A general introduction to the Hilbert scheme of points on the plane

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1 The Hilbert scheme of points as a set

There are several objects in mathematics which have the magical property of appearing in various areas and which are, therefore, of the greatest interest for mathematicians. One of those, of which we shall discuss in these notes, is the *Hilbert scheme of points* on a surface. Hilbert schemes were first introduced, in their broadest generality, by A. Grothendieck in his EGA, although we will only discuss a special case in these notes. The setting is the following.

Let X be any quasi-projective surface. Let us remind that, given any zero-dimensional subscheme $Z \subset X$, the *length* of Z is $\ell(Z) := h^0(X, \mathcal{O}_Z)$. Then the Hilbert scheme of n points on X, denoted by $\mathrm{Hilb}^n(X)$ or $X^{[n]}$ is, as a set:

$$\mathrm{Hilb}^n(X) = \{Z \subset X \mid Z \text{ is a zero-dimensional subscheme with } \ell(Z) = n\}$$

Now, the question is whether it is possible to introduce a natural scheme structure on the set $\mathrm{Hilb}^n(X)$, i.e. whether there exists a moduli space parametrizing such data. We will really only be interested in the case where X the complex plane, and in that case, it will turn out that the corresponding scheme will be smooth and symplectic. If we look carefully at the definition, we realize immediately that the support of any zero dimensional, length n subscheme consists of just n *points*, not necessarily distinct.
So there is a natural map:

$$\mathrm{Hilb}^n(\mathbb{A}^2) \longrightarrow \mathrm{Sym}^n(\mathbb{A}^2)\ ,\ Z \mapsto \sum_{x\in\mathbb{A}^2} (\mathrm{mult}_x Z)\cdot x,$$

where we write $\mathrm{mult}_x Z$ for the multiplicity of Z at the point $x \in X$, the summation sign should be understood formally.
This map is called *Hilbert-Chow morphism*, we will see that it is indeed a morphism of algebraic varieties and, moreover, is a birational equivalence between the Hilbert scheme and the symmetric product of $\mathbb{A}^2$.

Let us start with an easy example.

Example 1.1. Take $X = \mathbb{A}^2$ and $n = 2$. [1] Since everything is affine, the space we want to construct and study becomes:

$$\mathrm{Hilb}^n(\mathbb{A}^2) = \{I \subset \mathbb{C}[x, y] \mid I \text{ is an ideal with } \ell(I) = \dim_{\mathbb{C}}(\mathbb{C}[x, y]/I) = n\}$$

We want to get a grasp on how the subschemes corresponding to such ideals look like. If $I = \mathfrak{m}_{x_1} \otimes \mathfrak{m}_{x_2}$ for some distinct points x_1 and x_2, then its zero locus corresponds to the subscheme $\{x_1, x_2\} \subset \mathbb{A}^2$, which will be then in the Hilbert scheme. What happens if the two points collide? For each point $x \in X$ and a vector $v \neq 0 \in T_x\mathbb{A}^2$, we can define a length 2 ideal by setting $I = \{f \in \mathbb{C}[x, y] \mid f(x) = 0, df_x(v) = 0\}$. Hence any two points infinitesimally attached look like a point with the choice of a direction in the tangent space. Consider the product $\mathbb{A}^2 \times \mathbb{A}^2$. Moreover, we do not care about their order, therefore we take the quotient $\mathrm{Sym}^2(\mathbb{A}^2)) := (\mathbb{A}^2 \times \mathbb{A}^2)/\mathfrak{S}_2$ by the symmetric group of order 2. What we get, though, will have singularities precisely along the locus which is stabilized by the action of $\mathfrak{S}_2$, namely the diagonal $\Delta = \{(x, y) \mid x = y\}$. Therefore we blow up along the diagonal, and we obtain a nice, smooth variety

$$\mathrm{Hilb}^2(\mathbb{A}^2) := \mathrm{Bl}_\Delta((\mathbb{A}^2 \times \mathbb{A}^2)/\mathfrak{S}_2).$$

The variety $\mathrm{Hilb}^2(\mathbb{A}^2))$ is called *Hilbert scheme of two points on the plane.* Its points look like figures (1a) and (1b): if the two points were not coincident, they actually look like a pair of distinct points, while if they were, by blowing up we have attached a $\mathbb{P}^1$ to each point in the diagonal, therefore we want to add an arrow, to remind ourselves that in some sense, they have been infinitesimally attached in some direction.

(a) A reduced, length 2 subscheme

(b) A nonreduced, length 2 subscheme

The Hilbert- Chow morphism $\mathrm{Hilb}^2(\mathbb{A}^2) \longrightarrow \mathrm{Sym}^2(\mathbb{A}^2)$, in this case, sends (x, y) with $x \neq y$ to the corresponding cycle $\{x, y\}$, and a point as in figure (1b) to the cycle corresponding to the point with multiplicity two, namely $(x, \text{direction}) \mapsto \{x, x\}$.

Now, if we allow $n > 2$ the picture becomes a little more complicated, but the space $\mathrm{Hilb}^n(\mathbb{A}^2)$ can still be constructed, as we will see later. One also has a *universal (flat) family*

[1] Let us remind, for those who have never seen this notation, that by $\mathbb{A}^2 = \mathbb{A}^2_{\mathbb{C}}$ we mean the complex plane $\mathbb{C}^2$ seen as an algebraic variety.

$$\begin{array}{c} F_n \subset \mathrm{Hilb}^n(\mathbb{A}^2) \times \mathbb{A}^2 \\ \Big\downarrow{\scriptstyle \pi} \\ \mathrm{Hilb}^n(\mathbb{A}^2) \end{array}$$

which has the following property: for each point $Z = V(I) \in \mathrm{Hilb}^n(\mathbb{A}^2)$, the corresponding fiber is precisely Z seen as a subscheme of $\mathbb{A}^2$. It also satisfies a universal property: any family $Y \subset T \times \mathbb{A}^2$, finite, flat of degree n over a scheme T, is the pullback of F_n via a unique morphism $T \longrightarrow \mathrm{Hilb}^n(\mathbb{A}^2)$. For $n > 2$, we still have that the generic point in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is a subscheme consisting on n distinct points. An opposite, in a sense, case is that of subschemes defined by monomial ideals, i.e. ideals I spanned by monomials $x^i y^j$. Each monomial $x^i y^j$ that does not belong to I corresponds to a box (i, j) in the diagram of a partition ν of n. We will index the monomial ideals $I = I_\nu$ with the corresponding partitions. Now, there is a natural torus action on $\mathbb{C}[x, y]$, given by

$$(\mathbb{C}^*)^2 \ni \begin{pmatrix} t & 0 \\ 0 & q \end{pmatrix} \cdot x = tx,$$

$$(\mathbb{C}^*)^2 \ni \begin{pmatrix} t & 0 \\ 0 & q \end{pmatrix} \cdot y = qy,$$

which lift to the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{A}^2)$ and to the universal family. We have the following

Proposition 1.2. *The fixed points of the torus action on* $\mathrm{Hilb}^n(\mathbb{A}^2)$ *are the monomial ideals* I_ν.

