Chapter 17
Computation of Some Hodge Numbers

The Hodge numbers of a smooth projective algebraic variety are very useful invariants. By Hodge theory, these determine the Betti numbers. In this chapter, we turn to the practical matter of actually computing these for a number of examples such as projective spaces, hypersurfaces, and double covers. The GAGA theorem, Theorem 16.4.1, allows us to do this by working in the algebraic setting, where we may employ some of the tools developed in the earlier chapters.

17.1 Hodge Numbers of $\mathbb{P}^n$

Let $S = k[x_0, \ldots, x_n]$ and $\mathbb{P} = \mathbb{P}^n_k$ for some field k. We first need to determine the sheaf of differentials.

Proposition 17.1.1. *There is an exact sequence*

$$0 \to \Omega^1_{\mathbb{P}} \to \mathscr{O}_{\mathbb{P}}(-1)^{n+1} \to \mathscr{O}_{\mathbb{P}} \to 0.$$

Proof. Let $\Omega_S = \oplus S dx_i \cong S^{n+1}$ be the module of Kähler differentials of S. Construct the graded S-module

$$M = \Gamma_*(\Omega^1_{\mathbb{P}}) = \Gamma(\mathbb{A}^{n+1} - \{0\}, \pi^*\Omega^1_{\mathbb{P}}),$$

where $\pi : \mathbb{A}^{n+1} - \{0\} \to \mathbb{P}$ is the projection. This can be realized as the submodule Ω_S consisting of those forms that annhilate the tangent spaces of the fibers of π. The tangent space of the fiber over $[x_0, \ldots, x_n]$ is generated by the Euler vector field $\sum x_i \frac{\partial}{\partial x_i}$. Thus a 1-form $\sum f_i dx_i$ lies in M if and only if $\sum f_i x_i = 0$.

Next, we have to check the gradings. Ω_S has a grading such that the dx_i lie in degree 0. Under the natural grading of $M = \Gamma_*(\Omega^1_{\mathbb{P}})$, sections of $\Omega^1_{\mathbb{P}}(U_i)$ that are generated by

$$d\left(\frac{x_j}{x_i}\right) = \frac{x_i dx_j - x_j dx_i}{x_i^2}$$

should have degree 0. Thus the gradings on Ω_S and M are off by a shift of one.

To conclude, we have an exact sequence of graded modules

$$0 \to M \to \Omega_S(-1) \to m \to 0, \tag{17.1.1}$$

where $m = (x_0, \dots, x_n)$ and the first map sends dx_i to x_i. Since $m \sim S$, (17.1.1) implies the result. □

Dualizing yields the sequence of Example 15.3.10.

Proposition 17.1.2. *Suppose we are given an exact sequence of locally free sheaves*

$$0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0.$$

If $\mathscr{A}$ has rank one, then

$$0 \to \mathscr{A} \otimes \wedge^{p-1}\mathscr{C} \to \wedge^p \mathscr{B} \to \wedge^p \mathscr{C} \to 0$$

is exact for any $p \geq 1$. If $\mathscr{C}$ has rank one, then

$$0 \to \wedge^p \mathscr{A} \to \wedge^p \mathscr{B} \to \wedge^{p-1}\mathscr{A} \otimes \mathscr{C} \to 0$$

is exact for any $p \geq 1$.

Proof. We prove the first statement, where $\mathrm{rank}(\mathscr{A}) = 1$, by induction, leaving the second as an exercise. When $p = 1$ we have the original sequence. In general, the maps in the putative exact sequence need to be explained. The last map $\lambda : \wedge^p \mathscr{B} \to \wedge^p \mathscr{C}$ is the natural one. The multiplication $\mathscr{B} \otimes \wedge^{p-1}\mathscr{B} \to \wedge^p \mathscr{B}$ restricts to give $: \mathscr{A} \otimes \wedge^{p-1}\mathscr{B} \to \wedge^p \mathscr{B}$. We claim that factors through a map $\alpha : \mathscr{A} \otimes \wedge^{p-1}\mathscr{C} \to \wedge^p \mathscr{B}$. For this we can, by induction, appeal to the exactness of

$$0 \to \mathscr{A} \otimes \wedge^{p-2}\mathscr{C} \to \wedge^{p-1}\mathscr{B} \to \wedge^{p-1}\mathscr{C} \to 0.$$

Since $\mathscr{A}$ has rank one,

$$|_{\mathscr{A} \otimes (\mathscr{A} \otimes \wedge^{p-2}\mathscr{C})} = 0,$$

so factors as claimed. Therefore all the maps in the sequence are defined.

