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Partial immunization of trees

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ABSTRACT

For a graph G and a non-negative integer-valued function τ on its vertex set, a dynamic monopoly is a set of vertices of G such that iteratively adding to it vertices u of G that have at least $\tau(u)$ neighbors in it eventually yields the vertex set of G. We study the problem of maximizing the minimum order of a dynamic monopoly by increasing the threshold values of individual vertices subject to vertex-dependent lower and upper bounds, and fixing the total increase. We solve this problem efficiently for trees, which extends a result of Khoshkhah and Zaker (2015).

1. Introduction

As a simple model for an infection process within a network [1–3] one can consider a graph G in which each vertex u is assigned a non-negative integral threshold value $\tau(u)$ quantifying how many infected neighbors of u are required to spread the infection to u. In this setting, a dynamic monopoly of (G, τ) is a set D of vertices such that an infection starting in D spreads to all of G, and the smallest order $\mathrm{dyn}(G, \tau)$ of such a dynamic monopoly measures the vulnerability of G for the given threshold values.

Khoshkhah and Zaker [4] consider the maximum of $\mathrm{dyn}(G, \tau)$ over all choices for the function τ such that the average threshold is at most some positive real $\bar{\tau}$. They show that this maximum equals

$$\max\left\{k : \sum_{i=1}^{k}(d_G(u_i)+1) \le n(G)\bar{\tau}\right\}, \tag{1}$$

where $u_1, \ldots, u_{n(G)}$ is a linear ordering of the vertices of G with non-decreasing vertex degrees $d_G(u_1) \le \cdots \le d_G(u_{n(G)})$. To obtain this simple formula one has to allow $d_G(u)+1$ as a threshold value for vertices u,

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a value that makes these vertices completely immune to the infection, and forces every dynamic monopoly to contain them. Requiring $\tau(u) \leq d_G(u)$ for every vertex u of G leads to a harder problem; Khoshkhah and Zaker [4] show hardness for planar graphs and describe an efficient algorithm for trees. In the present paper we consider their problem with additional vertex-dependent lower and upper bounds on the threshold values. As our main result, we describe an efficient algorithm for trees based on a completely different approach than the one in [4].

In order to phrase the problem and our results exactly, and to discuss further related work, we introduce some terminology. Let G be a finite, simple, and undirected graph. A *threshold function* for G is a function from the vertex set $V(G)$ of G to the set $\mathbb{N}_0$ of non-negative integers. Let $\tau \in \mathbb{N}_0^{V(G)}$ be a threshold function for G. For a set D of vertices of G, the *hull* $H_{(G,\tau)}(D)$ *of* D *in* (G,τ) is the smallest set H of vertices of G such that $D \subseteq H$, and $u \in H$ for every vertex u of G with $|H \cap N_G(u)| \geq \tau(u)$. Clearly, the set $H_{(G,\tau)}(D)$ is obtained by starting with D, and iteratively adding vertices u that have at least $\tau(u)$ neighbors in the current set as long as possible. With this notation, the set D is a *dynamic monopoly of* (G,τ) if $H_{(G,\tau)}(D)$ equals the vertex set of G, and $\mathrm{dyn}(G,\tau)$ is the minimum order of such a set. A dynamic monopoly of (G,τ) of order $\mathrm{dyn}(G,\tau)$ is *minimum*. The parameter $\mathrm{dyn}(G,\tau)$ is computationally hard [5,6]; next to general bounds [7–9] efficient algorithms are only known for essentially tree-structured instances [5,6,10–12].

We can now phrase the problem we consider: For a given graph G, two functions $\tau, \iota_{\max} \in \mathbb{N}_0^{V(G)}$, and a non-negative integer *budget* b, let $\mathrm{vacc}(G,\tau,\iota_{\max},b)$ be defined as

$$\max\left\{\mathrm{dyn}(G,\tau+\iota) : \iota \in \mathbb{N}_0^{V(G)}, \iota \leq \iota_{\max}, \text{ and } \iota(V(G)) = b\right\}, \tag{2}$$

where inequalities between functions are meant pointwise, and $\iota(V(G)) = \sum_{u \in V(G)} \iota(u)$. The function ι is the *increment* of the original threshold function τ. The final threshold function $\tau+\iota$ must lie between τ and $\tau+\iota_{\max}$, which allows to incorporate vertex-dependent lower and upper bounds. Note that no such increment ι exists if $\iota_{\max}(V(G))$ is strictly less than b, in which case $\mathrm{vacc}(G,\tau,\iota_{\max},b)$ equals $\max\emptyset = -\infty$. Note that we require $\iota(V(G)) = b$ in (2), which determines the average final threshold as $(\tau(V(G))+b)/n(G)$. Since $\mathrm{dyn}(G,\rho) \leq \mathrm{dyn}(G,\rho')$ for every two threshold functions ρ and ρ' for G with $\rho \leq \rho'$, for $\iota_{\max}(V(G)) \geq b$, the value in (2) remains the same when replacing '$\iota(V(G)) = b$' with '$\iota(V(G)) \leq b$' provided that $b \leq \iota_{\max}(V(G))$.

