The Gysin map is compatible with mixed Hodge Structures

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1 Introduction

The language of Bivariant Theory was developed in [6] and it turns out to be extremely useful in Intersection Theory and in Riemann-Roch-type questions (cfr. [5]). The Chow-theoretic version associates a graded group $A^*(X \xrightarrow{f} Y)$ with a morphism $f : X \to Y$ of algebraic schemes.
There are a product structure, a proper push-forward and a pull-back.
In the case of the structural map $X \to Spec\, k$, we find the Chow groups $A^*(X \to Spec\, k) \simeq A_{-*}(X)$, and in the case of the identity map the groups $A^*(X \xrightarrow{Id} X)$ are called the Chow cohomology groups.
When X is nonsingular these latter groups agree with the Chow groups:

$$A^*(X \xrightarrow{Id} X) \simeq A_{\dim X - *}(X)$$

and the bivariant product structure agrees with the usual intersection product.

For varieties defined over the field of complex numbers, the topological counterpart, i.e. Topological Bivariant Theory, admits the characterization

$$H^i(X \xrightarrow{f} Y) \simeq Hom_{D^b(Y)}(Rf_!\mathbb{Q}_X, \mathbb{Q}_Y[i]),$$

from which it follows immediately that $H^*(X \to \{point\}) \simeq Hom(H^*_c(X), \mathbb{Q}) = H^{BM}_*(X)$, the Borel-Moore homology groups, and $H^i(X \xrightarrow{Id} X) = Hom_{D^b(Y)}(\mathbb{Q}_X, \mathbb{Q}_X[i]) = H^i(X, \mathbb{Q})$, the cohomology groups.

It is quite natural to expect that the topological bivariant groups can be endowed with natural mixed Hodge structures (MHS) and that the maps arising in the context of the bivariant formalism should be compatible with MHS.
It seems that this question can be satisfactorily dealt-with only after having developed the Hodge theory of maps, and in particular the Hodge-theoretic version of the Decomposition Theorem (cfr. [2]).

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In this note we shall be less ambitious and we shall deal with a very special yet useful case which does not seem to be available in the literature in the generality which is needed in some applications. See, for example, those contained in [1], where morphisms between cohomology groups associated with correspondences between global finite group quotients of smooth, not necessarily complete, varieties are stated to be maps of MHS.

Consider a codimension d regular embedding $h : Y \longrightarrow X$. There is a refined Gysin homomorphism $[h] \in H^{2d}(Y \stackrel{h}{\to} X) \simeq H^{2d}(X, X \setminus Y)$, cfr. [5], 19.2, which gives, for any map $X' \longrightarrow X$, the so-called refined Gysin maps $h^! : H_*^{BM}(X') \to H_{*-2d}^{BM}(Y \times_X X')$.

Theorem 4.1 states the compatibility of the Gysin map $h^!$ with the MHS involved and it is proved using the definition of the Gysin map via specialization to the normal cone ([5], [9]).

We work with algebraic varieties and algebraic schemes defined over the field of complex numbers. We use cohomology etc. with rational coefficients.

The first-named author dedicates this paper to the memory of Meeyoung Kim.

2 Remarks on mixed Hodge structures

Given a morphism $f : Y \to X$ of algebraic schemes, the relative cohomology groups $H^*(X \operatorname{mod} Y)$ are given a MHS which is functorial in the map f, in the sense that given a commutative diagram of maps of algebraic schemes

$$\begin{array}{ccc} Y' & \stackrel{i_1}{\to} & Y \\ \downarrow f' & & \downarrow f \\ X' & \stackrel{i_2}{\to} & X, \end{array}$$

the natural morphism $H^*(X \operatorname{mod} Y) \to H^*(X' \operatorname{mod} Y')$ is a morphism of Mixed Hodge structures (cfr. [3], Exemple 8.3.8; see also [4], §2). The relative cohomology groups are the cohomology groups of the simplicial scheme $C(f)$.
The diagram above gives rise to a map of simplicial schemes $C(f') \to C(f)$, hence the morphism of MHS.

Given an algebraic scheme U, one has, for every open immersion $U \longrightarrow U'$ into a proper algebraic scheme, $H_c^l(U) \simeq H^l(U', U' \setminus U)$.
It follows that cohomology with compact supports admits a natural MHS which is functorial for open immersions and proper maps.
Dually, since $H_l^{BM}(U) \simeq H_c^l(U)^\vee$, the same is true for Borel-Moore homology.

