# Pfaff, Johann Friedrich | Encyclopedia.com



(*b*. Stuttgart, Germany, 22 December 1765; *d*. Halle, Germany, 21 April 1825)

*mathematics*.

Pfaff came from a distinguished family of Wurttemberg civil servants. His father, Burkhard Pfaff, was chief financial councillor and his mother was the only daughter of a member of the consistory and of the exchequer; Johann Friedrich was the second of their seven sons.

The sixth son, Christoph Heinrich (1773–1852), did work of considerable merit in chemistry, medicine, and pharmacy. He also investigated "animal electricity" with Volta, Humboldt, and others. Pfaff's youngest brother, Johann Wilhelm ANdreas (1774–1835), distinguished himself in several areas of science, especially in mathematics, and became professor of mathematics at the universities of Wurzburg and Erlangen; but the rapid changes in his scientific interests prevented him from attaining the importance of Johann Friedrich.

As the son of a family serving the government of Württemberg, Pfaff went to the Hohe Karlsschule in Stuttgart at the age of nine. The school, which was well-administered but subject to a harsh military discipline, served chiefly to train Württemberg's government officials and superior officers. Pfaff completed his legal studies there in the fall of 1785.

On the basis of mathematical knowledge that he acquired by himself, Pfaff soon progressed to reading Euler's *Introductio in analysin infinitorum*. In the fall of 1785, at the urging of Karl Eugen, the duke of Wurttemberg, he began a journey to increase his scientific knowledge. He remained at the University of Göttingen for about two years, studying mathematics with A. G. Kaestner and physics with G. C. Lichtenberg. In the summer of 1787 he traveled to Berlin, in order to improve his skill in practical astronomy with J. E. Bode. While in Berlin, on the recommendationm of Lichtenberg, Pfaff was admitted to the circle of followers of the Enlightenment around Friedrich Nicolai. In the spring of 1788 he traveled to Vienna by way of Halle, Jena, Helmstedt, Gotha, Dresden, and Prague.

Through the recommendation of Lichtenberg, Pfaff was appointed full professor of mathematics at the University of Helmstedt as a replacement for Klügel, who had been called to Halle. Pfaff assumed the rather poorly paid post with the approval of the duke of Württemberg.

At first Pfaff directed all his attention to teaching, with evident success: the number of mathematics students grew considerably. Gauss, after completing his studies at Gottingen (1795–1798), attended Pfaff's lectures and, in 1798, lived in Pfaff's house. Pfaff recommended Gauss's doctoral dissertation and, when necessary, greatly assisted him; Gauss always retained a friendly memory of Pfaff both as a teacher and as a man.

While in Helmstedt, Pfaff aided students whose talents he recognized. For example, he was a supporter of Humboldt following his visit to Helmstedt and he recommended him to professors at Gottingen. During this period he also formed an enduring friendship with the historian G. G. Bredow. Their plan to edit all the fragments of Pappus of Alexandria progressed no further than a partial edition (Book 4 of the *Collectio*) done by Bredow alone.

In 1803 Pfaff married Caroline Brand, a maternal cousin. Their first son died young; the second, Carl, who edited a portion of his father's correspondence, became an historian, but his career was abbreviated by illness.

A serious threat to Pfaff's academic career emerged at the end of the eighteenth century, when plans were discussed for closing the University of Helmstedt. This economy measure was postponed—in no small degree as a result of Pfaff's interesting essay "Uber die Voteile, welche eine Universitat einem Lande gewahrt" (Haberlins *Staatsarchiv* [1796], no. 2)—but in 1810 the university was in the end closed. The faculty members were transferred to Gottingen, Halle, and Breslau. Pfaff went to Halle at his own request, again as professor of mathematics. After Klügel's death in 1812 he also took over the direction of the observatory there.

Pfaff's early work was strongly marked by Euler's influence. In his *Versuch einer neuen Summations-methode*... (1788) he uncritically employed divergent series in his treatment of Fourier expansions. In editing Euler's posthumous writings (1792) and in the inaugural essay traditionally presented by new professors at Helmstedt—"programma in inaugurale, in quo peculiars differentialis investigandi ratioex theoria functionum deducitur" (1788)—as well as in 1795, Pfaff investigated series of the form

A friend of K. F. Hindenburg, the leader of the German combinatorial school, Pfaff prepared a series of articles between 1794 and 1800 for *Archiv der reinen und angewandten Mathematik* and *Sammlung combinatorisch-analytischer Abhandlungen*, which were edited by Hindenburg. The articles consistently reflect the long-winded way of thinking and expression of Hindenburg's school, with the single exception of "Analysis einer wichtigen Aufgabe des Herrn La Grange" (1794), which sought to free the Taylor expansion (with the remainder in Lagrange's form) from the tradition that embedded it in the theory of combinations and instead to present it as a primary component of analysis.

In 1797 Pfaff published at Helmstedt the first and only volume of an introductory treatise on analysis written in the spirit of Euler: *Disquisitiones analyticae maxime ad calculum integralem et doctrinam serierum pertinentes*. In 1810 he participated in the soultion of a problem originating with Gauss that concerned the ellipse of greatest area that can be inscribed in a given quadrilateral. This led him to investigate conic pencils of rays.

