DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS SERIES B
Volume **25**, Number **2**, February **2020**

doi:10.3934/dcdsb.2019264

pp. **733–747**

ON THE APPROXIMATION OF FIXED POINTS FOR NON-SELF MAPPINGS ON METRIC SPACES

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Dedicated to Professor Juan J. Nieto on the occasion of his 60th birthday

Abstract. Starting from some classical results of R. Conti, A. Haimovici and K. Iseki, and from a more recent result of S. Reich and A.J. Zaslavski, we present several theorems of approximation of the fixed points for non-self mappings on metric spaces. Both metric and topological conditions are involved. Some of the results are generalized to the multi-valued case. An application is given to a class of implicit first-order differential systems leading to a fixed point problem for the sum of a completely continuous operator and a nonexpansive mapping.

1. **Introduction.** In 1960, R. Conti in [3] stated the following remark, which is presented below as a theorem, about the approximation of fixed points for continuous self mappings of a metric space, and discussed its applications to the approximation of solutions to the Cauchy problem.

Theorem 1.1 (R. Conti)**.** *Let (X,d) be a metric space and $T:X\to X$ be a continuous mapping. Assume that there exists a sequence $(x_n)_{n\geq 1}$ of elements of X such that:*

(i): *the set $\{Tx_n:\ n\geq 1\}$ is relatively compact;*

(ii): $d(Tx_n,x_n)\to 0$ *as* $n\to\infty$.

Then T has at least one fixed point, and each limit point of the sequence (x_n) is a fixed point of T.

Conti also noted that in case that T is *completely continuous* (i.e., continuous and with the property of sending bounded sets into relatively compact sets), then a sufficient condition for (i) to hold is that

(i'): the set $\{x_n;\ n\geq 1\}$ is bounded.

In 1961, independently of Conti, A. Haimovici [5] obtained a fixed point theorem similar to Theorem 1.1, with a concrete indication about the sequence (x_n). More

2010 *Mathematics Subject Classification.* Primary: 47H10; Secondary: 47H09.
Key words and phrases. Fixed point, metric space, generalized contraction mapping, measure of noncompactness, condensing mapping, nonexpansive mapping, multi-valued operator, implicit differential equation.

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exactly, x_n is assumed to be a fixed point of a mapping T_n which approximates T in the sense that

$$d(T, T_n) := \sup_{x \in X} d(Tx, T_n x) \to 0 \quad \text{as} \quad n \to \infty.$$

In fact, in [5], instead of condition (i), it was assumed that the set $\{x_n; n \geq 1\}$ is relatively compact. However, in virtue of the fact that

$$d(Tx_n, x_n) = d(Tx_n, T_n x_n) \to 0 \quad \text{as} \quad n \to \infty,$$

we have that $\{x_n : \ n \geq 1\}$ is relatively compact if and only if $\{Tx_n : \ n \geq 1\}$ is so. In addition, the condition (ii) holds too.

Notice that the proof of the continuation fixed point theorem for nonexpansive mappings in Hilbert spaces (see R. Precup [13] and [14]) offers an example of such a sequence (T_n) of mappings. Indeed, if H is a Hilbert space, U is an open bounded subset of H containing the origin and $T : \overline{U} \to H$ is nonexpansive such that the boundary condition

$$T(x) \neq \lambda x \quad \text{for all} \quad x \in \partial U \quad \text{and} \quad \lambda > 1$$

holds, then the mappings

$$T_n x = \left(1 - \frac{1}{n}\right) Tx \quad (n \geq 1)$$

approximate T, are contractions from $\overline{U}$ to H and also satisfy the boundary condition. Consequently, in view of the continuation principle for condensing mappings, T_n has a (unique) fixed point x_n. In addition, as proved in [13] and [14], the sequence (x_n) is convergent. Hence the assumptions of Haimovici's theorem are fulfilled. For an extension to complete CAT(0) spaces, see W.A. Kirk [8] and Theorem 9.12 in W. Kirk and N. Shahzad [9]. Another example is given in Section 5.

In 1962, again independently of Conti's work, K. Iseki in [6] extended Haimovici's result to the case where x_n is an approximate fixed point of T_n, in the sense that

$$d(T_n x_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

In this case, again $\{x_n : n \geq 1\}$ is relatively compact if and only if $\{Tx_n : n \geq 1\}$ is so, and (ii) holds, as follows from

$$d(Tx_n, x_n) \leq d(Tx_n, T_n x_n) + d(T_n x_n, x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Note that some concrete sequences (x_n) like the abstract Iseki's sequence, appear in Conti's paper in connection to different constructive schema for the Cauchy problem: the methods of Cauchy-Lipschitz, Tonelli, Severini and Picard-Peano.

Recently, S. Reich and A.J. Zaslavski [19] (see also Section 3.13 in the recent book [20]) considered a similar problem for non-self mappings and proved the following metric result.

Theorem 1.2 (Reich-Zaslavski). *Let (X, d) be a complete metric space, Y be a closed subset of X, and $T : Y \to X$ be a φ-contraction, in the sense that*

$$d(Tx, Ty) \leq \varphi(d(x, y)) \quad \text{for every} \quad x, y \in Y,$$

where φ is a comparison function, i.e., $\varphi : [0, \infty) \to [0, \infty)$ is increasing and $\varphi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. Assume that there exists a bounded sequence (y_n) such that $T^n y_n$ is defined for all $n \geq 1$. Then T has a unique fixed point x and $T^n y_n \to x$ as $n \to \infty$.

