"... the behavior of this spectral sequence ... is a bit like an Elizabethan drama, full of action, in which the business of each character is to kill at least one other character, so that at the end of the play one has the stage stewn with corpses and only one actor left alive (namely the one who has to speak the last few lines)."

J. F. Adams

(page 180) John McCleary, A User's Guide to Spectral Sequences, Second Edition.

Postnikov Towers, Whitehead Towers and their applications

MA598 Differential Topology in-class presentation

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$\mathbb{Z}_p$ means $\mathbb{Z}/p\mathbb{Z}$.

In algebraic topology, the notion of a tower is $\boxed{\text{not}}$ unfamiliar.

Let's consider how we build a CW complex.

$$
\begin{tikzcd}
 & X \\
 & \vdots \arrow[u, hook] \\
n\text{-skeleton} & X^{(n)} \arrow[u, hook] \arrow[r, two heads] & M(\bigoplus_{i\in I_n}, n-1) \\
(n-1)\text{-skeleton} & X^{(n-1)} \arrow[u, hook] \arrow[r, two heads] & M(\bigoplus_{i\in I_{n-1}} \mathbb{Z}, n-1) \\
 & \vdots \arrow[u, hook] \\
 & X^{(1)} \arrow[u, hook] \arrow[r, two heads] & M(\bigoplus_{i\in I_0} \mathbb{Z}, 1) \\
0\text{-skeleton} & X^{(0)} \arrow[u, hook]
\end{tikzcd}
$$

a set with discrete topology

Properties:

① Each $X^{(m-1)} \hookrightarrow X^{(m)}$ is an injection — cofibration

② cofiber = "cokernal" $:= X^{(m)}/X^{(m-1)}$

$\cong \bigvee_{i\in I_m} S^m$. So $\tilde{H}_*(X^{(m)}/X^{(m-1)}) = \begin{cases} \mathbb{Z}^{\oplus|I}, *= \\ 0, *\neq \end{cases}$

Moore space

③ $H_*(X^{(m)}) \cong \begin{cases} H_*(X) & \text{if } * < m \\ 0 & \text{if } * > m \end{cases}$

Now, we want to reverse everything and to use homotopy groups instead of homology groups.

Find an n, X is a ~~connected~~ path connected CW complex

$$
\begin{tikzcd}
 & X \arrow[d] \arrow[dd, bend left] \arrow[dddd, bend left] \arrow[ddddd, bend left] \\
K(\pi_n, n) \arrow[r] & P_n \arrow[d] \\
K(\pi_{n-1}, n-1) \arrow[r] & P_{n-1} \arrow[d] \\
 & \vdots \arrow[d] \\
K(\pi_1, 1) \arrow[r] & P_1 \arrow[d] \\
 & {*} = P_0
\end{tikzcd}
$$

Postnikov tower

We want:

① Each $P_m \longrightarrow P_{m-1}$ is a "surjection" — fibration

② fiber = "kernel" $= K(\pi_m(X), m)$

Eilenberg–Mac Lane spaces

③ $X \longrightarrow P_m$ induces isomorphisms of π_i when $i \le m$ and $\pi_i(P_m) = 0$ when $i > m$.

want to kill higher homotopy groups.

How do we construct a Postnikov tower?

— a naive but incorrect construction

Let $P_i := \prod_{j \le i} K(\pi_j, j)$.

$$\begin{array}{ccl}
K(\pi_n,n) \longrightarrow & P_n & = K(\pi_n,n) \times \cdots \times K(\pi_1,1) \\
& \downarrow & \\
& \vdots & \\
& \downarrow & \\
K(\pi_2,2) \longrightarrow & P_2 & = K(\pi_2,2) \times K(\pi_1,1) \\
& \downarrow & \\
& P_1 & = K(\pi_1,1)
\end{array}$$

Properties:

1. Each $P_m \to P_{m-1}$ is a fibration
2. fiber $= K(\pi_m, m)$
3. We do have $\pi_i(P_m) = \begin{cases} \pi_i, & i \le m \\ 0, & i > m \end{cases}$

but there is no map from $X \longrightarrow P_m$.
(The isomorphisms of π_i, $i \le m$ are not induced from $X \to P_m$.)

In general, X is not a direct product of $K(\pi,n)$'s, but a twisted product of $K(\pi,n)$'s.

We have to use other methods in building a Postnikov tower.

