithmatic set Z such that $Z \supseteq X$ and $Z \cap Y = \emptyset$*

Proof. See [10] Chapter 7B □

Theorem B.2.2.3. *Borel Sets* $= \bigcup_{g:\omega\to\omega}$ *Sets Hyperarithmatic in g*
$\boldsymbol{\Sigma^1_1} = \bigcup_{g:\omega\to\omega} \Sigma^1_1(g)$

Proof. This follows from Theorem 9.2.6.6. (See [10]) □

This give a concrete version of Theorem 9.2.6.5.

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