Motivated by the paper S. Reich and A.J. Zaslavski [19], and by its subsequent extensions A. Petruşel, I.A. Rus and M.-A. Şerban [12], I.A. Rus and M.-A. Şerban [22], M.-A. Şerban [23], we shall analyze the results of this type in connection with Conti's remark and we shall present new fixed point results for non-self mappings. Some extensions for multi-valued mappings are also given. Finally, we give an application to the initial value problem for an implicit first-order differential system leading to a fixed point problem for the sum of a completely continuous operator and a nonexpansive mapping.

2. Fixed point theorems for non-self mappings.

Conti's theorem is also true for a continuous non-self mapping $T : Y \to X$, where Y is any closed subset of X, if we assume that $x_n \in Y$ for all $n \geq 1$. Thus, we have the following theorem.

Theorem 2.1. *Let (X, d) be a metric space, $Y \subset X$ be a closed set and $T : Y \to X$ be a continuous mapping. Assume that there exists a sequence (x_n) of elements of Y such that:*

(i): *the set $\{Tx_n : \ n \geq 1\}$ is relatively compact;*
(ii): $d(Tx_n, x_n) \to 0$ *as* $n \to \infty$.

Then T has at least one fixed point, and each limit point of the sequence (x_n) is a fixed point of T.

Proof. Indeed, from (i), there exists a subsequence (Tx_{n_k}) of (Tx_n) which is convergent to some $x \in X$. Next from (ii),

$$d(x_{n_k}, x) \leq d(Tx_{n_k}, x_{n_k}) + d(Tx_{n_k}, x) \to 0 \text{ as } k \to \infty.$$

Hence $x_{n_k} \to x$ as $k \to \infty$. Since $x_{n_k} \in Y$ and Y is closed, one has $x \in Y$. Now, by the continuity of T, $Tx_{n_k} \to Tx$ as $k \to \infty$. Therefore $Tx = x$. □

From now on (X, d) is a metric space, $Y \subset X$ is closed and $T : Y \to X$ is continuous. The mapping T is said to be *condensing* (with respect to Hausdorff's measure of noncompactness α_H) if it is continuous and

$$\alpha_H(T(M)) < \alpha_H(M),$$

for any countable bounded set $M \subset Y$ with $\alpha_H(M) > 0$.

The next result gives a sufficient condition for (i) in Theorem 2.1 to hold.

Theorem 2.2. *Let (X, d) be a complete metric space, $Y \subset X$ be a closed set, and $T : Y \to X$ be a condensing mapping. If there exists a bounded sequence (x_n) of elements of Y such that $d(Tx_n, x_n) \to 0$ as $n \to \infty$, then T has at least one fixed point, and each limit point of the sequence (x_n) is a fixed point of T.*

The above result is a consequence of the following lemma.

Lemma 2.3. *If $(x_n), (y_n)$ are two bounded sequences of elements from the metric space (X, d) such that $d(x_n, y_n) \to 0$ as $n \to \infty$, then*

$$\alpha_H(\{x_n : n \geq 1\}) = \alpha_H(\{y_n : n \geq 1\}).$$

Proof. Denote $S := \{x_n : n \geq 1\}$, $S' := \{y_n : n \geq 1\}$, $\kappa := \alpha_H(S)$ and $\kappa' := \alpha_H(S')$. Assume the contrary, for instance that $\kappa < \kappa'$. Then, taking any $\varepsilon > 0$ with $\kappa + 2\varepsilon < \kappa'$, we can cover S by a finite number of balls of radius $\kappa + \varepsilon$. From $d(x_n, y_n) \to 0$ as $n \to \infty$, we find that starting to some index n_ε, the elements y_n belong to those balls enlarged to radius $\kappa + 2\varepsilon$. Thus S' can be covered by a finite number of balls of radius $\kappa + 2\varepsilon$. Then we derive $\kappa' = \alpha_H(S') \leq \kappa + 2\varepsilon < \kappa'$, a contradiction. □

Proof of Theorem 2.2. Let $S := \{x_n : n \geq 1\}$ and $S' = \{Tx_n : n \geq 1\}$. According to Lemma 2.3,

$$\alpha_H(S) = \alpha_H(S'). \tag{1}$$

On the other hand, if $\alpha_H(S) > 0$, then since $S' = T(S)$, we would have $\alpha_H(S') = \alpha_H(T(S)) < \alpha_H(S)$, which is excluded by (1). Hence $\alpha_H(S) = \alpha_H(S') = 0$, that is condition (i) holds. The result is now seen to be a consequence of Theorem 2.1. □

In particular Theorem 2.2 is applicable if T is *φ-condensing*, i.e.,

$$\alpha_H(T(M)) \leq \varphi(\alpha_H(M))$$

for any countable bounded set $M \subset Y$, where φ is a comparison function (see [22]). Indeed, since $\varphi^n(t) \to 0$ as $n \to \infty$, for all $t > 0$, one has that $\varphi(t) < t$ for all $t > 0$. Hence any φ-condensing mapping is condensing.

For example, the sum of a completely continuous mapping and a φ-contraction is a φ-condensing mapping.

Also recall that, if $\varphi(t) = at$, for $t \geq 0$ (where $a < 1$), then the φ-condensing property reduces to the *a-set-contraction* property, i.e.,

$$\alpha(T(M)) \leq a\alpha(M)$$

for any countable bounded set $M \subset Y$.

Theorem 2.2 yields the following topological version of Theorem 1.2.