In order to do that, let's introduce three tools.

① Replace a map by an injection / cofibration

$$M_f := A \times I \sqcup B / (a,1) \sim f(a)$$ mapping cylinder of f.

$$\begin{tikzcd} & M_f \arrow[dr, "\simeq\ \text{deformation retract}"] & \\ A \arrow[ur, hook] \arrow[rr, "f"] & & B \end{tikzcd}$$

$a \mapsto (a,0)$

B path connected

---

② Replace a map by a fibration

Given an injection (if it's not an injection, replace it by one)

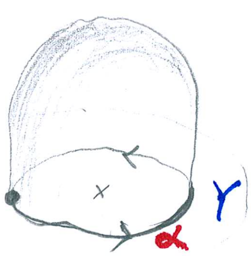
$f: A \hookrightarrow B$

$$\begin{tikzcd} & C_a \{\phi: I \to B \mid \phi(0) \in A\} \arrow[dr, "\text{fibration}"] & \\ A \arrow[ur, hook, "\simeq"] \arrow[rr, hook, "f"] & & B \end{tikzcd}$$

$a \mapsto \phi$, deformation retract $\phi \mapsto \phi(0)$; $\phi \mapsto \phi(1)$

---

③ Killing higher homotopy groups.



Given Y, want to kill $\pi_i(Y)$ for all $i > m$.

For each nontrivial element $[\alpha] \in \pi_{m+1}(Y)$, attach an $(m+2)$-cell to Y via α. The new space Y' has $\pi_{m+1}(Y') = 0$ but $\pi_i(Y') = \pi_i(Y)$ for $i \leq m$. (Prop 17.11) $Y \hookrightarrow Y'$

We can iterate this process until we kill all π_i, $i > m$.

We get $Y \hookrightarrow Y_m$ inducing isomorphisms on π_j, $j \leq m$ but $\pi_j(Y_m) = 0$ for $j > m$.

We are ready to construct a Postnikov tower.

Fix an n

$$
\begin{tikzcd}
& X \arrow[d] & & & & X \arrow[d] \arrow[d, bend left] \arrow[ddd, bend left] \arrow[ddddd, bend left] \\
\pi_{\geq n+1}(X) \text{ killed} & X_n \arrow[r, hook] \arrow[d] & P_n \arrow[d] & & F_n \arrow[r] & P_n \arrow[d] \\
\pi_{\geq n}(X_n) \text{ killed} & X_{n-1} \arrow[r, hook] \arrow[d, dotted] & P_{n-1} \arrow[d, dotted] & & F_{n-1} \arrow[r] & P_{n-1} \arrow[d, dotted] \\
\pi_{\geq 4}(X_4) \text{ killed} & X_3 \arrow[r, hook] \arrow[d] & P_3 \arrow[d] & & & \vdots \arrow[d] \\
\pi_{\geq 3}(X_3) \text{ killed} & X_2 \arrow[r, hook] \arrow[d] & P_2 \arrow[dl] & & F_2 \arrow[r] & P_2 \arrow[d] \\
P_1 = & X_1 = K(\pi_1, 1) & & & & P_1
\end{tikzcd}
$$

connected CW complex

Properties: ① Each $P_m \to P_{m-1}$ is a fibration by construct.

③ $X \hookrightarrow P_m$ induces isomorphism on π_i for $i \leq m$ and $\underline{\pi_i(P_m) = 0}$ for $\underline{i > m}$.

② The fiber $F_m = \boxed{K(\pi_m, m)}$

Proof of ②: Consider the long exact sequence of fibration

$$
\begin{tikzcd}
F_m \arrow[r] & P_m \arrow[d] \\
& P_{m-1}
\end{tikzcd}
$$

Case 1: $\boxed{i > m}$ $\quad \underbrace{\pi_{i+1}(P_{m-1})}_{0} \to \underbrace{\pi_i(F_m)}_{0} \to \underbrace{\pi_i(P_m)}_{0}$

$\boxed{i = m}$ $\quad \underbrace{\pi_{m+1}(P_{m-1})}_{0} \to \pi_m(F_m) \xrightarrow{\cong} \pi_m(P_m) \to \underbrace{\pi_m(P_{m-1})}_{0}$

$\boxed{i = m-1}$ $\quad \underbrace{\pi_m(P_{m-1})}_{0} \to \underbrace{\pi_{m-1}(F_m)}_{0} \to \pi_{m-1}(P_m) \xrightarrow{\cong} \pi_{m-1}(P_{m-1})$

$\boxed{i < m-1}$ $\quad \pi_{i+1}(P_m) \xrightarrow{\cong} \pi_{i+1}(P_{m-1}) \to \underbrace{\pi_i(F_m)}_{0} \to \pi_i(P_m) \xrightarrow{\cong} \pi_i(P_{m-1})$

Computation of $\pi_4(S^3)$ using Postnikov tower.