3 The deformation to the normal cone

For more details on what follows, see [5] and [9].

Let $h : Y \longrightarrow X$ be a closed embedding of algebraic schemes, and $\mathcal{I}_Y \subseteq \mathcal{O}_X$ be the corresponding sheaf of ideals.
The *normal cone to Y in X* is defined to be the algebraic scheme

$$C_Y X := Spec_{\mathcal{O}_Y} \bigoplus_{n \geq 0} \mathcal{I}_Y^n / \mathcal{I}_Y^{n+1}$$

There are natural maps $Y \longrightarrow C_Y X \longrightarrow Y$, where the second one is the (affine) cone-bundle projection to Y and the first one is the closed embedding of the zero-section.
If the embedding of Y in X is regular of codimension d, then $C_Y X$ is naturally identified with the rank d normal bundle $N_{Y,X}$ of Y in X.
The exceptional divisor of the blowing-up $Bl_Y X \longrightarrow X$ of X along Y is the projectivisation $P(C_Y X)$.

Let us recall how the deformation to the normal cone and the canonical maps which are associated with it are defined.

Let $M = Bl_{Y \times \{0\}} X \times \mathbb{A}^1$ be the blowing-up of $X \times \mathbb{A}^1$ along $Y \times \{0\}$.
The exceptional divisor can be naturally identified with $P(C_Y X \oplus \mathcal{O}_Y)$.
The resulting map $\tilde{p} : M \longrightarrow \mathbb{A}^1$ is flat.
We have that:
– the blowing-up $Bl_Y X$ is naturally embedded in $\tilde{p}^{-1}(0) \subseteq M$;
– $\tilde{p}^{-1}(0) = P(C_Y X \oplus \mathcal{O}_Y) \cup Bl_Y X$;
– $P(C_Y X \oplus \mathcal{O}_Y) \cap Bl_Y X$ can be viewed either as the exceptional divisor of the blowing-up $Bl_Y X$, or as the divisor at infinity of $P(C_Y X \oplus \mathcal{O}_Y)$;
– $C_Y X$ embeds in the exceptional divisor $P(C_Y X \oplus \mathcal{O}_Y)$ as the complement of the divisor at infinity;
– Y embeds in $C_Y X \subseteq P(C_Y X \oplus \mathcal{O}_Y)$ as the zero section;
– $Y \times \mathbb{A}^1$ embeds in M, compatibly with the projection to $\mathbb{A}^1$; in particular, over $\mathbb{A}^1 \setminus \{0\}$, it is just the product embedding into $X \times \mathbb{A}^1 \setminus \{0\}$; $Y \times \{0\}$ is embedded in $p^{-1}(0)$ as the zero-section of $C_Y X \subseteq P(C_Y X \oplus O_Y)$.

Let

$$M \setminus Bl_Y X =: M' \xrightarrow{p} \mathbb{A}^1$$

be the natural flat map. The map p is not proper, even if X is complete.

We say that the embedding $Y \subseteq X$ (i.e. the situation at $p^{-1}(t \neq 0)$) deforms to the embedding $Y \subseteq C_Y X$ (i.e. the situation at $p^{-1}(0)$).
This construction is called the *deformation to the normal cone*.

There are specialization maps ([9]) :

$$h^? : H_l^{BM}(X) \longrightarrow H_l^{BM}(C_Y X), \qquad h^{?\vee} : H_c^l(C_Y X) \longrightarrow H_c^l(X),$$

whose construction is recalled in the proof of 4.1.

In the case of a regular embedding the normal cone is in fact the normal bundle and the flat pull-back gives an isomorphism $H^{BM}_l(Y) \simeq H^{BM}_{l+2d}(C_Y X)$.

The Gysin map $h^!$ is, by definition, the composition $H^{BM}_l(X) \longrightarrow H^{BM}_l(C_Y X) \longrightarrow H^{BM}_{l-2d}(Y)$.

4 The main result

Theorem 4.1 *Let $h : Y \longrightarrow X$ and $C_Y X$ be as above. The natural maps*

$$h^? : H^{BM}_l(X) \longrightarrow H^{BM}_l(C_Y X), \qquad h^{?\vee} : H^l_c(C_Y X) \longrightarrow H^l_c(X)$$

are maps of MHS of type $(0,0)$.
If h is regular of pure codimension d, then the Gysin map and its dual

$$h^! : H^{BM}_l(X) \longrightarrow H^{BM}_{l-2d}(Y), \qquad h^{!\vee} : H^l_c(Y) \longrightarrow H^{l+2d}_c(X)$$

are maps of MHS of type $(-d,-d)$ and (d,d), respectively.