The results of Khoshkhah and Zaker [4] mentioned above can be phrased by saying

(i) that $\mathrm{vacc}(G,0,d_G+1,n(G)\bar{\tau})$ equals (1) whenever $n(G)\bar{\tau}$ is a non-negative integer at most $\sum_{u \in V(G)}(d_G(u)+1) = 2m(G)+n(G)$, where $m(G)$ is the size of G, and
(ii) that $\mathrm{vacc}(T,0,d_T,b)$ can be determined efficiently whenever T is a tree.

Our main result is the following.

Theorem 1.1. *For a given tuple $(T,\tau,\iota_{\max},b)$, where T is a tree of order n, $\tau,\iota_{\max} \in \mathbb{N}_0^{V(G)}$, and b is a non-negative integer with $b \leq \iota_{\max}(V(T))$, the value* $\mathrm{vacc}(T,\tau,\iota_{\max},b)$ *as well as an increment $\iota \in \mathbb{N}_0^{V(G)}$ with $\iota \leq \iota_{\max}$ and $\iota(V(G)) = b$ such that* $\mathrm{vacc}(T,\tau,\iota_{\max},b) = \mathrm{dyn}(T,\tau+\iota)$ *can be determined in time* $O\left(n^2\left(\min\{b,n^2\}+1\right)^2\right)$.

While our approach relies on dynamic programming, Khoshkhah and Zaker show (ii) using the following result in combination with a minimum cost flow algorithm.

Theorem 1.2 (*Khoshkhah and Zaker [4]*)**.** *For a given tree T, and a given non-negative integer b with $b \leq 2m(T)$, there is a matching M of T such that* $\mathrm{vacc}(T,0,d_T,b) = \mathrm{dyn}(G,\tau_M)$ *and $\tau_M(V(T)) \leq b$, where*

$$\tau_M : V(T) \to \mathbb{Z} : u \mapsto \begin{cases} d_T(u) & , u \text{ is incident with a vertex in } M, \text{ and} \\ 0 & , \text{otherwise.} \end{cases}$$

We believe that the threshold function τ_M considered in Theorem 1.2 is a good choice in general, and pose the following.

Conjecture 1.3. *For a given graph G, and a given non-negative integer b with $b \leq 2m(G)$, there is a matching M of G such that $\mathrm{vacc}(G, 0, d_G, b) \leq 2\mathrm{dyn}(G, \tau_M)$ and $\tau_M(V(G)) \leq b$, where τ_M is as in Theorem 1.2 (with T replaced by G).*

As a second result we show Conjecture 1.3 for some regular graphs.

Theorem 1.4. *Conjecture 1.3 holds if G is r-regular and $b \geq (2r-1)(r+1)$.*

Before we proceed to the proofs of Theorems 1.1 and 1.4, we mention some further related work. Centeno and Rautenbach [13] establish bounds for the problems considered in [4]. In [14], Ehard and Rautenbach consider the following two variants of (2) for a given triple (G, τ, b), where G is a graph, τ is a threshold function for G, and b is a non-negative integer:

$$\max\left\{\mathrm{dyn}(G - X, \tau) : X \in \binom{V(G)}{b}\right\} \qquad \text{and} \qquad \max\left\{\mathrm{dyn}(G, \tau_X) : X \in \binom{V(G)}{b}\right\},$$

where

$$\tau_X(u) = \begin{cases} d_G(u) + 1 & \text{, if } u \in X, \\ \tau(u) & \text{, if } u \in V(G) \setminus X, \end{cases}$$

and $\binom{V(G)}{b}$ denotes the set of all b-element subsets of $V(G)$. For both variants, they describe efficient algorithms for trees. In [15] Bhawalkar et al. study so-called anchored k-cores. For a given graph G, and a positive integer k, the *k-core* of G is the largest induced subgraph of G of minimum degree at least k. It is easy to see that the vertex set of the k-core of G equals $V(G) \setminus H_{(G,\tau)}(\emptyset)$ for the special threshold function $\tau = d_G - k + 1$. Now, the *anchored k-core problem* [15] is to determine

$$\max\left\{\left|V(G) \setminus H_{(G,\tau_X)}(\emptyset)\right| : X \in \binom{V(G)}{b}\right\}, \tag{3}$$

for a given graph G and non-negative integer b. Bhawalkar et al. show that (3) is hard to approximate in general, but can be determined efficiently for $k = 2$, and for graphs of bounded treewidth. Vaccination problems in random settings were studied in [1,16,17].

