Theodorus of Cyrene | Encyclopedia.com

(*b*. Cyrene, North Africa, ca. 465 b.c.; *d*. cyrene [?], after 399 b.c.)

mathematics.

Theodorus was the mathematical tutor of Plato and Theaetetus and is known for his contribution to the early development of the theory of irrational quantities. Iamblichus includes him in his catalog of Pythagoreans [1]. According to the account of Eudemus as preserved by Proclus [2], he was a contemporary of Hippocrates of Chios, and they both came after Anaxagoras and Oenopides of Chios. Diogenes Laërtius [3] states that he was the teacher of Plato; and Plato represents him as an old man in the *Theaetetus*, which is set in 399 b.c. Since Anaxagoras was born ca. 500 and Plato in 428 or 427, it is reasonable to suppose that Theodorus was born about 465. This would make him sixty-six years old in the fictive year of the *Theaetetus*. According to the dialogue he had been a disciple of Protagoras but had turned at an early age from abstract speculation to geometry [4]. He was in Athens at the time of the death of Socrates [5]. He is also made a character by Plato in the *Sophist* and the *Politicus*. Plato may have sat at his feet in Athens just before the death of Socrates or at Cyrene during his travels after that event. In the dialogue Theaetetus tells Socrates that he learned geometry, astronomy, harmony, and arithmetic from Theodorus [6]. As with Plato, this could have been at Athens or Cyrene.

In the dialogue [7] Theaetetus is made to relate how Theodorus demonstrated to him and to the younger Socrates, a namesake of the philosopher, that the square roots of 3,5, and so on up to 17 (excluding 9 and 16, it being understood) are incommensurable with the unit; and Theaetetus goes on to say how he and Socrates tried to find a general formula that would comprehend all square roots. plato clearly purports to be giving a historical account [8] and to be distinguishing the achievement of Theodorus from that of Theaetetus; and it would appear that Theodorus was the first to demonstrate the irrationality of . Two questions immediately arise. Why did he start at ? Why did he stop at ? The answer to the former question must be that the irrationality of was already known. It was, indeed, known to the earlier Pythagoreans; and there is a high probability that it was the discovery of the incommensurability of with the unit that revealed to the Greeks the existence of the irrational and made necessary a recasting of Greek mathematical theory [9]. After this discovery it would be natural for Theodorus and others to look for further examples of irrationality.

The answer to the second question depends on how Theodorus proved the irrationality of the numbers under examination, and is not so easy. We may rule out at once the suggestion of F. Hultsch that Theodorus tried the method of successively closer approximation, because it would never *prove* irrationality [10]. The answer is dependent also on the meaning given to the words πωζεὐυέσχετο. They ususlly have been translated "for some reason he stopped" [11]. A glance at the uses of ἐυέχειυ given in the lexicons, however, shows, as R. Hackforth first appreciated, that the Greek must mean "somehow he got into difficulties" [12].

This rules out the possibility that Theodorus stopped at 17 merely because he had to stop somewhere and felt he had proved enough [13]. It also rules out the possibility, despite the contention of A. Wasserstein, that Theodorus merely applied to 3, 5,. . .,17 the proof of the of the irrationality of [14]. This was known to Aristotle and is interpolated in the text of Euclid's Elements [15]; it may have been the way in which the irrationality of was originally demonstrated. in this proof it is shown that, if the diagonal of a square is commensurable with its side, the same number will be both odd and even [16]. This proof can be generalized for all square roots, and indeed for all roots, in the form "m is irrational unless N is the m-th power of an integer n" [17] Theodorus would soon have recognized the generality and would have run into no difficulties after 17.

It has been suggested that the Pythagorean devotion to the decad may have led Theodorus to stop where he did.[18] For can be represented as , and so on for all the odd numbers up to , at which point all the numerals from 1 to 9 would have been exhausted: Theodorus, however, would not have run into any difficulty in proceeding farther by this method, nor does it afford any proof of irrationality.

