The intersection game

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Spot it! is a fun card game made up of 55 cards each of which carries 8 pictures. What's special about this game is that any two cards have exactly one of the pictures in common. The aim of the game is to be the first to spot this common picture. Let's spot the nice mathematics at the core of this game.

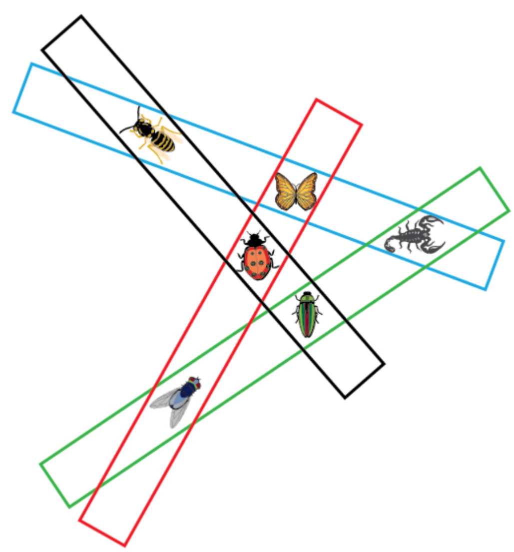
Spotting lines

How hard is it to make up 55 cards decorated with pictures such that

(I) any two cards have exactly one common picture?

Sounds tricky but it's actually a piece of cake. What mathematical objects have a similar property? Well, what about two non-parallel lines in a plane, they always intersect in a unique point, right?

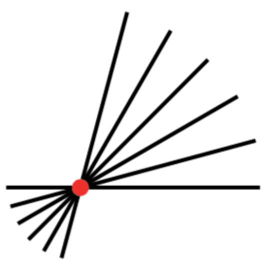
So, to speed-design a deck of cards with property (I) just draw 55 mutually non-parallel lines, label the resulting points of intersection with different pictures and gather all pictures on lines into cards. For example, in the following picture I've turned four intersecting lines into four playing cards carrying three pictures each.

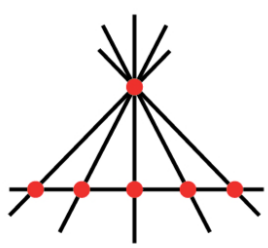


Pretty neat, but also a bit disappointing. Is there really no more to spot in *Spot it!* ? Well, let's have a closer look at what a brand new *Spot It!* deck constructed from 55 lines would look like.

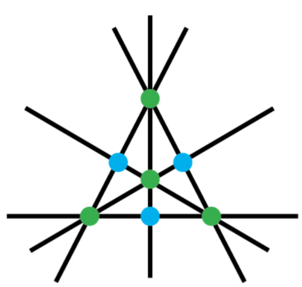
Using a random assortment of lines, chances are that, as in our small example, points of intersection corresponding to different pairs of lines end up being distinct. In other words, there will be as many points of intersection as there are ways to pair up our 55 lines, that is, 55 x 54 / 2 = 1485 different points corresponding to as many pictures. It is also not hard to see that there will be 54 pictures per card. Well, that's definitely not as compact as the commercial game–there are just way too many symbols and consequently our new game is probably not much fun to play!

To cut down on the number of pictures we have to make as many points of intersection as possible coincide. Let's give this a try with six lines. In this case our simple-minded approach would result in 15 pictures with 5 pictures per card. After experimenting a bit we are quite naturally led to the following two sneaky but (gamewise) useless ways of keeping down the number of intersections.



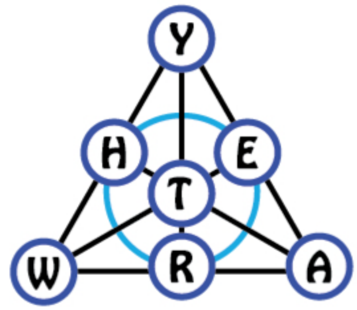


However, we also spot the following nice, balanced way corresponding to 7 pictures with 3 on every card.



Quite an improvement, and you cannot actually do better with six lines in a non-sneaky way. But here is an idea. The green points are each on three lines whereas the blue points are only on only two lines. So, let's see what happens when we collect the three blue points into a new seventh card. This can actually not be done with a straight line, but we don't really care.

Anyway, with this seventh card added and represented by a large circle we actually do arrive at a very compact 7-card version of *Spot it!* with only seven pictures. To add to the fun of playing this particular game we can choose as pictures the seven letters in the two words THEY WAR and distribute these letters as follows on the cards.



Then on the seven cards you can read the words in the sentence YEA WHY TRY HER RAW WET HAT. Here is a set of actual playing cards for you to actually play this game.

Spotting projective planes

As an abstract point-line geometry our new 7-card version of *Spot It!* is actually very famous. In the biz it is known as the *Fano plane* or the *smallest projective plane*. Let's explain what we mean by this.