Theorem 2.4. *Let (X, d) be a complete metric space, Y be a closed subset of X and $T : Y \to X$ be a condensing mapping. Assume that there exists a bounded sequence (y_n) such that $T^n y_n$ is defined for all $n \geq 1$ and*

$$d\left(T^n y_n, T^{n-1} y_n\right) \to 0 \quad \text{as} \quad n \to \infty. \tag{2}$$

Then T has at least one fixed point which is a limit point of the sequence $\left(T^{n-1} y_n\right)$.

Proof. Apply Theorem 2.2 to the sequence $x_n := T^{n-1} y_n$. □

Notice that, in the very particular case when T is a φ-contraction, the hypothesis (2) is trivially satisfied and Theorem 2.4 reduces to Theorem 1.2. Indeed, in this case,

$$d\left(T^n y_n, T^{n-1} y_n\right) \leq \varphi\left(d\left(T^{n-1} y_n, T^{n-2} y_n\right)\right) \leq \ \dots\ \leq \varphi^{n-1}\left(d\left(Ty_n, y_n\right)\right). \tag{3}$$

Also, the sequence (y_n) being assumed bounded and T being a φ-contraction, the sequence (Ty_n) is also bounded and thus there is a constant $c > 0$ with $d(Ty_n, y_n) \leq c$ for all $n \geq 1$. Then

$$\varphi^{n-1}(d(Ty_n, y_n)) \leq \varphi^{n-1}(c).$$

Now since $\varphi^{n-1}(c) \to 0$ as $n \to \infty$, we deduce that $\varphi^{n-1}(d(Ty_n, y_n)) \to 0$ as $n \to \infty$. In view of (3), we obtain (2).

The next result is the version for non-self mappings of the result of Haimovici-Iseki.

Theorem 2.5. *Let (X, d) be a metric space, $Y \subset X$ be a closed set, $T : Y \to X$ be a continuous mapping and $T_n : Y \to X$ (with $n \geq 1$) be a sequence of mappings with*

$$d(T_n, T) := \sup_{x \in Y} d(T_n x, Tx) \to 0 \quad \text{as} \quad n \to \infty. \tag{4}$$

Assume that there exists a sequence (x_n) *of elements of* Y *such that the set* $\{Tx_n : n \geq 1\}$ *is relatively compact and*

$$d(T_n x_n, x_n) \to 0 \quad \textit{as} \quad n \to \infty. \tag{5}$$

Then T *has at least one fixed point, and each limit point of the sequence* (x_n) *is a fixed point of* T.

Proof. From (4), (ii), we have

$$d(Tx_n, x_n) \leq d(Tx_n, T_n x_n) + d(T_n x_n, x_n) \leq d(T_n, T) + d(T_n x_n, x_n) \to 0$$

as $n \to \infty$. Then the conclusion follows from Theorem 2.1. □

Assuming that T is condensing, we have the following result whose proof makes use of Theorem 2.2.

Theorem 2.6. *Let* (X, d) *be a complete metric space,* $Y \subset X$ *be a closed set,* $T : Y \to X$ *be a condensing mapping and* $T_n : Y \to X$ *(with* $n \geq 1$*) be a sequence of mappings satisfying* (4). *If there exists a bounded sequence* (x_n) *of elements of* Y *such that condition* (5) *holds, then* T *has at least one fixed point, and each limit point of the sequence* (x_n) *is a fixed point of* T.

We conclude this section by another extension of Theorem 1.2 in the sense of the Haimovici-Iseki type condition.

Theorem 2.7. *Let* (X, d) *be a complete metric space,* Y *be a closed subset of* X, $T : Y \to X$ *be a condensing mapping and* $T_n : Y \to X$, $n \geq 1$, *be a sequence of mappings satisfying* (4). *Assume that there exists a bounded sequence* (y_n) *such that* $T_n^n y_n$ *is defined for all* $n \geq 1$ *and*

$$d\left(T_n^n y_n, T_n^{n-1} y_n\right) \to 0 \quad \textit{as} \quad n \to \infty. \tag{6}$$

Then T *has at least one fixed point which is a limit point of the sequence* $\left(T_n^{n-1} y_n\right)$.

Proof. Direct consequence of Theorem 2.5, with $x_n = T_n^{n-1} y_n$. □

Notice that the condition (6) holds in particular if T_n are φ-contractions with respect to the same comparison function φ.

3. Other contractive conditions.

3.1. **Kannan non-self mappings.** Let (X, d) be a metric space and $Y \subset X$ be a nonempty subset of it. An operator $T : Y \to X$ is an α-*Kannan mapping* for some $\alpha > 0$ if

$$d(Tx, Ty) \leq \alpha \left[d(x, Tx) + d(y, Ty)\right], \quad \text{for all} \quad x, y \in Y$$

(see R. Kannan [7]). In the case of the Kannan mappings we have the following general result.

Theorem 3.1. *Let* (X, d) *be a complete metric space,* $Y \subset X$ *be a closed subset and* $T : Y \to X$ *be a continuous* α*-Kannan mapping. If there exists a sequence* (x_n) *of elements of* Y *such that*

$$d(Tx_n, x_n) \to 0 \quad \textit{as} \quad n \to \infty,$$

then

(a): T *has a unique fixed point* x^*;
(b): $d(x_n, x^*) \leq (1 + \alpha) d(Tx_n, x_n)$.