Pfaff presented his most important mathematical achievement, the theory of Pfaffian forms, in "Methodus generalis, aequationes differentiarum partialium, necnon aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variabiles, complete integrandi," which he submitted to the printed in the *Abhandlumgen* of the Berlin Academy (1814–1815) and received an exceedingly favorable review by Gauss, the work did not become widely known. Its importance was not appreciated until 1827, when it appeared with a paper by Jacobi, "Über Pfaff's Methode, eine gewohnliche lineare Differentialgleichung zwischen 2 n Variabeln durch ein System von n Gleichungen zu integrieren" *(Journal für die reine und angewandte Mathematik*, **2** 347 ff.).

Pfaff's "Methodus" constituted the starting point of a basic theory of integration of partial differential equations which, through the work of Jacobi, Lie, and others, has developed into the modern Cartan calculus of extreme differential forms. (On this subject see, for example, C. Carathéodory, *Variationsrechnung und partielle Differentialglechungen 1. Ordnung*, I [Leipzig, 1956].)

The core of the method that Pfaff made available can be described as follows: In the title of the "Methodus " the expression "aequationes differentialis vulgares" appears; by this Pfaff meant equations of the form

the left side of which, in modern terminology, is a differential form in n variables (Pfaffian form) The equation itself is called a Pfaffian equation. Now, by means of a first-order partial differential equation in n + 1 variables,

$F(x_1, x_2, \ldots, x_n; z; p_1, p_2, \ldots, p_n) = 0,$

where the partial derivaties

one can easily transform the equation

into a Pfaffian equation in 2n variables by eliminating *dz*.

The significance of the reduction of a partial differential equation to a Pfaffian equation had previously been recognized by Euler and Lagrange. The reduction could not be exploited, however, for lack of an integration theory of the Pfaffian forms which would be valid for all n; it was this deficiency that Pfaff's Methodus in large measure remedied. Gauss justifiably emphasized this aspect of Pfaff's work in his review in *Göttingische gelehrte Anzeigen* (1815).

Pfaff's theory is based on a transformation theorem that in current terminology, and going a little beyond Pfaff, can be stated in the following manner: A Pfaffian form with an even number of variables can be transormed, by means of a factor $\varrho(x_1, x_2, \ldots, x_n)$ into a Pfaffian form of n—1 variables. Moreover, for the case n = 2, $\varrho$ is simply the Euler multiplier or integrating factor of the differential equation $\phi_1 dx_1 + \phi_2 dx_2 = 0$. For therefore, there is a multiplier $\varrho$, so that can be written in the form and they are independent functions of $x_1, \ldots, x_4$.

For a Pfaffian form with an odd number of variables ther is in general no corresponding multiplier that will enable one to reduce the number of variables.

In the 1827 article cited above, Jacobi later provided a sutiable method of reduction: a Pfaffian form with an odd number of variables can,through subtraction of differential dw, which is always reducible by means of the transformation $x_i=f_1(y_1,y_2 \ldots y_{n-1},t) i=1,2\ldots,n$ be brought to the form where the are functions of the y s.

Through alternately employing transformation (following Pfaff) and reduction (following Jacobi) one can finally bring ever Pfaffian form with arbitrary number of variables into a canonical form: n = 2p, into the form $z_1dz_2+z_3dz_4+\ldots+dz_{2p-1}dz_{2p}$ for $n=2p+1$,into the form $dz_1+dz_2,dz_3+\ldots+z_{2p}dz_{2p+1}$

Lie later gave the relationship between partial differential equations and Pfaffian forms a geometrical interpretation that possessed a greater intuitive clarity than the analytic approach.

# BIBLIOGRAPHY

I. Original Works. There is a list of Pfaff's writings in Poggendorff, II, cols. 424–x425. They include *Versuch einer neuen Summationsmethode nebst anderen damit zusammenhängeden analytischen Bemerkungen* (Berlin, 1788); "Analysis einer wichtigen Aufgable des Herrn La Grange," in Hindenburg's *Archive der reinen und angewandten Mathematik*, 1 (1794), 81–84; *Disquistiones analyticae maxime ad calculum integralem et doctrinam serierum pertinentes* "Methodus genralis, aequationes differentiarum partialium, neconon aequationes differentiales vulgares, utrasque primi ordinis, inter quotcunque variablies complete integrandi," in *abhandlungen der Preussischen Akademie der wissenschaften* (1814–1815), 76-135 also translated into German by G. Kowalewski as no. 129 in Ostwald's Klassiker der exakten Wissenschaften (Leipzig, 1902); and *Sammlung von Briefen, gewechselt zwischen Johann Friedrich Pfaff Carl Pfaff, ed*. (Leipzig, 1853).

II. Secondary Literature. See G. Kowalewski, Grosse Mthematiker, 2nd ed. (Munich-Berlin, 1939), 228–247; and Carl Pfaff's biographical introduction to his ed. of his father's correspondence, pp. 1–35. Also see articles of Pfaff in *neuer Nekrolog der Deutschen*, **3** (1825), 1415–1418; and *Allgemeine deutsche Biographie, XXV* (Leipzig, 1887), 592-593.

H. Wussing