NUMERICAL INTEGRATION FOR THE REAL TIME PRODUCTION OF FUNDAMENTAL EPHEMERIDES OVER A WIDE TIME SPAN

ALDO VITAGLIANO
Dipartimento di Chimica, Università di Napoli 'Federico II', via Mezzocannone 4, 80134 Napoli, Italy (e-mail: aldov@chemna.dichi.unina.it)

(Received: 21 May 1996; accepted: 4 December 1996)

Abstract. A simplified model of the solar system has been developed along with an integration method, enabling to compute planetary and lunar ephemerides to an accuracy better than 1 and 2 milliarcsecs, respectively. On current personal computers, the integration procedure (SOLEX) is fast enough that by using a relatively small ($\sim$ 20 Kbytes/Cy) database of starting conditions, any epoch in the time interval (up to $\pm$100 Cy) covered by the database can be reached by the integrator in a few seconds. This makes the algorithm convenient for the direct computation of high precision ephemerides over a time span of several millennia.

Key words: Ephemerides, extrapolation methods, numerical integration, orbit computation, solar system

1. Introduction

Numerical integration of the equations of motion is currently the most accurate method of computing fundamental ephemerides. Accordingly, the planetary and lunar ephemerides compiled by the Astronomical Almanac are based on the DE200 integration carried out at the Jet Propulsion Laboratory in the early 80s (Newhall *et al.*, 1983; Standish, 1982). On the other hand, numerical integration is normally *not* used by routine procedures computing planetary and lunar positions. When low to moderate precision is enough, algorithms are used (Meeus, 1991) which are based on truncated versions of the huge numerical series produced by analytical or semi-analytical theories (Bretagnon and Francou, 1988; Chapront-Touzé and Chapront, 1983, 1988). If high precision is demanded, Chebyshev polynomials can be used, interpolating the numerical integration output over short and contiguous intervals of time (Newhall, 1989; Kammeyer, 1989). In the latter case, a drawback is that only a limited span of time can be covered by the stored data, unless a large amount of disk memory is used. For example, a database of 840 Kbytes is used to store the Chebishev coefficients for the DE200 ephemerides, covering about 250 years (Kammeyer, 1989). The question arises if the *direct use* of a numerical integrator could be more convenient for the quick routine computation of ephemerides than the interpolation of pregenerated data. It is the purpose of this paper to discuss the above question and to propose a numerical algorithm which can conveniently perform the required task.

Celestial Mechanics and Dynamical Astronomy **66:** 293–308, 1997.

2. Direct Integration *versus* Interpolation of Pre-generated Data

The main disadvantage of numerical integration for the computation of ephemerides is that it cannot immediately find the position of a celestial body at a given instant. To do the job, a suitable set of starting conditions has to be fed to the integrating procedure, and the program has to run step-by-step until the desired epoch is reached. At that point, *all* the planetary positions will be known at once, and if a table of ephemerides is desired, subsequent calculations will produce it with a speed comparable if not better than any other method. The initial overcost of running the integrator up to the desired epoch can be reduced by creating and storing on disk memory a database of precalculated starting conditions, among which the program will chose the one which is closest in time to the target epoch. Of course the time spacing between stored starting points can be made small enough to make the maximum initial 'go to epoch' time as short as desired for a given machine. This is thus a typical case in which speed can be bought with disk memory and vice-versa, according to the individual hardware and preferences.

Suppose you have got a suitable integration program already, and have built a database of starting conditions smaller than the database of Chebyshev coefficients which would be needed to cover the same range of epochs. The integration algorithm can be convenient against the Chebyshev interpolation only if the average 'go to epoch' waiting time is some small, tolerable value, say a fraction of a minute. Whether this condition is matched or not is of course dependent on the machine running the program. As a consequence, Chebyshev interpolation is best suited for slow machines, while direct integration would be best suited for fast machines. Optimization of speed is thus very important for the convenience of the direct integration approach, and the selection of a mathematical model as simple as possible and of an integration method as fast as possible appears to be essential.

3. Mathematical Model

A reasonably accurate model of the Solar System should include, besides the purely Newtonian accelerations, the additional perturbations due to general relativistic effects, figure effects and tidal forces. A model taking full account of all the above contributions, as the one developed at JPL (Newhall *et al.*, 1983) would require too much computational effort to be suitable for the practical purpose of this work. Indeed, this model has been recently implemented (Moshier, 1992) in a program (de118i) running on personal computers and available on computer networks. In a test run on a PC equipped with a 75 MHz 486 dx4 processor, about hundred seconds are required to integrate one year of motion of the solar system. Assuming a tolerable 'go to epoch' waiting time of 5 seconds, a database of starting conditions recorded at one month intervals should be provided to the program. This would be about twice larger than the corresponding database of Chebyshev coefficients, which would make the direct use of this integrator inconvenient.

Following the above considerations, it seemed attractive to find out if a conveniently simple model could reproduce the planetary and lunar ephemerides (DE200) to an accuracy of the order of a few milliarcseconds. The strategy was to start with a simple model, fitting it to the DE200 positions, and then to improve it, until an acceptable accuracy (revealed by the post-fit residuals) was reached. A discussion follows about how the various perturbations have been considered in the final model.