Proof. An ideal $I \subset \mathbb{C}[x, y]$ is fixed if and only if it is doubly homogeneous, i,e, if and only if it is monomial. □

If one wants to formally construct moduli spaces like the Hilbert scheme, one can use some GIT techniques, which is precisely what we are going to do next.

2 GIT and categorical quotients

We want to give a couple of reminders what it means to take a quotient of an algebraic variety (or, more in general, an algebraic scheme) by a group acting on it. First of all, let us remind what a reductive group is.

Definition 2.1. A *reductive group* over $\mathbb{C}$ is an affine algebraic group G such that any rational representation of G is completely reducible.

Any semisimple algebraic group is reductive, as is any algebraic torus $(\mathbb{C}^*)^n$ and any general linear group. Of course, any finite group is reductive.

Now, let G be a reductive group acting on an *affine* algebraic variety X. We want to produce a well-behaved quotient, more precisely, to give an algebraic variety structure to the space of closed orbits of G. In other words, we want to find a variety Y whose closed points are in one-to-one correspondence with the closed G-orbits, and such that there exists a G-invariant surjective morphism $\pi : X \longrightarrow Y$. The idea is the following: consider the ring $\mathbb{C}[X]$ of regular functions on X and its subring of G-invariant functions $\mathbb{C}[X]^G$. Clearly, there is an inclusion

$$i : \mathbb{C}[X]^G \hookrightarrow \mathbb{C}[X]$$

which induces a morphism of the respective spectra:

$$\pi : \underbrace{\mathrm{Spec}\mathbb{C}[X]}_{=X} \longrightarrow \mathrm{Spec}\mathbb{C}[X]^G$$

A theorem by Hilbert tells us that the ring $\mathbb{C}[X]^G$ is finitely generated as an algebra over $\mathbb{C}$.

Definition 2.2. In the above setting, we define

$$X//G := \mathrm{Spec}\mathbb{C}[X]^G$$

to be the *categorical quotient* or *affine algebro-geometric* quotient of X with respect to the G-action.

The quotient morphism$\pi : X \longrightarrow X//G$ has some good properties proved by Hilbert:

Theorem 2.3. *Let X be an affine algebraic variety, G be a reductive algebraic group acting on X and $\pi : X \longrightarrow X//G$ denote the quotient morphism. Then the following is true:*

(i) π is surjective.

(ii) Every fiber of π contains a single closed orbit.

(iii) Let $X_1 \subset X$ be a closed subvariety (or, more generally, a closed subscheme). Then $\pi(X_1) \subset X//G$ is closed and a natural morphism $X_1//G \to \pi(X_1)$ is an isomorphism.

We also have that the closure of any orbit of a reductive group action in an affine variety contains a single closed orbit, since the closure $\overline{G \cdot x}$ obviously contains a closed orbit, and it is contained in a single fiber.
Let us give a trivial example of a categorical quotient:

Example 2.4. Let $X = \mathbb{C}^n$ and $G = \mathbb{C}^*$ be the one-dimensional torus acting diagonally, i.e. $(t, (x_1, ..., x_n)) \mapsto (tx_1, ..., tx_n)$. Then $\mathbb{C}[X] = \mathbb{C}[x_1, ..., x_n]$ and $\mathbb{C}[X]^G = \mathbb{C}$, since there are no non-constant invariant polynomials. Therefore $\mathbb{C}^n//\mathbb{C}^* = \mathrm{Spec}\mathbb{C}[X]^G = \mathrm{Spec}\mathbb{C} \cong pt$.

This example, as well as many others, show that categorical quotients are not always the best that can be done: sometimes parametrizing closed orbits is useless, and we may want to parametrize other kinds of orbits. A more refined and, usually, effective construction is a slight modification of the one we just saw.

The first, intuitive idea would be to take some affine, G-stable open affine subsets $X_1, ..., X_n$ of X, take their categorical quotients with respect to the G-action and then glue the quotients together. However, the space we obtain is not always a nice space, but it turns out that if we carefully choose *which* affine open subsets to consider, then the object we get is a well-behaved geometrical object.

First, we choose a character $\chi : G \longrightarrow \mathbb{C}^*$. Then, a function $f \in \mathbb{C}[X]$ is called χ^n-semiinvariant if it satisfies $f(g \cdot x) = \chi(g)^n \cdot f(x)$ for some non-negative power n. The χ^n-semiinvariant functions span a ring,

$$\bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n},$$

which is a finitely generated graded algebra.

Definition 2.5. A point $x \in X$ is called *χ-semistable* if there exists a χ^n-semi invariant f such that $f(x) \neq 0$. A point which is not semistable is called *unstable.*

Call X^{ss} the locus of semistable points in X (we write $X^{\chi-ss}$ when we want to indicate the dependence on χ). By definition, X^{ss} can be covered by open sets of the form $X_f := \{x \in X | f(x) \neq 0\}$ with $f \in \mathbb{C}[X]^{G,\chi^n}$ for some n, and each of this open sets is acted on by G.

Definition 2.6. The *GIT quotient*[2] of X with respect to G and χ is

$$X//^\chi G := \operatorname{Proj}\left(\bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n}\right).$$

Notice that if $\chi = 0$, then the functions we consider are just the invariant functions. Therefore the GIT quotient is a generalization of the categorical quotient.
We remark that, for $f \in \mathbb{C}[X]^{G,\chi^n}$, the subvariety X_f is affine and G-stable so we can take the categorical quotient $X_f//G$. For $f_1 \in \mathbb{C}[X]^{G,\chi^{n_1}}$, $f_2 \in \mathbb{C}[X]^{G,\chi^{n_2}}$, we have G-equivariant inclusions $X_{f_i} \subset X_{f_1 f_2} = X_{f_1} \cap X_{f_2}$ that give rise to morphisms $X_{f_i}//G \to X_{f_1 f_2}//G$. By the very definition of Proj, the variety $X//^\chi G$ is glued from the open subsets $X_f//G$ along the gluing morphisms of the form $X_{f_1}//G \to X_{f_1 f_2}//G$. The Proj construction guarantees that the result of the gluing is an algebraic variety. Also we would like to point out that there is a natural quotient morphism $X^{\chi-ss} \to X//^\chi G$.
The inclusion $j : \mathbb{C}[X]^G \hookrightarrow \bigoplus_{n \geq 0} \mathbb{C}[X]^{G,\chi^n}$ induces a natural projective morphism

$$\Psi : \quad \begin{tikzcd} X//_\chi G \arrow[r] \arrow[d, equal] & X//G \arrow[d, equal] \\ \operatorname{Proj}\left(\bigoplus_{n\geq 0} \mathbb{C}[X]^{G,\chi^n}\right) \arrow[r] & \operatorname{Spec}\mathbb{C}[X]^G \end{tikzcd}$$

This morphism makes the following diagram commutative

$$\begin{tikzcd} X^{\chi-ss} \arrow[r] \arrow[d] & X \arrow[d] \\ X//^\chi G \arrow[r] & X//G \end{tikzcd}$$

[2]Here GIT is an acronym for "Geometric Invariant Theory". Please do not consult the English dictionary for this word

The top horizontal map is the inclusion and the vertical maps are quotient morphisms. This morphism will be very important to us later on.

