ON REMARKABLE CHARACTERISTICS OF QUANTIZED ELECTRON ORBITS

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A translation of:
Über eine bemerkenswerte Eigenschaft der Quantenbahnen eines einzelnen Elektrons
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In WEYL's[1] space-time geometry there appears, in addition to the well known *quadratic* differential form, which specifies the metric at each point, another linear form:

$$\varphi_0 dx_0 + \vec{\varphi} \cdot d\vec{x} = \varphi \cdot d\mathbf{x},$$

which specifies the metrical interrelationship between points. Its geometrical significance is that, it gives the change in the "measure" (Masszahl) of the displacement l (the absolute square of the interval) under "congruent displacement" to neighboring points by

$$dl = -l\varphi \cdot d\mathbf{x}. \tag{1}$$

WEYL discovered that through both together (metric and metrical interrelationship) an affine connection for the world (i.e., a definition of vector parallel transport) is determined if one also demands that for parallel transport of a vector, its modulus is to be congruently transported. For congruent transport of a displacement along a world line — e.g., for parallel transport of a vector along such a displacement — the measure is multiplied by the factor

$$e^{-\int \varphi \cdot d\mathbf{x}}, \tag{2}$$

where the line integral is to be taken naturally along the world line, upon which it significantly depends, so that the quantities

$$\mathbf{f} = \nabla \times \varphi, \tag{3}$$

do not vanish identically. As Physics, the affine connection given above constitutes the gravitational field and $\mathbf{f}$ are the electromagnetic fields. If the situation, including the choice of coordinates, is such that in a world patch, at least approximately, x_0 is the time (in seconds) and $\vec{x}$ are Cartesian coordinates (in cm), then φ, up to a constant, is proportional to the usual electromagnetic potentials:

$$V, -\frac{\vec{A}}{c}. \tag{4}$$

Let us write the constant as $\gamma^{-1}e$ where e is the electron charge in electrostatic units; that is:

$$\varphi_0 = \gamma^{-1} eV, \quad \vec{\varphi} = -\gamma^{-1}\frac{e}{c}\vec{A},$$

so that φ_0 has the units of $\sec^{-1}$, eV of 'energy' and γ of 'work' (g cm^2 sec^{-1}). The "displacement factor," Eq. (2) becomes:

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[1] See: WEYL, H., *Raum, Zeit, Materie*, (Springer, Berlin, 1921). Denoted as "RZM" below.

$$e^{-\frac{e}{\gamma}\int(v dt - \vec{A}\cdot dx)}. \tag{5}$$

The characteristic of quantum orbits, mentioned in the title, and to me especially noteworthy, is that those which are "real," that is that lead to observed energies and spectral lines, are also those for which the 'displacement factor,' Eq. (5), arises as an integer factor of $\gamma^{-1}h$ (which from the above considerations turns out to be a whole number) for all approximate periods of the system. I wish to consider first some particlar examples as there are a number of conditions to be added to the story in its simple form given above. Then I would like to discuss the significance of these conditions, although, to be honest, I have not yet achieved much.

A. Undisturbed Kepler orbits.[2] Let us begin ignoring relativity, which shall be considered below (under ***E).*** For this case, the only 'real' quantum condition[3] is

$$J = 2\tau\bar{T} = nh, \tag{6}$$

where τ in the period, $\bar{T}$ is the average kinetic energy.

Let V be the potential of the positive charge of a hydrogen nucleus, which then vanishes at infinity. Then, as is well known, (we take e as a whole number)

$$\bar{T} = \frac{1}{2}e\bar{V}, \tag{7}$$

which in Eq. (6) gives:

$$e\tau\bar{V} = e\int_0^\tau V dt = nh. \tag{8}$$

Thus, the exponent of the displacement factor, Eq. (5), is $-nh/\gamma$ for one period. The only extra condition in this case is the normalization of the additive constant in V.