Consider

$$
\begin{tikzcd}
 & S^3 \arrow[d] & & \\
K(\pi_4,4) \arrow[r] & P_4 \arrow[d] & K(\pi_4,4) \arrow[r] & P_4 \arrow[d] \\
K(\pi_3,3) \arrow[r] & P_3 \simeq K(\underbrace{\pi_3}_{\mathbb{Z}},3) \arrow[d] & & K(\mathbb{Z},3). \\
* \simeq K(\underset{0}{\pi_2},2) \arrow[r] & P_2 \arrow[d] & & \\
* \simeq K(\underset{0}{\pi_1},1) \arrow[r, equal] & P_1 & &
\end{tikzcd}
$$

What we know about the fiber $K(\pi_4,4)$:

By Hurewicz : $H_i(K(\pi_4,4)) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & 0<i<4 \\ \pi_4 & i=4 \\ \text{we don't care} & i>4. \end{cases}$

What we know about the total space P_4:

$$P_4 \simeq S^3 \cup e^6 \cup \cdots$$

① So $H_4(P_4) \cong H_4(S^3) = 0.$

② By an application of the Mayer–Vietoris sequence

$$\underbrace{H_5(\underbrace{S^3 \cup e^6 \setminus \{\text{origin}\}}_{A \,\simeq\, S^3}) \oplus H_5(\underbrace{e^6_{\text{radius} \le \frac{1}{2}}}_{B \,\simeq\, *})}_{0} \to H_5(\underbrace{S^3 \cup e^6}_{A \cup B}) \to \overset{0}{H_4(\underbrace{A \cap B}_{\partial e^6 \simeq S^5})}$$

$$H_5(S^3 \cup e^6) \cong 0$$

By induction and the fact that attaching e^n, $n>6$ doesn't affect H_5, $\boxed{H_5(P_4) = 0.}$

What we know about the base $K(\mathbb{Z},3)$:

On Tuesday, Byeongho computed the cohomology of $K(\mathbb{Z},3)$ using
$$\begin{array}{ccc} K(\mathbb{Z},2)=\Omega K(\mathbb{Z},3) \longrightarrow & PK(\mathbb{Z},3) & \\ & \downarrow & : \\ & K(\mathbb{Z},3) & \end{array}$$

i	0	1	2	3	4	5	6	7	8
H^i	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$
H_i	$\mathbb{Z}$	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	0

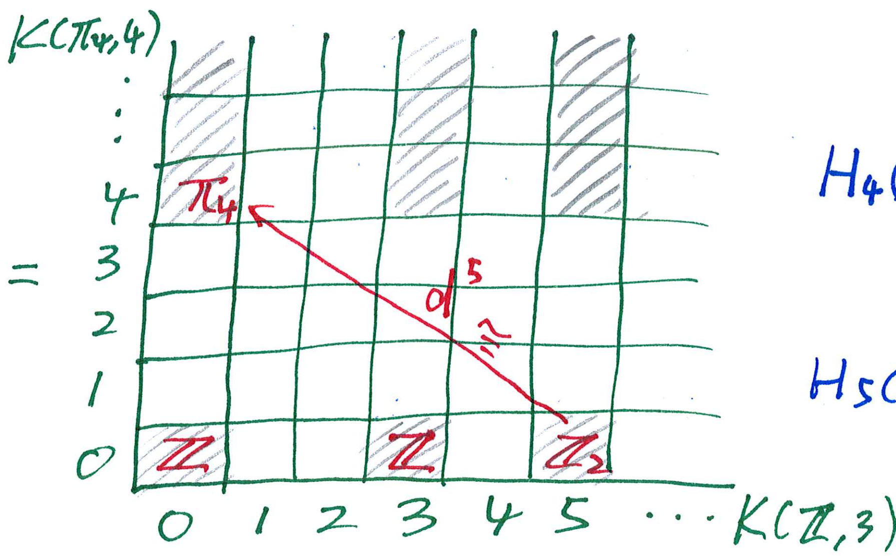
Recall the Universal Coefficient Theorem for cohomology:

$$H^i(X;\mathbb{Z}) \cong \underbrace{Hom(H_i(X;\mathbb{Z});\mathbb{Z})}_{\text{only free part survives}} \oplus \underbrace{Ext(H_{i-1}(X;\mathbb{Z});\mathbb{Z})}_{\text{only the torsion part survives}}$$