Proof. By the relation

$$d(Tx_n, Tx_m) \leq \alpha [d(x_n, Tx_n) + d(x_m, Tx_m)] \to 0 \text{ as } n, m \to \infty,$$

we have that (Tx_n) is a Cauchy sequence, and so it is convergent. Thus the set $\{Tx_n : n \geq 1\}$ is relatively compact. Applying Theorem 2.1 we get that T has at least one fixed point in Y and each limit point of the sequence (x_n) is a fixed point of T. Since T is a Kannan mapping, it has at most one fixed point x^*. We also have

$$\begin{aligned} d(x_n, x^*) &\leq d(x_n, Tx_n) + d(Tx_n, x^*) = d(x_n, Tx_n) + d(Tx_n, Tx^*) \\ &\leq d(x_n, Tx_n) + \alpha [d(x_n, Tx_n) + d(x^*, Tx^*)] \\ &= (1+\alpha) d(Tx_n, x_n). \end{aligned}$$

□

The next result is an extension of Theorem 3.1 in Haimovici-Iseki's approximation spirit.

Theorem 3.2. *Let (X, d) be a complete metric space, $Y \subset X$ be a closed subset and $T : Y \to X$ be a continuous α-Kannan mapping. Let $T_n : Y \to X$ (where $n \geq 1$) be a sequence of mappings satisfying (4). If there exists a sequence (x_n) of elements of Y such that the condition (5) holds, then*

(a): *T has a unique fixed point x^*;*
(b): $d(x_n, x^*) \leq (1+\alpha) d(Tx_n, x_n)$;
(c): $d(x_n, x^*) \leq (1+\alpha) [d(T_n, T) + d(T_n x_n, x_n)]$.

Proof. Since

$$d(Tx_n, x_n) \leq d(Tx_n, T_n x_n) + d(T_n x_n, x_n) \leq d(T, T_n) + d(T_n x_n, x_n) \to 0$$

as $n \to \infty$, the conclusion follows by Theorem 3.1. □

By the *maximal displacement functional* corresponding to T we understand the functional $E_T : P(Y) \to \mathbb{R}_+ \cup \{+\infty\}$ defined by

$$E_T(A) := \sup \{d(x, Tx) \mid x \in A\}. \tag{7}$$

In the case of the Kannan mappings we have the following version of Theorem 1.2. A similar result was obtained in I.A. Rus and M.-A. Şerban [22].

Theorem 3.3. *Let (X, d) be a complete metric space, $Y \subset X$ a nonempty closed subset and $T : Y \to X$ a continuous mapping. Assume that the following conditions are satisfied:*

(i): *T is an α-Kannan mapping with $\alpha < 1/2$;*
(ii): *there exists a bounded sequence (y_n) in Y such that $T^n y_n$ is defined for every $n \geq 1$;*
(iii): $E_T(Y) < +\infty$.

Then

(a): *T has a unique fixed point x^*;*
(b): *$T^{n-1} y_n \to x^*$ and $T^n y_n \to x^*$ as $n \to +\infty$;*
(c): $d(T^{n-1} y_n, x^*) \leq (1+\alpha) (\alpha / (1-\alpha))^{n-1} d(y_n, Ty_n)$.

Proof. From

$$d(T^n y_n, T^{n-1} y_n) \leq \alpha \left[d(T^{n-1} y_n, T^n y_n) + d(T^{n-2} y_n, T^{n-1} y_n) \right],$$

we obtain that

$$\begin{aligned} d\left(T^n y_n, T^{n-1} y_n\right) &\leq \frac{\alpha}{1-\alpha} d\left(T^{n-2} y_n, T^{n-1} y_n\right) \\ \cdots &\leq \left(\frac{\alpha}{1-\alpha}\right)^{n-1} d\left(y_n, T y_n\right) \leq \left(\frac{\alpha}{1-\alpha}\right)^{n-1} E_T\left(Y\right). \end{aligned}$$

Since $\alpha < 1/2$, one has $\alpha/\left(1-\alpha\right) < 1$ and so

$$d\left(T^n y_n, T^{n-1} y_n\right) \to 0 \quad \text{as} \quad n \to +\infty.$$

The conclusion now follows from Theorem 3.1 applied to the sequence $x_n := T^{n-1} y_n$. □

Another Haimovici-Iseki type result for non-self Kannan mappings is the following one.

Theorem 3.4. *Let (X, d) be a complete metric space, $Y \subset X$ a nonempty closed subset, $T : Y \to X$ a continuous mapping and $T_n : Y \to X$, $n \geq 1$, a sequence of mappings. Assume that the following conditions are satisfied:*

(i): *T is an α-Kannan mapping with $\alpha < 1/2$;*
(ii): *the mappings T_n $(n \geq 1)$ satisfy (4);*
(iii): *there exists a bounded sequence (y_n) in Y, such that $T_n^n y_n$ is defined for every $n \geq 1$;*
(iv): $E_T\left(Y\right) < +\infty$.

Then

(a): *T has a unique fixed point x^*;*
(b): *$T_n^{n-1} y_n \to x^*$ and $T_n^n y_n \to x^*$ as $n \to +\infty$;*
(c): $d\left(T_n^{n-1} y_n, x^*\right) \leq \alpha^{n-1}\left(1-\alpha\right)^{1-n}\left(1+\alpha\right)\left[d\left(y_n, T y_n\right) + d\left(T y_n, T_n y_n\right)\right].$

Proof. From (i) and (ii) we have

$$\begin{aligned} d\left(T_n^n y_n, T_n^{n-1} y_n\right) &\leq \frac{\alpha}{1-\alpha} d\left(T_n^{n-2} y_n, T_n^{n-1} y_n\right) \\ \cdots &\leq \left(\frac{\alpha}{1-\alpha}\right)^{n-1} d\left(y_n, T_n y_n\right) \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^{n-1} \left[d\left(y_n, T y_n\right) + d\left(T y_n, T_n y_n\right)\right] \\ &\leq \left(\frac{\alpha}{1-\alpha}\right)^{n-1} \left[E_T\left(Y\right) + d\left(T, T_n\right)\right]. \end{aligned}$$