Example 2.7 ($\mathbb{P}^1$)**.** We can consider the diagonal action of $\mathbb{C}^*$ on $\mathbb{A}^2$ by $t\cdot(x_1,x_2)=(tx_1,tx_2)$. So for $f\in\mathbb{C}[x_1,x_2]$, we have $(t\cdot f)(x_1,x_2)=f(t^{-1}x_1,t_2^{-1}x_2)$. If $\chi(t)=t^{-1}$, then $\mathbb{C}[X]^{G,\chi^n}=\mathbb{C}[x_1,x_2]^n$ and $X//^{\chi}G=\mathbb{P}^1$. If $\chi(t)=t$, then $\mathbb{C}[X]^{G,\chi^n}=0$, and $X^{\chi-ss}=\varnothing$.

3 The Hilbert scheme of points on $\mathbb{A}^2$ as a GIT quotient

3.1 $\mathrm{Hilb}^n(\mathbb{A}^2)$ as a GIT quotient

Let us now focus on the case where $X=\mathbb{A}^2$. It turns out that the Hilbert scheme of points $\mathbb{A}^{2[n]}$ has a natural and handy description.

Theorem 3.1. *Let*

$$\begin{aligned}\bar{H}&:=\{(B_1,B_2,i)\in\mathrm{End}(\mathbb{C}^n)^{\oplus 2}\oplus\mathbb{C}^n|[B_1,B_2]=0\},\\ \widetilde{H}&:=\{(B_1,B_2,i)\in\bar{H}|\mathbb{C}[B_1,B_2]i=\mathbb{C}^n\}.\end{aligned}$$

Consider the natural action of $\mathrm{GL}_n(\mathbb{C})$ *over* $\widetilde{H}$ *given by* $g\cdot(B_1,B_2,i)=(gB_1g^{-1},gB_2g^{-1},gi)$. *Then* $\widetilde{H}=\bar{H}^{\det^{-1}-ss}$, *the* $\mathrm{GL}_n(\mathbb{C})$*-action on* $\widetilde{H}$ *is free and the GIT quotient*

$$H:=\bar{H}//^{\det^{-1}}\mathrm{GL}_n(\mathbb{C})$$

is in a natural bijection with $\mathrm{Hilb}^n(\mathbb{A}^2)$.

It is easy to see the action of $\mathrm{GL}_n(\mathbb{C})$ is free: if $gB_1g^{-1}=B_1$, $gB_2g^{-1}=B_2$ and $gi=i$, then g acts by the identity on $\mathbb{C}[B_1,B_2]i$ that coincides with the whole space $\mathbb{C}^n$. On the other hand, the action of $\mathrm{GL}_n(\mathbb{C})$ on $\bar{H}$ is no longer free, because $(0,0,0)\in\bar{H}$.

Let us make a couple of remarks.

1) First, let us try to understand what is $\widetilde{H}$. As a subset,

$$\begin{aligned}\widetilde{H}&\subset\mathrm{End}(\mathbb{C}^n)\times\mathrm{End}(\mathbb{C}^n)\times\mathbb{C}^n\\ &\cong\mathbb{C}^{n^2}\times\mathbb{C}^{n^2}\times\mathbb{C}^n\\ &\cong\mathbb{C}^{2n^2+n}.\end{aligned}$$

Now, the condition $[B_1,B_2]=0$ is clearly a closed condition of rank at most n^2, which makes the set $\bar{H}$ into an algebraic subvariety of $\mathbb{C}^{2n^2+n}$ of dimension at least n^2+n (in fact, we will see that it has dimension n^2+2n). The condition specifying $\tilde{H}$ inside of $\bar{H}$ (to be referred to as the *cyclicity condition*), on the other hand, is clearly open: it can be easily translated into a condition of linear independence, which is open because it is given by the non-vanishing of some determinant of the form

$$\det(B_1^{a_1}B_2^{b_1}i,B_1^{a_2}B_2^{b_2}i,\ldots,B_1^{a_n}B_2^{a_n}i).\tag{1}$$

2) Now let us establish a bijection between the set of the $\mathrm{GL}_n(\mathbb{C})$-orbits in $\widetilde{H}$ and $\mathrm{Hilb}^n(\mathbb{A}^2)$. As we noticed above, a point in $\mathrm{Hilb}^n(\mathbb{A}^2)$ is given by an ideal $I \subset \mathbb{C}[x,y]$ such that $\dim_{\mathbb{C}}(\mathbb{C}[x,y]/I) = n$. On $\mathbb{C}[x,y]/I \cong_{Vect_{\mathbb{C}}} \mathbb{C}^n$ we have two natural endomorphisms given by the multiplication by each of the two variables modulo I:

$$B_1 : \mathbb{C}[x,y]/I \longrightarrow \mathbb{C}[x,y]/I \quad ; \quad f(x,y) \mapsto m_x(f(x,y)) := x \cdot f(x,y) \bmod I$$

$$B_2 : \mathbb{C}[x,y]/I \longrightarrow \mathbb{C}[x,y]/I \quad ; \quad f(x,y) \mapsto m_y(f(x,y)) := y \cdot f(x,y) \bmod I$$

The condition $[B_1, B_2] = 0$ obviously holds. Finally, define $i : \mathbb{C} \longrightarrow \mathbb{C}[x,y]/I$ by $i(1) = 1 \bmod I$. It is then clear that the cyclicity condition holds: the only subspace which is invariant under multiplication by x and y and which contains the constants is $\mathbb{C}[x,y]/I$ itself.

Vice versa, consider a triple $(B_1, B_2, i) \in \tilde{H}$. We need to produce an ideal $I \subset \mathbb{C}[x,y]$. Consider the homomorphism

$$\phi : \mathbb{C}[x,y] \longrightarrow \mathbb{C}^n$$

given by $f \mapsto f(B_1, B_2)i$. It is clear that ϕ is surjective, by the definition of $\widetilde{H}$. The ideal we are looking for will be now given by $\mathrm{Ker}\phi$, which satisfies all the required conditions. In fact, due to the cyclicity condition, $\mathrm{Ker}\phi = \{f | f(B_1, B_2) = 0\}$, it is an exercise to check this.