---

homology spectral sequeme

$$\begin{array}{ccc} K(\pi_4,4) \longrightarrow & P_4 \\ & \downarrow \\ & K(\mathbb{Z},3) \end{array}$$

$E^2 =$ 

$H_4(P_4)=0 \Rightarrow d^5$ is surjecti

$H_5(P_4)=0 \Rightarrow d^5$ is injecti

So $\pi_4(S^3)=\mathbb{Z}_2$

We can compute $\pi_4(S^3)$ using another tower:

$\boxed{\text{Whitehead tower}}$

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
K(\pi_n, n-1) \arrow[r] & W_n \arrow[d] \\
K(\pi_{n-1}, n-2) \arrow[r] & W_{n-1} \arrow[d, dotted] \\
K(\pi_2, 1) \arrow[r] & W_2 \arrow[d] \\
K(\pi_1, 0) \arrow[r] & W_1 \arrow[d] \\
 & X
\end{tikzcd}$$

$X \leftarrow$ connected

We want:

① Each $W_m \longrightarrow W_{m-1}$ is a fibration

② fiber $= K(\pi_m, m-1)$ ← shifted (comes from some loop operation) ; brings convenience

③ $\pi_i(X_m) = \begin{cases} 0 & i \leq m \\ \pi_i(X) & i > m \end{cases}$

lower homotopy groups are killed (especially convenient: Hurewicz)

Recall what a Postnikov tower is:

Fix n.

$$\begin{tikzcd}
 & X \arrow[d] \\
K(\pi_n, n) \arrow[r] & P_n \arrow[d] \\
K(\pi_{n-1}, n-1) \arrow[r] & P_{n-1} \arrow[d, dotted] \\
K(\pi_2, 2) \arrow[r] & P_2 \arrow[d] \\
K(\pi_1, 1) = & P_1
\end{tikzcd}$$

$X \leftarrow$ connected

We have:

① Each $P_m \longrightarrow P_{m-1}$ is a fibration.

② fiber $= K(\pi_m, m)$

③ $\pi_i(P_m) \cong \begin{cases} \pi_i(X) & i \leq m \\ \boxed{0 \quad i > m} \end{cases}$

The isomorphisms $\underset{i \leq m}{\pi_i(P_m) \cong \pi_i(X)}$ were induced by $X \longrightarrow P_m$

Let's construct the Whitehead tower.

$* \in X \hookrightarrow X_1$: kill all $\pi_{\geq 2}(X)$. So $X_1 = K(\pi_1, 1)$.

$$\begin{tikzcd} K(\pi_1, 0) = \Omega X_1 \arrow[r] & W_1 := \{\phi : I \to X_1 \mid \phi(0) = * \,;\, \phi(1) \in X\} \arrow[d, "\phi \mapsto \phi(1)", "\text{fibration}"'] \\ & X \end{tikzcd}$$

$$\pi_1(\cancel{K(\pi_1, 0)})^{0} \to \underset{\overset{\|}{0}}{\pi_1(W_1)} \to \underset{\pi_1}{\underline{\pi_1(X)}} \xrightarrow{\partial\ \cong} \pi_0(\underset{=\pi_1}{K(\pi_1, 0)})$$

$$i \geq 2: \quad \pi_i(\underbrace{K(\pi_1, 0)}_{0}) \to \pi_i(W_1) \xrightarrow{\cong} \pi_i(X) \xrightarrow{\partial} \underbrace{\pi_{i-1}(K(\pi_1, 0))}_{0}$$

In general, given W_{n-1}, we have $W_{n-1} \hookrightarrow \cancel{X}_n$: kill all $\pi_{\geq n+1}(W_{n-1})$.
So $\cancel{X}_n = K(\pi_n, n)$.