The above hypothesis is similar to one propounded by an anonymous commentator on the *Theaetetus*[19]. He first says that Plato made Theaetetus start with because he had already shown in the *Meno* that the square on the diagonal of a square is double that on the side. He then proceeded to point out that Theaetetus was both a geometer and a student of musical theory. The tone interval has the ratio 9:8. If we double the two numbers we have 18:16; and between these two numbers the arithmetic mean is 17, dividing the extremes into unequal ratios, "as is shown in the commentaries on the *Timaeus*". The comment of Proclus on Plato, *Timaeus* 35B (*Commentarium in Timaeum, 195A*), is relevant, but we need not pursue it because it is clearly a rather farfetched hypothesis to explain why Theodorus stopped at 17.

An ingenious theory has been put forward by J. H. Anderhub [20]. If a right-angled isosceles triangle with unit sides is set out as in Figure 1, its hypotenuse is . If at one extremity of the hypotenuse a perpendicular of unit length is erected, we have a second triangle with hypotenuse = . The process can be continued with all the hypotenuses radiating from a common point, and the angles at the common point can be shown to be 45°, 35° 15', and so on. The total of all the angles up to hypotenuse = is approximately 351°10's, and the total up to is approximately 364°48'–that is, after the circle has been completed and the triangles begin to overlap. But although this would have given Theodorus a reason for stopping, he would have had no difficulty in going on; and the method does not prove the irrationality of any hypotenuse.

There is one theory, put forward by H. G. Zeuthen, that satisfies the requirements that there shall be a separate proof for each number as Plato's text suggests, and that after the proof will encounter difficulties.[21] Zeuthen's suggestion is that Theodorus used the process of finding the greatest common measure of two magnitudes as set out in Euclid's *Elements*, **X.2** , and actually made a test of incommensurability by Euclid: "If when the lesser of two unequal magnitudes is continually subtraced from the greater, the remainder never measures the one before it, the magnitudes will be incommensurable."[22] The method may conveniently be illustrated from itself. Let *ABC* be a right-angled triangle in which *AB* = 1, *BC* = 4, so that *CA* = . Let *CD* be cut off from *CA* equal to *CB* so that , and let *DE* be drawn at right angles to *CA*. The triangles *CDE*, *CBE* are equal and therefore

DE = *EB*. The triangles *ADE, ABC* are similar and *DE* = 4*AD*. We therefore have *DE* = 4*AD* = 4 (– 4). Now from *EA* let *EF* be cut off equal to ED and at *F* let the perpendicular *FG* be drawn. Then by parity of reasoning

AF = *AB* – *BF* = *AB* – 2*DE*

Obviously, the process can be continued indefinitely, so that *ABC, ADE, AFG*, . . . is a diminishing series of triangles such that

and we shall never be left with a magnitude that exactly measures *CA*, which is accordingly incommensurable.

Theodorus would certainly have used a geometrical proof, but the point can be made as shown below in modern arithmetical notation. The process of finding the greatest common measure of 1 and (if any) may be set out as follows:

1)

The next stage in the process would be to divide into , but this is the same as dividing into 1, which was the previous step. The process is therefore periodic and will never end, so that 1 and do not have a greatest common measure. It will be recognized as the same process as that for finding a continued fraction equal to .

It is a powerful argument in favor of this theory that Plato, in the passage of the *Theaetetus* under discussion, for the first time in Greek literature uses the term οὐ σύμμετρος ("incommensurable") for what had previously been described as ἀρρητος ("inexpressible"). This strongly reinforces the conviction that he was doing something new, and that the novelty consisted in using the test of incommensurability later found in Euclid.

These proofs, geometrical and arithmetical, are simple; and the former would certainly have been within the grasp of Theodorus. So would the earlier proofs for , and so on.[23] The next case, , would not call for investigation since = ; but presents difficulties at which even a modern mathematician may quail. Recurrence does not take place until after six stages, which, on the basis of the exposition of B.L. van der Waerden, may be set out as follows.[24] We start by subtracting the appropriate multiple of 1 from and get a remainder . We now divide into 1. But

We treat and 3 in exactly the same way, subtracting 2.3 from and getting . Now,

and we subtract the 5 from , getting , and divide this into 5. But

and after subtracting 3 · 2 from , we get again. But

and subtracting 5 from yields . Now

any by subtracting 2 · 3 from we obtain . But

subtracting 4.1 from leaves us with , and dividing 1 into brings us back where we started. The process is therefore periodic and will never end, so that is incommensurable with 1.

