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A MEAN VALUE THEOREM IN GEOMETRY OF NUMBERS

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1. Let R be the space of the n-dimensional real vectors x, with $n > 1$, denote by $\{dx\}$ the euclidean volume element in R and consider a bounded function $f(x)$ which is integrable in the Riemann sense and vanishes everywhere outside a bounded domain in R. Recently E. Hlawka[1] proved the following remarkable proposition:

For any arbitrarily small positive ϵ there exists a real n-rowed matrix A of determinant $|A| = 1$ such that

$$\sum_{g \neq 0} f(Ag) \leqq \int_R f(x)\,\{dx\} + \epsilon, \tag{1}$$

where the summation is carried over all integral vectors $g \neq 0$.

As a consequence of his theorem Hlawka deduced an assertion of Minkowski which had remained unproved for more than fifty years:

If B is an n-dimensional star domain of volume $< \zeta(n)$, then there exists a lattice of determinant 1 such that B does not contain any lattice point $\neq 0$.

This statement had been announced by Minkowski on several occasions,[2] and he observed: "Der Nachweis dieses Satzes erfordert eine arithmetische Theorie der Gruppe aus allen linearen Transformationen." Later this arithmetical theory was created in the shape of Minkowski's method of reduction of positive quadratic forms; but he did not come back to his assertion on star domains, except for the special case connected with the closest packing of spheres.

Hlawka's proof is as simple and straightforward as one might wish; however, it does not make clear the relation to the fundamental domain of the unimodular group which was in Minkowski's mind. This relation will become obvious in the theorem which we are going to state.

2. Let Ω_1 denote the multiplicative group of all real n-rowed A with $|A| = 1$. The $(n^2 - 1)$-dimensional group space Ω_1 possesses an invariant volume element $d\omega$, unique up to a constant factor. The proper unimodular group Γ_1 is the subgroup consisting of all integral A in Ω_1. We shall define on Ω_1 a fundamental region F with respect to Γ_1, and we shall prove, as an immediate consequence of Minkowski's reduction theory, that the volume of F is finite. Now we determine the arbitrary factor in the definition of $d\omega$ by the condition that F has the volume 1. The connection between Hlawka's theorem (1) and Minkowski's reduction theory is provided by the following

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[1] Edmund Hlawka, *Zur Geometrie der Zahlen*, Math. Zeitschr. 49 (1944), pp. 285–312.

[2] Hermann Minkowski, *Gesammelte Abhandlungen*, vol. I, p. 265, p. 270, p. 277.

THEOREM: *Let g run over all integral vectors $\neq 0$, then*

$$\int_F \sum_{g\neq 0} f(Ag)\, d\omega = \int_R f(x)\, \{dx\}. \tag{2}$$

It follows immediately from (2) that (1) holds for a suitably chosen A in F, even with $\epsilon = 0$.

It is worth notice that the proof of (2) also leads to the value of the volume of F, in terms of an independently defined volume element on Ω_1. The result is closely related to Minkowski's well known formula for the volume of the domain of reduced positive quadratic forms with determinant $\leqq 1$; it seems that our method presents the most satisfactory way of proving this formula.

3. Now consider the group Ω of all non-singular real n-rowed matrices Y. The differential matrix $M = (dY)Y^{-1}$ is invariant under all mappings $Y \to YC$, $C \in \Omega$, of the group space Ω onto itself, and the positive definite quadratic differential form $\sigma(M'M)$ defines on Ω a right-invariant Riemannian metric. Plainly this metric induces on the subgroup space Ω_1 a right-invariant $(n^2 - 1)$-dimensional volume element. It is practical to define a certain constant multiple $d\omega_1$ of this volume element in the following way.

Let G be a subset of Ω_1 which is measurable in the Jordan sense, and denote by $\bar{G}$ the cone over the base G consisting of all matrices $Y = \lambda A$, where $0 < \lambda < 1$ and $A \in G$. If $\{dY\}$ is the volume element in the euclidean metric defined on Ω by $ds^2 = \sigma(dY'dY)$, then

$$V(G) = \int_{\bar{G}} \{dY\} \tag{3}$$

is the euclidean volume of $\bar{G}$. Since the linear transformation $Y \to YC$ has the jacobian $|C|^n$, it follows that $V(GC) = V(G)$, for all C in Ω_1; consequently the formula

$$V(G) = \int_G d\omega_1$$

defines an invariant volume element on Ω_1. If $\psi(A)$ is an integrable function on Ω_1, then we obtain

$$\int_G \psi(A)\, d\omega_1 = \int_{\bar{G}} \psi(|Y|^{-1/n} Y)\, \{dY\}. \tag{4}$$