Proof. We first consider a commutative diagram of maps of algebraic schemes

$$\begin{array}{ccccc} U & \stackrel{j}{\longrightarrow} & \mathcal{X} & \stackrel{i}{\longleftarrow} & B \\ & p \searrow & \downarrow \pi & \swarrow b & \\ & & S & & \end{array}$$

where j is an open immersion, $B := \mathcal{X} \setminus U$ and π is proper.
We have a distinguished triangle in the bounded derived category $D^b_{cc}(\mathcal{X})$ of the category of constructible complexes of rational vector spaces on $\mathcal{X}$:

$$Rj_! j^! \mathbb{Q}_{\mathcal{X}} (\simeq Rj_! \mathbb{Q}_U) \longrightarrow \mathbb{Q}_{\mathcal{X}} \longrightarrow Ri_* \mathbb{Q}_B \stackrel{+1}{\longrightarrow},$$

to which we apply $R\pi_* \simeq R\pi_!$ and get

$$Rp_! \mathbb{Q}_U \longrightarrow R\pi_* \mathbb{Q}_{\mathcal{X}} \longrightarrow Rb_* \mathbb{Q}_B \stackrel{+1}{\longrightarrow} .$$

It follows that

$$\mathbb{H}^l(S, Rp_! \mathbb{Q}_U) \simeq H^l(\mathcal{X}, B), \quad \forall l.$$

Let $s \in S$ be a point. By the remarks in §2, the map of pairs:

$$(\pi^{-1}(s), b^{-1}(s)) \longrightarrow (\mathcal{X}, B)$$

induces natural maps $H^l(\mathcal{X}, B) \to H^l(\pi^{-1}(s), b^{-1}(s))$ of MHS for every l. Since π is proper, $\pi^{-1}(s)$ is compact and the latter group can be identified with $H^l_c(p^{-1}(s))$. This endows these latter groups with MHS.

To summarize, the natural maps

$$\mathbb{H}^l(S, Rp_!\mathbb{Q}_U) \to H^l_c(p^{-1}(s))$$

are of MHS, for every $s \in S$ and for every l.

We apply what above to the following situation.
Let $Y' \subseteq X'$ be an algebraic compactification of $Y \subseteq X$, $\mathcal{X} := Bl_{Y' \times \{0\}}(X' \times \mathbb{A}^1)$, $B := (X' \setminus X) \times (\mathbb{A}^1 \setminus \{0\}) \coprod Bl_{Y'} X' \times \{0\} \coprod C_{Y' \setminus Y}(X' \setminus X)$, $U := \mathcal{X} \setminus B$. Let $\pi : \mathcal{X} \longrightarrow \mathbb{A}^1$ be the natural map and p and b the induced maps.
Note that $p^{-1}(\{0\}) = C_Y X$.
We have maps of MHS by choosing $s = 0$ and $s = s_0 \neq 0$:

$$H^l_c(p^{-1}(0)) \xleftarrow{a} \mathbb{H}^l(Rp_!\mathbb{Q}_U) \simeq H^l(\mathcal{X}, B) \longrightarrow H_c(p^{-1}(s_0)).$$

CLAIM: a is an isomorphism of MHS for every l.
Proof. It is enough to prove that a is an isomorphism. Consider the fundamental system of neighborhoods of the point $\{0\} \in \mathbb{A}^1$ given by disks $D_r = \{z : |z| < r\}$. Let $\mathcal{X}_r := \pi^{-1}(D_r)$, $U_r : p^{-1}(D_r)$, $B_r = b^{-1}(D_r)$, $p_r := p_{|U_r}$. Since $\mathcal{X}$ is a product away from $\pi^{-1}(\{0\})$, the homotopy axiom for relative cohomology ensures that the restriction maps $\mathbb{H}^l(\mathbb{A}^1, Rp_!\mathbb{Q}_U) \simeq H^l(\mathcal{X}, B) \longrightarrow H^l(\mathcal{X}_r, B_r) \simeq \mathbb{H}^l(U_r, Rp_{r!}\mathbb{Q}_{U_r})$ are isomorphisms for every $r > 0$.
The complex $Rp_!\mathbb{Q}_U$ is constructible with respect to the stratification $(\mathbb{A}^1 \setminus \{0\}, \{0\})$ of $\mathbb{A}^1$ and, by [7], §1.4, this implies that the restriction map $H^l(\mathcal{X}, B) \simeq \mathbb{H}^l(\mathbb{A}^1, Rp_!\mathbb{Q}_U)) \xrightarrow{a} H^l_c(p^{-1}(0))$ is an isomorphism.