3.1. GENERAL RELATIVISTIC EFFECTS

The approach was to consider the relativistic perturbations in a two-body approximation, and to simulate the major one through the addition of a dipole-like potential (Nobili and Roxburgh, 1986)

$$R = \frac{3(GM_o)^2}{c^2 r^2}. \tag{1}$$

As described by Nobili and Roxburgh (1986) this reproduces the secular advance of perihelion predicted by general relativity*exactly*, neglecting a small contraction of the orbit given in Equation 2 and an even smaller periodic oscillation around the average orbit having an amplitude given in Equation 3. The constant orbital contraction given in Equation 2 is easily simulated by a corresponding fractional correction to the mass of the Sun (Equation (4)).

$$\left(\frac{\Delta a}{a}\right)_c \cong -\frac{3GM_o}{c^2 a}(1 + \tfrac{5}{2}e^2), \tag{2}$$

$$\left(\frac{\Delta a}{a}\right)_p \cong \frac{GM_o}{c^2 a}\tfrac{1}{2}e^2, \tag{3}$$

$$\frac{\Delta M_o}{M_o} = -\frac{9GM_o}{c^2 a} + O(e^2). \tag{4}$$

According to the above considerations, the following Equation 5 was adopted for computing the acceleration of the ith body due to the Sun, while the pure Newtonian field was used for computing the perturbative planetary accelerations.

$$\ddot{\mathbf{r}}_{o_i} = -\frac{GM_o}{|\mathbf{r}_{o_i}|^3}(\mathbf{r}_i - \mathbf{r}_o)\left(1 - \frac{9GM_o}{c^2 a_i} + \frac{6GM_o}{c^2|\mathbf{r}_{o_i}|}\right). \tag{5}$$

The accuracy of the above model was tested by comparison with the full PPN relativistic model implemented by Moshier (1992) over a time span of up to hundred centuries (Figure 1). No relevant secular drifts were observed, and the maximum deviation was a periodic residual with an amplitude of ~ 1 milliarcsec in the longitude of Mercury. It is worth to point out that, differently from a full or even a simplified PPN formulation (Quinn *et al.*, 1991), the acceleration computed in Equation 5 is not dependent on the velocity. This gives the advantage of allowing the use of a simpler and comparatively more efficient integration algorithm (Press *et al.*, 1992).

3.2. FIGURE AND TIDAL EFFECTS

Figure effects strongly influence the lunar orbit. Seeking for a reasonably good compromise between computational effort and precision the semiempirical model depicted in Equations (6–9) was developed. In a geocentric equatorial reference frame, x y and z are the coordinates of the Moon, r is the Earth–Moon distance, $\ddot{x}$ $\ddot{y}$ and $\ddot{z}$ are the components of the acceleration due to the Earth and acting on the Moon, ε is the obliquity of the ecliptic, $z_{\rm ec}$ is the geocentric ecliptic z coordinate of the Moon and r_s is the Sun–Moon distance.

$$\ddot{x} = \frac{-\mu_e}{r^3}\left\{[1+S]x + Q_t\frac{x+y\delta}{r^5}\right\}, \tag{6}$$

$$\ddot{y} = \frac{-\mu_e}{r^3}\left\{[1+S]y + \frac{2}{5}\frac{Q_m}{r^2}z_{\rm ec}\sin\varepsilon + Q_t\frac{y-x\delta}{r^5}\right\}, \tag{7}$$

$$\ddot{z} = \frac{-\mu_e}{r^3}\left\{[1+S]z - \tfrac{2}{5}(Q_e z + Q_m z_{\rm ec}\cos\varepsilon)\frac{1}{r^2} + Q_t\frac{z}{r^5}\right\}, \tag{8}$$

$$S = Q_o + \frac{1}{r^2}\left(Q_1 + Q_e\frac{z^2}{r^2} + Q_m\frac{z_{\rm ec}^2}{r^2} + Q_2\frac{1}{r_s^2}\right). \tag{9}$$

The six Q terms appearing in Equations (6–9) are semiempirical parameters whose value were obtained from the fitting procedure. Their meaning is as follows.

Q_e represents the coefficient of the second order zonal harmonic of the Earth figure. Its expected value is

$$Q_e = -\tfrac{15}{2}J_2^e r_e^2 = -3.303177\cdot 10^5\,\mathrm{km}^2.$$

The value resulting from the fitting procedure was $-3.303240\cdot 10^5 \pm 0.1\,\mathrm{km}^2$.

Q_m represents the coefficient of the second order zonal harmonic of the Moon figure. The approximation was made of considering the Moon equator coinciding with the ecliptic plane. The expected value is

$$Q_m = -\tfrac{15}{2}J_2^m r_m^2 = -4.6176\cdot 10^3\,\mathrm{km}^2.$$

The value resulting from the fitting was $-6.3974\cdot 10^3 \pm 0.1\,\mathrm{km}^2$. The discrepancy most likely arises from the fact that tesseral harmonics were not explicitly considered in the model. Therefore the optimized value of Q_m includes the contribution of second order tesseral harmonics, averaged with respect to the longitude.

Q_1 represents the coefficient for the latitude independent parts of the second order zonal harmonics of the Earth and the Moon, collected together. Its expected value is

$$Q_1 = -\tfrac{1}{5}(Q_e + Q_m) = 6.698706 \cdot 10^4 \,\mathrm{km}^2.$$

The fitting gave $Q_1 = 6.48447 \cdot 10^4 \pm 0.1\,\mathrm{km}^2$. As before, the omission from the model of relevant perturbations (tesseral harmonics of the Moon figure, higher order zonal harmonics of the Earth figure, relativistic effects in the Earth–Moon interaction) most likely account for the observed discrepancy.

Q_t is the coefficient for the tidal acceleration, where δ is the phase angle. The expected value is

$$Q_t = 3\frac{\mu_m}{\mu_e} k_1 r_e^5 = 1.16847 \cdot 10^{17}\,\mathrm{km}^5,$$

when the same values are given to the Love number k_1 and to the phase angle δ as those adopted by Moshier (1992). The fitting gave $Q_t = 1.16849 \cdot 10^{17}\,\mathrm{km}^5$.

Q_2 is an empirical parameter. The corresponding correction is probably related to the main term of the libration series (the one dependent on the Earth's mean anomaly). It has the effect of about halving the average residuals, stably removing a periodical deviation having a period of 1.0 y. The fitting gave $Q_2 = 2.649 \cdot 10^7\,\mathrm{km}^2$.

Q_0 is an empirical parameter which is needed to minimize the residuals in the Earth–Moon distance. The fitting gave $Q_0 = -5.34126 \cdot 10^{-8}$.

The effect of the figure of the Earth has been taken in account also when computing the Earth–Sun acceleration. Thus a further term

$$\frac{Q_e}{r^2}\left(\frac{z^2}{r^2} - \frac{K}{5}\right)$$

has been added to the summation in Equation 5 in the case of i = Earth ($K = 1$ for the x and y components, $K = 3$ for the z component).