2. Proofs of Theorems 1.1 and 1.4

Throughout this section, let T be a tree rooted in some vertex r, and let $\tau, \iota_{\max} \in \mathbb{N}_0^{V(T)}$ be two functions. For a vertex u of T, and a function $\rho \in \mathbb{N}_0^{V(T)}$, let V_u be the subset of $V(T)$ containing u and its descendants, let T_u be the subtree of T induced by V_u, and let $\rho^{\to u} \in \mathbb{N}_0^{V(T)}$ be the function with

$$\rho^{\to u}(v) = \begin{cases} \rho(v) & \text{, if } v \in V(T) \setminus \{u\}, \text{ and} \\ \max\{\rho(v) - 1, 0\} & \text{, if } v = u. \end{cases}$$

Below we consider threshold functions of the form $\rho|V_u + \rho'|V_u$ for the subtrees T_u, where ρ and ρ' are defined on sets containing V_u. For notational simplicity, we omit the restriction to V_u and write '$\rho + \rho'$' instead of '$\rho|V_u + \rho'|V_u$' in these cases. For an integer k and a non-negative integer b, let $[k]$ be the set of positive integers at most k, and let

$$\mathcal{P}_k(b) = \left\{(b_1, \ldots, b_k) \in \mathbb{N}_0^k : b_1 + \cdots + b_k = b\right\}$$

be the set of ordered partitions of b into k non-negative integers.

Our approach to show Theorem 1.1 is similar as in [14] and relies on recursive expressions for the following two quantities: For a vertex u of T and a non-negative integer b, let

- $x_0(u,b)$ be the maximum of $\text{dyn}(T_u, \tau+\iota)$ over all $\iota \in \mathbb{N}_0^{V_u}$ with $\iota(v) \leq \iota_{\max}(v)$ for every $v \in V_u$, and $\iota(V_u) = b$, and
- $x_1(u,b)$ be the maximum of $\text{dyn}\left(T_u, (\tau+\iota)^{\to u}\right)$ over all $\iota \in \mathbb{N}_0^{V_u}$ with $\iota(v) \leq \iota_{\max}(v)$ for every $v \in V_u$, and $\iota(V_u) = b$.

The increment ι captures the local increases of the thresholds within V_u. The value $x_1(u,b)$ corresponds to a situation, where the infection reaches the parent of u before it reaches u, that is, the index 0 or 1 indicates the amount of help that u receives from outside of V_u. Note that going from $\tau+\iota$ to $(\tau+\iota)^{\to u}$, the value at the vertex u is only reduced by 1 if $\tau(u)+\iota(u)$ is positive.

If $b > \iota_{\max}(V_u)$, then no function ι as in the definition of $x_0(u,b)$ and $x_1(u,b)$ exists, that is, these two values are $\max\emptyset = -\infty$. Conversely, if $b \leq \iota_{\max}(V_u)$, then there are feasible choices for ι, and $x_0(u,b)$ and $x_1(u,b)$ are both non-negative integers. In this case we fix optimal choices for the function ι. More precisely, if $b \leq \iota_{\max}(V_u)$, then let $\iota_0(u,b), \iota_1(u,b) \in \mathbb{N}_0^{V_u}$ with $\iota_j(u,b) \leq \iota_{\max}$, and $\iota_j(u,b)(V_u) = b$ for both $j \in \{0,1\}$, be such that

$$x_0(u,b) = \text{dyn}\Big(T_u, \tau+\iota_0(u,b)\Big) \text{ and}$$
$$x_1(u,b) = \text{dyn}\Big(T_u, \Big(\tau+\iota_1(u,b)\Big)^{\to u}\Big).$$

Whenever this is possible, we choose $\iota_0(u,b)$ equal to $\iota_1(u,b)$. As we show in Corollary 2.4, $\iota_0(u,b)$ always equals $\iota_1(u,b)$, which is rather surprising and a key fact for our approach.

Lemma 2.1. *$x_0(u,b) \geq x_1(u,b)$, and if $x_0(u,b) = x_1(u,b)$, then $\iota_0(u,b) = \iota_1(u,b)$.*

Proof. If $x_1(u,b) = -\infty$, then the statement is trivial. Hence, we may assume that $x_1(u,b) > -\infty$, which implies that the function $\iota_1(u,b)$ is defined. Let D be a minimum dynamic monopoly of $(T_u, \tau+\iota_1(u,b))$. By the definition of $x_0(u,b)$, we have $x_0(u,b) \geq |D|$. Since D is a dynamic monopoly of $(T_u, (\tau+\iota_1(u,b))^{\to u})$, we obtain $x_0(u,b) \geq |D| \geq \text{dyn}\left(T_u, (\tau+\iota_1(u,b))^{\to u}\right) = x_1(u,b)$. Furthermore, if $x_0(u,b) = x_1(u,b)$, then $x_0(u,b) = |D| = \text{dyn}\left(T_u, \tau+\iota_1(u,b)\right)$, which implies $\iota_0(u,b) = \iota_1(u,b)$. □

Lemma 2.2. *If u is a leaf of T, and b is a non-negative integer with $b \leq \iota_{\max}(u)$, then, for $j \in \{0,1\}$,*

$$x_j(u,b) = \begin{cases} 0 & \text{, if } \tau(u)+b-j \leq 0, \\ 1 & \text{, otherwise, and} \end{cases}$$
$$\iota_j(u,b)(u) = b.$$

Proof. These equalities follow immediately from the definitions. □

The following rather technical lemma is the core statement for our dynamic programming approach (see Fig. 1).