The first assertion of the Theorem follows.

If the embedding $Y \subseteq X$ is regular of codimension d, then $C_Y X$ is the normal bundle of Y in X and we have an isomorphism of MHS $H^{BM}_l(Y) \simeq H^{BM}_{l+2d}(C_Y X)$ of type (d, d) and the Gysin map $h^!$ being, by definition, the composition $H^{BM}_l(X) \longrightarrow H^{BM}_l(C_Y X) \longrightarrow H^{BM}_{l-2d}(Y)$, is therefore of MHS of type $(-d, -d)$. □

Remark 4.2 After simple modifications to the statement and to the proof, Theorem 4.1 holds when h is a local complete intersection morphism.

The following corollary can be used when dealing with correspondences:

Corollary 4.3 *Let $X \xleftarrow{p} \Gamma \xrightarrow{q} Y$ be algebraic maps of algebraic varieties. Assume that the graph embedding $h : \Gamma \longrightarrow \Gamma \times X$ is regular and that q is proper. Then the natural map*

$$\Gamma_* : H^{BM}_\bullet(X) \longrightarrow H^{BM}_{\bullet+2(\dim\Gamma - \dim X)}(Y)$$

is of MHS of type $(\dim\Gamma - \dim X, \dim\Gamma - \dim X)$.

Proof. We have $\Gamma_*(x) = q_*(h^!([\Gamma] \times x))$ and all operations involved are compatible with MHS. □

Remark 4.4 The same statement above holds when X is assumed to be smooth or a quotient $X = X'/G$ of a smooth variety by a finite group, in which case one modifies the statement by working with X' and with the distinguished irreducible component of $\Gamma \times_X X'$. We leave this task to the reader.

5 Examples

The paper [1] contains several examples of correspondences, stemming from maps between surfaces, from Hilbert schemes of points on surfaces, semi-small resolutions of singularities etc. In what follows, for the reader's convenience, we offer two of the applications of Theorem 4.1 contained in [1].

5.1 Small resolutions

Let $f_i : X_i \longrightarrow Y$, $i = 1, 2$, be *small* resolutions of the singularities of an algebraic variety Y. This means that the X_i are nonsingular, the f_i are proper and birational morphisms, and the not necessarily irreducible algebraic schemes $X_i \times_Y X_i$ have exactly one component of maximal dimension $\dim Y$. This component is the unique component dominating the spaces X_i under either projection. See [1] for more on this notion and more references. Note that the same will be true for $X_1 \times_Y X_2$.

Let $D_{12} \subseteq X_1 \times_Y X_2$ be the unique irreducible component of maximal dimension $\dim Y$.

Theorem 4.1 and a simple calculation using the calculus of correspondences imply the following

Proposition 5.1 *The maps*

$$D_{ij_*} : H^{BM}_{\bullet}(X_i) \longrightarrow H^{BM}_{\bullet}(X_j)$$

are isomorphisms of MHS with inverse D_{ji_*}.
In particular, the virtual Hodge-Deligne numbers of the varieties X_i, $i = 1, 2$, *coincide.*

Proof. See [1], §3. □

5.2 Wreath products, rational double points and orbifolds

The combinatorial details of this example are rather lengthy. We omit them in favor of the end result. The interested reader can see [1], §7.3.

Let Y' be a smooth algebraic surface and $G \subseteq SL_2(\mathbb{C})$ be a finite group acting on Y' with only isolated fixed points.

Let $Y := Y'/G$ and $f : X \longrightarrow Y$ be its minimal resolution.

The semi-direct product G_n (called the Wreath product) of G^n with the symmetric group in n letters S_n acts on Y'^n.

The Hilbert scheme of $n-$points $X^{[n]}$ is a semi-small resolution of the singularities of Y'^n/G_n.

There is an explicit collection of correspondences that arises in this situation.

There is the notion of orbifold cohomology groups $H^*(Y'^n/G_n)_{orb}$ for the pair (Y'^n, G_n). These groups carry a natural MHS.

Theorem 4.1 allows to prove, using the aforementioned collection of correspondences, the following

Proposition 5.2 *There is a canonical isomorphism of MHS*

$$H^*(Y'^n/G_n)_{orb} \simeq H^*(X^{[n]}).$$

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