3.3. PERTURBATIONS BY THE ASTEROIDS

In the DE200 ephemerides, the forces on each planet and on the Moon due to the five asteroids Ceres, Pallas, Vesta, Iris and Bamberga have been considered. In the very recent DE403 ephemerides (Standish *et al.*, 1995), the forces on each planet and on the Moon due to the three major asteroids, plus the forces on Earth, Moon and Mars due to almost three hundred minor asteroids have been taken in account. In both cases, the asteroids' orbits were considered as fixed Keplerian ellipses. The effect of the above approximation, as well as the effect of neglecting some or all of the asteroids has been investigated on a fictitious solar system from which the Moon had been removed and only Earth, Mars, Jupiter and Saturn were left as planets. The positions computed by a full numerical integration including the five asteroids were considered as 'true', and the other simplified models were optimized by fitting them to these 'true' positions along a time span of 100–400 years. The models fitted along a 400 y time span were then integrated to a total time of 8000 y and the deviations of Earth, Mars and Jupiter from the 'true' positions were compared. Over the time span of one century, the model with Keplerian asteroids gives a maximum error in the longitude of Mars of the order of 1 milliarcsec, compared with 4 milliarcsecs for a fully integrated model including only Ceres, Pallas and Vesta and about 9 milliarcsecs for a model neglecting all the asteroids. However, the Keplerian model produces better results only if the time span is limited to a few centuries. After about 15 centuries, a fully integrated model considering only Ceres, Pallas and Vesta gives better precision, and beyond 60 centuries even a model neglecting all the asteroids is about as good, giving a longitude deviation in the orbit of Mars of about 0.4 arcsecs.

In the present model only the three major asteroids were included and they were treated in the same way as the planets, i.e. all the mutual newtonian interactions were taken in account and their orbits were numerically integrated.

3.4. FITTING THE MODEL: SHORT AND LONG TERM PRECISION

The model described in the above section was fitted to the DE200 ephemerides generated by the program de118i (Moshier, 1992), using the numerical integrator described in the next section and a Newton–Gauss least squares method. A numerical precision of 19 decimal digits (80 bits IEEE format) was used. The fitting was performed along a time span of about 500 y, starting from the same point as DE200 (June 28, 1969, JDE 2440400.5) and going backwards in time. A 'short' fitting was also performed along a time span of 50 y and going forward in time. The starting coordinates and velocities of the nine planets and the Moon, plus the six Q parameters described in section 3.2 were optimized. Starting conditions for the three asteroids Ceres, Pallas, Vesta were first determined from their osculating elements at the epoch November 13, 1996, JDE 2450400.5 (Batrakov and Shor, 1995) and then extrapolated back at the starting epoch JDE 2440400.5 by numerical integration. Their positions computed along the decade 1986–1996 resulted in perfect agreement with the data given by the Astronomical Almanac. Consistently with Moshier (1992), precession angles were computed using Laskar's formulae (1986). Nutation was computed using only the first term of the IAU nutation series (Hohenkerk *et al.*, 1992). To avoid unnecessary computing effort and possible accumulation of roundoff error, the precession and nutation adjustments of the coordinates were made at intervals of 60 d.

Figure 1 shows a plot of the right ascension deviation of the present model from the DE200 values for the Moon, Mercury, Venus, Earth and Mars. In order to avoid a long-term discrepancy in the planetary positions arising from the different treatment of the asteroids' orbits (see later), the comparison is made for the planets

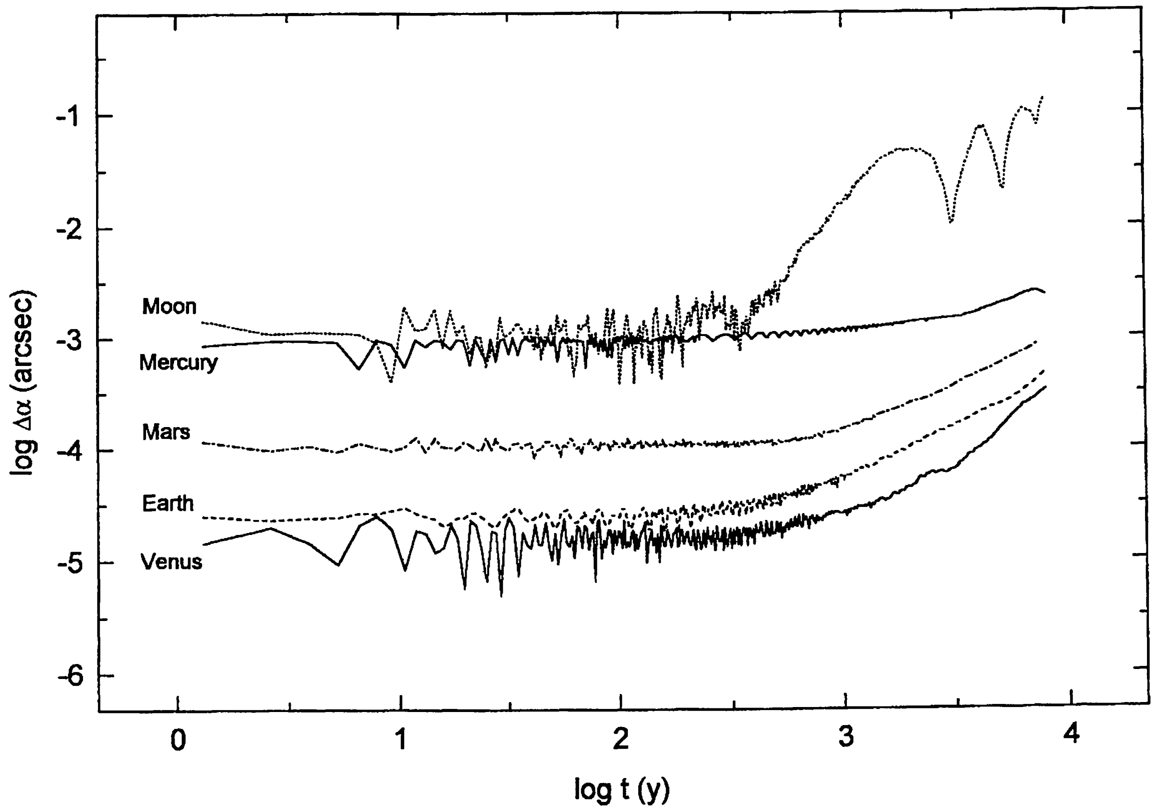

Figure 1. Maximum local (within 2 percent from given time) right ascension deviations from the reference data. The data for the planets are computed omitting the asteroids from both the reference and the tested model. Time is backwards from JDE 2440400.5.