The *point set* of an abstract point-line geometry can be any set whatsoever and its line set is a set of subsets of the point set. In this context we call the elements of the first set the POINTS and those of the second set the LINES of our geometry. Then it should be clear what it means for a POINT to be on a LINE, or for two POINTS being connected by a LINE, etc.

So, for example, we can interpret any deck of *Spot it!* cards as a point-line geometry simply by making the pictures into the POINTS and the cards into the LINES. In the case of our Fano plane {T,H,E,Y,W,A, R} is the point set and {{Y,E,A}, {W,H,Y}, {T,R,Y}, {HER}, {R,AW}, {W,E,T}, {H,A,T}} is the line set.

Property (I) makes the resulting geometries into special point-line geometries called *dual linear spaces* which are characterized by the fact that

(I) any two LINES intersect in a unique *POINT*.

Projective planes like the Fano plane are super special dual linear spaces which also have the property that

(C) any two POINTS are part of exactly one LINE.

In the case of the Fano plane this means that any two of the letters in THEY WAR are contained simultaneously in exactly one of the words in YEA WHY TRY HER RAW WET HAT.

Now, one of the fundamental theorems of the theory linear spaces discovered by de Bruijn and Erdős (of Erdős number fame) tells us that for (non-sneaky) *Spot it!* decks there are always at least as many pictures as there are cards. In addition, any optimal *Spot it!* deck with the same number of cards and symbols is automatically an incarnation of a finite projective plane, the number of cards is of the form n^2+n+1 and there are $n+1$ pictures on every card and every picture appears on $n+1$ cards.

For example, in the case of the Fano plane game both the number of cards and pictures is $2^2+2+1=7$ and there are $2+1=3$ pictures on every card.

In a commercial *Spot it!* deck the number of symbols is $7^2+7+1=57$ and there are $7+1=8$ pictures on every card. And that's not a coincidence. On closer inspection it becomes clear that commercial *Spot it!* with its 55 cards is a projective plane with two lines missing. (The original French name for *Spot It!* is *Dobble* and the double 5 in 55 appealed marketingwise and also happened to be an ideal number of cards taking into account manufacturing considerations.)

It is actually quite straightforward to reconstruct the missing two cards and to add them to the deck. In fact, to do so makes a lot of sense because with the completed deck a whole set of new games based on property (C) become possible.

Spot It! cubed

Recently I received an e-mail in which someone asked whether it is possible to make up decks in which any three cards have exactly one picture in common. The answer to this quite natural question is “Yes”.

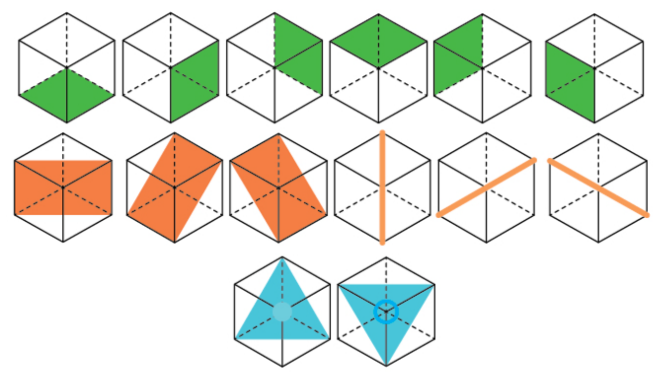
Any three mutually non-parallel planes in space intersect in a unique point or in a line. So, just choose a bunch of mutually non-parallel planes no three of which pass through a common line (which is not a problem!). Then label the resulting points of intersection with different pictures and gather all pictures on the different planes into cards and voila instant *Spot It cubed*!

It is just as easy to make up *Spot It*^{4}, *Spot It*^{5}, etc. sets using intersecting hyperplanes. However, as in the case of regular *Spot it!* when designing these new decks the real trick is to keep the number of pictures small. Luckily, there are mathematical objects that can be translated into these new types of decks.

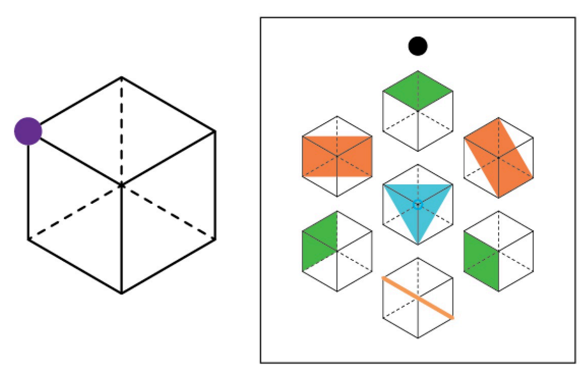
Here is a nice simple one. It's a point-circle geometry whose POINTS are the 8 vertices of the cube and whose CIRCLES are 14 sets of 4 vertices each, represented by the 6 faces (green) of the cube, the six diagonal rectangle cuts (orange) and the two tetrahedra (blue) inscribed into the cube. The diagram shows one rep each of the three different types of four vertex sets. Convince yourself that just like in the geometry of circles on a sphere any three POINTS in our mini geometry are contained in exactly one of the CIRCLES.