$$\begin{tikzcd} K(\pi_n, n-1) = \Omega X_n \arrow[r] & W_n := \{\phi : I \to X_n \mid \phi(0) = * \,;\, \phi(1) \in W_{n-1}\} \arrow[d] \\ & W_{n-1} \end{tikzcd}$$

$$i \geq n+1: \quad \pi_i(K(\pi_n, \underset{0}{n-1})) \to \pi_i(W_n) \xrightarrow{\cong} \pi_i(W_{n-1}) \xrightarrow{\partial} \pi_{i-1}(\underset{0}{K(\pi_n, n-1)})$$

$$i \leq n-2: \quad \pi_i(K(\underset{0}{\pi_n}, n-1)) \to \pi_i(\underset{\overset{\|}{0}}{W_n}) \to \underset{0}{\pi_i}(W_{n-1})$$

Finally, $\pi_n(K(\underset{0}{\pi_n}, n-1)) \to \pi_n(\underset{\overset{\|}{0}}{W_n}) \to \pi_n(W_{n-1}) \xrightarrow[\cong]{\partial} \pi_{n-1}(K(\pi_n, n-1)) \to \underset{\overset{\|}{0}}{\pi_{n-1}(W_n)} \to \underset{0}{\pi_{n-1}(W_{n-1})}$

$$\pi_i(W_n) = 0 \qquad i \leq n$$

$$\pi_i(W_n) \cong \pi_i(X) \quad i > n.$$

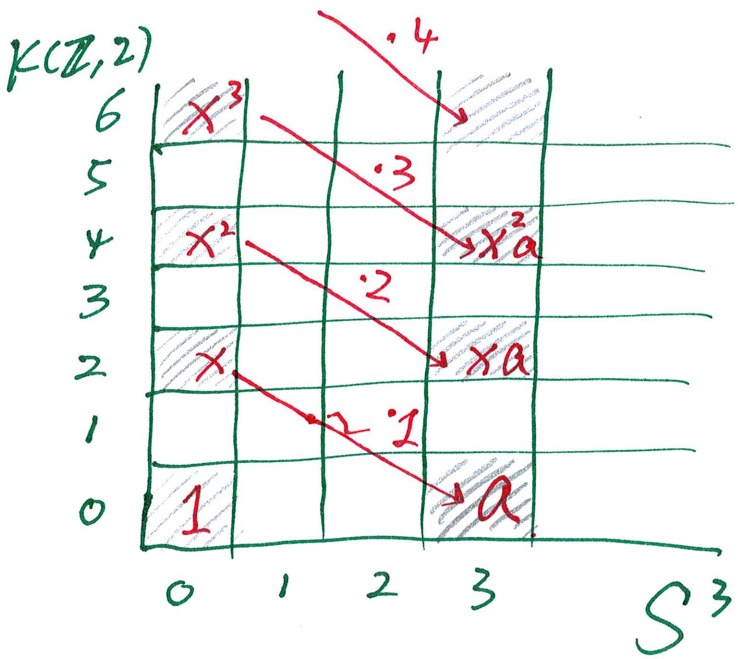
$\pi_4(S^3)$ again.

$$\begin{tikzcd}
 & K(\pi_3,2) \arrow[r] & W_3 \arrow[d] & \\
* \simeq & \underset{\overset{\shortparallel}{0}}{K(\pi_2,1)} \arrow[r] & W_2 \simeq S^3 \arrow[d] & \\
* \simeq & \underset{\overset{\shortparallel}{0}}{K(\pi_1,0)} \arrow[r] & W_1 \arrow[d] & \\
 & & S^3 &
\end{tikzcd}$$

$$\begin{tikzcd}
\underset{\mathbb{Z}}{K(\pi_3,2)} \arrow[r] & W_3 \arrow[d] \\
 & S^3
\end{tikzcd}$$

We need $\pi_4(S^3) \underset{\text{Whitehead}}{\cong} \pi_4(W_3) \underset{\text{Hurewicz}}{\cong} H_4(W_3)$

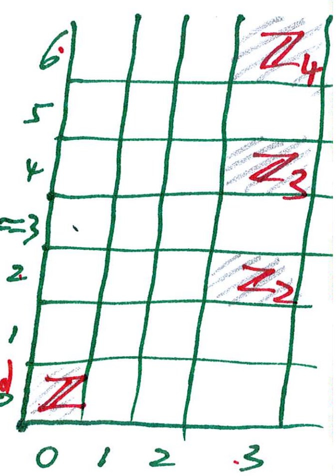
To find $H_4(W_3)$, we compute $H^*(W_3)$ first, because we can use the product structure



$H^3(W_3;\mathbb{Z})=0$

$d_n x^n = n x^{n-1} a$

$= E_2 = E_3 \xrightarrow{\ker d_3 / \operatorname{im} d_3} E_4 = E_\infty =$



$H^q(K(\mathbb{Z},2);\mathbb{Z})$ f.g. free $\mathbb{Z}$-mod

S^3 is 2-connected.