Thus

$$d\left(T_n^n y_n, T_n^{n-1} y_n\right) \to 0 \quad \text{as} \quad n \to +\infty,$$

and so the conclusion follows by Theorem 3.2 applied to the sequence $x_n := T_n^{n-1} y_n$. □

3.2. **Ćirić-Reich-Rus non-self mappings.** Let (X, d) be a metric space and $Y \subset X$ be a nonempty subset of it. An operator $T : Y \to X$ is said to be an (a, b)-*Ćirić-Reich-Rus mapping* for some numbers $a, b \in \mathbb{R}_+$, if

$$d\left(Tx, Ty\right) \leq a d\left(x, y\right) + b\left[d\left(x, Tx\right) + d\left(y, Ty\right)\right], \quad \text{for all} \quad x, y \in Y$$

(see L.B. Ćirić [2], S. Reich [17] and I.A. Rus [21]).

Theorem 3.5. *Let (X,d) be a complete metric space, $Y \subset X$ a closed subset and $T : Y \to X$ a continuous (a,b)-Ćirić-Reich-Rus mapping with $a < 1$. If there exists a sequence (x_n) of elements of Y such that*

$$d\left(Tx_n, x_n\right) \to 0 \quad as \quad n \to +\infty,$$

then

(a): *T has a unique fixed point x^*;*
(b): $d\left(x_n, x^*\right) \leq (1+b)\left(1-a\right)^{-1} d\left(Tx_n, x_n\right).$

Proof. From

$$\begin{aligned} d\left(Tx_n, Tx_m\right) &\leq ad\left(x_n, x_m\right) + b\left[d\left(x_n, Tx_n\right) + d\left(x_m, Tx_m\right)\right] \\ &\leq ad\left(Tx_n, Tx_m\right) + (a+b)\left[d\left(x_n, Tx_n\right) + d\left(x_m, Tx_m\right)\right], \end{aligned}$$

we obtain

$$d\left(Tx_n, Tx_m\right) \leq \frac{a+b}{1-a}\left[d\left(x_n, Tx_n\right) + d\left(x_m, Tx_m\right)\right] \to 0 \ \text{ as } n, m \to +\infty.$$

Hence (Tx_n) is a Cauchy sequence and so it converges. Thus the set $\{Tx_n : n \geq 1\}$ is relatively compact. Applying Theorem 2.1 we get that T has in Y at least one fixed point x^* and each limit point of the sequence (x_n) is a fixed point of T. Since T is a Ćirić-Reich-Rus mapping, it has at most one fixed point. Hence T has a unique fixed point x^*. We also have

$$\begin{aligned} d\left(x_n, x^*\right) &\leq d\left(x_n, Tx_n\right) + d\left(Tx_n, x^*\right) \\ &\leq (1+b) d\left(Tx_n, x_n\right) + ad\left(x_n, x^*\right), \end{aligned}$$

which implies

$$d\left(x_n, x^*\right) \leq \frac{1+b}{1-a} d\left(Tx_n, x_n\right).$$

□

Theorem 3.6. *Let (X,d) be a complete metric space, $Y \subset X$ a closed subset and $T : Y \to X$ a continuous (a,b)-Ćirić-Reich-Rus mapping with $a < 1$. Let $T_n : Y \to X$ $(n \geq 1)$ be a sequence of mappings satisfying the condition (4). If there exists a sequence (x_n) of elements of Y such that (5) holds, then*

(a): *T has a unique fixed point x^*;*
(b): $d\left(x_n, x^*\right) \leq (1+b)\left(1-a\right)^{-1} d\left(Tx_n, x_n\right);$
(c): $d\left(x_n, x^*\right) \leq (1+b)\left(1-a\right)^{-1}\left[d\left(T_n, T\right) + d\left(T_n x_n, x_n\right)\right].$

Proof. We have

$$d\left(Tx_n, x_n\right) \leq d\left(Tx_n, T_n x_n\right) + d\left(T_n x_n, x_n\right) \leq d\left(T_n, T\right) + d\left(T_n x_n, x_n\right) \to 0$$

as $n \to \infty$. Thus the conclusion follows from Theorem 3.5. □

In the case of the Ćirić-Reich-Rus mappings we also have the following version of Theorem 1.2. A similar result was obtained in I.A. Rus and M.-A. Şerban [22].

Theorem 3.7. *Let (X,d) be a complete metric space, $Y \subset X$ a nonempty closed subset and $T : Y \to X$ a continuous mapping. Assume that the following conditions are satisfied:*

(i): *T is an (a,b)-Ćirić-Reich-Rus mapping with $a + 2b < 1$;*
(ii): *there exists a bounded sequence (y_n) in Y such that $T^n y_n$ is defined for every $n \geq 1$;*

(iii): $E_T(Y) < +\infty$.

Then

(a): *T has a unique fixed point x^*;*

(b): *$T^{n-1}y_n \to x^*$ and $T^n y_n \to x^*$ as $n \to +\infty$;*

(c): $d\left(T^{n-1}y_n, x^*\right) \leq (1+b)(1-a)^{-1}(a+b)^{n-1}(1-b)^{1-n} d(y_n, Ty_n)$.