Lemma 2.3. *Let u be a vertex of T that is not a leaf, and let b be a non-negative integer. If $v_1,\ldots,v_k$ are the children of u, and $\iota_0(v_i,b_i) = \iota_1(v_i,b_i)$ for every $i \in [k]$ and every non-negative integer b_i with $b_i \leq \iota_{\max}(V_{v_i})$, then, for $j \in \{0,1\}$,*

$$x_j(u,b) = z_j(u,b), \text{ and} \tag{4}$$
$$\iota_0(u,b) = \iota_1(u,b), \text{ if } b \leq \iota_{\max}(V_u), \tag{5}$$

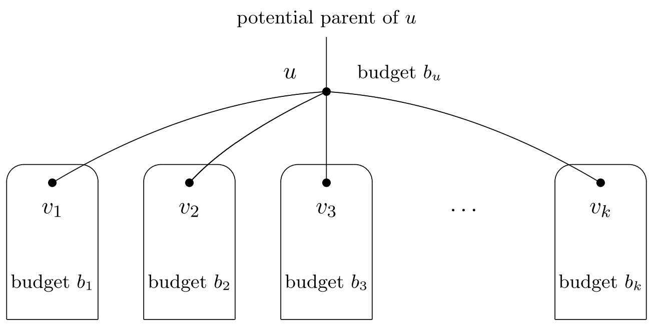

Fig. 1. The situation considered in Lemma 2.3. The total budget of b available on T_u is distributed to u and to the subtrees of T_u rooted in the children of u. The vertex u may have a parent.

where $z_j(u,b)$ is defined as

$$\max\left\{\delta_j(b_u,b_1,\dots,b_k)+\sum_{i=1}^{k}x_1(v_i,b_i) : (b_u,b_1,\dots,b_k)\in\mathcal{P}_{k+1}(b) \text{ with } b_u\le\iota_{\max}(u)\right\},$$

and, for $(b_u,b_1,\dots,b_k)\in\mathcal{P}_{k+1}(b)$ with $b_u\le\iota_{\max}(u)$,

$$\delta_j(b_u,b_1,\dots,b_k):=\begin{cases}0 & \text{, if } \left|\left\{i\in[k] : x_0(v_i,b_i)=x_1(v_i,b_i)\right\}\right|\ge\tau(u)+b_u-j, \text{ and}\\ 1 & \text{, otherwise.}\end{cases}$$

Proof. By symmetry, it suffices to consider the case $j=0$.

First, suppose that $b>\iota_{\max}(V_u)$. If $(b_u,b_1,\dots,b_k)\in\mathcal{P}_{k+1}(b)$ with $b_u\le\iota_{\max}(u)$, then $b_i>\iota_{\max}(V_{v_i})$ for some $i\in[k]$, which implies $z_0(u,b)=-\infty=x_0(u,b)$.

Now, let $b\le n(T_u)$, which implies $x_0(u,b)>-\infty$. The following two claims complete the proof of (4).

Claim 1. $x_0(u,b)\ge z_0(u,b)$.

Proof of Claim 1. It suffices to show that $x_0(u,b)\ge\delta_0(b_u,b_1,\dots,b_k)+\sum_{i=1}^{k}x_1(v_i,b_i)$ for every choice of $(b_u,b_1,\dots,b_k)$ in $\mathcal{P}_{k+1}(b)$ with $b_u\le\iota_{\max}(u)$ and $b_i\le\iota_{\max}(V_{v_i})$ for every $i\in[k]$. Let $(b_u,b_1,\dots,b_k)$ be one such an element. Let $\iota_u\in\mathbb{N}_0^{V_u}$ be defined as

$$\iota_u(v)=\begin{cases}b_u & \text{, if } v=u, \text{ and}\\ 0 & \text{, otherwise,}\end{cases}\tag{6}$$

and let $\iota=\iota_u+\sum_{i=1}^{k}\iota_1(v_i,b_i)$, where $\iota_1(v_i,b_i)(u)$ is set to 0 for every $i\in[k]$. Since $\iota(V_u)=b$ and $0\le\iota\le\iota_{\max}$, we have $x_0(u,b)\ge\mathrm{dyn}(T_u,\tau+\iota)$.