This is formidable enough in modern notation, and impossible to set out in a drawing, particularly a drawing in sand. If this is the method that Theodorus used, it is therefore fully understandable why he stopped at .

Although this is only a hypothesis, there is no other that fits the facts so well; and if his pupil Theaetetus developed a theory of proportion based on the method of finding the greatest common measure, as is argued in the article devoted to him in this *Dictionary*, it becomes virtually certain that this is the method employed by Theodorus.

Proclus, in analyzing curves in the manner of Geminus, criticizes “Theodorus the mathematician” for speaking of “blending” in lines.[25] He is probably to be identified with Theodorus of Cyrene, since in his only other reference Proclus describes the subject of this article. He may also be identified with the Theodorus whom Xenophon held up as a model of a good mathematician.[26]

NOTES

1. Iamblichus, *De vita Pythagorica*, 267; L. Deubner, ed. (Leipzig, 1937), p. 146.8–9.

2. Proclus, *In primum Euclidis*, G. Friedlein, ed. (Leipzig, 1873); repr. Hildesheim, 1967), 65.21–66.7.

3. Diogenes Laërtius, *Vitae philosophorum*, **II** 103, **III** .6; H. S. Long, ed., **I** (Oxford, 1964), 100.9–13, 123.18.

4. Plato, *Theaetetus* 164E–165A, in *Platonis opera*, J. Burnet, ed., **I** (Oxford, 1899: frequently repr.).

5. It is not obvious why James Gow. *A Short History of Greek Mathematics* (Cambridge, 1884), 164, should flatly contradict the evidence of Plato’s dialogue and say, “He does not seem to have visited Athens.”

6. Plato, *Theaetetus*, 145c–D.

7.*Ibid*., 147D–148B.

8. Jean Itard, *Les livres arithmètiques d’Euclide*, Histoire de la Pensèe, **X** (Paris, 1961), is exceptional in regarding as tenable the view that Theodorus and Theaetetus may not be historical persons but “personnages composites nès dans l’esprit méme de Platon.”

9. The fullest account of this subject is in Kurt von Fritz, “The Discovery of Incommensurability by Hippasus of Metapontium,” in *Studies in Presocratic Philosophy*, David J. Furley and R. E. Allen, eds., **I** (London–New York, 1970), 382–412. But in his earlier paper with the same title in *Annals of Mathematics*, **46** (1945), 242–264, Fritz exposed himself to some strictures, of which he has not taken notice, from A. Wasserstein, “Theaetetus and the Theory of Numbers.” **165** , n.3.

10. F. Hultsch, “Die Näherungswerthe irrationaler Quadratwurzeln bei Archimedes,” in *Nachrichten von der königlich Gesellschaft der Wissenschaften zu Göttingen*, **22** (1893), 368–428. Hultsch received some support from T. L. Heath in his early work, *The Works of Archimedes* (Cambridge, 1897), Ixxix–Ixxx, in which he regarded it as “pretty certain” that Theodorus, like Archimedes after him, represented geometrically as the perpendicular from an angular point of an equilateral triangle to the opposite side. He also presumed that Theodorus would start from the identity 3 = 48/16 = (49-1)/16, so that

but in his later work, *A History of Greek Mathematics*, **I** (Oxford, 1921), 204, he realized that this “would leave Theodorus as far as ever from *proving* that is incommensurable.” These approximations may, of course, have played a part in Theodorus’ researches until he found a demonstrative proof.

11. “There he stopped” (B. Jowett); “here he somehow came to a pause” (B. J. Kennedy); “il s’ètait, je ne sais pourquoi, arrêtè lá” (A. Diès): “there, for some reason, he stopped” (F. M. Cornford); “at that he stopped” (H. N. Fowler); “qui, non so come, si fermó” (M. Timpanaro Cardini). But the latest translator, J. McDowell (Oxford, 1973), has the sense right–“at that point he somehow got tied up.”