Exactness can be checked on stalks. For this the sheaves can be replaced by free modules. Let $\{b_0, b_1, \dots\}$ be a basis for $\mathscr{B}$ with b_0 spanning $\mathscr{A}$. Then the images $\bar{b}_1, \bar{b}_2, \dots$ give a basis for $\mathscr{C}$. Then the above maps are given by

$$\alpha : b_0 \otimes \bar{b}_{i_1} \wedge \cdots \wedge \bar{b}_{i_{p-1}} \mapsto b_0 \otimes b_{i_1} \wedge \cdots \wedge b_{i_{p-1}},$$
$$\lambda : b_{i_1} \wedge \cdots \wedge b_{i_p} \mapsto \bar{b}_{i_1} \wedge \cdots \wedge \bar{b}_{i_p},$$

where $\cdots > i_2 > i_1 > 0$. The exactness is immediate. □

Corollary 17.1.3. *There is an exact sequence*

$$0 \to \Omega_{\mathbb{P}}^p \to \mathscr{O}_{\mathbb{P}}(-p)^{\binom{n+1}{p}} \to \Omega_{\mathbb{P}}^{p-1} \to 0$$

and in particular,

$$\Omega_{\mathbb{P}}^n \cong \mathscr{O}_{\mathbb{P}}(-n-1).$$

Proof. This follows from the above proposition and Proposition 17.1.1, together with the isomorphism

$$\wedge^p[\mathscr{O}_{\mathbb{P}}(-1)^{n+1}] \cong \mathscr{O}_{\mathbb{P}}(-p)^{\binom{n+1}{p}}. \qquad \square$$

This corollary can be understood from another point of view. Using the notation introduced in the proof of Proposition 17.1.1, we can extend the map $\Omega_S(-1) \to m$ to an exact sequence

$$0 \to [\wedge^{n+1}\Omega_S](-n-1) \xrightarrow{\delta} \cdots \to [\wedge^2\Omega_S](-2) \xrightarrow{\delta} \Omega_S(-1) \xrightarrow{\delta} m \to 0, \qquad (17.1.2)$$

where

$$\delta(dx_{i_1} \wedge \cdots \wedge dx_{i_p}) = \sum (-1)^p x_{i_j} dx_{i_1} \wedge \cdots \wedge \hat{dx}_{i_j} \wedge \cdots \wedge dx_{i_p}$$

is contraction with the Euler vector field. The sequence is called the Koszul complex, and it is one of the basic workhorses of homological algebra [33, Chapter 17]. The associated sequence of sheaves is

$$0 \to [\wedge^{n+1}\mathscr{O}_{\mathbb{P}}^{n+1}](-n-1) \to \cdots \to [\wedge^2\mathscr{O}_{\mathbb{P}}^{n+1}](-2) \to [\mathscr{O}_{\mathbb{P}}^{n+1}](-1) \to \mathscr{O}_{\mathbb{P}} \to 0.$$

If we break this up into short exact sequences, then we obtain exactly the sequences in Corollary 17.1.3.

Proposition 17.1.4.

$$H^q(\mathbb{P}, \Omega_{\mathbb{P}}^p) = \begin{cases} k & \textit{if } p = q \le n, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof. When $p = 0$, this follows from Theorem 16.2.1. In general, the same theorem together with Corollary 17.1.3 implies

$$H^q(\Omega_{\mathbb{P}}^p) \cong H^{q-1}(\Omega_{\mathbb{P}}^{p-1}).$$

Therefore, we get the result by induction. $\square$

When $k = \mathbb{C}$, this gives a new proof of the formula for Betti numbers of $\mathbb{P}^n$ given in Section 7.2. By a somewhat more involved induction we can obtain the following theorem of Raoul Bott:

Theorem 17.1.5 (Bott). $H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}^n}(r)) = 0$ *unless*

(a) $p = q$, $r = 0$,
(b) $q = 0$, $r > p$,
(c) or $q = n$, $r < -n + p$.