Let D be a minimum dynamic monopoly of $(T_u,\tau+\iota)$, that is, $|D|\le x_0(u,b)$. For each $i\in[k]$, it follows that the set $D_i=D\cap V_{v_i}$ is a dynamic monopoly of $\left(T_{v_i},(\tau+\iota)^{\to v_i}\right)$. Since, restricted to V_{v_i}, the two functions $(\tau+\iota)^{\to v_i}$ and $(\tau+\iota_1(v_i,b_i))^{\to v_i}$ coincide, we obtain

$$|D_i|\ge\mathrm{dyn}\Big(T_{v_i},\big(\tau+\iota_1(v_i,b_i)\big)^{\to v_i}\Big)\ge x_1(v_i,b_i).$$

If $\delta_0(b_u,b_1,\dots,b_k)=0$, then $|D|\ge\sum_{i=1}^{k}|D_i|\ge\delta_0(b_u,b_1,\dots,b_k)+\sum_{i=1}^{k}x_1(v_i,b_i)$. Similarly, if $u\in D$, then $|D|=1+\sum_{i=1}^{k}|D_i|\ge\delta_0(b_u,b_1,\dots,b_k)+\sum_{i=1}^{k}x_1(v_i,b_i)$. Therefore, we may assume that $\delta_0(b_u,b_1,\dots,b_k)=1$ and that $u\notin D$. This implies that there is some $\ell\in[k]$ with $x_0(v_\ell,b_\ell)>x_1(v_\ell,b_\ell)$ such that $D_\ell=D\cap V_{v_\ell}$ is a dynamic monopoly of $\left(T_{v_\ell},\tau+\iota\right)$. Since, by assumption, $\iota_0(v_\ell,b_\ell)=\iota_1(v_\ell,b_\ell)$, we obtain that, restricted to V_{v_ℓ}, the two functions $\tau+\iota$ and $\tau+\iota_0(v_\ell,b_\ell)$ coincide, which implies $|D_\ell| \geq \mathrm{dyn}\left(T_{v_\ell},\tau+\iota_0(v_\ell,b_\ell)\right) = x_0(v_\ell,b_\ell) \geq 1+x_1(v_\ell,b_\ell)$. Therefore, also in this case, $|D| = |D_\ell| + \sum_{i\in[k]\setminus\{\ell\}}|D_i| \geq \delta_0(b_u,b_1,\ldots,b_k) + \sum_{i=1}^k x_1(v_i,b_i)$. $\square$

Claim 2. $x_0(u,b) \leq z_0(u,b)$.

Proof of Claim 2. Let $\iota = \iota_0(u,b)$, that is, $x_0(u,b) = \mathrm{dyn}(T_u,\tau+\iota)$. Let $b_i = \iota(V_{v_i})$ for every $i\in[k]$, and let $b_u = b - \sum_{i=1}^k b_i$. Clearly, $(b_u,b_1,\ldots,b_k)\in\mathcal{P}_{k+1}(b)$ and $b_u \leq \iota_{\max}(u)$. Let D_i be a minimum dynamic monopoly of $\left(T_{v_i},(\tau+\iota)^{\to v_i}\right)$ for every $i\in[k]$. By the definition of $x_1(v_i,b_i)$, we obtain $|D_i| \leq x_1(v_i,b_i)$. Let $D = \{u\}\cup\bigcup_{i=1}^k D_i$. The set D is a dynamic monopoly of $(T_u,\tau+\iota)$, which implies $x_0(u,b)\leq |D|$.

If $\delta_0(b_u,b_1,\ldots,b_k)=1$, then

$$x_0(u,b) \leq |D| = 1+\sum_{i=1}^k |D_i| \leq \delta_0(b_u,b_1,\ldots,b_k)+\sum_{i=1}^k x_1(v_i,b_i) \leq z_0(u,b).$$

Therefore, we may assume that $\delta_0(b_u,b_1,\ldots,b_k)=0$. By symmetry, we may assume that $x_0(v_i,b_i) = x_1(v_i,b_i)$ for every $i\in[\tau(u)+b_u]$. Let D_i' be a minimum dynamic monopoly of $\left(T_{v_i},\tau+\iota\right)$ for every $i\in[\tau(u)+b_u]$. By the definition of $x_0(v_i,b_i)$, we obtain $|D_i'| \leq x_0(v_i,b_i) = x_1(v_i,b_i)$. Let $D' = \bigcup_{i\in[\tau(u)+b_u]} D_i' \cup \bigcup_{i\in[k]\setminus[\tau(u)+b_u]} D_i$. The set D' is a dynamic monopoly of $(T_u,\tau+\iota)$. This implies

$$x_0(u,b) \leq |D'| = \sum_{i\in[\tau(u)+b_u]} |D_i'| + \sum_{i\in[k]\setminus[\tau(u)+b_u]} |D_i| \leq \sum_{i\in[k]} x_1(v_i,b_i) \leq z_0(u,b),$$

which completes the proof of the claim. $\square$

It remains to show (5). If $x_0(u,b)=x_1(u,b)$, then (5) follows from Lemma 2.1. Hence, we may assume that $x_0(u,b)>x_1(u,b)$. Since, by definition,

$$\delta_1(b_u,b_1,\ldots,b_k) \leq \delta_0(b_u,b_1,\ldots,b_k) \leq \delta_1(b_u,b_1,\ldots,b_k)+1$$

for every $(b_u,b_1,\ldots,b_k)\in\mathcal{P}_{k+1}(b)$ with $b_u\leq\iota_{\max}(u)$, we obtain $z_1(u,b)\leq z_0(u,b)\leq z_1(u,b)+1$. Together with (4), the inequality $x_0(u,b)>x_1(u,b)$ implies that

$$\begin{aligned} x_0(u,b) &= z_0(u,b) > z_1(u,b) = x_1(u,b) \text{ and} \\ z_1(u,b) &= z_0(u,b)-1. \end{aligned}$$