between models from which the asteroids have been removed. It is apparent from Figure 1 that the errors due to the approximate treatment of relativistic perturbations may have some little relevance only in the case of Mercury, while for the other planets they are so small to be totally negligible. Indeed, for the planets from Venus through Saturn, the effect of missing or imperfect treatment of the perturbations due to the asteroids is much larger.

Figure 2 shows the short term right ascension deviations of Earth, Mars and Moon from the DE200 values, inclusive of the asteroids perturbations, as resulting from the 'short' fitting.

Figure 3 shows the analogous deviations of Earth and Mars as resulting from the 'long' fitting and the successive extrapolation extending to 80 Cy. The solid lines (a) in Figure 3 refer to a model including five asteroids (Iris and Bamberga in addition to the three major ones), while the dotted lines (b) refer to a model containing only the major planets. The 'short' fitting (Figure 2) gives an agreement similar to that in Figure 1, while for the 'long' fitting (Figure 3) the deviations are much larger than the corresponding ones plotted in Figure 1. It is very significant that, when the asteroids are included in the model, a better agreement with the DE200 model is obtained only over a relatively short time span. In the long run, the deviations are smaller for a model *neglecting* the asteroids. This indicates that

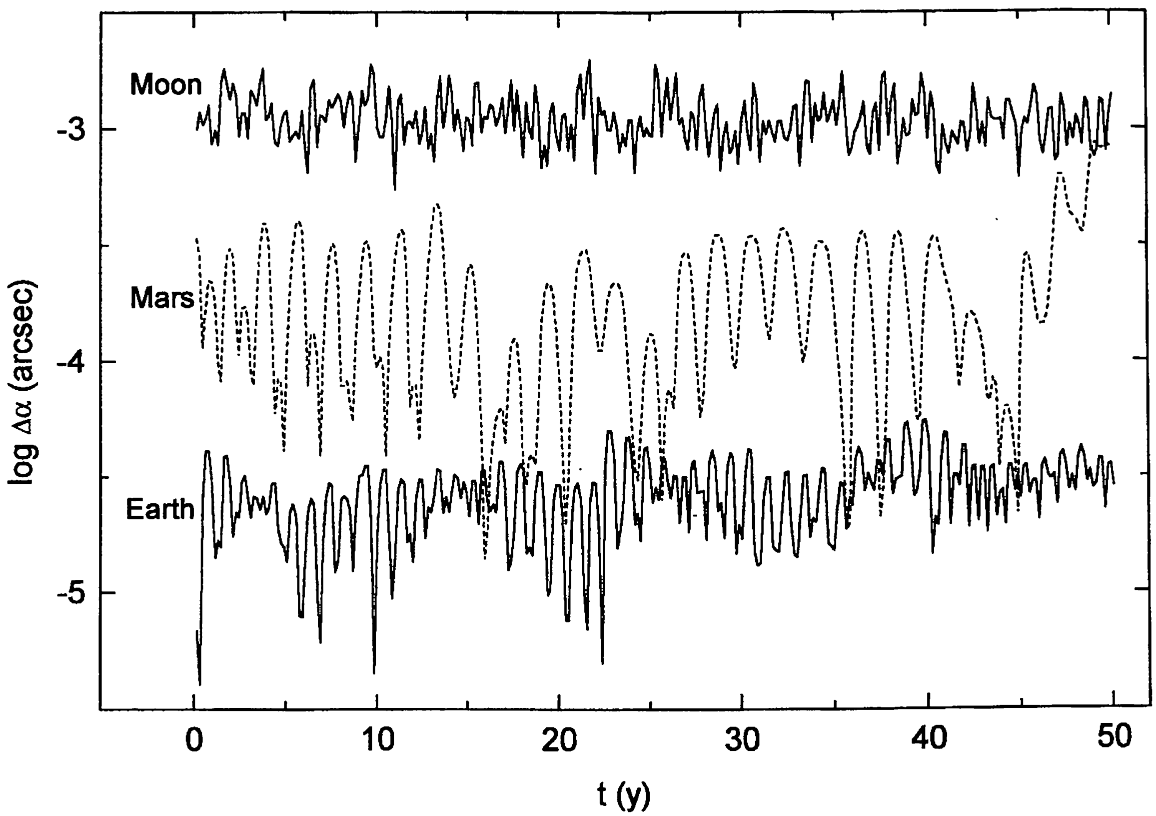

Figure 2. Maximum bimonthly right ascension deviations from DE200 after a short-term fitting. Only the three major asteroids are included in the tested model.

the larger discrepancies (compare the plots in Figure 3 and Figure 1) reflect the imperfect treatment of the asteroids' perturbations by the DE200 model, which probably adequately describes the asteroids' positions for not longer than a few decades. It is worth to note in this respect that the secular drift of the perihelion of Pallas (as computed by the present integration) is $-0.32°$/Cy. It is thus apparent that, in order to produce high precision 'long' ephemerides without integrating the asteroid's orbits, at least the secular variation of their orbital elements should be taken in account. In summary, compared to the DE403 model over the range of about a century, the planetary positions computed by the present model and the DE200 model are both less precise, since they both neglect a large number of minor asteroid perturbations. In the long run, however, the accuracy of the DE403 model degrades because of the approximation of considering Keplerian orbits for the asteroids, and the present model is likely to be more accurate.