We now have a correspondence:

$$\{I \subset \mathbb{C}[x,y] \mid \ell(I) = n\} \quad \longrightarrow \quad \left\{ (B_1, B_2, i) \;\middle|\; \begin{array}{l} B_1 = m_x \bmod I \\ B_2 = m_y \bmod I \\ i \text{ such that } i = 1 \bmod I \end{array} \right\}$$

$$\{\mathrm{Ker}\phi \text{ where } \phi(f) = f(B_1, B_2)i\} \quad \longleftarrow \quad \{(B_1, B_2, i) \mid \text{as above}\}.$$

It is an exercise that these two maps are mutually inverse bijections between the set of ideals in $\mathbb{C}[x,y]$ of codimension n and the set of G-orbits in $\widetilde{H}$.

What remains to do to prove the theorem above is to establish the equality $\bar{H}^{\det^{-1}-ss} = \widetilde{H}$. The inclusion $\tilde{H} \subset \bar{H}^{\det^{-1}-ss}$ follows from the observation that the functions in (1) actually belong to $\mathbb{C}[\bar{H}]^{G,\det^{-1}}$. The other inclusion is more subtle. One can show that $\bigoplus_{n\geqslant 0} \mathbb{C}[\bar{H}]^{G,\det^{-n}}$ is generated by $\mathbb{C}[\bar{H}]^{G,\det^{-1}}$ as a $\mathbb{C}[\bar{H}]^G$-algebra, this follows from the 1st and 2nd fundamental theorems of Invariant theory, see [PV, Section 9]. This claim easily implies the required inclusion.
We are going to use a different approach based on the Hilbert-Mumford theorem (no, this wasn't a joint work).

Theorem 3.2. *Let a reductive group G act on an affine algebraic variety X. Pick $x \in X$. Then there is a one-parametric subgroup $\gamma : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \gamma(t) \cdot x$ exists and lies in a unique closed orbit in $\overline{Gx}$.*

This theorem is very useful because actions of one-dimensional tori are much easier than of general reductive groups.
There is also the following useful but easy observation that gives an alternative description of $X^{\chi-ss}$. Namely, let $\mathbb{C}_\chi$ be the 1-dimensional representation of G with action given by χ.

Lemma 3.3. *$X^{\chi-ss}$ coincides with the set of all points $x \in X$ such that the closure of the G-orbit of $(x, 1) \in X \times \mathbb{C}_\chi$ (with diagonal G-action) does not intersect $X \times \{0\} \subset X \times \mathbb{C}_\chi$.*

The proof is pretty basic and is left as an exercise.

Proposition 3.4. *We have $\bar{H}^{\chi-ss} \subset \widetilde{H}$.*

Proof. We just need to show that $\overline{G(x,1)} \cap (\bar{H} \times \{0\}) = \varnothing$ for any $x = (B_1, B_2, i) \in \widetilde{H}$. Assume the converse. The Hilbert-Mumford theorem implies that there is a one-parametric subgroup $\gamma : \mathbb{C}^* \to G$ such that $\lim_{t\to 0} \gamma(t) \cdot (B_1, B_2, i, 1)$ exists and lies in $\bar{H} \times \{0\}$. This means that the limits as $t \to 0$ of $\gamma(t)B_1\gamma(t)^{-1}, \gamma(t)B_2\gamma(t)^{-1}, \gamma(t)i$ exist while $\det(\gamma(t)) = t^m$ with $m < 0$. One can diagonalize $\gamma(t)$: in some basis $\gamma(t) = \mathrm{diag}(t^{m_1}, \dots, t^{m_n})$. Let $V_{\geqslant 0}$ denote the span of eigenvectors corresponding to non-negative degrees m_i. Then it is an exercise to check that B_1, B_2 map $V_{\geqslant 0}$ to $V_{\geqslant 0}$, while $i \in V_{\geqslant 0}$. The cyclicity condition then implies that $V_{\geqslant 0} = \mathbb{C}^n$, which contradicts $\det(\gamma(t)) = t^m$ with $m < 0$. □

This completes the proof of Theorem 3.1.

3.2 $\mathrm{Sym}^n(\mathbb{A}^2)$ as a categorical quotient

We also can realize the symmetric product $\mathrm{Sym}^n(\mathbb{A}^2)$ as a categorical quotient.

Theorem 3.5. *There exists a bijection*

$$\mathrm{Sym}^n(\mathbb{A}^2) \cong \bar{H}//\mathrm{GL}_n(\mathbb{C}).$$

Proof. The proof follows from the fact that an orbit is closed if and only if it contains an element of the form $(B_1, B_2, 0)$ such that both B_1 and B_2 are simulataneously diagonalizable. Knowing this, by associating the orbit with the set of simultaneous eigenvalues of (B_1, B_2) we will have the claimed bijection. To prove this fact, we first observe that for closed orbits, we must have $i = 0$ because of the actions of constant matrices. It is a well-known fact from linear algebra that commuting matrices can be simultaneously conjugated to the triangular form. Then we can take a suitable one-parametric subgroup of diagonal matrices and act on B_1 and B_2 such that the limit is a pair of diagonal matrices. Since the orbit of (B_1, B_2) is closed, the limit lies in the orbit, and hence B_1, B_2 are simulataneously diagonalizable.
Now let us show that the orbit $G(B_1, B_2, 0)$ is closed provided B_1, B_2 are diagonal. As we have seen above, $\overline{G(B_1, B_2)}$ contains a pair (B_1', B_2') of diagonal matrices. But the simultaneous eigenvalues of (B_1, B_2) and (B_1', B_2') have to coincide and hence (B_1, B_2) is conjugate to (B_1', B_2'). □

One can actually show that the map above is an isomorphism of algebraic varieties. A key ingredient is the fact that $\mathbb{C}[\bar{H}]^G$ is generated by polynomials of the form $\mathrm{tr}(B_1^a B_2^b)$. To deduce the claim about an isomorphism from here is an exercise.
Now we can interpret the Hilbert-Chow map.