Proof. From

$$\begin{aligned} d\left(T^n y_n, T^{n-1} y_n\right) &\leq a d\left(T^{n-1}y_n, T^{n-2}y_n\right) \\ &+ b\left[d\left(T^{n-1}y_n, T^n y_n\right) + d\left(T^{n-2}y_n, T^{n-1}y_n\right)\right], \end{aligned}$$

we obtain that

$$\begin{aligned} d\left(T^n y_n, T^{n-1} y_n\right) &\leq \frac{a+b}{1-b} d\left(T^{n-2}y_n, T^{n-1}y_n\right) \\ \cdots &\leq \left(\frac{a+b}{1-b}\right)^{n-1} d(y_n, Ty_n) \leq \left(\frac{a+b}{1-b}\right)^{n-1} E_T(Y). \end{aligned}$$

Since $a + 2b < 1$, one has $(a+b)(1-b)^{-1} < 1$. Consequently,

$$d\left(T^n y_n, T^{n-1}y_n\right) \to 0 \quad \text{as} \quad n \to +\infty.$$

The conclusion follows from Theorem 3.5 applied to the sequence $x_n := T^{n-1}y_n$. $\square$

The following theorem is a Haimovici-Iseki type result for non-self Ćirić-Reich-Rus mappings.

Theorem 3.8. *Let (X, d) be a complete metric space, $Y \subset X$ a nonempty closed subset, $T : Y \to X$ a continuous mapping and $T_n : Y \to X$, $n \geq 1$, a sequence of mappings. Assume that the following conditions are satisfied:*

(i): *T is an (a, b)-Ćirić-Reich-Rus mapping with $a + 2b < 1$;*

(ii): *the mappings T_n $(n \geq 1)$ satisfy (4);*

(iii): *there exists a bounded sequence (y_n) in Y such that $T_n^n y_n$ is defined for every $n \geq 1$;*

(iv): $E_T(Y) < +\infty$.

Then

(a): *T has a unique fixed point x^*;*

(b): *$T_n^{n-1}y_n \to x^*$ and $T_n^n y_n \to x^*$ as $n \to +\infty$;*

(c): $d\left(T_n^{n-1}y_n, x^*\right) \leq \frac{1+b}{1-a} \cdot \left(\frac{a+b}{1-b}\right)^{n-1} \cdot \left[d(y_n, Ty_n) + d(Ty_n, T_n y_n)\right]$.

Proof. The conclusion follows from the estimate

$$\begin{aligned} d\left(T_n^n y_n, T_n^{n-1} y_n\right) &\leq \frac{a+b}{1-b} d\left(T_n^{n-2}y_n, T_n^{n-1}y_n\right) \\ \cdots &\leq \left(\frac{a+b}{1-b}\right)^{n-1} d(y_n, T_n y_n) \\ &\leq \left(\frac{a+b}{1-b}\right)^{n-1} \left[d(y_n, Ty_n) + d(Ty_n, T_n y_n)\right] \\ &\leq \left(\frac{a+b}{1-b}\right)^{n-1} \left[E_T(Y) + d(T, T_n)\right], \end{aligned}$$

and from Theorem 3.6 applied to the sequence $x_n := T_n^{n-1}y_n$. $\square$

4. **Some extensions to the multi-valued case.** If (X,d) is a metric space, then we denote by $P(X)$ the family of all nonempty subsets of X, by $P_{cl}(X)$ the family of all closed nonempty subsets of X, and by D the *gap functional*, i.e.,

$$D(A,B) := \inf\{d(a,b) \ : \ a \in A, \ b \in B\}, \text{ for } A, B \in P(X).$$

We also denote by H the *Hausdorff-Pompeiu pseudometric* on $P(X)$ expressed by

$$H(A,B) := \max\{\sup_{a\in A} D(a,B), \sup_{b\in B} D(b,A)\}.$$

See J.-P. Aubin and H. Frankowska [1] for related properties of the above functionals.

Our first result in this section is a multi-valued version of Theorem 2.1.

Theorem 4.1. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $T : Y \to P(X)$ be a multi-valued non-self operator with closed graph. Assume that there exists a sequence $(x_n) \subset Y$ such that:*

(i): $\bigcup\limits_{n\geq 1} Tx_n$ *is relatively compact;*

(ii): $D(x_n, Tx_n) \to 0$ *as* $n \to \infty$.

Then, there exists at least one fixed point for T and each limit point of the sequence (x_n) is a fixed point of T.

Proof. By (ii) there exists a sequence (u_n) in X such that $u_n \in Tx_n$ for $n \geq 1$ and $d(x_n, u_n) \to 0$ as $n \to \infty$. Since $\{u_n \ : n \geq 1\} \subset \bigcup\limits_{n\geq 1} Tx_n$, there exists a subsequence (u_{n_k}) of (u_n) which converges to an element $x \in X$ as $k \to \infty$. Since $u_{n_k} \in Tx_{n_k}$ for all $k \geq 1$ we have $d(x_{n_k}, u_{n_k}) \to 0$ as $k \to \infty$. Thus,

$$d(x_{n_k}, x) \leq d(x_{n_k}, u_{n_k}) + d(u_{n_k}, x) \to 0 \text{ as } n \to \infty.$$

Hence, x_{n_k} converges to x and, since $(x_{n_k}) \subset Y$ and Y is closed in (X,d), we obtain that $x \in Y$. We notice now that $u_{n_k} \in Tx_{n_k}$ for $k \geq 1$, $x_{n_k} \to x \in Y$ and $u_{n_k} \to x$, together with the hypothesis that T has closed graph, implies that $x \in Tx$. □

An example where the assumption (i) of the above theorem is satisfied is given by the following result.

