Math Gems

An assortment of mathematical marvels.

$$\begin{array}{r} 12345679 \\ \times\ 9 \\ \hline 111111111 \\ \times\ 111111111 \\ \hline 12345678987654321 \end{array}$$

$$\frac{1}{9} = .111111\ldots$$

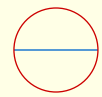
Fun arithmetic with the number nine.

$$142857 \times 2 = 285714$$
$$142857 \times 3 = 428571$$
$$142857 \times 4 = 571428$$
$$142857 \times 5 = 714285$$
$$142857 \times 6 = 857142$$
$$\frac{1}{7} = .142857\ldots$$

Fun arithmetic with the number seven.

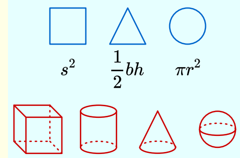
16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

A magic square. All rows, columns, and diagonals have the same sum.



$$\pi = 3.14159\ldots$$

The ratio of the circumference of a circle to its diameter is pi. Pi is transcendental, i.e., irrational and non-algebraic.



s^2 $\quad \frac{1}{2}bh$ $\quad \pi r^2$

s^3 $\quad \pi r^2 h$ $\quad \frac{\pi}{3} r^2 h$ $\quad \frac{4}{3}\pi r^3$

Area and volume formulas. Archimedes solved the sphere.

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots$$

$$\frac{\pi}{2} = \frac{2}{1} \times \frac{2}{3} \times \frac{4}{3} \times \frac{4}{5} \times \frac{6}{5} \times \cdots$$

Pi, expressed as an infinite series and an infinite product.

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

The sum of the numbers from 1 to n.

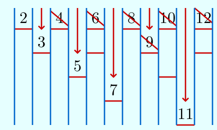
$$\prod_{i=1}^{n} i = 1 \times 2 \times 3 \times \cdots \times n = n!$$

The product of the numbers from 1 to n is called n factorial.

$$n! \approx \left(\frac{n}{\mathrm{e}}\right)^n \sqrt{2\pi n}$$
$$\Gamma(n+1) = n!$$
$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Stirling's approximation of n factorial. Euler's gamma function gives factorials for integers but has surprising values for fractions.

2, 3, 5, 7, 11, 13, 17, 19, …



A prime number is divisible only by one and itself. The sieve of Eratosthenes finds primes.

$$\pi(x) \approx \frac{x}{\log x}$$

The prime number theorem of Gauss and Legendre approximates the number of primes less than x.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \frac{1}{1 - \frac{1}{p^s}}$$

The zeta function of Euler and Riemann, expressed as an infinite series and a curious product over all primes.

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$
$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

The binomial theorem expands powers of sums. The binomial coefficient is the number of ways to choose k objects from a set of n objects, regardless of order.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
…

Pascal's triangle shows the binomial coefficients.

Suppose $\sqrt{2}$ were rational

→ $\sqrt{2} = \frac{n}{m}$, reduced

→ $\left(\frac{n}{m}\right)^2 = 2$ → $n^2 = 2m^2$

→ n^2 is even → n is even

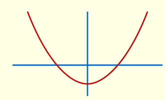
→ n^2 is divisible by 4

→ m^2 is even → m is even

→ $\frac{n}{m}$ is not reduced

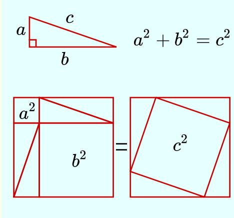
→ $\sqrt{2}$ is not rational

Proof that the square root of two is irrational.



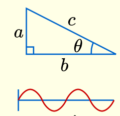
$$ax^2 + bx + c = 0$$
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The quadratic equation defines a parabola.



$$a^2 + b^2 = c^2$$

The Pythagorean theorem. A proof by rearrangement.



$$\sin\theta = \frac{a}{c} \qquad \cos\theta = \frac{b}{c} \qquad \tan\theta = \frac{a}{b}$$

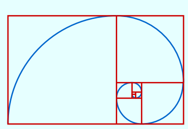
$$y = \sin x$$

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$
$$\sin^2\theta + \cos^2\theta = 1$$

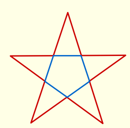
The trigonometric functions. Another form of the Pythagorean theorem.

$$\phi = \frac{a+b}{a} = \frac{a}{b}$$
$$\phi = 1 + \frac{1}{\phi}$$
$$\phi = 1.618\ldots \quad \frac{1}{\phi} = 0.618\ldots$$

The golden ratio, phi. The ratio of a whole to its larger part equals the ratio of the larger part to the smaller. phi is irrational and algebraic.