So $H^p(S^3, H^q(K(\mathbb{Z},2);\mathbb{Z})) \cong H^p(S^3;\mathbb{Z}) \otimes H^q(K(\mathbb{Z},2);\mathbb{Z})$

So for W_3:

i	0	1	2	3	4	5	6	7	8
H^i	$\mathbb{Z}$	0	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	0
H_i	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$	0	$\mathbb{Z}_4$

$$H_i(W_3) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i=2 \\ \mathbb{Z}_n & i=2n,\ n>1 \\ 0 & \text{otherwise.} \end{cases}$$

In particular $\pi_4(S^3) \cong H_4(W_3) \cong \mathbb{Z}_2$

We can do Exercise 18.24 now.

"Given a prime p, find the $\boxed{\text{least } q}$ such that $\pi_q(S^3)$ has p-torsion."

Answer: $q = 2p$.

More precisely, we have

The p-torsion subgroup of $\pi_i(S^3)$ is 0 for $i < 2p$ and $\mathbb{Z}_p$ for $i = 2p$.

Proof: Let $\mathcal{C}_p$ be the $\boxed{\text{(Serre)}}$ class of torsion abelian groups such that one cannot find $\mathbb{Z}_{p^i}$ for any $i \geq 1$ in $\mathcal{C}_p$. Kirk & Davis, Lecture Notes in Algebraic Topology.

So $H_i(W_3) \in \mathcal{C}_p$ for $0 < i < 2p$.

By $\boxed{\text{mod } \mathcal{C}_p \text{ Hurewicz theorem,}}$

1. $\pi_i(W_3) \in \mathcal{C}_p$: $\pi_i(W_3)$ doesn't contain $\mathbb{Z}_{p^i}$, $i \geq 1$ for $i < 2p$

2. $\pi_{2p}(W_3) \underset{\text{mod } \mathcal{C}_p}{\cong} H_{2p}(W_3) \cong \mathbb{Z}_p$

ker, coker $\in \mathcal{C}_p$.

$\oplus$ lower primary groups

$$\Rightarrow \pi_{2p} \underset{\text{honest isomorphism}}{\cong} H_{2p}(W_3) \cong \mathbb{Z}_p \oplus \text{lower primary groups}$$

$\blacksquare$

Now, we know $\pi_4(S^3) \cong \mathbb{Z}_2$ and $\pi_6(S^3) \cong \underbrace{\mathbb{Z}_3 \oplus \cdots}_{\text{actually } \mathbb{Z}_{12}}$

What is $\pi_5(S^3)$?

$$\begin{array}{ccc} K(\underbrace{\pi_4}_{\mathbb{Z}_2}, 3) & \longrightarrow & W_4 \\ & & \downarrow \\ & & W_3 \end{array}$$