B. Zeeman effect. Mechanically this amounts to introducing the LARMOR pression with frequency (number of complete precession rotations per second)

$$\frac{1}{\vartheta} = \frac{eH}{4\pi mc}. \tag{9}$$

Quantum theoretically the above condition persists and secures the "whole number character of the displacement exponent" (as a brierf manner of speaking) for the first period τ, at least approximately. Closer examination shows that Eq. (8) remains valid up to quadratic terms in H in so far as the LARMOR theorem holds still for this approximation for the rotated coordinate origin both mechanically and quantum theoretically just as it does for the motionless case ***A***. If we now examine whether whole integers pertain to the second period ϑ, we can ignore the V term, as it gives a whole number contribution (namely, as many times — nh, as the simple Larmor cycle rotates). Now, the second quantum condition is known, that is, from the surface moment about the field axis

$$2m\frac{f}{\tau} = \frac{n'h}{2\pi}, \tag{10}$$

[2]Prof. WEYL has written me that this proposition was known to Prof. FOKKER, already two years ago, and that it lead him to imaginary values of φ. See below.

[3]Here we are following BOHR's formulation, in particular his theory of perturbed periodic systems, as they were treated in part II of the still uncompleted Proceedings of the Copenhagen Akdemy, *Naturw. u. Mathem. Art.* 8 Series **IV, 1, 2** (1918). Cited below as "BOHR l.c."

where f is the projection of the elliptical surface on the plane of the equator. From Eq. (9) and (10) one gets

$$Hf \cdot \frac{\vartheta}{\tau} \cdot \frac{e}{c} = n'h. \tag{11}$$

Hf is the work (Kraftflus) through the ellipse, so that

$$Hf = \int (\nabla \times \vec{\mathbf{A}})_n df = \int_\tau \vec{A} \cdot dx, \tag{12}$$

gives, with Eq. (11), for the whole LARMOR cycle

$$\frac{e}{c} \int_{(\vartheta)} \vec{A} \cdot \overrightarrow{dx} = n'h; \tag{13}$$

that is, the additional quantum condition supports the exact "whole number character" of the extra magnetic terms in the displacement exponent, extended over the LARMOR period.

C. Stark effect.[4] Mechanically in this case there appears a secular variation in both the position and form of the KEPLER orbit; still the secular variation (in the approximation in which the experiment is analyzed) is periodic, that is, when the KEPLER ellipse has completed a secular cycle and retaken the same form, it finds itself at the same position. The cycle path can be best described as follows. If one specifies the center-of-mass of the usual KEPLER orbit taking into consideration the dwell time of the electron at various positions along the orbit (i.e., the "electrical center-of-mass"), then one finds the middle point of the line between the charges. The 'electrical center of mass' executes harmonic, in general elliptical, motion in a plane perpendicular to the field direction. Then, according to the above, the form of the KEPLER must change, not in its size (i.e., neither the energy nor cycle time), rather only its eccentricity, which is determined by the electrical 'center of gravity.' The then position of the orbital plane is determined in so far as, although the total angular momentum and eccentricity vary, the components in the field direction remain constant.

The supplementary quantum condition consists, then, in, that the distance of the nucleus from the above mentioned plane perpendicular to the field direction, in which the electrical 'center of mass' executes its secular harmonic motion, can take on only certain discrete values. A more convenient formulation of this quantum condition can be gotten from BOOR'S theory of periodic systems. The additional energy, which simply equals the potential energy of the electron in an external field for a KEPLER period (the average is a secular constant) — the additional energy, in my view, is related to the secular period ϑ in exactly the same ration as the total energy to the period of an harmonic oscillator, that is

$$\triangle E = n'h\frac{1}{\vartheta}, \tag{14}$$

where $\triangle E$ is the additional energy, and n' is a whole number. Now when V', the potential of the exterior field, under the conditions mention above is so normed that it vanishes in the nucleus, then one sees easily that

$$\triangle E = -e'\bar{V}' = -\frac{e}{\tau}\int_t^{t+\tau} V'dt. \tag{15}$$

From Eqs. (14) and (15) it follows that

$$\frac{e\vartheta}{\tau}\int_t^{t+\tau} V'dt = e\int_t^{t+\tau} V'dt = -n'h. \tag{16}$$

[4] BOHR, *l.c.* ¶4, p. 69.

A glance at Eq. (5) reveals that with the "whole number" character of the electronic term in the displacement exponent is demonstrated for the secular STARK period — in complete analogy to the result for the LARMOR period in the ZEEMAN effect.