Put $Y'Y = S = (s_{kl})$; this is a mapping of Ω into the space P of all positive real symmetric n-rowed matrices. On the other hand, the equation $Y_1'Y_1 = S$ has for any $S \in P$ a solution $Y_1 \in \Omega$, and the general solution is $Y = OY_1$, with an arbitrary orthogonal matrix O. We introduce in P the euclidean volume element $\{dS\} = \prod_{k\leqq l} ds_{kl}$. Let Q be a measurable set in P, and Q^* the set in Ω which is mapped into Q. If $h(S)$ is any integrable function in P, then

$$\int_{Q^*} h(Y'Y)\, \{dY\} = a_n \int_Q h(S)\, |S|^{-\frac{1}{2}}\, \{dS\}, \qquad a_n = \prod_{k=1}^{n} \frac{\pi^{k/2}}{\Gamma\left(\frac{k}{2}\right)}. \tag{5}$$

We denote by D and T the diagonal matrices $[t_1, \cdots, t_n]$ with positive diagonal elements $t_1, \cdots, t_n$ and the triangular matrices (t_{kl}) with $t_{kl} = 0$ $(1 \leqq l < k \leqq n)$, $t_{kk} = 1$ $(k = 1, \cdots, n)$, t_{kl} real $(1 \leqq k < l \leqq n)$. The Jacobi transformation of quadratic forms leads to the decomposition $S = D[T] = T'DT$, and this defines a one-to-one mapping of P into the product space of all D and T. Putting $\{dD\} = dt_1 \cdots dt_n$, $\{dt\} = \prod_{k<l} dt_{kl}$, we obtain

$$\{dS\} = \{dD\}\, \{dT\} \prod_{k=1}^{n} t_k^{n-k}.$$

Instead of $t_1, \cdots, t_n$ we introduce the $n - 1$ ratios $t_k/t_{k+1} = q_k$ $(k = 1, \cdots, n - 1)$ and the determinant $q_n = \prod_{k=1}^{n} t_k = |S| = |Y|^2$; then

$$\frac{t_k}{t_n} = q_k \cdots q_{n-1}, \qquad q_n = t_n^n \prod_{k=1}^{n-1} q_k^k, \qquad \frac{d(q_1, \cdots, q_n)}{d(t_1, \cdots, t_n)} = n\, \frac{t_1}{t_n},$$

$$|S|^{-\frac{1}{2}} \{dS\} = \frac{1}{n} \{dT\} q_n^{(n/2)-1}\, dq_n \prod_{k=1}^{n-1} (q_k^{(k/2)(n-k)-1}\, dq_k). \tag{6}$$

We call $q_1, \cdots, q_n$ and $t_{kl} (1 \leqq k < l \leqq n)$ the normal coordinates of S. It is clear that S and λS have the same normal coordinates, with the exception of q_n, for all positive scalar factors λ.

4. The group Γ of all unimodular n-rowed matrices U has in P the discontinuous representation $S \to S[U]$; plainly, U and $-U$ define the same mapping in P. A well known result of Minkowski's reduction theory states that this representation possesses in P a fundamental region K which is a convex pyramid with the vertex in the point $S = 0$, and that the normal coordinates, with the exception of q_n, are bounded in K. Now consider the corresponding domain K^* in Ω; this is a fundamental region in Ω for the representation $Y \to \pm YU$ of the factor group of Γ obtained by identifying U and $-U$. By the additional condition $\sigma(Y) \geqq 0$ we define one half of K^* as a fundamental region H for Γ itself. Finally, let F be the intersection of Ω_1 with H; then F obviously is a fundamental domain on Ω_1 for the proper unimodular group Γ_1. On the cone $\bar{F}$ we have $q_n = |Y|^2 < 1$, so also q_n is bounded there. Since the exponents of $q_1, \cdots, q_n$ in (6) are > -1, it follows from (3), (5), (6) that the volume

$$V_n = V(F) = \int_F d\omega_1$$

is finite.