What we have to do to make this geometry work for us is to *dualize* it: We make the different CIRCLES into the pictures and have one card per POINT. Here the card corresponding to a POINT contains all the CIRCLES through the POINT. Confusing?

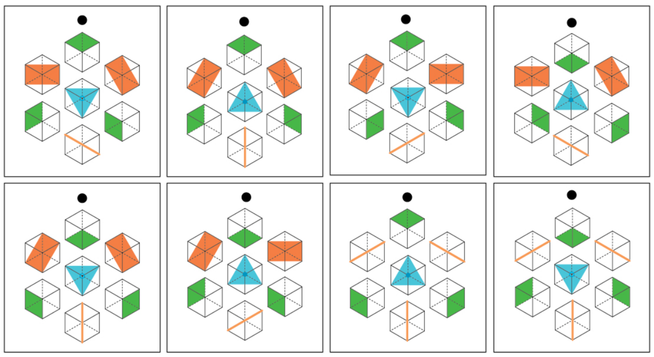
Okay, so let's actually do this. Using a regular hexagon picture of the cube we can draw the 14 CIRCLES as follows. These will be the pictures on our cards.



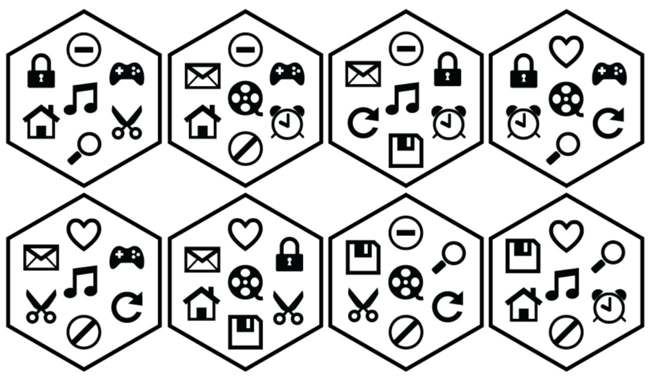
Now there will be one card for every POINT. For example, the card corresponding to the purple POINT contains the pictures of all the CIRCLES that contain this vertex.



Here is the complete set of cards.



But of course you can also use symbols that don't encode any of the geometry, like the following.



Spot a biplane!

If you want to be an innovative *Spot It!* game designer the mathematical objects to check out are the so-called t-(v,k,λ) (block) designs. A t-(v,k,λ) design has a point set consisting of v POINTS and BLOCKS consisting of k POINTS each such that any t POINTS are contained in exactly λ blocks.

For example, the projective planes are just the 2-$(n^2+n+1,\ n+1,\ 1)$ designs and our little geometry on the cube is a 3-(8,4,1) design. Dualizing any t-(whatever, whatever, 1) design as we've just demonstrated gives a deck of cards in which any t cards have exactly one picture in common. The problem with larger t is that no matter how hard you try you'll always end up with lots of pictures on your cards.

For example, starting with the amazingly compact 5-(12,6,1) Mathieu design we'd get a deck of 12 cards with 66 pictures on each card out of a total of 132 pictures. Pretty mindboggling but almost certainly no fun to play.

But, instead of just trying to up t, why not play around with the other parameters in the definition of block design. In particular, some of the 2-(whatever, whatever, 2) designs translate into interesting decks of cards with small numbers of pictures.

These very rare and exotic designs are called *biplanes.* Similar to projective planes any two POINTS of a biplane are contained in two BLOCKS and (thrown in for free!) any two BLOCKS intersect in two POINTS. This intersection property means that you don't have to worry about dualizing a biplane. Just make the points into the pictures and the blocks into the cards and you've got a *Spot a biplane!* deck.

For example, the blocks of the unique 2-(7,4,2) biplane are just the complements of the LINES of the Fano plane within its point set. Here is the corresponding *Spot a biplane*! deck.

Ready to play? Spot the two common letters on the red and green card. Spot the two cards that contains the two letters WA.

Further Reading

The Mathematics of "SPOT IT! by Rebekah Coggin and Anthony Meyer, ΠME Journal 13(8), 459-467 has more details on projective planes, as do a number of articles online.

Check out *Spot It!* for mobile devices. It incorporates (complete!) decks corresponding to projective planes with 31, 57, and 91 cards.

For some more info about the Fano plane and some other geometry based card games google "Ed Pegg, Fano plane".

If you are interested in making your own serious *Spot a biplane*! deck google "Gordon Royle, biplane".

Burkard Polster specializes in fun mathematics and you may be familiar with some of his books like The Mathematics of Juggling or Math goes to the Movies (with Marty Ross). Burkard teaches math(s) at Monash University and together with his colleague Marty writes the weekly "Maths Masters" column for The AGE in Melbourne, Australia.

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