Theorem 4.2. *Let (X,d) be a metric space, $Y \in P_{cl}(X)$ and $T : Y \to P(X)$ be a multi-valued non-self operator with closed graph. Assume that:*

(i): *T has the property of sending bounded sets into relatively compact sets;*

(ii): *there exists a bounded sequence $(x_n) \subset Y$ such that $D(x_n, Tx_n) \to 0$ as $n \to \infty$.*

Then, there exists at least one fixed point for T and each limit point of the sequence (x_n) is a fixed point of T.

Proof. Since $Z := \{x_n : n \geq 1\}$ is bounded, by (i), the set $T(Z) = \bigcup_{n\geq 1} Tx_n$ is relatively compact. Thus, the conclusion follows by Theorem 4.1. □

Another example when the hypothesis (i) of Theorem 4.1 is fulfilled involves the Hausdorff measure of noncompactness.

Theorem 4.3. *Let (X,d) be a complete metric space, $Y \in P_{cl}(X)$ and $T : Y \to P(X)$ be a multi-valued non-self operator with closed graph. Assume that:*

(i): *T is condensing with respect to Hausdorff's measure of noncompactness, i.e., $\alpha_H(T(M)) < \alpha_H(M)$, for each bounded and countable $M \in P(Y)$ with $\alpha_H(M) > 0$;*

(ii): *there exists a bounded sequence $(x_n) \subset Y$ such that $D(x_n, Tx_n) \to 0$ as $n \to \infty$;*

Then, there exists at least one fixed point for T and each limit point of the sequence (x_n) is a fixed point of T.

Proof. We will show that, by our hypotheses, the first assumption of Theorem 4.1 holds. As before, by (ii), there exists a sequence (u_n) in X such that $u_n \in Tx_n$ for $n \geq 1$ and $d(x_n, u_n) \to 0$ as $n \to \infty$. Thus, by Lemma 2.3, we have

$$\alpha_H(\{x_n \ : \ n \geq 1\}) = \alpha_H(\{u_n \ : \ n \geq 1\}).$$

If we denote by $S := \{x_n \ : \ n \geq 1\}$ and by $W := \{u_n \ : \ n \geq 1\}$, then we observe that

$$W \subset T(S) = \bigcup_{n \geq 1} Tx_n.$$

Let us suppose, by *reductio ad absurdum*, that $\alpha_H(S) > 0$. Then

$$\alpha_H(W) \leq \alpha_H(T(S)) < \alpha_H(S),$$

a contradiction with the above equality. Hence

$$\alpha_H(S) = \alpha_H(W) = \alpha_H(\bigcup_{n \geq 1} T(x_n)) = 0,$$

showing that $\bigcup_{n \geq 1} Tx_n$ is relatively compact. Now the conclusion follows by Theorem 4.1. □

A multi-valued variant of the Haimovici-Iseki fixed point theorem is the following result.

Theorem 4.4. *Let (X, d) be a complete metric space, $Y \in P_{cl}(X)$, $T : Y \to P(X)$ a multi-valued non-self operator with closed graph and $T_n : Y \to P(X)$ $(n \geq 1)$ a sequence of multi-valued non-self operators. Assume that the following conditions are satisfied:*

(i): *T is condensing with respect to Hausdorff's measure of noncompactness;*

(ii): *there exists a bounded sequence $(x_n) \subset Y$ such that $D(x_n, T_n x_n) \to 0$ as $n \to \infty$;*

(iii): *$\tilde{H}(T_n, T) := \sup_{x \in Y} H(T_n x, Tx) \to 0$ as $n \to \infty$.*

Then, there exists at least one fixed point for T and each limit point of the sequence (x_n) is a fixed point of T.

Proof. By (ii) there exists a sequence (u_n) in X such that $u_n \in T_n x_n$ for $n \geq 1$ and $d(x_n, u_n) \to 0$ as $n \to \infty$. Then we have

$$\begin{aligned} D(x_n, Tx_n) &\leq d(x_n, u_n) + D(u_n, Tx_n) \\ &\leq d(x_n, u_n) + H(T_n x_n, Tx_n) \\ &\leq d(x_n, u_n) + \tilde{H}(T_n, T) \to 0 \text{ as } n \to \infty. \end{aligned}$$

The conclusion follows now from Theorem 4.2. □

For related fixed point results involving multi-valued operators, see A. Petruşel [11].

5. **An application.** Consider the initial value problem for an implicit first-order differential system

$$\left\{\begin{array}{l} u' = f(t,u) + g(t,u') \quad \text{for a.a. } t \in (0,1) \\ u(0) = u^0, \end{array}\right. \tag{8}$$

where $f, g : [0,1] \times \mathbb{R}^N \to \mathbb{R}^N$ are Carathéodory functions and $u^0 \in \mathbb{R}^N$.

Assume that there exist positive constants a, b, c, p with $c < 1$ and $p < 1$ such that the following conditions are satisfied for all $u, v \in \mathbb{R}^N$ and a.a. $t \in (0,1)$:

- **(h1):** $|f(t,u)| \leq a|u|^p + b$
- **(h2):** $|g(t,u) - g(t,v)| \leq |u - v|$
- **(h3):** $\langle g(t,u) - g(t,v),\ u - v\rangle \leq c|u - v|$.

Theorem 5.1. *Under assumptions* (h1)-(h3)*, problem* (8) *has at least one solution* $u \in H^1\left(0,1;\mathbb{R}^N\right)$.