With the ZEEMAN effect we were able, by virtue of the especially simple character of the secular perturbation from the whole number character of the additional term, to infer the whole number character of the displacement exponent. Here that inference would be too hasty, as the average of the nuclear potential, V, over a KEPLER ellipse suffers perturbations of the first order, which during a secular period, ϑ, could accumulate to a finite contribution.[5] To be completely certain, let us return to the explicit form of the central quantum condition of the perturbed case. Let $\mathbf{q}$ be the electron's orthogonal coordinates, $\mathbf{p}$ the momentum, so that

$$\int_t^{t+\vartheta} (\mathbf{p}\cdot\mathbf{q})dt = nh, \tag{17}$$

where ϑ now — more precisely — is the an exact quasi period of the system after which to high precision the coordinates and momenta repeat. As a consequence it must be the case that

$$\int_t^{t+\tau} \frac{d}{dt}(\mathbf{p}\cdot\mathbf{q})\,dt = 0.$$

Thus, in stead of Eq. (17), one can also write:

$$\int_t^{t+\tau} (\mathbf{q}\cdot\dot{\mathbf{p}})dt = -nh; \tag{18}$$

or, if

$$U = -e(V+V'),$$

the potential energy is, as follows because of the equation of motion from Eq. (18)

$$\int_t^{t+\tau} (\mathbf{q}\cdot\nabla U)dt = nh. \tag{19}$$

The two terms of U are homogeneous functions of $\mathbf{q}$, in fact V is homogeneous of degree -1 and V of degree $+1$. Therefore it follows from Eq. (19) that

$$\int_t^{t+\tau} e(V-V')dt = nh. \tag{20}$$

Considering Eq. (16), it follows

$$\int_t^{t+\tau} e(V+V')dt = (n-2n')h, \tag{21}$$

which completes the proof. — Regarding the STARK effect, we emphasize again the necessity of normalization of the potential with which it vanishes in the nucleus.

[5] Note that BOHR has shown — as follows directly from the constancy of $\bar{V}'$ —, that the average of the total energy function of the unperturbed KEPLER orbit suffers only second order perturbations. Here, however, the issue concerns only potential energy, and the perturbing field destroys the simple relation, Eq. (7) between the two energy contributions.

D. Combined STARK *and* ZEEMAN *effect with parallel axes*[6]. According to BOHR's theory of perturbed periodic systems, one gets for superimposed homogeneous electric and homogenous magnetic fields, when the perturbation from both fields is of roughly the same magnitude, well defined discrete quantized orbits only when the field axes are parallel.[7] We restrict ourselves, therefore, to this case. In mechanical terms we have here simply one of the above considered rotations of the STARK effect cycle with respect to a LARMOR rotation, Eq. (9), of the axis crossing, where it is to be kept in mind that the LARMOR frequency depends only from the electron constants and magnetic field strength and not from form or orientation of the orbit so that LARMOR rotation is uniform. In addition the quantum conditions are superposed, so to speak. The large half axes of the KEPLER ELLIPSES are allowed to have the same value, as unperturbed atoms, for the distance from the nucleus at which the electrical center-of-mass oscillates, i.e., the same value as in the pure STARK effect; the magnetic field constrains the components of angular momentum in the field direction (which in the pure STARK effect were constant but not quantized) now to be whole number multiples of $\hbar$ as in the ZEEMAN effect. The total perturbation is, naturally,no longer purely periodic, rather there arises two, in general unreconcilable, secular periods of nearly the same magnitude: in one LARMOR precession of the comoving coordinates reproduce the form and orientation of KEPLER ellipses with period ϑ_s of the STARK effect, while the ellipses harmonically pass through the electrical center-of-mass with LARMOR period ϑ_l about the field direction. In that with relations to the rotating system and quantum mechanically exactly the same relationships prevail as for the pure STARK effect with respect to a stationary system, and insofar as the electric field by cause of LARMOR rotation is transformed into itself, one easily sees that the first two quantum conditions lead to

$$\int_t^{t+\vartheta_s} e(V+V')dt = nh. \tag{22}$$

With regard to magnetic quantization, recall that both KEPLER periods and the angular momentum [Flaächenmoment] in the field direction, and therefore also the projection of the KEPLER ellipse on the equatorial plane or the magnetic work around the KEPLER ellipse are secular constants. From this is follows from the magnetic quantization condition just as in §B, that

$$\frac{e}{c}\int_{(\vartheta_l)} \vec{A}\cdot\vec{dx} = n'h, \tag{23}$$

when integrated over a LARMOR cycle. That the KEPLER ellipse does not return to its original form and orientation, does not affect matters.