Proposition 3.6. *Under the identifications $\bar{H}//^{\det^{-1}}G \cong \mathrm{Hilb}^n(\mathbb{A}^2), \bar{H}//G \cong \mathrm{Sym}^n(\mathbb{A}^2)$ as above, the Hilbert-Chow morphism is identified with the natural projective morphism $\bar{H}//^{\det^{-1}}G \to \bar{H}//G$.*

Proof. As above, we can simultaneously conjugate B_1, B_2 to triangular matrices, let $(\lambda_1, \ldots, \lambda_n)$, $(\mu_1, \ldots, \mu_n)$ be the eigenvalues. By the commutative diagram describing the morphism $\bar{H}//^{\det}G \to \bar{H}//G$, it maps the orbit of (B_1, B_2, i) to the closed orbit in its closure, meaning to that of $(\mathrm{diag}(\lambda_1, \ldots, \lambda_n), \mathrm{diag}(\mu_1, \ldots, \mu_n), 0)$.
Now let I be the ideal corresponding to (B_1, B_2, i), i.e., consisting of all polynomials f with $f(B_1, B_2) = 0$. Take a composition series of $\mathbb{C}[x,y]/I$, with constituents $S_1, ..., S_n$. By Nullstellensatz, simple modules over $\mathbb{C}[x,y]$ are one-dimensional and correspond to points in $\mathbb{A}^2$, and the image of $\mathbb{C}[x,y]/I$ under the Hilbert-Chow morhism are precisely the points appearing in the composition series. By putting the matrices B_1, B_2 into upper triangular form, it is easy to see that this constituents correspond to the (λ_i, μ_i)'s. Now we are done by the definition of the Hilbert-Chow map. □

3.3 Tautological bundle and the universal property

Above, we had the universal property for $\mathrm{Hilb}^n(\mathbb{C}^2)$. Let us show why the variety we have constructed satisfies the universal property.
We have a *tautological bundle* on the Hilbert scheme of points. Take the trivial, rank n bundle on $\widetilde{H}$:

$$\begin{tikzcd} \widetilde{H} \times \mathbb{C}^n \arrow[d, "\pi"] \\ \widetilde{H} \end{tikzcd}$$

The action of $G = \mathrm{GL}_n(\mathbb{C})$ on $\widetilde{H}$ lifts to the diagonal action on $\widetilde{H} \times \mathbb{C}^n$, so that the projection π is G-equivariant. Therefore we can take the quotient:

$$\begin{tikzcd} (\widetilde{H} \times \mathbb{C}^n)//G \arrow[d, "\pi"] & \\ \widetilde{H}//G & \cong \mathrm{Hilb}^n(\mathbb{A}^2) \end{tikzcd}$$

The rank n vector bundle we obtain on $\mathrm{Hilb}^n(\mathbb{A}^2)$ has a distinguished section, namely $\sigma : \mathrm{Hilb}^n(\mathbb{A}^2) \longrightarrow (\widetilde{H} \times \mathbb{C}^n)//G$, i.e. $\sigma(B_1, B_2, i) = (B_1, B_2, i, i)$. We can repeat the construction by imposing

$$\begin{tikzcd} \widetilde{H} \times \operatorname{End}(\mathbb{C}^n) \arrow[d, "\pi"] \\ \widetilde{H} \end{tikzcd}$$

where now G acts on $\operatorname{End}(\mathbb{C}^n)$ by conjugation. Again, the projection is equivariant and again we have two distinguished sections, i.e. $\sigma'(B_1, B_2, i) = (B_1, B_2, i, B_1)$ and $\sigma''(B_1, B_2, i) = (B_1, B_2, i, B_2)$, that are equivariant, as well. Therefore we can take the quotient, and we have thus produced a rank n vector bundle $\mathcal{T}$ on H with a distinguished section, say i, and two distinguished endomorphisms, say $\mathsf{B}_1, \mathsf{B}_2$.
We remark that the bundle we have constructed is actually a bundle of algebras. Indeed, by the construction we have an epimorphism $\mathbb{C}[x, y] \times H \to \mathcal{T}$ of vector bundles on H that maps a section f to $f(\mathsf{B}_1, \mathsf{B}_2)\mathsf{i}$. By the construction, the kernel is the sheaf of ideals, and so $\mathcal{T}$ becomes a sheaf of algebras. Let $\mathscr{H}$ be the relative spectrum of $\mathcal{T}$. By the construction, this is a subscheme in $H \times \mathbb{C}^2$, flat, finite of degree n over H.
Let us show that $\mathscr{H} \to H$ satisfies the universal property from Section 1. Namely, let T be a scheme and $Y \subset T \times \mathbb{C}^2$ be flat, finite of degree n over T. Let $\mathcal{F}$ be the sheaf on T obtained by pushing forward $\mathcal{O}_Y$. This bundle comes equipped with a distinguished section i' corresponding to 1 and with two distinguished endomorphisms, $\mathsf{B}_1', \mathsf{B}_2'$. Consider the principal G-bundle, say $\mathcal{G}$, on T of frames in $\mathcal{F}$. There is a natural map $\mathcal{G} \to \tilde{H}$ that to a frame assigns the matrices of $\mathsf{B}_1', \mathsf{B}_2', \mathsf{i}'$ in that basis. This map upgrades to a morphism of schemes, this is, try to guess, an exercise. Taking the quotient by the G-action, we get a morphism $T \to H$ such that Y is the pull-back of $\mathscr{H}$.

4 Hamiltonian reductions

4.1 Moment maps

Now we are going to show that $\operatorname{Hilb}^n(\mathbb{A}^2)$ is a smooth and symplectic variety. For this, we will realize $\operatorname{Hilb}^n(\mathbb{A}^2)$ as a *GIT Hamiltonian reduction.*

4.2 Moment maps

Before realizing the Hilbert scheme as a Hamiltonian reduction, we want to remind the basics of Symplectic geometry and Hamiltonian reductions. It only makes sense to speak about Hamiltonian reductions for symplectic varieties equipped with *Hamiltonian* G-actions.
Let M be an algebraic variety. Recall that a *symplectic form* on M will be any closed, non-degenerate 2-form ω. The existence of such a form surprisingly makes the geometry of M extremely rigid but, on the other hand, it implies a certain number of good properties.
Here is our main starting example of symplectic varieties.

Example 4.1 (Cotangent bundles)**.** Take any n-dimensional variety Z and consider its cotangent bundle $M = T^*Z$. We claim that M always has a symplectic structure. First, consider a canonical 1-form α defined as follows. Any point in M can be thought of as a pair $p = (x, y)$ where $x \in Z$ and y is a vector in $T^*_x Z$. Consider the projection

$$\pi : M \longrightarrow Z \ , p = (x, y) \mapsto x$$

and define α point-wise via

$$\alpha_p = (d\pi_p)^* y$$

or, equivalently,

$$\alpha_p(v) = y((d\pi_p)(v)).$$

Now define

$$\omega = -d\alpha.$$

Now we want to compute the form in coordinates. If we worked in the $\mathbb{C}^\infty$ or analytic setting, we could use the usual coordinates, while in our present setting we want everything to be algebraic, so we need to use étale coordinates, instead. Namely, we can introduce étale coordinates in a neighborhood of each point $x \in Z$. It is an easy exercise to show that, if $\{x_1, ..., x_n\}$ are étale coordinates and $\{y_1, ..., y_n\}$ are their corresponding fiberwise coordinates, then $\alpha = \sum_{i=1}^n y_i dx_i$ and $\omega = \sum_{i=1}^n dx_i \wedge dy_i$.