Turning to the Moon, the short term angular deviation of the Moon (Figure 2) is kept within 2 milliarcsecs (average deviation $\sim$ 0.6 milliarcsec). Although being about 20 times larger than the uncertainty of the best present observational data (Standish, et al., 1995), this is still about 20 times smaller than the error carried by analytical theories (Chapront–Touzé and Chapront, 1983) and is quite adequate for most purposes. In the long run ($\sim$ 80 Cy, Figure 1), the discrepancy from

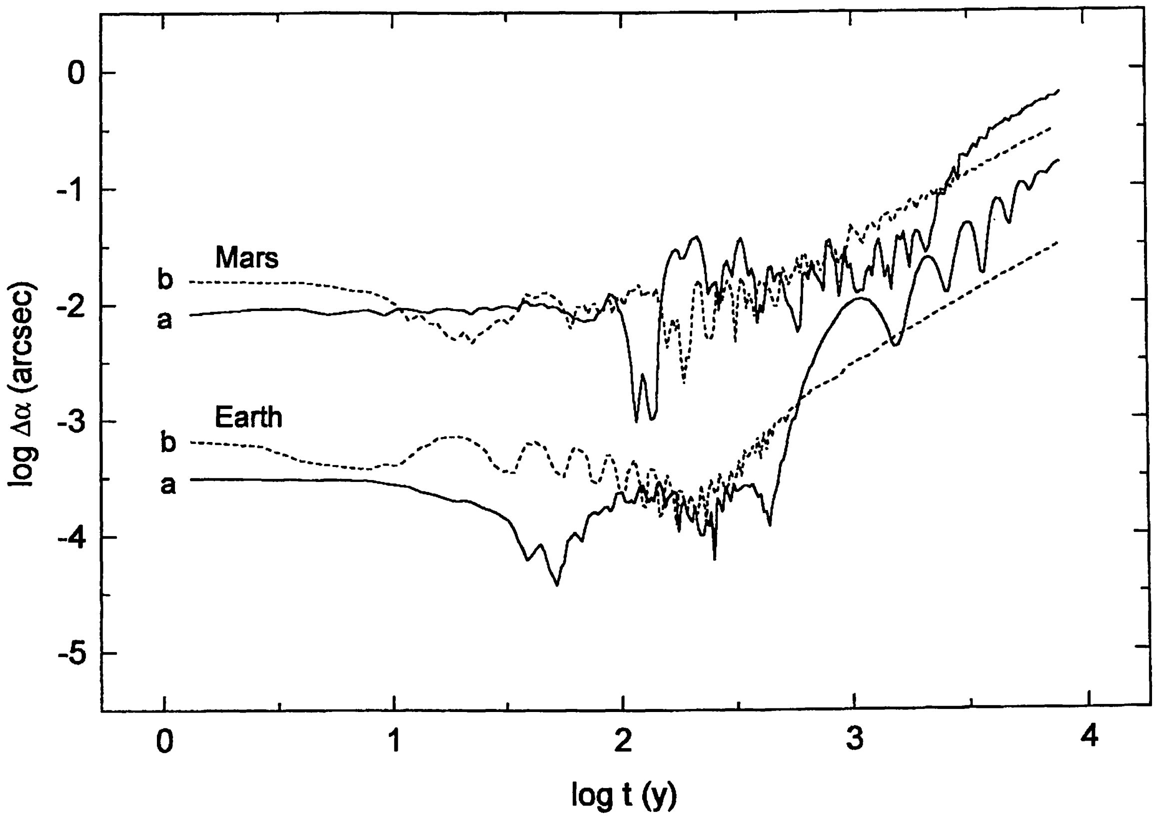

Figure 3. Maximum local (within 2 percent from given time) right ascension deviations from DE200, when (a) the five asteroids are included in the tested model and (b) the asteroids are omitted from the tested model.

DE200 increases by about a factor of 50, still remaining about two orders of magnitude smaller than that of improved analytical theories (Moshier, 1992). On the other hand, the intrinsic uncertainty of long term extrapolation of observational data (especially considering the quadratically growing effect of tidal acceleration) makes this model practically as adequate as the JPL DE200 and DE403 models for the computation of ephemerides in the far past or future.

4. Numerical Integrator

Among the various integration techniques, extrapolation methods are becoming increasingly popular because of their high accuracy, efficiency, flexibility and easiness to program (Press *et al.*, 1992; Fukushima, 1989). After some experiments with various algorithms, the selected method was the one given by Press et al. (1992) for second-order conservative equations. In this method, n trial values of the dependent variables are computed at the end of a large interval H, by using each time a different stepsize $h_1, \ldots, h_n$, where the h_j are submultiples of H such as $H, H/2, H/3$ and so on. The computation is made according to Equations (10–12), where x_0 and x_m are the positional coordinates at the beginning and at the end of the interval H, respectively, with $H = mh$.

$$\Delta_0 = h(\dot{x}_0 + \tfrac{1}{2}h\ddot{x}_0); \qquad x_1 = x_0 + \Delta_0, \tag{10}$$

$$\Delta_k = \Delta_{k-1} + h^2\ddot{x}_k; \qquad x_{k+1} = x_k + \Delta_k \quad k = 1, \ldots, m-1, \tag{11}$$

$$\dot{x}_m = \frac{\Delta_{m-1}}{h} + \frac{h}{2}\ddot{x}_m. \tag{12}$$

As shown by Gragg (1965), each estimate x_j can be expressed as an even power series of h