Eqs. (22) and (23) each represent only a part of the "displacement factor," (22) is he electric contribution and (23) the magnetic part. Moreover they pertain to entirely different time intervals, ϑ_s and ϑ_l for which neither constitutes a quantum period. The latter can and does in general comprise to a certain approximation multiples of these periods whenever the following nearly holds:

$$n_s\vartheta_s = n_l\vartheta_l = \vartheta.$$

If here we chose n_s as a whole number, and then n_l so that this relationship is satisfied exactly, multiply Eq. (22) by n and (23) by n_l and subtract them, we get

$$e\int_{(\vartheta)} (V+V')dt - \frac{1}{c}\vec{A}\cdot\vec{dx} = (n_s n - n_l n')h. \tag{24}$$

[6] BOHR, N., *l.c.*,p. 91.

[7] BOHR, N., *l.c.*, p. 93.

Here we find on the left (up to the factor -γ^{-1}) the whole displacement factor for the quasi period ϑ; on the right there is a whole multiple of h, that is a whole number to the same approximation as allowed by the same approximation of ϑ. While n' is the usual magnetic quantum number, thus, at least for low lying orbits, a small whole number; small deviations of n_l away from being a whole number will be by multiplication by n' insignificantly enlarged. (This is not so for n, which is a very large number of the order of the number of KEPLER rotations during a STARK period; this does no damage as n_s is exactly a whole number and must be so chosen that the phase that within a KEPLER orbit reproduce themselves.) It seems somewhat unsatisfying that the derivation of Eq. (24) requires a certain linear combination of both the "true" (i.e., necessary to determine the energy) conditions Eqs. (22) and (23). Thus, it seems to me that Eqs. (22) and (23) are singularly necessary to determine Eq. (24) for each quasi-period. For example if $n_s = 7$, $n_l = 12$ gives a quasi-period, then in general not $n_s = 70$, $n_l = 120$, rather maybe $n_s = 69$, $n_l = 118$ another, about ten times larger. However, one may not take high multiples of the secular period for such considerations, as quadratic members of fields do not enter, where less the approximations comes into question than the physical coherence of a quantum orbit.

E. Relativistic mass alteration. Iin sections §B, C, D we have neglected to consider what happens in these cases, to expose the electron to the perturbation from the external field taken as large in comparison to the relativistic mass perturbation caused by the pure periodic KEPLER orbit. Taking it into the calculation, the force free atom already has two periods, the short, KEPLER one τ, and the period ϑ of the perihelion rotation. For τ the "whole number character of the displacement factor" is satisfied by the same quantum conditions as in the non relativistic case. The question arises, whether it pertains to ϑ. If one fixes ϑ more sharply, as a quasi-period, that is, so that the coordinates and momenta nearly reproduce themselves, then the following, expressed in polar coordinates, holds:

$$\int_t^{t+\vartheta} (p_r \dot{r} + p_\varphi \dot{\varphi}) dt = n'h, \tag{25}$$

[r, φ are polar coordinates, p_r, p_φ the corresponding momenta; Eq. (25) is a whole number linear combination of the usual "radial" and "azimuthal" quantum conditions, and is the number of the φ-rotations exactly one larger than the r-oscillations]. The integrand is invariant under point transformations, thus the Euclidean variant is

$$\int_t^{t+\vartheta} (p_x \dot{x} + p_y \dot{y}) dt = n'h. \tag{26}$$

For this one can, given that $(xp_x + yp_y)$ returns to its initial value, write:

$$\int_t^{t+\vartheta} (x\dot{p}_x + y\dot{p}_y) dt = -n'h, \tag{27}$$

$\dot{p}_x, \dot{p}_y$ are in relativistic mechanics the negative partial derivatives of the potential energy, namely $-eV$ and are homogeneous in x, y of the first degree. Therefore it follows from Eq. (27)

$$\int_t^{t+\vartheta} eV dt = n'h. \tag{28}$$

This demonstrates the rectitude of our proposition for unperturbed relativistic orbits.