Suppose now that the variety M is acted on by a reductive group G. Then we have that there is a natural map that takes an element is the Lie algebra $\mathfrak{g}$ of G and to it associates a complete vector field on M, namely $\mathfrak{g} \longrightarrow \mathrm{Vect}(M)$, $\xi \mapsto \xi^M$, where ξ^M is the derivation on $\mathbb{C}[M]$ corresponding to ξ under the induced representation of $\mathfrak{g}$. An important special case: if V is a vector space and the G-action is linear, then $\xi_v^V = \xi \cdot v$, the image of v under the operator corresponding to ξ. Also, by the nondegeneracy of ω, we get an isomorphism

$$TM \longrightarrow T^*M \ , \ X \mapsto i_X\omega$$

where $i_X\omega$ denotes the contraction of the 2-form ω via the vector field X. This implies that for every one form β, there exists a unique vector field X such that $i_X\omega = \beta$. In particular, for any $f \in \mathbb{C}[M]$, there exists a unique vector field, which we call X_f, such that $i_{X_f}\omega = df$. It is an easy computation to show that, in the Darboux coordinates,

$$X_f = \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} \right).$$

We thus get, for free, a Poisson bracket on the algebra $C^\infty(M)$ by setting

$$\{f, g\} := \omega(X_f, X_g).$$

4.3 Hamiltonian reductions

Let us give the following

Definition 4.2. Let G be a reductive group and (M, ω) a symplectic G-variety. Then the action of G is called *Hamiltonian* (and the variety (M, ω, G, μ) is called *Hamiltonian G-space*) if there exists a map $\mu : M \longrightarrow \mathfrak{g}^*$ such that

1. For every $\xi \in \mathfrak{g}$, if $\mu^\xi : M \to \mathbb{C}$ is defined by $\mu^\xi(p) := \langle \mu(p), \xi \rangle$, then

$$i_{\xi^M}\omega = d\mu^\xi.$$

2. μ is equivariant with respect to the action of G on M and the co-adjoint action of G on $\mathfrak{g}^*$.

It is easy to see that the form in the l.h.s. in (1) is closed. Then (1) says that it is exact. In our case, this is the only additional condition: since we require that G is reductive, we can always achieve (2) (by averaging μ with respect to the G-action).
The morphism μ above is called a *moment map*.

Example 4.3. Consider the example above, namely $M = T^*Z$ for some smooth variety Z. We have a chain of maps:

$$\mathfrak{g} \longrightarrow \mathrm{Vect}(Z) \hookrightarrow \mathbb{C}[M],\ \xi \mapsto \xi^Z \mapsto H^{\xi^M}$$

where H^{ξ^Z} is the function $H^{\xi^Z}(x,y) = \langle y, \xi^Z_x \rangle$. Then we can construct a moment map by setting:

$$\langle \mu(p), \xi \rangle = H^{\xi^Z}(p). \tag{2}$$

It is a useful exercise to check that this map indeed satisfies required axioms.

The map $\mu^* : \mathfrak{g} \to \mathbb{C}[M]$, $\xi \mapsto H^{\xi^Z}$, dual to μ in the sense of (2), is called a *comoment map*. The following is the main result of this subsection.

Theorem 4.4. *Let M be a smooth affine algebraic variety equipped with an action of a reductive group G such that the action on $\mu^{-1}(0)$ is free. Then the following is true:*

1. *$\mu^{-1}(0)$ is a smooth subvariety of codimension* $\dim G$.
2. *Every fiber of the quotient morphism $\pi : \mu^{-1}(0) \to M_{red} := \mu^{-1}(0)//G$ is a single orbit.*
3. *M_{red} is a smooth variety of dimension* $\dim M - 2\dim G$.
4. *Let $\iota : \mu^{-1}(0) \hookrightarrow M$ denote the inclusion. Then there is a unique 2-form ω_{red} on M_{red} such that $\pi^*\omega_{red} = \iota^*\omega$. The form ω_{red} is symplectic.*

To prove the theorem we need the following lemma describing the image and the kernel of $d_x\mu : T_xM \to \mathfrak{g}^*$. We write $\mathfrak{g}_x$ for $\{\xi \in \mathfrak{g} | \xi^M_x = 0\}$, this is the tangent algebra of the stabilizer G_x.

Lemma 4.5. *We have $\operatorname{im} d_x\mu = \mathfrak{g}_x^\perp$ and $\ker d_x\mu = (T_xGx)^\perp$ (the first $\perp$ means the annihilator in the dual space, the second is a skew-orthogonal complement w.r.t. the symplectic form).*

The proof is an exercise.

Proof of the theorem. Since the G-action on $\mu^{-1}(0)$ is free, we see that $\mathfrak{g}_x = \{0\}$ for all $x \in \mu^{-1}(0)$. It follows that μ is a submersion at all points of $\mu^{-1}(0)$. (1) follows.
To prove (2) recall that every fiber of π contains a unique closed orbit. But since the action is free, all orbits have the same dimension (equal $\dim G$) and hence are closed.
To show (3) we will show that M_{red} is smooth of required dimension as a complex analytic manifold. For this we take a small transversal slice to an orbit. Shrinking it, we can assume that π embeds it into M_{red} as an open complex submanifold. This slice will be a coordinate chart in M_{red}. Our claim follows.
Let us prove (4). The form $\iota^*\omega$ vanishes on T_xGx for any $x \in \mu^{-1}(0)$, this is an exercise, and is G-invariant. The existence (and uniqueness) of ω_{red} follows. The claim that this form is symplectic is an exercise. $\square$

The variety M_{red} is called *Hamiltonian reduction* of M. Its ring of function is given by $(\mathbb{C}[M]/\mu^*(\mathfrak{g})\mathbb{C}[M])^G$: this also allows us to give a Poisson bracket to the quotient M_{red}, as well. Indeed, we define $\{f + \mu^*(\mathfrak{g})\mathbb{C}[M], g + \mu^*(\mathfrak{g})\mathbb{C}[M]\} := \{f, g\} + \mu^*(\mathfrak{g})\mathbb{C}[M]$. It can be checked that the bracket is well defined, and that this gives a Poisson structure to the quotient. We remark that in order to define this bracket we do not need to assume that the G-action on $\mu^{-1}(0)$ is free (and that M is symplectic, it is enough to assume that M is Poisson).
The theorem above carries to the setting of GIT quotients in a straightforward way. Namely, we need to assume that G acts freely on $\mu^{-1}(0)^{\chi-ss}//G$. Then straightforward analogs of (1)-(4) for $\mu^{-1}(0)^{\chi-ss}$ instead of $\mu^{-1}(0)$ and $\mu^{-1}(0)//^{\chi}G$ instead of $\mu^{-1}(0)//G$ still hold.