$$x_j = a_0 + a_1 h_j^2 + \cdots + a_{n-1} h_j^{2(n-1)} + \cdots \tag{13}$$

(Equation 13), where a_0 is the true value of the variable x, which can thus be computed by extrapolating the series of n trial x_j values to $h = 0$. The conventional way of performing the polynomial extrapolation (Deuflhard, 1985) is based on a recursive algorithm which requires a computing time which is not negligible compared to the time spent on function evaluation. In order to save as much computing time as possible an alternative approach was adopted, which does not seem to have been used before. The extrapolation to $h = 0$ is equivalent to the solution for the variable a_0 of the system of n linear equations (13) $(j = 1, \ldots, n)$, where by a proper choice of the time unit the h_j can be expressed as rational numbers. The solution of Equations (13) gives a_0 (actually, its extrapolated value ${}^n a_0$ based on n trials) as a linear combination of the computed x_j values of the type

$${}^n a_0 = \frac{1}{{}^n b_0} \sum_{j=1}^{n} {}^n b_j x_j, \tag{14}$$

where the coefficients ${}^n b_j$ are integer numbers of alternating sign such that $\Sigma^n b_j = {}^n b_0$. If the coefficients ${}^n b_j$ are independently determined for a given sequence of stepsizes h_j and a given number of trials n (see Table I), the extrapolated value ${}^n a_0$ can be straigthforwardly calculated by Equation (14). By using this approach, the extrapolation procedure becomes almost costless in computing time, and for a system of 11 bodies, about 25 percent of total computing time is saved (more than 50 percent for a two-body system). The roundoff error implied in Equation (14) depends on the magnitude of the coefficients ${}^n b_j$ relative to ${}^n b_0$, which in turn depends on the adopted sequence of stepsizes. If each $|{}^n b_j|$ does not exceed ${}^n b_0$ by more than about one order of magnitude, the roundoff error does not exceed the one involved in the conventional extrapolation algorithm. A convenient sequence of stepsizes fulfilling the above condition is: $h_j = H/m_j$, where $m_j = 1, 2, 3, 4, 5, 6, 8, 10, 12$ (see Table I).

Table I

Coefficients of the extrapolation series (Equation 14) for the sequence of stepsizes H/m, with $m = 1, 2, 3, 4, 5, 6, 8, 10, 12$ (n = number of terms included in the series, m_0 = starting value of m).

$m_0 = 1$	$n = 9$	$n = 8$	$n = 7$	$n = 6$	$n = 5$	$n = 4$	$n = 3$	$n = 2$
${}^nb_0$	45850332528000	49037788800	637491254400	19958400	362880	2520	120	3
${}^nb_1$	170	−26	33462	−66	42	−7	5	−1
${}^nb_2$	−30805632	1153152	−359783424	168960	−24576	896	−128	4
${}^nb_3$	24149210481	−387420489	50924270943	−9743085	531441	−6561	243	
${}^nb_4$	−1682398248960	14394851328	−982448603136	92274688	−2097152	8192		
${}^nb_5$	25177001953125	−128173828125	4998779296875	−244140625	1953125			
${}^nb_6$	−100460715600960	322333846848	−7449493349376	181398528				
${}^nb_7$	411217348788224	−549755813888	4020089389056					
${}^nb_8$	−830078125000000	390625000000						
${}^nb_9$	541653102231552							
$m_0 = 2$								
${}^nb_0$		57891834000	681080400	32432400	166320	15120	105	5
${}^nb_1$		−29172	12012	−13728	1056	−768	28	−4
${}^nb_2$		27103491	−4782969	2302911	−72171	19683	−243	9
${}^nb_3$		−1991475200	187432960	−46858240	720896	−81920	320	
${}^nb_4$		30517578125	−1708984375	244140625	−1953125	78125		
${}^nb_5$		−123320884050	4352501790	−368465760	1469664			
${}^nb_6$		511101108224	−7516192768	201326592				
${}^nb_7$		−1037597656250	5371093750					
${}^nb_8$		679156088832						

In the numerical integration of the solar system the stepsize can be kept constant without substantial loss of efficiency, because the orbit of the Moon, which determines the optimal stepsize, is only moderately eccentric. Therefore the extrapolation method can be used without need of getting an error estimate along with each extrapolated value. Nevertheless it is worth to point out that, if in a more general case an adaptive stepsize is required, a conservative error estimate on each variable can be obtained by evaluating ${}^n a_{n-1}$, the highest order coefficient in expansion 13, and setting the error to ${}^n a_{n-1} h_n^{2(n-1)}$, where h_n is the smallest stepsize in the adopted sequence of stepsizes. The coefficient ${}^n a_{n-1}$ can in turn be evaluated in the same way as ${}^n a_0$, via a preliminary solution of the linear equations (13), giving the integer coefficients ${}^n c_j$ such that

$$ {}^n a_{n-1} = \frac{1}{{}^n c_0} \sum_{j=1}^{n} {}^n c_j x_j, \tag{15} $$

where $\Sigma_{j=1}^{n} {}^n c_j = 0$ (Table II).

The maximum number $n_{\max}$ of trial estimates (i.e. the maximum number of terms used in Equations (13) and (14) which is meaningful to perform depends on the arithmetic precision carried by the computer. Thus, $n_{\max} = 9$ if extended precision arithmetics (80 bits IEEE format) is used, while $n_{\max} = 8$ if standard double precision arithmetic (64 bits IEEE format) is adopted. Larger values of n will cause unnecessary waste of computing time and accumulation of roundoff error.

If the orbit controlling the optimal stepsize has a low to moderate eccentricity, as is the case with the moon orbit, an interesting property of the extrapolation series (13) allows a further improvement of the integrator efficiency. Computer experiments show that, when different values of n are used, and provided that the stepsize does not exceed a critical value above which numerical instability starts to appear, the truncation errors in longitude have opposite signs when even or odd values of n are used. That is, the estimated position ${}^n a_0$ at the end of each large stepsize H is slightly ahead of the true position when n is even and slightly behind it when n is odd. This means that the errors tend to cancel if the estimates ${}^n a_0$ and ${}^{n-1} a_0$ (obtained using the two sequences of n stepsizes $H, H/2, H/3 \ldots$ and $n-1$ stepsizes $H/2, H/3, \ldots$, respectively) are combined with appropriate weights. This combination can be performed at no extra cost by appropriately weighting the ${}^n b_j$ and ${}^{n-1} b_j$ coefficients (Equation 14, Table I) resulting from the two sequences of stepsizes. Optimal combination weights for many different stepsizes were found by computer experiments. The general finding was that for any given stepsize (not exceeding a critical value corresponding to an angular distance of about 45°), the truncation error can be reduced by as much as four orders of magnitude by an appropriate choice of weights, leading to an overall improvement of the efficiency of up to 60 percent (see Figure 4).