Let g run over all integral vectors $\neq 0$ and define

$$\varphi(\lambda, A) = \lambda^n \sum_{g \neq 0} f(\lambda A g), \qquad \phi(\lambda, A) = \lambda^n \sum_{g \neq 0} \operatorname{abs} f(\lambda A g), \tag{7}$$

where $0 < \lambda \leqq 1$ and $A \in \Omega_1$. The function $f(x)$ has the former meaning, viz., it is bounded, integrable in the Riemann sense and 0 everywhere outside a bounded domain in R; consequently the function $\varphi(\lambda, A)$ is integrable in Ω_1.

LEMMA: *There exists an integrable function $m(A)$, independent of λ, so that $\phi(\lambda, A) < m(A)$, everywhere in Ω_1, and that the integral*

$$J = \int_F m(A)\, d\omega_1$$

converges.

PROOF: Since $f(x)$ is bounded and $f(x) = 0$ outside a certain sphere $x'x < r^2$, it suffices to prove the assertion for the characteristic function of this sphere, namely

$$f(x) = 1 \quad (x'x < r^2), \qquad f(x) = 0 \quad (x'x \geqq r^2).$$

Put $A'A = S = D[T]$ and consider the integral solutions g of the inequality $0 < S[g] < \rho^2$, for any given positive ρ. If $g_1, \cdots, g_n$ are the coordinates of g, then

$$S[g] = D[Tg] = \sum_{k=1}^{n} t_k \left(g_k + \sum_{l=k+1}^{n} t_{kl}\, g_l \right)^2;$$

hence g_k lies in an interval of length $2\rho t_k^{-\frac{1}{2}}$, and the number of solutions g has the value

$$\alpha(\rho, A) < \prod_{k=1}^{n} (1 + 2\rho t_k^{-\frac{1}{2}}).$$

This estimate implies that, for $0 < \lambda \leqq 1$,

$$\phi(\lambda, A) = \lambda^n \alpha(\lambda^{-1} r, A) < \prod_{k=1}^{n} (\lambda + 2r t_k^{-\frac{1}{2}}) \leqq \prod_{k=1}^{n} (1 + 2r t_k^{-\frac{1}{2}}) = m(A), \tag{8}$$

say. Plainly the function $m(A)$ depends only upon $S = A'A$ and r.

For any A in Ω_1 there exists a uniquely determined integer $\nu = 0, 1, \cdots, n$ such that $t_k < 1 (k = 1, \cdots, \nu)$ and $t_{\nu+1} \geqq 1$; this means in case $\nu = 0$ that $t_1 \geqq 1$, and in case $\nu = n$ that $t_k < 1$ $(k = 1, \cdots, n)$. Let F_ν be the set of all A in F with given ν, and put

$$J_\nu = \int_{F_\nu} m(A)\, d\omega_1 \qquad (\nu = 0, \cdots, n);$$

then $J = J_0 + \cdots + J_n$, and it remains to prove the convergence of the integrals J_ν.

Since $S = A'A$ lies in the reduced domain K, for all A in F, it follows that the ratios $q_k = t_k/t_{k+1} (k = 1, \cdots, n - 1)$ are bounded; hence $t_k^{-\frac{1}{2}}$ is bounded in F_ν, for $k = \nu + 1, \cdots, n$, and

$$m(A) < c \prod_{k=1}^{\nu} t_k^{-\frac{1}{2}} = c \prod_{k=1}^{\nu} (q_k \cdots q_{n-1})^{-\frac{1}{2}} \prod_{k=1}^{n-1} q_k^{k\nu/2n}, \tag{9}$$

by (8), where c depends only on n and r. Now we change the notation and define $A = |Y|^{-1} Y$, $S = Y'Y$; this does not affect the coordinates $q_1, \cdots, q_{n-1}$. If Y lies in the cone $\bar{F}_\nu$, then (6) and (9) lead to the inequality

$$m(|Y|^{-1} Y)\, |S|^{-\frac{1}{2}} \{dS\} < \frac{c}{n} \{dT\} q_n^{(n/2)-1}\, dq_n \prod_{k=1}^{n-1} q_k^{\alpha_k - 1}\, dq_k, \tag{10}$$

with $\alpha_k = \frac{k}{2}(n - k - 1 + \nu/2n) > 0$, for $1 \leqq k \leqq \min(\nu, n - 1)$, and $\alpha_k = \frac{k}{2}(n - k + \nu/2n) - \nu/2 > 0$, for $\nu < k \leqq n - 1$. Formulae (4), (5), (10) imply the convergence of J_ν; q.e.d.