Proof. A complete proof will be left for the exercises. We give the proof for $p \leq 1$. For $p = 0$, this is a consequence of Theorem 16.2.1. We now turn to $p = 1$. Corollary 17.1.3 implies that

$$H^{q-1}(\mathscr{O}(r)) \to H^q(\Omega^1(r)) \to H^q(\mathscr{O}(r-1))^e \tag{17.1.3}$$

is exact. We use constants $e, e', \ldots$ for exponents whose exact values are immaterial for the argument. The sequence (17.1.3), along with Theorem 16.2.1, forces $H^q(\Omega^1(r)) = 0$ in the following four cases: $q = 0, r < 1$; $q = 1, r < 0$; $1 < q < n$; $q = n, r \geq -n+1$. The remaining cases are $q = 0, r = 1$ and $q = 1, r > 0$. The trick is to apply Corollary 17.1.3 with other values of p. This yields exact sequences

$$H^q(\mathscr{O}(r-2))^e \to H^q(\Omega^1(r)) \to H^{q+1}(\Omega^2(r)) \to H^{q+1}(\mathscr{O}(r-2))^e,$$
$$H^{q+1}(\mathscr{O}(r-3))^{e'} \to H^{q+1}(\Omega^2(r)) \to H^{q+2}(\Omega^3(r)) \to H^{q+2}(\mathscr{O}(r-3))^{e'},$$
$$\ldots$$

leading to isomorphisms

$$H^0(\Omega^1(1)) \cong H^1(\Omega^2(1)) \cong \cdots \cong H^{n-1}(\Omega^n(1)) = H^{n-1}(\mathscr{O}(-n)) = 0.$$

Likewise, for $r > 0$,

$$H^1(\Omega^1(r)) \cong H^2(\Omega^2(r)) \cong \cdots \cong H^n(\Omega^n(r)) = H^n(\mathscr{O}(-n-1+r)) = 0. \qquad \square$$

Exercises

17.1.6. Finish the proof of Proposition 17.1.2.

17.1.7. Given an exact sequence $0 \to \mathscr{A} \to \mathscr{B} \to \mathscr{C} \to 0$ of locally free sheaves, prove that the top exterior power $\det \mathscr{B}$ is isomorphic to $(\det \mathscr{A}) \otimes (\det \mathscr{C})$. Use this to rederive the formula for $\Omega^n_{\mathbb{P}^n}$.

17.1.8. Finish the proof of Theorem 17.1.5.

17.2 Hodge Numbers of a Hypersurface

We now let $X \subset \mathbb{P} = \mathbb{P}^n_k$ be a nonsingular hypersurface defined by a degree-d polynomial.

Proposition 17.2.1. *The restriction map*

$$H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}}) \to H^q(X, \Omega^p_X)$$

is an isomorphism when $p+q < n-1$.

We give two proofs, one now, over $\mathbb{C}$, and another later for general k.

Proof. Let $k = \mathbb{C}$. The weak Lefschetz theorem, Theorem 14.3.1, implies that the restriction map $H^i(X_{an}, \mathbb{C}) \to H^i(Y_{an}, \mathbb{C})$ is an isomorphism for $i < n-1$. The proposition is a consequence of this together with the the canonical Hodge decomposition (Theorem 12.2.4) and GAGA (Theorem 16.4.1). □

As a corollary, we can calculate many of the Hodge numbers of X.

Corollary 17.2.2. *The Hodge numbers* $h^{pq}(X)$ *equal* δ_{pq}, *where* δ_{pq} *is the Kronecker symbol, when* $n-1 \neq p+q < 2n-2$.

Proof. We give a proof when $k = \mathbb{C}$. For $p+q < n-1$, this follows from the above proposition and Proposition 17.1.4. For $p+q > n-1$, this follows from GAGA and Corollary 10.2.3. □

We prepare for the second proof by establishing a few key lemmas.