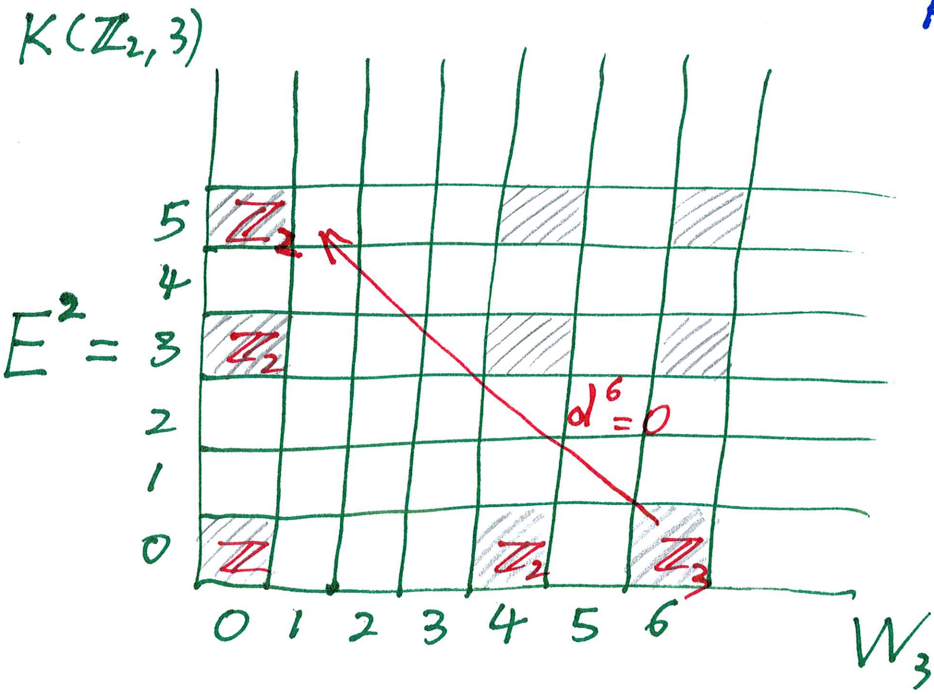
one floor higher

Then $\pi_5(S^3) \underset{\text{Whitehead}}{\cong} \pi_5(W_4) \underset{\text{Hurewicz}}{\cong} H_5(W_4)$.

	i	0	1	2	3	4	5	6
base	$H_i(W_3)$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_3$
fiber	$H^i(K(\mathbb{Z}_2,3))$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$.
	$H_i(K(\mathbb{Z}_2,3))$	$\mathbb{Z}$	0	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_2$	$\cdots$.

Exercise 18.16 $\mathbb{C}P^\infty \cong K(\mathbb{Z}_2,1) \to PK(\mathbb{Z}_2,2)$, $\downarrow$ $K(\mathbb{Z}_2,2)$ | $K(\mathbb{Z}_2,2) \to PK(\mathbb{Z}_2,$ $\downarrow$ $K(\mathbb{Z}_2,3$

$E^2 =$



$\pi_5(S^3) \cong H_5(W_4) \cong \mathbb{Z}_2$

Serre's Theorem

① For sphere of odd dimension ($n = \text{odd}$), $\pi_i(S^n)$ is ~~torsion~~ finite except when $i = n$.

② For even dimensional sphere ($n = \text{even}$), $\pi_i(S^n)$ is ~~torsion~~ finite. except when $i = n,\ 2n-1$.

Proof: We state without proof that $\pi_i(S^n)$ are finitely generated. (Again, Serre's mod $\mathcal{C}$ theory)

By Hurewicz, we only need to look at $i > n$.

We will use Whitehead towers:

$$\begin{array}{ccc} & & \vdots \\ K(\pi_{n+1}, n) & \longrightarrow & W_{n+1} \\ & & \downarrow \\ K(\mathbb{Z}, n-1) & \longrightarrow & W_n \\ & & \downarrow \\ & & S^n \end{array}$$

For $\pi_{i+1}(S^n)\otimes\mathbb{Q} \cong \pi_{i+1}(W_n)\otimes\mathbb{Q} \cong H_{i+1}(W_n)\otimes\mathbb{Q}$
$i \geq n$

$$\cong H_{i+1}(W_n;\mathbb{Q}).$$

We can show by induction that

$\boxed{\text{Exercise 18.12}}$ $H^*(K(\mathbb{Z}, n);\mathbb{Q}) = \begin{cases} \mathbb{Q}[x] & x\text{-degree } n \quad n \text{ even.} \\ \mathbb{Q}[x]/\langle x^2\rangle & x\text{-degree } n \quad n \text{ odd} \end{cases}$

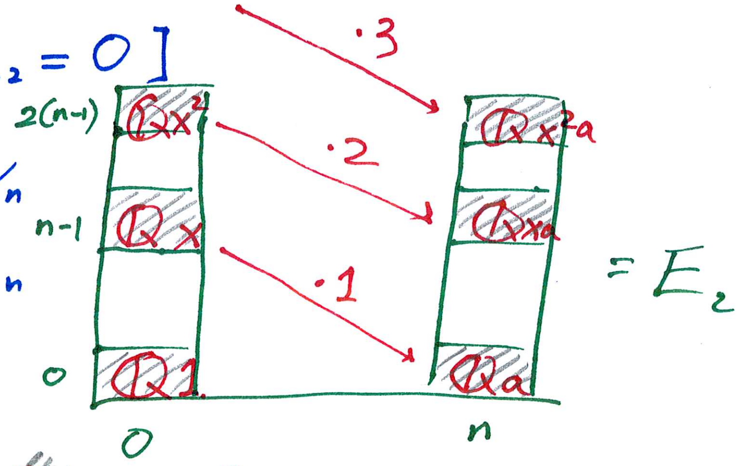
We can show by induction and $H^*(K(\mathbb{Z}_q, 1);\mathbb{Z}) = \begin{cases} \mathbb{Z} & * = \\ \mathbb{Z}_q & * = \\ 0 & \text{othe} \end{cases}$