12.*A Greek-English Lexicon*, H. G. Liddell and R. Scott, eds., new ed. by H. Stuart Jones (Oxford, 1940), see ἐνἐχω **II** , 565. The general meaning of the passive and middle is “to be held. caught, entangled in”: and a particularly relevant example is given in **II** .2, [κῦρος] ἐνειχετοϱιη,ςι: Heredotus 1.190. *Theatetus* 147D is the only passage quoted for the meaning “come to a standstill” (11.5), and therefore it can hardly determine the meaning of that passage.

R. Hackforth, “Notes on Plato’s *Theaetetus*,” **128** . It is significant that Hackforth’s interest is purely literary, and his interpretation is therefore free from any bias in favor of some particular mathematical solution. He is supported by Malcolm S. Brown, “*Theaetetus*: Knowledge as Continued Learning,” in Journal of the History of Philosophy, **7** (1969), 367.

13. Wasserstein, *op. cit*., **165** , makes this suggestion without necessarily endorsing it. G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 4th ed. (Oxford, 1960), 43, in the light of their views about the difficulty of generalizing the Pythagorean proof for (see note 14), regard the suggestion as credible.

14. It is the main burden of Wasserstein’s paper (cited above) that this is precisely what Theodorus did. He argues that the difficulties of effecting a valid generalization are such as would have been perceived by Theodorus and that “it was precisely this refusal of the rigorous mathematician to enumerate a general theory based on doubtful foundations that led his pupil Theaetetus to investigate not only the problem of irrationality but also the more fundamental arithmetical questions.” Although

it may be conceded that Theodorus was an acute mathematician, it is most unlikely that he, or any other ancient mathematician, would have thought about this problem in the manner of G. H. Hardy and E. M. Wright. (see note 16.) Wasserstein's thesis is controverted in detail by Brown, *op cit*., 366–367, but in part for an irrelevant reason. Brown accepts van der Waerden's view that bk. **VII** of Euclid's *Elements* was already in "apple-pie order" before the end of the fifth century, whereas Wasserstein, like Zeuthen, regards it as the work of Theaetutus; but Wasserstein's contention that Theodorus applie the traditional proof for can be detached from this belief.

15. Aristotle, *Prior Analytics* 1.23.41a23–30; W. D. Ross, ed. (Oxford. 1949, corr. repr. 1965), English trans, by A. J. W. D. Ross, ed., **I** (Oxford, 1928; repr. 1968). Euclid, *Elements*, **X** , app. 27; *Euclidis opera omnia*, J. L. Heiberg and H. Menge, eds., **III** (Leipzig, 1886), 408.1–410, 16; E. S. Stamatis, ed., *in Euclidis Elementa post J. L. Heiberg*, **III** (Leipzig, 1972), 231.10–233.13. An alternative proof is also given. In earlier eds. the proof was printed as Euclid, *Elements*, **X** , 117, but it is now recognized as an interpolation.

16. T. L. Heath, *A History of Greek Mathematics*, **I** (Oxford, 1921), 205, purports to give a fairly easy generalization; but it is logically defective in that he assumes that "if $m^2 = N.n^2$, therefore m^2 is divisible by N, so that m also is a multiple of N," which is true only if N is not itself the multiple of a square number. Wasserstein, *op cit*., 168–169, corrects Heath. Hardy and Wright, *op. cit*., 40, show that the generalization is not so simple as Heath represented it to be and "requires a good deal more than a "trivial" variation of the Pythagorean proof." This is true; but Hardy and Wright are working to standards of logical rigor far beyond what any mathematician of the fifth century B. C. would have demanded, and Theodorus could easily have found a generalization that would have satisfied his own standards.

17. Hardy and Wright, *op cit*., 41. The authors discuss Theodorus' work helpfully in the light of modern mathematics on 42–45.

18. See Brown, *op cit*., 367–368, and his n. 26.

19.*Anonymer Kommentar zu Platons Theaeter*, H. Diels and W. Schubart, eds., Berliner Klassikertexte, **II** (Berlin, 1905). The passage is reproduced and translated, as is Proclus' commentary on the relevant passage in the *Timaeus*, with illuminating notes, in Wasserstein, *op cit*., 172–179. The commentator does not appear to accept his own suggestion, saying that Theodorus stopped at 17 because that is the first number after 16, and 16 is the only square in which the number denoting the sum of the sides is equal to the number denoting the area (4 + 4 + 4 + 4 = 4 × 4).