Let $(b_u,b_1,\ldots,b_k)\in\mathcal{P}_{k+1}(b)$ with $b_u\leq\iota_{\max}(u)$ be such that

$$z_0(u,b) = \delta_0(b_u,b_1,\ldots,b_k)+\sum_{i=1}^k x_1(v_i,b_i).$$

We obtain

$$\begin{aligned} z_1(u,b) &\geq \delta_1(b_u,b_1,\ldots,b_k)+\sum_{i=1}^k x_1(v_i,b_i) \\ &\geq \delta_0(b_u,b_1,\ldots,b_k)-1+\sum_{i=1}^k x_1(v_i,b_i) \\ &= z_0(u,b)-1 \\ &= z_1(u,b), \end{aligned}$$

which implies $z_1(u,b) = \delta_1(b_u,b_1,\dots,b_k) + \sum_{i=1}^k x_1(v_i,b_i)$, that is, the same choice of $(b_u,b_1,\dots,b_k)$ in $\mathcal{P}_{k+1}(b)$ with $b_u \leq \iota_{\max}(u)$ maximizes the terms defining $z_0(u,b)$ and $z_1(u,b)$.

Since $z_0(u,b) > z_1(u,b)$, we obtain $\delta_1(b_u,b_1,\dots,b_k) = 0$ and $\delta_0(b_u,b_1,\dots,b_k) = 1$, which, by the definition of δ_j, implies that there are exactly $\tau(u)+b_u-1$ indices i in $[k]$ with $x_0(v_i,b_i) = x_1(v_i,b_i)$. By symmetry, we may assume that $x_0(v_i,b_i) = x_1(v_i,b_i)$ for $i \in [\tau(u)+b_u-1]$ and $x_0(v_i,b_i) > x_1(v_i,b_i)$ for $i \in [k] \setminus [\tau(u)+b_u-1]$.

Let $\iota = \iota_u + \sum_{i=1}^k \iota_0(v_i,b_i)$, where $\iota_0(v_i,b_i)(u)$ is set to 0 for every $i \in [k]$ and ι_u is as in (6). Note that, by assumption, we have $\iota = \iota_u + \sum_{i=1}^k \iota_1(v_i,b_i)$. Let D be a minimum dynamic monopoly of $(T_u, \tau+\iota)$. By the definition of $x_0(u,b)$, we have $|D| \leq x_0(u,b)$. Let $D_i = D \cap V_{v_i}$ for every $i \in [k]$. Since D_i is a dynamic monopoly of $\left(T_{v_i}, (\tau+\iota)^{\to v_i}\right)$ for every $i \in [k]$, we obtain $|D_i| \geq x_1(v_i,b_i)$. Note that

- either $u \in D$,
- or $u \notin D$ and there is some index $\ell \in [k] \setminus [\tau(u)+b_u-1]$ such that $D_\ell = D \cap V_{v_\ell}$ is a dynamic monopoly of $(T_{v_\ell}, \tau+\iota)$.

In the first case, we obtain

$$z_0(u,b) = x_0(u,b) \geq |D| = 1 + \sum_{i=1}^k |D_i| \geq 1 + \sum_{i=1}^k x_1(v_i,b_i) = z_0(u,b),$$

and, in the second case, we obtain $|D_\ell| \geq x_0(v_\ell,b_\ell) \geq x_1(v_\ell,b_\ell)+1$, and, hence,

$$z_0(u,b) = x_0(u,b) \geq |D| = |D_\ell| + \sum_{i \in [k]\setminus\{\ell\}} |D_i| \geq 1 + \sum_{i=1}^k x_1(v_i,b_i) = z_0(u,b).$$

In both cases we obtain $|D| = x_0(u,b)$, which implies that $\iota_0(u,b)$ may be chosen equal to ι.

Now, let D^- be a minimum dynamic monopoly of $(T_u, (\tau+\iota)^{\to u})$. By the definition of $x_1(u,b)$, we have $|D^-| \leq x_1(u,b)$. Let $D_i^- = D^- \cap V_{v_i}$ for every $i \in [k]$. Since D_i^- is a dynamic monopoly of $\left(T_{v_i}, (\tau+\iota)^{\to v_i}\right)$ for every $i \in [k]$, we obtain $|D_i^-| \geq x_1(v_i,b_i)$. Now,

$$z_1(u,b) = x_1(u,b) \geq |D^-| \geq \sum_{i=1}^k x_1(v_i,b_i) = z_1(u,b),$$

which implies that $|D^-| = x_1(u,b)$, and that $\iota_1(u,b)$ may be chosen equal to ι. Altogether, the two functions $\iota_0(u,b)$ and $\iota_1(u,b)$ may be chosen equal, which implies (5). $\square$

Applying induction using Lemmas 2.2 and 2.3, we obtain the following.