Lemma 17.2.3. *There is an exact sequence*

$$0 \to \Omega^p_{\mathbb{P}}(-d) \to \Omega^p_{\mathbb{P}} \to \Omega^p_{\mathbb{P}}|_X \to 0.$$

(Recall that $\Omega^p_{\mathbb{P}}|_X$ *is shorthand for* $i_* i^* \Omega^p_{\mathbb{P}}$, *where* $i : X \to \mathbb{P}$ *is the inclusion.)*

Proof. Tensor

$$0 \to \mathcal{O}_{\mathbb{P}}(-d) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_X \to 0$$

with $\Omega^p_{\mathbb{P}}$ to get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^p_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}}(-d) & \longrightarrow & \Omega^p_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{P}} & \longrightarrow & \Omega^p_{\mathbb{P}} \otimes \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\
0 & \longrightarrow & \Omega^p_{\mathbb{P}}(-d) & \longrightarrow & \Omega^p_{\mathbb{P}} & \longrightarrow & \Omega^p_{\mathbb{P}}|_X & \longrightarrow & 0
\end{array}$$

For the last isomorphism, it is simply a matter of expanding the notation. Observe that if $\iota : X \to \mathbb{P}$ is the inclusion, then

$$\Omega^p_{\mathbb{P}}|_X = \iota_* \iota^* \Omega^p_{\mathbb{P}} = \iota_*(\iota^{-1} \Omega^p_{\mathbb{P}} \otimes \mathcal{O}_X) \cong \Omega^p_{\mathbb{P}} \otimes \mathcal{O}_X.$$ □

Lemma 17.2.4. $0 \to \mathcal{O}_X(-d) \to \Omega^1_{\mathbb{P}}|_X \to \Omega^1_X \to 0$ *is exact.*

Proof. We have a natural epimorphism $\Omega^1_{\mathbb{P}}|_X \to \Omega^1_X$ corresponding to restriction of 1-forms. We just have to determine the kernel. Let f be a defining polynomial of X, and let $M = \Gamma_*(\Omega^1_{\mathbb{P}})$ and $\overline{M} = \Gamma_*(\Omega^1_X)$. We embed M as a submodule of $\Omega_S(-1)$ as in the proof of Proposition 17.1.1. In particular, the symbols dx_i have degree 1. Then $\ker[M/fM \to \overline{M}]$ is a free $S/(f)$-module generated by

$$df = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$

Thus it is isomorphic to $S/(f)(-d)$. □

Corollary 17.2.5. $0 \to \Omega^{p-1}_X(-d) \to \Omega^p_{\mathbb{P}}|_X \to \Omega^p_X \to 0.$

Proof. Apply Proposition 17.1.2 to the lemma. □

For the second proof of Proposition 17.2.1, it is convenient to prove something stronger.

Proposition 17.2.6. *If $p+q < n-1$, then*

$$H^q(X, \Omega^p_X(-r)) = \begin{cases} H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}}) & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$

Proof. We prove this by induction on p. For $p = 0$, this follows from the long exact sequences associated to

$$0 \to \mathscr{O}_{\mathbb{P}}(-d-r) \to \mathscr{O}_{\mathbb{P}}(-r) \to \mathscr{O}_X(-r) \to 0$$

and Theorem 16.2.1.

In general, by induction and Corollary 17.2.5 we deduce

$$H^q(X, \Omega^p_X(-r)) = H^q(\Omega^p_{\mathbb{P}}(-r)|_X)$$

for $r \geq 0$ and $p+q < n-1$. Lemma 17.2.3 and Theorem 17.1.5 give

$$H^q(\Omega^p_{\mathbb{P}}(-r)|_X) = \begin{cases} H^q(\mathbb{P}^n, \Omega^p_{\mathbb{P}}) & \text{if } r = 0, \\ 0 & \text{if } r > 0. \end{cases}$$ □

Exercises

17.2.7. Using Exercise 17.1.7, deduce a version of the adjunction formula $\Omega^{n-1}_X \cong \mathscr{O}_X(d-n-1)$ with X as above.

17.2.8. Compute $H^q(X, \Omega^{n-1}_X)$ for all q.

17.3 Hodge Numbers of a Hypersurface II

As in the previous section, $X \subset \mathbb{P}^n$ is a nonsingular degree-d hypersurface. By Corollary 17.2.2, the Hodge numbers $h^{pq}(X)$ equal δ_{pq} when $n-1 \neq p+q < 2(n-1)$. So the only thing left to compute are the Hodge numbers in the middle. The formulas simplify a bit by setting $h_0^{pq}(X) = h^{pq}(X) - \delta_{pq}$. These can be expressed by the Euler characteristics:

Lemma 17.3.1. $h_0^{p,n-1-p}(X) = (-1)^{n-1-p}\chi(\Omega_X^p) + (-1)^n$.