20. Both B. L. van der Waerden, *Science Awakening*, English trans, by Arnold Dresden of *Ontwakende Wetenschap*, 2nd ed., **I** (Groningen, n.d.), 143; and Arpàd Szabò, *Anfänge der griechischen Mathematik* (Munich–Vienna, 1969). 70, attribute this interesting construction to J. H. Anderhub, *Joca-Seria; Aus den Papieren eines reisenden Kaufmunnes*, but I have not been able to obtain a copy.

21. H. G. Zeuthen, "Sur la constitution des livres arithmétioues des Elèments d'Euclide et leur rapport á la question de l'irrationalitè," in *Oversigt over det K. Danske Videnskabernes Selskabs Forhandlinger* for 1910 (1910–1911), 422–426. The theory is taken up again in articles in the same periodical in the volumes for 1913 and 1915. Zeuthen's thesis is supported by O. Toeplitz, 28–29; and by Brown, *op cit*.

22.*Euclidis opera omnia*, J. L. Heiberg and H. Menge, eds., **III** (Leipzig, 1886), 6.12–8,13; E. C. Stamatis, ed., in *Euclidis Elementa post J. L. Heiberg*, **III** (Leipzig, 1972), 3.19–4.19.

23. The cases of and are similar to . The case of is a little more difficult, involving one more step before recurrence takes place. The case of is difficult, as may be seen from the process of expressing it as a continued fraction given by G. Chrystal, *Algebra*, **II** (Edinburgh, 1889), 401–402, where it is shown that recurrence occurs only after five partial quotients:

Perhaps it was Theodorus' experience with that made him unwilling to embark on . The same method can be applied to , although it is probable that was originally proved irrational not by this method but by that referred to by Aristotle (see note 15). Zeuthen, *loc. cit*., and Heath, *loc. cit*., give the proofs for and ; and Heath adds a geometrical proof for ; Hardy and Wright, *op. cit*., give proofs for and . The method is used by Kurt von Fritz in Studies in Presocratic Philosophy, **I** , 401–406, to prove the incommensurability of the diagonal of a regular pentagon in relation to its side.

24. Van der Waerden, *op. cit*., 144–146. The method is a simplification of the process of finding the greatest common measure as used in the text for by taking new ratios equal to the actual ratios of the process; but van der Waerden's exposition is rather ellipical, and it may more clearly be set out as here.

25. Proclus, *In primum Euclidis*, G. Friedlein, ed., 118.7–9. (The reference in Friedlein's index is incorrect). In favor of identifying him with Theodorus of Cyrene is the fact that Plato in one place (*Theaetetus* 143B) calls the Cyrenaic "Theodorus the geometer": and Diogenes Laërtius, *op. cit*., also calls him "the Cyrenaic geometer" and "the mathematician." Van der Waerden, *op. cit*., 146, accepts the identification, but it is rejected (by implication) by H. Diels and W. Kranz, eds., *Die Fragmente der Vorsokratiker*, 6th ed. (Dublin-Zurich, 1954; repr. 1969) and most writers; Glenn R. Morrow, *Proclus: A Commentary on the First Book of Euclid's Elements* (Princeton, 1970), 95, n. 70, thinks the reference is to Theodorus of Soli,

who is cited by Plutarch on certain mathematical difficulties in the *Timaeus*. Diogenes Laërtius, *loc. cit*., refers to twenty persons with the name Theodorus, and Pauly-Wissowa lists no fewer than 203. There was even a second Theodorus of Cyrene, a philosopher of some repute, who flourished at the end of the fourth century B.C. In the passage under discussion proclus is reproducing Geminus' classification of curves; and in treating mixed curves he says the mixing can come about through "composition," "fusing," or "blending." According to Geminus and Proclus, but no Theodorus, planes can be blended but lines cannot.

26. Xenophon, *Memorabilia*, which is *Commentarii*; **IV** .2, 10; *Xenophontis opea omnia*, E. C. Marchant, ed., **II** (Oxford, 1901; 2nd ed., 1921: repr. 1942), **II** . 25–26.

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