Table II

Coefficients of the error series (Equation 15) for the sequence of stepsizes H/m, with $m = 1, 2, 3, 4, 5, 6, 8, 10, 12$ (n = number of terms included in the series, m_0 = starting value of m).

$m_0 = 1$	$n = 8$	$n = 7$	$n = 6$	$n = 5$	$n = 4$	$n = 3$
${}^n c_0$	113809007812500000	124141197852672	2327947776	9843750	17920	270
${}^n c_1$	2002	3146	66	42	7	5
${}^n c_2$	−22198176	−8456448	−42240	−6144	−224	−32
${}^n c_3$	3314597517	531972441	1082565	59049	729	27
${}^n c_4$	−69275222016	−5772935168	−5767168	−131072	−512	
${}^n c_5$	394775390625	18798828125	9765625	78125		
${}^n c_6$	−689436283536	−19454992128	−5038848			
${}^n c_7$	661424963584	5905580032				
${}^n c_8$	−300781250000					

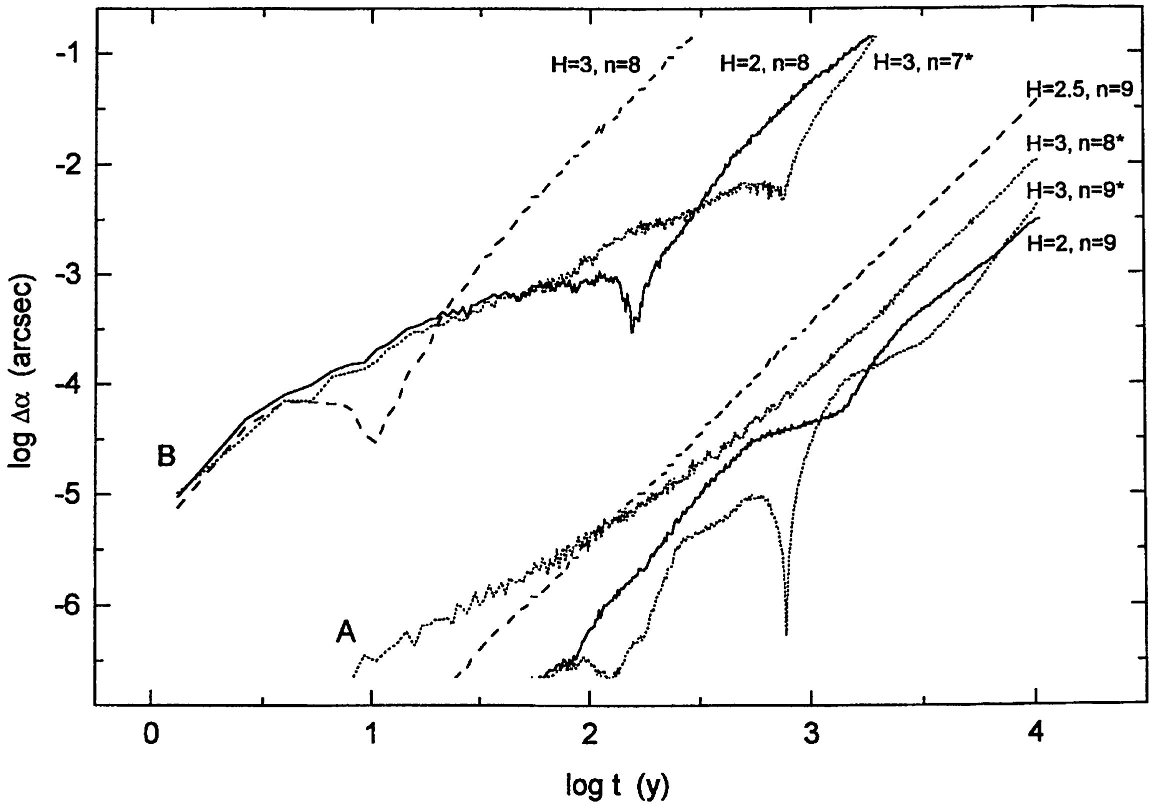

Figure 4. Integration error on the Moon's orbit, using different stepsizes H (days) and number of trial estimates n, with 80 bits arithmetics (A) and 64 bits arithmetics (B). Asterisked values of n (dotted lines) refer to the use of combined extrapolation series (see text and Table III). The plotted data are the right ascension deviations from a reference integration run with $H = 1.0$ d and $n = 9$.

Combination weights which were found convenient in the integration of the moon orbit (using a stepsize $H = 3.0$ d and extended precision arithmetics) are 1 and $\frac{1}{8}$, for the series with $n = 9, m_0 = 1$ and $n = 8, m_0 = 2$, respectively. The above choice gives a precision which is close to the limit of roundoff error for this integration method (using 80 bits arithmetics). If the precision requirements are slightly relaxed, a 30 percent faster integration can be obtained by combining the series with $n = 8, m_0 = 1$ and $n = 7, m_0 = 2$, with combination weights 33 and 263/18, respectively (this was the choice adopted while fitting the present model). The coefficients of a few combined extrapolation series (optimized for the integration of the solar system) are given in Table III. Figure 4 displays the integration error on the right ascension of the Moon in a few selected cases (the deviation of the present model from the reference data is plotted in Figure 1).

The performance of the above method has been compared with that of other extrapolation methods tested by Montenbruck (1992), using 64 bits double precision arithmetics for consistency of the comparison. On a test two-body system with small eccentricity ($e = 0.1$), a standard length of 3.2 revolutions is covered with a relative accuracy of $5 \cdot 10^{-13}$ and 1080 function calls. This result compares

Table III
Combination weights and corresponding coefficients of the combined extrapolation series, optimized for the integration of the solar system with stepsize $H = 3.0$ d.