$\boxed{\text{Exercise 18.9}}$ that $H^*(K(\mathbb{Z}_q, n);\mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0 \\ 0 & \text{otherwise,} \end{cases}$

First consider the case when n = odd

Assume $n \geq 3$. [S^1 has $\pi_{\geq 2} = 0$]

Consider $\begin{array}{ccc} K(\mathbb{Z}, \underbrace{n-1}_{\text{even}}) & \longrightarrow & W_n \\ & & \downarrow \\ & & S^n \end{array}$



W_n is n-connected $\Rightarrow H^{\leq n}(W_n) = 0 \Rightarrow H^{n-1}(W_n) = H^n(W_n) = 0$.

So $d_n(x) = a$. Thus, all $\cdot i$ are isomorphisms. It follows $\tilde{H}^*(W_n;\mathbb{Q}) = 0$

Hence, $\tilde{H}_*(W_n;\mathbb{Q}) = 0$.

$\pi_{n+1}(S^n) \otimes \mathbb{Q} = 0$

So $\pi_{n+1}(S^n)$ is pure torsion.

$\begin{array}{ccc} K(\underbrace{\pi_{n+1}}_{\text{torsion}}, n) & \longrightarrow & W_{n+1} \\ & & \downarrow \\ & & W_n \end{array}$

$\tilde{H}^*(\uparrow) = 0$

So $\tilde{H}_*(K(\pi_{n+1}, n);\mathbb{Q}) \cong \tilde{H}_*(W_n;\mathbb{Q}) = 0$

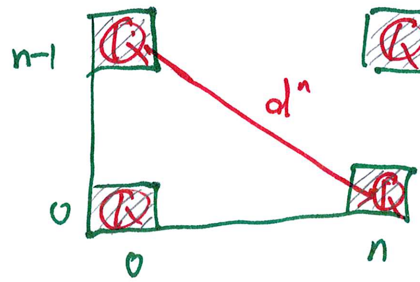
$\Rightarrow \tilde{H}_*(W_{n+1};\mathbb{Q}) \cong 0$

So $\pi_{n+2}(S^n)$ is pure torsion.

Inductively, π_i is torsion for all $i > n$.

Let n be even.

$$K(\mathbb{Z}, n-1) \longrightarrow W_n \\ \downarrow \\ S^n$$



↙ cannot kill

$= E^2$

W_n is n-connected

W_n is n-connected $\Longrightarrow$ d^n is an isomorphism

So $H_*(W_n; \mathbb{Q}) = \begin{cases} \mathbb{Q} & * = 0,\ 2n-1 \\ 0 & \text{otherwise}. \end{cases}$

$\pi_{n+1} \otimes \mathbb{Q} \cong H_{n+1}(W_n; \mathbb{Q})$.

$n > 2 \Rightarrow n+1 < 2n-1$. So $\pi_{n+1} = H_{n+1}(W_n)$ is torsion

Then consider

$$\underset{\text{torsion}}{K(\pi_{n+1}, n)} \longrightarrow W_{n+1} \\ \downarrow \\ W_n$$

$\Longrightarrow H^*(W_{n+1}; \mathbb{Q}) \cong H^*(W_n; \mathbb{Q})$

Inductively, π_i is torsion for $n < i < 2n-1$

and π_{2n-1} has one infinite cyclic generator.

Now we can assume $n \geq 2$.

$$K(\pi_{2n-1}, 2n-2) \longrightarrow W_{2n-1} \\ \downarrow \\ W_{2n-2}$$

$H^*(\downarrow) \cong \mathbb{Q}[x]$.

Then the proof goes like that of the odd case.

So $\pi_i(S^n) =$ torsion for all $i > 2n-1$. ▨

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Thank You!

Have a nice break!