4.4 The Hilbert scheme and the symmetric product as Hamiltonian reductions

Let us now start working in the setting we are interested in. Consider the vector space $\mathbb{C}^n$, and let $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ be the Lie algebra of $G := \mathrm{GL}_n(\mathbb{C})$. Define the following space:

$$\mathscr{M} := \{(B_1, B_2, i, j) \in \mathfrak{g} \times \mathfrak{g} \times \mathbb{C}^n \times (\mathbb{C}^n)^* \mid [B_1, B_2] + ij = 0\}$$

Via the perfect pairing $\mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$, $(A, B) \mapsto \mathrm{Tr}(AB)$, we can see the space $\mathfrak{g} \times \mathfrak{g} \times \mathbb{C}^n \times (\mathbb{C}^n)^*$ as the cotangent bundle $T^*\mathfrak{B}$, where $\mathfrak{B} := \mathfrak{g} \times \mathbb{C}^n$, which has a natural symplectic structure. The group G acts diagonally on $T^*\mathfrak{B}$ via

$$g \cdot (B_1, B_2, i, j) = (gB_1g^{-1}, gB_2g^{-1}, gi, jg^{-1})$$

for each $g \in G$.
We want to do the following three things:

1. Show that the action is Hamiltonian (which is equivalent, as we saw, to produce a moment map μ);

2. Study the quotients $\mu^{-1}(0)//G$ and $\mu^{-1}(0)//^{\chi}G$ for some character χ;

3. Deduce properties of the above quotients from the properties of μ and of $\mu^{-1}(0)$.

1) The G-action is Hamiltonian. Indeed, we have shown that actions of reductive groups on cotangent bundles admit a natural comoment map, namely

$$\mu^* \mathfrak{g} \to \mathbb{C}[T^* Z] \ , \ \xi \mapsto H^{\xi^Z}$$

where $H^{\xi^Z}(x, y) = \langle y, \xi_x^Z \rangle$. Moreover, since in our case the group G acts diagonally on vector spaces, we have that $\xi_x^Z = \xi \cdot x$. Hence we defined μ via

$$\langle \mu(x, y), \xi \rangle = \mu^*(\xi)(x, y) = \langle y, \xi \cdot x \langle .$$

Claim: The moment map associated to the diagonal action described above on $T^* \mathfrak{B}$ is:

$$\mu : T^* \mathfrak{B} \longrightarrow \mathfrak{g} \cong \mathfrak{g}^* \ , \ \mu(B_1, B_2, i, j) = [B_1, B_2] + ij \ .$$

Proof. We just use a straightforward property of the moment map (whose proof follows from the definition and, in any case, is left as an exercise):

Lemma 4.6. *If any algebraic group G acts diagonally on the variety $M \times N$, then we can write the moment map as $\mu_G(m, n) = \mu_G(m) + \mu_G(n)$.*

We have that:

$$\begin{aligned}
\langle \mu(B_1, B_2, i, j), \xi \rangle &= \langle (B_2, j), \xi \cdot (B_1, i) \rangle \\
&= \langle (B_2, j), [\xi, B_1] + \xi i \rangle \\
&= \mathrm{Tr}(B_2[\xi, B_1] + j\xi i) \\
&= \mathrm{Tr}([B_1, B_2]\xi + ij\xi) \\
&= \langle [B_1, B_2] + ij, \xi \rangle
\end{aligned}$$

so indeed we have that $\mu(B_1, B_2, i, j) = [B_1, B_2] + ij$. □

We can thus write $\mathscr{M} = \mu^{-1}(0)$.

2) Now that we have explicitly produce a moment map, we want to realize the Hilbert scheme of point and the symmetric quotient as Hamiltonian reductions.

(a) $\mathbf{Sym^n}(\mathbb{A}^\mathbf{2})$ **as a categorical quotient.** We need the following:

Lemma 4.7. *Let B_1, B_2 be any two linear operators in $\mathfrak{g}$ such that $[B_1, B_2]$ is a nilpotent rank one operator. Then B_1 and B_2 are simultaneously triangularizable.*

Hence if $B_1, B_2 \in \mathscr{M}$, then they can be put into their triangular forms simultaneously. Then by sending an element (B_1, B_2, i, j) to the joint spectrum $(\mathrm{Spec}B_1, \mathrm{Spec}B_2) \in \mathbb{A}^n \times \mathbb{A}^n$, we get a well-defined morphism

$$\mathscr{M}//G \longrightarrow (\mathbb{A}^n \times \mathbb{A}^n)/\mathfrak{S}_n$$

where the symmetric group $\mathfrak{S}_n$ acts diagonally. The map is clearly surjective, hence dominant. Therefore, we have an induced morphism on the level of rings of functions $\mathbb{C}[\mathbb{A}^n \times \mathbb{A}^n]^{\mathfrak{S}_n} \longrightarrow \mathbb{C}[\mathscr{M}]^G$ which turns out to be an algebra isomorphism. Therefore, we conclude that the morphism $\mathscr{M}//G \longrightarrow (\mathbb{A}^n \times \mathbb{A}^n)/\mathfrak{S}_n$ is actually an isomorphism. But now $(\mathbb{A}^n \times \mathbb{A}^n)/\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts diagonally, is isomorphic to $(\underbrace{\mathbb{A}^2 \times ... \times \mathbb{A}^2}_{\text{n times}})/\mathfrak{S}_n$ where now the symmetric groups acts by permuting the pairs in the n copies $\mathbb{A}^2$; this is just the symmetric product $\mathrm{Sym}^{\,n}(\mathbb{A}^2)$, therefore we conclude that:

$$\mathscr{M}//G \cong \mathrm{Sym}^{\,n}(\mathbb{A}^2).$$

(b) $\mathbf{Hilb^n}(\mathbb{A}^\mathbf{2})$ **as a GIT quotient.** Now we fix the same stability condition as before, i.e. the one corresponding to the character $\chi = \det$. We have already seen that this stability condition is equivalent to the condition of i being cyclic. Actually, something more happens:

Lemma 4.8. *If $(B_1, B_2, i, j) \in \mathscr{M}$, then j vanishes on $\mathbb{C}[B_1, B_2]i$.*

Proof. It is known that since the rank of the matrix $[B_1, B_2$ is at most 1, then B_1 and B_2 can be simultaneously conjugated into upper triangular matrices. Therefore, let us assume that B_1 and B_2 are both upper triangular. Hence, for any $A \in \mathbb{C}[B_1, B_2]$ we have:

$$jAi = \mathrm{Tr}(Aij) = -\mathrm{Tr}(A[B_1, B_2]) = 0$$

since we can assume A to be upper triangular and $[B_1, B_2]$ is strictly upper triangular. □

Hence, we have that the set of χ-semistable points in $\mathscr{M}$ is

$$\mathscr{M}^{\chi-ss} = \{(B_1, B_2, i, j) \mid [B_1, B_2] = 0,\ j = 0,\ i \text{ is a cyclic vector for } (B_1, B_2)\} \cong \widetilde{H}$$

where $\widetilde{H}$ is the commuting variety we defined at the beginning of Section 3. From this identification, we get:

$$\mathscr{M}^{\chi-ss}//^{\det^{-1}}G \cong \widetilde{H}//^{\det^{-1}}G \cong \mathrm{Hilb}^n(\mathbb{A}^2).$$

Our conclusion is that $\mathrm{Hilb}^n(\mathbb{C}^2)$ is a symplectic smooth variety.