	$n = 9$	$n = 8$	$n = 7$
w_n; w_{n-1}	1; 1/8	33; 263/18	1; 8786/19656
${}^nb_0$	5158162409400	2334743611200	922442320800
${}^nb_1$	17	−858	33462
${}^nb_2$	−3369366	50690640	−480397632
${}^nb_3$	2683245609	−17816559525	71157646989
${}^nb_4$	−187955429376	672209567744	−1394145099776
${}^nb_5$	2819824218750	−6027587890625	7143798828125
${}^nb_6$	−11266948312191	15215848829064	−10686833516736
${}^nb_7$	46181635850240	−26048976650240	5788944826368
${}^nb_8$	−93280029296875	18541015625000	
${}^nb_9$	60888955502592		

favourably with the $\sim$ 1700 function calls required to achieve the same accuracy by the best extrapolation method (ODEX2) tested by Montenbruck (1992).

It is worth mentioning one final convenience of the described integration method, which is connected to the purpose of this paper. If equidistant ephemerides are required at time intervals smaller than the optimal stepsize, the order of the method can be easily scaled down to avoid unnecessary computing effort and accumulation of roundoff error. That is, n can be reduced down to 2, using in Equation (13) the appropriate coefficients ${}^nb_j$, according to Table I. For example, if a time spacing as little as 1 min is required, a good accuracy is obtained for the lunar orbit using $n = 2$ (which gives roughly the same performances of a 4th order Runge–Kutta method).

Executable and source code (SOLEX) implementing the above model and integration method are available from the author.

5. Conclusions

The simplified model of the solar system and the integration algorithm which have been developed constitute a convenient engine (SOLEX) for the quick computation of high precision ephemerides, if a suitable database of starting conditions is available to the running program. Over a short time range (±3 Cy from J2000) the precision of the computed lunar positions is intermediate between that of the JPL models (DE200 and DE403) and that carried by current analytical theories (Chapront Touzé and Chapront, 1983, 1988). Over a long time range (±100 Cy from J2000), the intrinsic uncertainty due to extrapolating the orbit far beyond the available observational data and, above all, the uncertainty in the tidal parameters, makes the positions computed by this model not worse than those given by the best model so far available. The planetary positions, with the exception of Mars over a short time span, are computed with a precision comparable to that of the JPL models.

References

Batrakov, Y. V. and Shor, V. A.: 1995, 'Catalogue of Orbital Elements and Photometric Parameters of 6160 Minor Planets, *Inst. Theor. Astron.*, St. Petersbourg, Russia.

Bretagnon, P. and Francou, G.: 1988, Planetary Theories in Rectangular and Spherical Variables. VSOP87 Solution, *Astron. Astrophys.* **202**, 309–315.

Chapront-Touzé, M. and Chapront, J.: 1983, The Lunar Ephemeris ELP 2000, *Astron. Astrophys.* **124**, 50–62.

Chapront-Touzé, M. and Chapront, J.: 1988, ELP 2000-85: a Semi-analytical Lunar Ephemeris Adequate for Historical Times, *Astron. Astrophys.* **190**, 342–352.

Deuflhard, P.: 1985, Recent Progress in Extrapolation Methods for Ordinary Differential Equations, *SIAM Review* **27**, 505–535.

Fukushima, T.: 1989, Tests of the Extrapolation Method for the Numerical Integration of the Keplerian Motion, *Cel. Mech.* **45**, 73–76.

Gragg, W.: 1965, On Extrapolation Algorithms for Ordinary Initial Value Problems, *SIAM J. Numer. Anal.* **2**, 384–403.

Hohenkerk, C. Y., Yallop, B. D., Smith, C. A. and Sinclair, A. T.: 1992, Celestial Reference Systems, in *Explanatory Supplement to the Astronomical Almanac*, University Science Books, Mill Valley, CA, pp. 111–114.

Kammeyer, P.: 1989, Compressed Planetary and Lunar Ephemerides, *Cel. Mech.* **45**, 311–316.

Laskar, J.: 1986, Secular Terms of Classical Planetary Theories Using the Results of General Theories, *Astron. Astrophys.* **157**, 59–70.

Meeus, J.: 1991, *Astronomical Algorithms*, Willmann-Bell, Richmond, Virginia.

Montenbruck, G.: 1992, Numerical Integration Methods for Orbital Motion, *Cel. Mech.* **53**, 59–69.

Moshier, S.: 1992, Comparison of a 7000-year Lunar Ephemeris with Analytical Theory, *Astron. Astrophys.* **262**, 613–616.

Newhall, X. X., Standish, E. M. and Williams, J. G.: 1983, DE102: a Numerically Integrated Ephemeris of the Moon and Planets Spanning Forty-Four Centuries, *Astron. Astrophys.* **125**, 150–167.

Newhall, X. X.: 1989, Numerical Representation of Planetary Ephemerides, *Cel. Mech.* **45**, 305–310.

Nobili, A. M. and Roxburgh, I. W.: 1986, Simulation of General Relativistic Corrections in Long Term Numerical Integration of Planetary Orbits, in *Relativity in Celestial Mechanics and Astrometry*, Reidel Pu. Co., Dordrecht, Holland, pp. 105–111.

Press, W. H., Flannery, B. P., Teukolsky, S. A. and Vetterling, W. T.: 1992, in *Numerical Recipes (Fortran)*, Cambridge Univ. Press, Cambridge, MA, pp. 726–727.

Quinn, T. R., Tremaine, S. and Duncan, M.: 1991, A Three Million Year Integration of the Earth's Orbit, *Astron. J.* **101**, 2287–2305.

Standish, E. M.: 1982, The JPL Planetary Ephemerides, *Cel. Mech. J.* **26**, 181–186.

Standish, E. M., Newhall, X. X., Williams, J. G. and Folkner, W. M.: 1995, JPL Planetary and Lunar Ephemerides, DE403/LE403, *JPL IOM* 314.10–127.