The golden rectangle, a classical aesthetic ideal. Cutting off a square leaves another golden rectangle. A logarithmic spiral is inscribed.



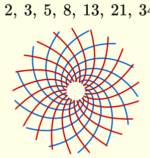
$$\phi = \frac{1+\sqrt{5}}{2}$$

The pentagram contains many pairs of line segments that have the golden ratio.

$$\phi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}$$

The golden ratio, expressed as a continued fraction.

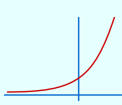
1, 1, 2, 3, 5, 8, 13, 21, 34, …



Each Fibonacci number is the sum of the previous two. The number of spirals in a sunflower or a pinecone is a Fibonacci number.

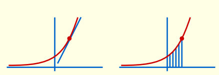
$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \phi$$
$$F_n = \frac{\phi^n - \left(\frac{-1}{\phi}\right)^n}{\sqrt{5}}$$

The ratio of successive Fibonacci numbers approaches the golden ratio. An exact formula for the nth Fibonacci number.



$$y = \mathrm{e}^x \quad x = \log y$$
$$\mathrm{e} = 2.71828\ldots$$

Napier's constant, e, is the base of natural logarithms and exponentials. e is transcendental.



$$\frac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x \qquad \int \mathrm{e}^x\,\mathrm{d}x = \mathrm{e}^x$$

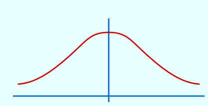
Calculus, developed by Newton and Leibniz, is based on derivatives (slopes) and integrals (areas) of curves. The derivative of e^x is e^x. The integral of e^x is e^x.

$$\mathrm{e} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$
$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

e, expressed as a limit and an infinite series.

$$\mathrm{e}^{x\sqrt{-1}} = \cos x + \sqrt{-1}\sin x$$
$$\mathrm{e}^{\pi\sqrt{-1}} = -1$$

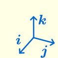
Euler's formula relating exponentials to sine waves. A special case relating the numbers pi, e, and the imaginary square root of −1.



$$y = \frac{1}{\sigma\sqrt{2\pi}}\,\mathrm{e}^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

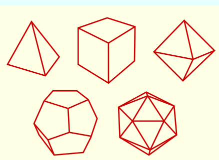
The Gaussian or normal probability distribution is a bell-shaped curve.

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



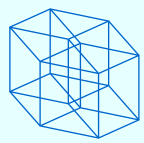
$$\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$$
$$\nabla U \quad \nabla \cdot \vec{V} \quad \nabla \times \vec{V}$$
$$\nabla^2 U \quad \Box U$$

Gibbs's vector cross product. Del operates on scalar and vector fields in 3D, box in 4D.



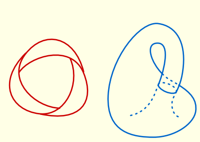
$$v - e + f = 2$$

The five regular polyhedra. Euler's formula for the number of vertices, edges, and faces of any polyhedron.

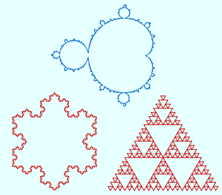


$$v - e + f - c = 0$$

The hypercube. Schläfli's formula for vertices, edges, faces, and cells of any 4-dimensional polytope.



The Möbius strip has only one side. The Klein bottle's inside is its outside.



Fractals of Mandelbrot, Koch, and Sierpinski have infinite levels of detail.

Imagine listing all real numbers between 0 and 1 in any order.

1 → .⑧4 9 7 3 8…
2 → . 1⑦9 3 8 0…
3 → . 1 0③4 2 1…
4 → . 3 5 6①2 2…

You can always make an unlisted real number by changing every digit on the diagonal, e.g., change .8731… to .9842…

Cantor's proof that the infinity of real numbers is greater than the infinity of integers.

$$(\exists y)(x) \sim \mathrm{Dem}(x, y)$$
$$\supset$$
$$(x) \sim \mathrm{Dem}(x, \mathrm{sub}(n, 13, n))$$

[from Nagel and Newman, *Gödel's Proof*]

Gödel proved that if arithmetic is consistent, it must be incomplete, i.e., it has true propositions that can never be proved.

To find out more, look it up on the web or in the library.