The ZEEMAN effect with Relativity taken into account[8] is rather simple, it brings up simply the relativistic rosette in the LARMOR rotation. There are two secular periods, as in §D with two parallel fields.The treatment there is fully analogue, that it repeats—it can be understood without calculation and leads naturally again to a verification of our propagation.

The STARK effect with relativity, which KRAMERS[9] treated recently in a beautiful paper, I have net yet analyzed from this viewpoint; nevertheless, there can be little doubt that the situation is very similar to that for the ZEEMAN effect, and in §D.

The case considered in §D with Relativity, to my knowledge, has not been investigated, however (because of rotational symmetry) must lead to well defined quantum orbits. It is still of little interest.

Discussion and conclusions. In summary, we have considered the following. If an electron on its way along an orbit were to take an "interval" along with it, which with the motion did not change, then for the interval, considered from an arbitrary starting point, its measure would appear to be multiplied by whole multiple of the factor

$$e^{\frac{h}{\gamma}}, \tag{29}$$

once for each passage of the starting point.

It is difficult to imagine, that this result is only an accidental consequence of the quantum conditions and without deeper physical significance. The somewhat imprecise form of the approximation, which has emerged, changes nothing; we do know, that the quantum orbits are physically defined lacking complete precision[10] for two reasons: first because of radiation reaction, which surely does not exist in the classical electrodynamic form, but which is surely of the same order quantum mechanically also, otherwise the decay time could not be calculate correctly[11]. [For example, in the ZEEMAN effect the field strength quadratic terms can be ignored *in principle;* and in the STARK effect belongs, if relativity is taken into account, no longer to the separable problems[12].]

That an electron really carries an "interval" along, is more than questionable. It is well possible, that it is "frozen" in its progression in the sense used by WEYL[13]. It can be, that the condition of our proposition is to be found, in the fact that for electrons not all tempos are equally possible, but depend somehow on the quasi-period of the orbit.

One feels tempted to guess which value the universal constant γ might have. There are two well known constants with the units of action, namely h and e^2/c (for my part, I am convinced that they are mutually independent). Were $\gamma \sim e^2/c$, the universal fact (29) would be a very large number[14] on the order of e^{1000}. the other possibility, $\gamma \simeq h$, suggests the imaginary value

$$\gamma = \frac{h}{2\pi\sqrt{-1}},$$

[8] A. SOMMERFELD, *Phys.Z.* **17**, 491 (1916), was first to treat this problem, see also: P. DEBYE, *evenda* p. 507.

[9] H. KRAMERS, *Z. f. Phys.* **3**, 199 (1920).

[10] N. BOHR, *l.c.* pp. 50,61,66,97.

[11] A. SOMMERFELD and W. HEISENBERG, *ZS. f. Phys.* **10**, 393 (1922).

[12] H. A. KRAMERS, *ZS. f. Phys.* **3**, 201 (1920).

[13] WEYL, , RZM, p. 280.

[14] $2\pi e^2/(hc)$ is the so-called fine structure constant equaling 7.29×10^{-3}.

where then the universal factor (29) would be nearly unity and the measure of an interval carried along would repeat itself each quai-period. I do not presume to decide if this sort of thing is meaningful in the WEYL world geometry.

Moreover, it is natural to consider that e, h, c are not the only universal constants. If one adds to this list the gravitational constant k and a universal mass, that of the electron, m_e, say, then[15]

$$\frac{e^2}{km_e^2} \simeq 10^{+40}.$$

This would render

$$\frac{he^2}{km_e^2}$$

a "universal quantum of action" of the order of 10^{13} erg./sec. — We wish also recall that out of just dimensional considerations alone, little can be determined.

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[15]See: WEYL, RZM p. 238.