We can calculate these Hodge numbers by hand using the following recurrence formulas.

Proposition 17.3.2.

(a)
$$\chi(\Omega_{\mathbb{P}}^p(i)) = \sum_{j=0}^{p}(-1)^j\binom{n+1}{p-j}\binom{i-p+j+n}{n}.$$

(b)
$$\chi(\mathscr{O}_X(i)) = \binom{i+n}{n} - \binom{i+n-d}{n}.$$

(c)
$$\chi(\Omega_X^p(i)) = \chi(\Omega_{\mathbb{P}}^p(i)) - \chi(\Omega_{\mathbb{P}}^p(i-d)) - \chi(\Omega_X^{p-1}(i-d)).$$

Proof. Corollary 17.1.3 yields the recurrence

$$\chi(\Omega_{\mathbb{P}}^p(i)) = \binom{n+1}{p}\chi(\mathscr{O}_{\mathbb{P}}(i-p)) - \chi(\Omega_{\mathbb{P}}^{p-1}(i)).$$

Therefore (a) follows by induction on p. The base case was obtained previously in (16.2.4).

Lemma 17.2.3 and Corollary 17.2.5 imply

$$\begin{aligned}\chi(\Omega_X^p(i)) &= \chi(\Omega_{\mathbb{P}}^p(i)|_X) - \chi(\Omega_X^{p-1}(i-d))\\ &= \chi(\Omega_{\mathbb{P}}^p(i)) - \chi(\Omega_{\mathbb{P}}^p(i-d)) - \chi(\Omega_X^{p-1}(i-d)).\end{aligned}$$

When $p=0$, the right side can be evaluated explicitly to obtain (b). □

Corollary 17.3.3. *The Hodge numbers of X depend only on d and n, and are given by polynomials in these variables.*

In principle, formulas for all the Hodge numbers can be calculated using the above recurrence formulas. For example,

$$h^{0,n-1}(X) = (-1)^n\binom{n-d}{n} = \binom{d-1}{n}, \tag{17.3.1}$$

$$h_0^{1,n-2}(X) = (-1)^n\left[(n+1)\binom{n-1}{n} - (n+1)\binom{n-d-1}{n} + \binom{n-2d}{n}\right].$$

But this gets quite messy as p increases. So, instead, we give a closed form for the generating function below. Let $h^{pq}(d)$ denote the pqth Hodge number of a smooth hypersurface of degree d in $\mathbb{P}^{p+q+1}$. Define the formal power series

$$H(d) = \sum_{pq}(h^{pq}(d) - \delta_{pq})x^p y^q$$

in x and y.

Theorem 17.3.4 (Hirzebruch).

$$H(d) = \frac{(1+y)^{d-1} - (1+x)^{d-1}}{(1+x)^d y - (1+y)^d x}.$$

Corollary 17.3.5. *Hodge symmetry $h^{pq} = h^{qp}$ holds for smooth hypersurfaces in projective space over arbitrary fields.*

Remark 17.3.6. Hodge symmetry can fail for arbitrary smooth projective varieties in positive characteristic [90].

Corollary 17.3.7. *If $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ has degree 2, then $b_n(X) = 0$ if n is odd; otherwise, $b_n(X) = h^{n/2,n/2}(X) = 2$.*

Proof.

$$H(2) = \frac{1}{1-xy}.$$ □

By expanding the series $H(3)$ for a few terms, we obtain the following corollary:

Corollary 17.3.8. *If $X \subset \mathbb{P}^{n+1}$ has degree 3, the middle hodge numbers are*

$$1,1$$
$$0,7,0$$
$$0,5,5,0$$
$$0,1,21,1,0$$
$$0,0,21,21,0,0$$

for $n \leq 5$.