Corollary 2.4. *$\iota_0(u,b) = \iota_1(u,b)$ for every vertex u of T, and every non-negative integer b with $b \leq \iota_{\max}(V_u)$.*

Apart from the specific values of $x_0(u,b)$ and $x_1(u,b)$, the arguments in the proof of Lemma 2.3 also yield feasible recursive choices for $\iota_0(u,b)$. In fact, if

$$x_0(u,b) = \delta_0(b_u,b_1,\dots,b_k) + \sum_{i=1}^k x_1(v_i,b_i) > -\infty$$

for $(b_u,b_1,\dots,b_k) \in \mathcal{P}_{k+1}(b)$ with $b_u \leq \iota_{\max}(u)$, and ι_u is as in (6), then $\iota_u + \sum_{i=1}^k \iota_0(v_i,b_i)$ is a feasible choice for $\iota_0(u,b)$.

Our next lemma explains how to efficiently compute the expressions in Lemma 2.3.

Lemma 2.5. *Let u be a vertex of T that is not a leaf, let b be a non-negative integer with $b \leq \iota_{\max}(V_u)$, and let $v_1, \ldots, v_k$ be the children of u. If the values $x_1(v_i, b_i)$ are given for every $i \in [k]$ and every non-negative integer b_i with $b_i \leq \iota_{\max}(V_{v_i})$, then $x_0(u, b)$ and $x_1(u, b)$ can be computed in time $O\left(k^2(b+1)^2\right)$.*

Proof. By symmetry, it suffices to explain how to compute $z_0(u, b)$.

For $p \in \{0\} \cup [k]$, an integer $p_=$, an integer $b' \in \{0\} \cup [b]$, and $b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b'\}]$, let $M(p, p_=, b', b_u)$ be defined as the maximum of the expression $\sum_{i=1}^{p} x_1(v_i, b_i)$ over all $(b_1, \ldots, b_p) \in \mathcal{P}_p(b' - b_u)$ such that $p_=$ equals $\left|\left\{i \in [p] : x_0(v_i, b_i) = x_1(v_i, b_i)\right\}\right|$. Clearly, $M(p, p_=, b', b_u) = -\infty$ if $p < p_=$ or $p_= < 0$ or $b' - b_u > \sum_{i=1}^{p} \iota_{\max}(V_{v_i})$, and

$$M(0, 0, b', b_u) = \begin{cases} 0 & \text{, if } b' = b_u \text{, and} \\ -\infty & \text{, otherwise.} \end{cases}$$

For $p \in [k]$, the value of $M(p, p_=, b', b_u)$ is the maximum of the following two values:

- The maximum of $M(p-1, p_= - 1, b_{\leq p-1}, b_u) + x_1(v_p, b_p)$ over all $(b_{\leq p-1}, b_p) \in \mathcal{P}_2(b' - b_u)$ with $x_0(v_p, b_p) = x_1(v_p, b_p)$, and
- the maximum of $M(p-1, p_=, b_{\leq p-1}, b_u) + x_1(v_p, b_p)$ over all $(b_{\leq p-1}, b_p) \in \mathcal{P}_2(b' - b_u)$ with $x_0(v_p, b_p) > x_1(v_p, b_p)$,

which implies that $M(p, p_=, b', b_u)$ can be determined in $O(b'+1)$ time given the values

$$M(p-1, p_=, b_{\leq p-1}, b_u),\ M(p-1, p_= - 1, b_{\leq p-1}, b_u),\ x_0(v_p, b_p), \text{ and } x_1(v_p, b_p).$$

Altogether, the values $M(k, p_=, b, b_u)$ for all $p_= \in \{0\} \cup [k]$ can be determined in time $O\left(k^2(b+1)\right)$.

For $b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b\}]$, let $m(b_u)$ be the maximum of the two expressions

$$1 + \max\left\{M(k, p_=, b, b_u) : p_= \in \{0\} \cup [\tau(u) - b_u - 1]\right\}$$

and

$$\max\left\{M(k, p_=, b, b_u) : p_= \in [k] \setminus [\tau(u) - b_u - 1]\right\}.$$

Now, by the definition of $\delta_0(b_u, b_1, \ldots, b_k)$, the value of $z_0(u, b)$ equals $\max\left\{m(b_u) : b_u \in \{0\} \cup [\min\{\iota_{\max}(u), b\}]\right\}$. Hence, $z_0(u, b)$ can be computed in time $O\left(k^2(b+1)^2\right)$. $\square$

We proceed to the proof of our first theorem.

Proof of Theorem 1.1. Let $(T, \tau, \iota_{\max}, b)$ be given as in the statement. Let

$$b^* = \sum_{u \in V(T)} \min\left\{\iota_{\max}(u), \max\left\{d_T(u) + 1 - \tau(u), 0\right\}\right\}.$$

Since $d_T(u) + 1 \leq n$ and $\tau(u) \geq 0$ for every vertex u of T, we have $b^* \leq n^2$.