Put

$$\int_R f(x) \{dx\} = \gamma;$$

then

$$\lim_{\lambda \to 0} \varphi(\lambda, A) = \lim_{\lambda \to 0} \lambda^n \sum_g f(\lambda A g) = \int_R f(Ax) \{dx\} = \gamma, \tag{11}$$

by virtue of the definition of the integral. On the other hand, we infer from the lemma that the integral

$$\psi(\lambda) = \int_F \varphi(\lambda, A)\, d\omega_1 = \int_F \lambda^n \sum_{g \neq 0} f(\lambda A g)\, d\omega_1 \tag{12}$$

converges, that

$$\psi(\lambda) = \lambda^n \sum_{g \neq 0} \int_F f(\lambda A g)\, d\omega_1 \tag{13}$$

and that, by (11),

$$\lim_{\lambda \to 0} \psi(\lambda) = \int_F \lim_{\lambda \to 0} \varphi(\lambda, A)\, d\omega_1 = \gamma V_n . \tag{14}$$

5. In this section we investigate the sum

$$\chi(\lambda) = \sum_g{}' \int_F f(\lambda A g)\, d\omega_1 , \tag{15}$$

extended over all primitive g, i.e., over all integral g with the greatest common divisor $(g_1, \cdots, g_n) = 1$.

We complete the primitive g to a proper unimodular matrix $U = U_g$ with the first column g; then

$$\int_F f(Ag)\, d\omega_1 = \int_{FU} f(AU^{-1}g)\, d\omega_1 = \int_{\overline{FU}} f(|Y|^{-1}x) \{dY\}, \tag{16}$$

where x denotes the first column of the variable matrix Y in the cone $\overline{FU}$. The unimodular matrices of the particular form

$$U_1 = \begin{pmatrix} 1 & u' \\ 0 & \hat{U} \end{pmatrix},$$

with an arbitrary $(n - 1)$-dimensional integral vector u and an arbitrary proper unimodular $(n - 1)$-rowed matrix $\hat{U}$, constitute a subgroup Δ of Γ_1. The left cosets of Δ, relative to Γ_1, are $U_g\Delta$, where g runs exactly over all primitive n-dimensional vectors; consequently the union of all FU_g is a fundamental domain $F(\Delta)$ for Δ on Ω_1, and

$$\chi(1) = \int_{\overline{F(\Delta)}} f(\,|\,Y\,|^{-1}x)\,\{dY\}, \tag{17}$$

by (15), (16).

Completing x to a matrix W_x in Ω_1 with the first column x, we obtain the decomposition

$$Y = W_x Y_1, \qquad Y_1 = \begin{pmatrix} 1 & y' \\ 0 & \hat{Y} \end{pmatrix}, \tag{18}$$

with a real $(n - 1)$-dimensional vector y and a real non-singular $(n - 1)$-rowed matrix $\hat{Y}$; plainly,

$$|\,Y\,| = |\,Y_1\,|, \qquad \{dY\} = \{dx\}\{dy\}\{d\hat{Y}\}. \tag{19}$$

The mapping $Y \rightarrow YU_1$ is the same as $\hat{Y} \rightarrow \hat{Y}\hat{U}, y \rightarrow \hat{U}'y + u$; this shows that another fundamental domain G for Δ on Ω_1 can be defined in the following way: Write the general element $Y = A$ of Ω_1 in the form (18), restrict $\hat{Y} = \hat{A}$ to the fundamental region F of the group $\hat{\Gamma}_1$ of all proper unimodular $(n - 1)$-rowed matrices $\hat{U}$, in the space $\hat{\Omega}_1$ of all $(n - 1)$-rowed matrices $\hat{A}$ with $|\,\hat{A}\,| = 1$, and restrict the coordinates $y_1, \cdots, y_{n-1}$ of y to the $(n - 1)$-dimensional unit cube $0 \leqq y_k \leqq 1$ $(k = 1, \cdots, n - 1)$. In view of (17), (19), we obtain

$$\chi(1) = \int_{\bar{\hat{F}}} \left(\int_R f(\,|\,\hat{Y}\,|^{-1}x)\,\{dx\} \right) \{d\hat{Y}\} = \gamma \int_{\bar{\hat{F}}} |\,\hat{Y}\,|\,\{d\hat{Y}\}.$$