Although this result can be deduced from the previous formulas in principle, Hirzebruch [63, 22.1.1] obtained this from his general Riemann–Roch theorem. His original formula gave a generating function for $\chi(\Omega_X^p)$; the above form can be obtained by a change of variables, cf. [30, Example XI Corollary 2.4]. Similar formulas are available for complete intersections. We will be content to work out the case $y = 0$. On one side we have the generating function $\sum h^{n0}(d)x^n$. By (17.3.1), this equals

$$\sum \binom{d-1}{n+1} x^n = \frac{(1+x)^{d-1} - 1}{x},$$

which is what one gets by substituting $y = 0$ into Hirzebruch's formula.

Exercises

17.3.9. Prove Lemma 16.2.4.

17.3.10. Calculate the Hodge numbers of a degree-d surface in $\mathbb{P}^3$ (a) using the recurrence formulas, (b) using the generating function. Compare the expressions.

17.3.11. Prove that for every fixed d and p, there exists q_0 such $h^{pq}(d) = 0$ for $q \geq q_0$.

17.4 Double Covers

Our goal is to compute the Hodge numbers for another natural class of examples that generalize hyperelliptic curves. Let $f(x_0,\dots,x_n) \in \mathbb{C}[x_0,\dots,x_n]$ be a homogeneous polynomial of degree $2d$ such that the hypersurface $D \subset \mathbb{P}^n = \mathbb{P}$ defined by $f = 0$ is nonsingular. Let $\pi : X \to \mathbb{P}$ be the double cover branched along D (Example 3.4.9). By construction, this is gotten by gluing the affine varieties defined by $y_i^2 = f(x_0,\dots,1,\dots,x_n)$ over U_i. It follows that X is nonsingular. Using these coordinates, it is also clear that the local coordinate ring $\mathscr{O}_X(\pi^{-1}U_i)$ is a free $\mathscr{O}(U_i)$-module generated by 1 and y_i. Globally, we have

$$\pi_*\mathscr{O}_X \cong \mathscr{O}_{\mathbb{P}} \oplus L,$$

where L is the line bundle locally generated by y_i. The ratios y_i/y_j give a cocycle for L, from which it easily follows that $L = \mathscr{O}(\pm d)$. To get the correct sign, we need to observe that L is a nontrivial ideal in $\pi_*\mathscr{O}_X$, so it has no nonzero global sections. Therefore we obtain the following:

Lemma 17.4.1.

$$\pi_*\mathscr{O}_X \cong \mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}(-d).$$

It is worth observing that the summands $\mathscr{O}_{\mathbb{P}}$ and $\mathscr{O}_{\mathbb{P}}(-d)$ are exactly the invariant and anti-invariant parts under the action of the Galois group, which is generated by the involution $\sigma : y_i \mapsto -y_i$. A more abstract, but less ad hoc, argument involves observing that $\mathscr{O}_{\mathbb{P}} \oplus \mathscr{O}_{\mathbb{P}}(-d)$ is a sheaf of algebras, and defining X as its relative spectrum [35]. Then the lemma becomes a tautology. We need an extension of the previous lemma to forms:

Lemma 17.4.2 (Esnault–Viehweg). *There are isomorphisms*

$$\pi_*\Omega_X^p \cong \Omega_{\mathbb{P}}^p \oplus (\Omega_{\mathbb{P}}^p(\log D) \otimes \mathscr{O}(-d))$$

for every p.

Proof. We sketch the proof. See [35, pp. 6–7] for further details. The Galois group acts on $\pi_*\Omega_X^p$. We check that the invariant and anti-invariant parts correspond to

$\Omega_{\mathbb{P}}^{p}$ and $\Omega_{\mathbb{P}}^{p}(\log D)\otimes\mathscr{O}(-d)$ respectively. It is enough to do this for the associated analytic sheaves. By the implicit function theorem, we can choose new analytic local coordinates such that X is given locally by $y^2 = x_1$. Then $y, x_2, \ldots, x_n$ are coordinates on X, so that their derivatives locally span Ω_X^1. It follows that a local basis for $\pi_*\Omega_X^1$ is given by

$$\underbrace{ydy = \frac{1}{2}dx_1, dx_2, \ldots, dx_n,}_{\text{invariant}} \quad \underbrace{dy = y\frac{dx_1}{2x_1}, ydx_2, \ldots, ydx_n}_{\text{anti-invariant}}.$$