3) Consider now the following spaces:

$$\mathscr{M}'_k \overset{def}{=} \left\{ (B_1, B_2, i, j) \in \mathscr{M} \;\middle|\; \begin{array}{rl} (i) & B_2 \text{ has pairwise distinct eigenvalues} \\ (ii) & \dim(\mathbb{C}[B_1, B_2]i) = n - k \\ (iii) & \dim(j\mathbb{C}[B_1, B_2]) = k \end{array} \right\}.$$

for $k = 0, ..., n$ and let $\mathscr{M}_k$ be the closure of $\mathscr{M}'_k$ in $\mathscr{M}$. We want to prove the following:

Theorem 4.9. *(i) $\mathscr{M}$ is a complete intersection in $\mathfrak{g} \times \mathfrak{g} \times \mathbb{C}^n \times (\mathbb{C}^n)^*$, i.e.,* $\dim \mathscr{M} = n^2 + 2n$*;*

(ii) The irreducible components of $\mathscr{M}$ are $\mathscr{M}_0, ..., \mathscr{M}_k$, they have dimension $n^2 + 2n$;

(iii) $\mathscr{M}$ is reduced.

Proof. Our first observation is that $\mu^{-1}(0)$ is the union of conormal bundles to the G-orbits in $\mathfrak{B}$. Indeed, $\mu^{-1}(0)$ is the set of zeroes of the polynomials H_ξ given by $H_\xi(x, y) = y(\xi_x^{\mathfrak{B}})$. So the fiber of the projection $\mu^{-1}(0)$ over $x \in \mathfrak{B}$ consists of all 1-forms vanishing on $T_x(Gx)$. Our claim follows. The dimension of each conormal bundle is $\dim \mathfrak{B} = n^2 + n$. It follows that $\dim \mu^{-1}(0) = n^2 + n + d$, where, somewhat informally, d is the number of parameters describing the G-orbits in $\mathfrak{B}$. More formally, suppose that we have a decomposition of $\mathfrak{B} = \sqcup_k \mathfrak{B}_k$ into the union locally closed irreducible G-stable subvarieties $\mathfrak{B}_k$ such that all G-orbits in $\mathfrak{B}_k$ have the same codimension, say d_k. Then $\dim \mu^{-1}(0) \cap p^{-1}(\mathfrak{B}_i) = n^2 + n + d_k$, here we write p for the canonical projection $T^*\mathfrak{B} \to \mathfrak{B}$.

We decompose $\mathfrak{B} = \mathrm{End}(\mathbb{C}^n) \oplus \mathbb{C}^n = \{(B_1, i)\}$ according to the Jordan type of B_1 and the dimension of $Z_G(B_1)i$, where $Z_G(\bullet)$ stands for the centralizer in G (and then we still need to take irreducible components, but that's a technicality). Here are $n + 1$ strata that are most important for us: $\mathfrak{B}_k$ consists of all (B_1, i), where B_1 has n distinct eigenvalues and $\dim \mathbb{C}[B_1]i = n - k$ (we remark that in thus case $Z_G(B_1)$ consists of all non-degenerate matrices in $\mathbb{C}[B_1]$). We will explicitly describe such stratification in the case when $n = 2$ in Example (4.11). The stratum $\mathfrak{B}_k, k = 0, \ldots, n$, is easily seen to be irreducible, and $d_k = n$. Indeed, one needs n different parameters to describe to describe the orbit of B_1 and the number of $Z_G(B_1)$-orbits in $\mathbb{C}^n$ is finite.In fact, the latter is a general result.

Lemma 4.10. *For any $B_1 \in \mathrm{End}(\mathbb{C}^n)$, the number of $Z_G(B_1)$-orbits in $\mathbb{C}^n$ is finite.*

The proof is an exercise (a hint: do a single Jordan block first).

It follows that for all other strata $\mathfrak{B}_k$ in our stratification we have $d_k < n$. We remark that $\mu^{-1}(0) \cap p^{-1}(\mathfrak{B}_k)$ is precisely $\mathscr{M}'_k$ from above provided $0 \leqslant k \leqslant n$. So $\dim \mathscr{M} \leqslant n^2 + 2n$. On the other hand, since $\mathscr{M}$ is given by n^2 equation, the dimension of each component is at least $n^2 + 2n$. (i) and (ii) follow.

Also it is easy to see that there is a free G-orbit in each $\mathscr{M}_k, k = 0, \ldots, n$ (well, yet again, an exercise). From the properties of $d\mu$, it follows that there is a point in each component of $\mu^{-1}(0)$, where μ is a submersion. It follows that, as a scheme, $\mu^{-1}(0)$ is generically reduced. By a theorem of Serre, a generically reduced complete intersection is reduced, and we've proved (iii). $\square$

Let us give an example of such stratification.

Example 4.11. Take $n = 2$. Then we have three main strata:

$$\mathfrak{B}_0 = \{(B_1, i) \mid B_1 \text{ has two distinct eigenvalues and } \dim \mathbb{C}[B_1]i = 2\},$$
$$\mathfrak{B}_1 = \{(B_1, i) \mid B_1 \text{ has two distinct eigenvalues and } \dim \mathbb{C}[B_1]i = 1\},$$
$$\mathfrak{B}_2 = \{(B_1, i) \mid B_1 \text{ has two distinct eigenvalues and } \dim \mathbb{C}[B_1]i = 0\}.$$

For instance, we can take the pairs

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ with } \lambda \neq \mu, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) \in \mathfrak{B}_0$$

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ with } \lambda \neq \mu, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) \in \mathfrak{B}_1$$

and

$$\left(\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \text{ with } \lambda \neq \mu, \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right) \in \mathfrak{B}_2.$$

Moreover, $\mathfrak{B}_i, i = 0, 1, 2$, are the unions of the orbits of the pairs above.
Plus, we have five more strata given by the other Jordan types, namely the type $\left\{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right\}$ and the type $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}$. Those of type $\left\{\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\right\}$ define three different strata: again, take the vector i to be, respectively, $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and see that $\dim Z_G(B_1)i$ changes every time. Finally, we have two strata corresponding to the Jordan type $\left\{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right\}$, since $\dim Z_G(B_1)i = 0$ for $i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $\dim Z_G(B_1)i = 1$ for $i \neq 0$.

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