We consider two cases.

Case 1 $b \geq b^*$.

Let $\iota^* \in \mathbb{N}_0^{V(T)}$ be such that

$$\iota^*(u) = \min\left\{\iota_{\max}(u), \max\left\{d_T(u) + 1 - \tau(u), 0\right\}\right\}$$

for every vertex u of T. Since $b \leq \iota_{\max}(V(T))$, there is a function $\iota \in \mathbb{N}_0^{V(T)}$ with $\iota^* \leq \iota$ and $\iota(V(T)) = b$. We claim that

$$\mathrm{vacc}(T, \tau, \iota_{\max}, b) = \mathrm{dyn}(T, \tau + \iota^*) = \mathrm{dyn}(T, \tau + \iota), \tag{7}$$

that is, the function ι that can be computed in linear time optimally solves the problem, and $\mathrm{vacc}(T,\tau,\iota_{\max},b)$ can be computed in linear time using the linear time algorithms for $\mathrm{dyn}(T,\tau+\iota^*)$ given in [5,6].

In order to show (7), let u be a vertex of T. If $\iota_{\max}(u) \leq \max\Big\{d_T(u)+1-\tau(u),0\Big\}$, then $\iota(u)=\iota^*(u)=\iota_{\max}(u)$, which is the largest possible choice for $\iota(u)$ permitted by $\iota_{\max}$. If $\iota_{\max}(u) > \max\Big\{d_T(u)+1-\tau(u),0\Big\}$, then $\tau(u)+\iota(u)\geq d_T(u)+1$. Since every threshold value for u that is at least $d_T(u)+1$ is equivalent for the infection process captured by the considered model, it follows that $\iota(u)$ is either equal to the upper bound $\iota_{\max}(u)$ or such that larger values would not affect the optimal size of the dynamic monopolies. Altogether, we obtain (7), which completes the proof in this case.

Case 2 $b < b^*$.

Lemmas 2.2–2.5 imply that the values of $x_0(u,b')$ and of $x_1(u,b')$ for all $u\in V(T)$ and all $b'\in\{0\}\cup[b]$ can be determined in time

$$O\left(\sum_{u\in V(T)} d_T(u)^2(b+1)^2\right).$$

It is a simple folklore exercise that $\sum_{u\in V(T)} d_T(u)^2 \leq n^2-n$ for every tree T of order n, which implies that the overall running time is $O\left(n^2(b+1)^2\right)$. Since $\mathrm{vacc}(T,\tau,\iota_{\max},b)=x_0(r,b)$, the statement about the value of $\mathrm{vacc}(T,\tau,\iota_{\max},b)$ follows. The statement about the increment ι follows easily from the remark after Corollary 2.4 concerning the function $\iota_0(u,b)$, and the proof of Lemma 2.5, where, next to the values $M(p,p_=,b',b_u)$, one may also memorize suitable increments. This completes the proof in this case.

Since in Case 1, we have a linear running time, and in Case 2, we have running time $O\left(n^2(b+1)^2\right)$ and $b<b^*\leq n^2$, the overall running time is $O\left(n^2\Big(\min\{b,n^2\}+1\Big)^2\right)$. □

We conclude with the proof of our second theorem.

Proof of Theorem 1.4. Let G be an r-regular graph of order n, and let b be an integer with $(2r-1)(r+1)\leq b\leq rn=2m(G)$.

Let $\iota\in\mathbb{N}_0^{V(G)}$ with $\iota\leq d_G$ and $\iota(V(G))=b$ be such that $\mathrm{vacc}(G,0,d_G,b)=\mathrm{dyn}(G,\iota)$. By a result of Ackerman et al. [7],

$$\mathrm{vacc}(G,0,d_G,b)=\mathrm{dyn}(G,\iota)\leq\sum_{u\in V(G)}\frac{\iota(u)}{d_G(u)+1}=\frac{\iota(V(G))}{r+1}=\frac{b}{r+1}.$$

First, suppose that the matching number ν of G satisfies $2r\nu>b$. In this case, G has a matching M with $\tau_M(V(G))=2r|M|\leq b$ and $2r(|M|+1)\geq b+1$, where τ_M is as in the statement. We obtain $2\mathrm{dyn}(G,\tau_M)\geq 2|M|\geq 2\left(\frac{b+1}{2r}-1\right)\geq\frac{b}{r+1}\geq\mathrm{vacc}(G,0,d_G,b)$. Next, suppose that $2r\nu\leq b$. If M is a maximum matching and D is a minimum vertex cover, then $|D|\leq 2|M|$. Since D is a dynamic monopoly of (G,d_G), we obtain $2\mathrm{dyn}(G,\tau_M)\geq 2|M|\geq|D|\geq\mathrm{dyn}(G,d_G)\geq\mathrm{vacc}(G,0,d_G,b)$, that is, $2\mathrm{dyn}(G,\tau_M)\geq\mathrm{vacc}(G,0,d_G,b)$ holds in both cases. □

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