The forms in the first group are invariant and give a local basis for $\Omega_{\mathbb{P}}^1$. The remainder are anti-invariant and form a local basis for $\Omega_{\mathbb{P}}^1(\log D)\otimes\mathscr{O}(-d)$. By taking wedge products, we get a similar decomposition for p-forms. □

Corollary 17.4.3.

$$H^q(X,\Omega_X^p) \cong H^q(\mathbb{P},\Omega_{\mathbb{P}}^p) \oplus H^q(\mathbb{P},\Omega_{\mathbb{P}}^p(\log D)\otimes\mathscr{O}(-d)).$$

Proof. Let $\{U_i\}$ be the standard affine cover of $\mathbb{P}^n$. Then $\tilde{U}_i = \pi^{-1}U_i$ gives an affine cover of X. We can compute $H^q(X,\Omega_X^p)$ using the Čech complex

$$\check{C}(\{\tilde{U}_i\},\Omega_X^p) = \check{C}(\{U_i\},\Omega_{\mathbb{P}}^p) \oplus \check{C}(\{U_i\},\Omega_{\mathbb{P}}^p(\log D)),$$

which decomposes into a sum. This decomposition passes to cohomology. □

Corollary 17.4.4. *We have*

$$\begin{aligned} h^{pq}(X) = \delta_{pq} &+ \dim\operatorname{coker}[H^{q-1}(\Omega_D^{p-1}(-d)) \to H^q(\Omega_{\mathbb{P}}^p(-d))] \\ &+ \dim\ker[H^q(\Omega_D^p(-d)) \to H^{q+1}(\Omega_{\mathbb{P}}^p(-d))] \end{aligned}$$

in general, and $h^{pq}(X) = \delta_{pq}$ *if* $p+q<n$.

Proof. This follows from (12.6.4) from §12.6 together with Bott's vanishing theorem, Theorem 17.1.5. □

We can obtain more explicit formulas by combining this with earlier results.

Exercises

17.4.5. When $n = 2$, check that $h^{02}(X) = \frac{(d-1)(d-2)}{2}$ and $h^{11}(X) = 3d^2 - 3d + 2$.

17.4.6. Verify that $h^{pq}(X) = \delta_{pq}$ also holds when $p+q>n$.

17.5 Griffiths Residues*

In this section, we describe an alternative method for computing the Hodge numbers of a hypersurface due to Griffiths [48], although the point is really that the method gives more, namely a method for computing the Hodge structure (or more precisely the part one gets by ignoring the lattice). Further details and applications can be found in books of Carlson, Peters, Müller-Stach [17, §3.2] and Voisin [116, Chapter 6] in addition to Griffiths' paper. We work over $\mathbb{C}$ in this section.

Suppose that $X \subset \mathbb{P} = \mathbb{P}^{n+1}$ is a smooth hypersurface defined by a polynomial $f \in \mathbb{C}[x_0, \ldots x_{n+1}]$ of degree d. Let $U = \mathbb{P} - X$. The exact sequence (12.6.5) yields

$$H^{n-1}(X) \to H^n(\mathbb{P}) \to H^{n+1}(U) \to H^n(X) \to H^{n+2}(\mathbb{P}).$$

The first map is an isomorphism by weak Lefschetz. Therefore $H^{n+1}(U)$ maps isomorphically onto the primitive cohomology $P^n(X) = \ker[H^n(X) \to H^{n+2}(\mathbb{P})]$. This is the same as $H^n(X)$ if n is odd, and has dimension one less if n is even. The Hodge filtration on $F^pH^{n+1}(U)$ maps onto the Hodge filtration on X with a shift $F^{p-1}P^n(X)$. We refer to Section 12.6 for the definition of this and of the pole filtration $\mathrm{Pole}^pH^{n+1}(U)$. The key step is to compare these filtrations.