If μ is any positive scalar factor, then

$$\int_{\mu\bar{\hat{F}}} \{d\hat{Y}\} = \mu^{(n-1)^2} \int_{\bar{\hat{F}}} \{d\hat{Y}\} = \mu^{(n-1)^2} V_{n-1},$$

and partial integration leads to the formula

$$\int_{\bar{\hat{F}}} |\,\hat{Y}\,|\,\{d\hat{Y}\} = (n - 1) \int_0^1 u^{n-1}\,du V_{n-1} = \frac{n-1}{n} V_{n-1}.$$

This proves that

$$\chi(1) = \frac{n-1}{n} \gamma V_{n-1}. \tag{20}$$

Replacing $f(x)$ by $f(\lambda x)$, we infer that

$$\chi(\lambda) = \lambda^{-n}\chi(1). \tag{21}$$

6. If g runs over all primitive vectors and l over all natural numbers, then lg runs exactly over all integral vectors $\neq 0$. Therefore, by (13), (20), (21),

$$\psi(\lambda) = \lambda^n \sum_{l=1}^{\infty} \chi(l\lambda) = \chi(1)\zeta(n); \tag{22}$$

this shows that $\psi(\lambda)$ is independent of λ. From (14), (20), (22) we deduce the recursion formula

$$nV_n = (n-1)V_{n-1}\zeta(n). \tag{23}$$

Since $V_1 = 1$, it follows that

$$nV_n = \prod_{k=2}^{n} \zeta(k). \tag{24}$$

Minkowski's formula for the volume of the domain of all reduced positive S with $|S| \leqq 1$ is a simple consequence of (5) and (24).

On the other hand, by (12), (14),

$$\psi(1) = \int_F \sum_{g \neq 0} f(Ag)\, d\omega_1 = \gamma V_n .$$

Defining $d\omega = V_n^{-1}\, d\omega_1$, we have

$$\int_F d\omega = 1, \qquad \int_F \sum_{g \neq 0} f(Ag)\, d\omega = \int_R f(x)\, \{dx\},$$

and this is the assertion of the theorem.

From (15), (20) and (23) we deduce the additional result that

$$\zeta(n) \int_F \sum_g{}' f(Ag)\, d\omega = \int_R f(x)\, \{dx\}. \tag{25}$$

Now let B be a star domain in R, i.e., a point set which is measurable in the Jordan sense and which contains with any point x the whole segment λx, $0 < \lambda < 1$. Suppose that for each A in Ω_1 the domain $A^{-1}B$ contains an integral point $g \neq 0$; then it contains also a primitive g. If $f(x)$ denotes the characteristic function of the set B, then we obtain

$$\sum_g{}' f(Ag) = \sum_{g \in A^{-1}B}{}' 1 \geqq 1$$

and

$$\int_R f(x)\, \{dx\} \geqq \zeta(n),$$

in virtue of (25); this is Minkowski's assertion concerning star domains.

Our theorem may be generalized in various directions:

1) We may drop the restriction that the integrable function $f(x)$ vanishes everywhere outside a bounded domain and replace it, e.g., by the weaker condition that $(x'x)^s f(x)$ is bounded in R, for some fixed $s > n/2$.

2) Instead of the function $f(x)$ of a single vector we may introduce an integrable function $f(x_1, \cdots, x_m)$ of m vectors, with $1 \leqq m \leqq n - 1$. The corresponding generalization of (2) is the formula

$$\int_F \sum_{g_1, \cdots, g_m} f(Ag_1, \cdots, Ag_m)\, d\omega = \int_{R^m} f(x_1, \cdots, x_m) \{dx_1\} \cdots \{dx_m\},$$

where the summation is carried over all systems of linearly independent integral vectors $g_1, \cdots, g_n$.

3) We may consider certain other discrete subgroups of topological groups, instead of Ω_1 and Γ_1, e.g., the real symplectic group and the modular group of degree n. In my researches on symplectic geometry, I have already applied the method of the present paper to the determination of the volume of the fundamental domain of the modular group of degree n. Another and more general example is provided by the group of units of the simple order J_n, $n > 1$, consisting of all n-rowed matrices $A = (\alpha_{kl})$, where the elements $\alpha_{kl}(k, l = 1, \cdots, n)$ belong to a given order J_1 in a division algebra which is of finite rank in the field of rational numbers; this comprises in particular the group of n-rowed unimodular matrices in an arbitrary algebraic number field.

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