Theorem 17.5.1 (Griffiths). *The Hodge filtration $F^pP^n(X)$ coincides with the shifted pole filtration $\mathrm{Pole}^{p+1}H^{n+1}(U)$. This can, in turn, be identified with the quotient*

$$\frac{H^0(\Omega^{n+1}_{\mathbb{P}}((n-p+1)X))}{dH^0(\Omega^n_{\mathbb{P}}((n-p)X))}.$$

Proof. For $p = n$, this is immediate because

$$F^nP^n(X) = F^{n+1}H^{n+1}(U) = H^0(\Omega^{n+1}_{\mathbb{P}}(\log X)) = H^0(\Omega^{n+1}_{\mathbb{P}}(X)).$$

For $p = n-1$, we use the exact sequence

$$0 \to \Omega^n_{\mathbb{P},cl}(\log X) \to \Omega^n_{\mathbb{P}}(X) \xrightarrow{d} \Omega^{n+1}_{\mathbb{P}}(2X) \to 0,$$

where $\Omega_{\mathbb{P},cl}(\ldots)$ is the subsheaf of closed forms. Then

$$H^0(\Omega^n_{\mathbb{P}}(X)) \xrightarrow{d} H^0(\Omega^{n+1}_{\mathbb{P}}(2X)) \to H^1(\Omega^n_{\mathbb{P},cl}(\log X)) \to H^1(\Omega^n_{\mathbb{P}}(X)).$$

On the right, the group $H^1(\Omega_{\mathbb{P}}(X))$ is equal to $H^1(\Omega^n_{\mathbb{P}}(d)) = 0$ by Bott's Theorem 17.1.5. Thus $F^nH^{n+1}(U) = H^1(\Omega^n_{\mathbb{P},cl}(\log X))$ is isomorphic to the cokernel of the first map labeled by d. This proves the theorem for this case. The remaining p's can be handled by a similar argument, which is left for the exercises. □

This can be made explicit using the following lemma:

Lemma 17.5.2. *Let*

$$\omega = \sum (-1)^i x_i dx_0 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$

Then

$$H^0(\Omega_{\mathbb{P}}^{n+1}(kX)) = \left\{ \frac{g\omega}{f^k} \mid g \textit{ homogeneous of } \deg = kd - (n+2) \right\}.$$

With these identifications, elements of $P^n(X)$ can be represented by homogeneous rational differential forms $g\omega/f^k$ modulo exact forms. Set

$$R = \frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{(\partial f/\partial x_1,\dots,\partial f/\partial x_{n+1})}.$$

The ring inherits a grading $R = \oplus R_i$ from the polynomial ring. We define a map $P^n(X) \to R$ by sending the class of $g\omega/f^k$ to the class of g.

Theorem 17.5.3 (Griffiths). *Under this map, the intersection of $H^{n-p}(X,\Omega_X^p)$ with $P^n(X)$ maps isomorphically to $R_{\tau(p)}$, where $\tau(p) = (n-p+1)d - (n+2)$.*

This leads to an alternative method for computing the Hodge numbers of a degree-d hypersurface.

Corollary 17.5.4. *$h^{pq}(d) - \delta_{pq}$ is the coefficient of $t^{\tau(p)}$ in $(1+t+\cdots+t^{d-2})^{n+2}$.*

Proof. We can assume that $f = x_0^d + x_1^d + \cdots + x_{n+1}^d$ is the Fermat equation. It suffices to prove that the Poincaré series of R, which is the generating function $p(t) = \sum \dim R_i t^i$, is given by

$$p(t) = (1+t+\cdots+t^{d-2})^{n+2}.$$

Note that

$$R = \frac{\mathbb{C}[x_0,\dots,x_{n+1}]}{(x_0^{d-1},x_1^{d-1},\dots)} \cong \frac{\mathbb{C}[x]}{(x^{d-1})} \otimes \frac{\mathbb{C}[x]}{(x^{d-1})} \otimes \cdots \quad (n+2 \text{ times}).$$

Since Poincaré series for graded rings are multiplicative for tensor products, $p(t)$ is the $(n+2)$ power of the Poincaré series of $\mathbb{C}[x]/(x^{d-1})$, and this is given by the above formula. □

Exercises

17.5.5. Using exact sequences

$$0 \to \Omega^{n-i}_{\mathbb{P},cl}((j-1)X) \to \Omega^{n-i}_{\mathbb{P}}((j-1)X) \to \Omega^{n-i+1}_{\mathbb{P}}(jX) \to 0$$

and identifications

$$F^p H^{n+1}(U) \cong H^{n+1-p}(\Omega^p_{\mathbb{P},cl}(\log X)) \cong H^{n+1-p}(\Omega^p_{\mathbb{P},cl}(X)),$$

finish the proof of Theorem 17.5.1.