Proof. If we denote $x = u'$, the the problem is equivalent to the fixed point equation

$$x = Tx, \quad x \in L^2\left(0,1;\mathbb{R}^N\right),$$

where

$$(Tx)(t) = f\left(t, u^0 + \int_0^t x(s)\,ds\right) + g(t, x(t)).$$

We have $T = A + B$, where

$$(Ax)(t) = f\left(t, u^0 + \int_0^t x(s)\,ds\right), \quad (Bx)(t) = g(t, x(t)).$$

In virtue of (h2), the operator B is nonexpansive from $L^2\left(0,1;\mathbb{R}^N\right)$ to itself, while A is completely continuous due to the compact embedding of $H^1\left(0,1;\mathbb{R}^N\right)$ into $L^2\left(0,1;\mathbb{R}^N\right)$.

Our **first step** is to find an *a priori* bound of solutions. We claim that there exists a constant $R > 0$ such that $\|x\|_{L^2} < R$ for every solution of the equation

$$x = \lambda Ax + \mu Bx, \tag{9}$$

and every $\lambda, \mu \in (0,1)$. Note that, from (h3),

$$\langle g(t, x(t)),\ x(t)\rangle \leq (c + |g(t,0)|)\,|x(t)|.$$

Then, if we multiply (9) by $x(t)$, and we take also into account (h1), we obtain

$$\begin{aligned} |x(t)|^2 &= \lambda\left\langle f\left(t, u^0 + \int_0^t x(s)\,ds\right),\ x(t)\right\rangle + \mu\langle g(t, x(t)),\ x(t)\rangle \quad (10)\\ &\leq \left(a\left|u^0 + \int_0^t x(s)\,ds\right|^p + b\right)|x(t)| + (c + |g(t,0)|)\,|x(t)|. \end{aligned}$$

Using the inequality $(\alpha + \beta)^p \leq \alpha^p + \beta^p$ and Hölder's inequality, we have

$$\begin{aligned} \left|u^0 + \int_0^t x(s)\,ds\right|^p &\leq \left|u^0\right|^p + \left|\int_0^t x(s)\,ds\right|^p \\ &\leq \left|u^0\right|^p + \|x\|_{L^2}^p. \end{aligned}$$

This together with integration of (10) gives

$$\begin{aligned}
\|x\|_{L^2}^2 &\leq \left[a\left(\left|u^0\right|^p+\|x\|_{L^2}^p\right)+b\right]\int_0^1 |x(t)|\,dt+\int_0^1 (c+|g(t,0)|)\,|x(t)|\,dt \\
&\leq \left[a\left(\left|u^0\right|^p+\|x\|_{L^2}^p\right)+b+\|c+|g(t,0)|\|_{L^2}\right]\|x\|_{L^2} \\
&= a\,\|x\|_{L^2}^{p+1}+\widetilde{b}\,\|x\|_{L^2}\,,
\end{aligned}$$

where $\widetilde{b}=a\left|u^0\right|^p+b+\|c+|g(t,0)|\|_{L^2}$, or

$$\|x\|_{L^2}\leq a\,\|x\|_{L^2}^p+\widetilde{b}. \tag{11}$$

Since $p<1$, there is $R>0$ (independent of x, λ and μ) such that $\|x\|_{L^2}<R$, as claimed.

The second step is to construct an approximation sequence of operators (T_n) for T, and to find a sequence (x_n) of their fixed points as required by Haimovici's result. The operators are

$$T_n=A+\left(1-\frac{1}{n}\right)B,\quad n=1,2,\cdots,$$

which obviously are condensing. On the other hand, in virtue of the result from the first step, one has $\|x\|_{L^2}<R$ for every solution of the equations

$$x=\lambda T_n x,\quad \text{for}\ \ \lambda\in(0,1)\,.$$

Then, by Leray-Schauder continuation principle (see, e.g., R. Precup [15, Theorem 4.1]), or by Mönch fixed point theorem for non-self mappings (see K. Deimling [4, Theorem 18.1] or D. O'Regan and R. Precup [10, Theorem 5.3]), T_n has a fixed point x_n. Also, the distance between T_n and T on the ball of $L^2(0,1;\mathbb{R}^N)$ of radius R tends to zero as $n\to\infty$. The conclusion will follow from Theorem 2.5 once we prove that the set $\{Tx_n:\ n\geq 1\}$, equivalently $\{x_n:\ n\geq 1\}$ is relatively compact in $L^2(0,1;\mathbb{R}^N)$.

This is our aim for the **third step**. The idea of the proof comes from the paper R. Precup [14] (see also D. O'Regan and R. Precup [10, p. 46]). Since A is completely continuous, passing to a subsequence, we may assume that the sequence (Ax_n) is convergent. We show that the corresponding subsequence of (x_n), still denoted by (x_n), is Cauchy. From

$$x_n=Ax_n+(1-1/n)\,Bx_n,$$

we deduce

$$\frac{n}{n-1}x_n-\frac{k}{k-1}x_k=\frac{n}{n-1}Ax_n-\frac{k}{k-1}Ax_k+Bx_n-Bx_k.$$

Denote $a_n=(n-1)^{-1}$. Then

$$x_n-x_k+a_nx_n-a_kx_k=na_nAx_n-ka_kAx_k+Bx_n-Bx_k,$$

whence

$$\begin{aligned}
\langle a_nx_n-a_kx_k,\ x_n-x_k\rangle_{L^2} &= \langle Bx_n-Bx_k,\ x_n-x_k\rangle_{L^2}-\|x_n-x_k\|_{L^2}^2 \\
&\quad+\langle na_nAx_n-ka_kAx_k,\ x_n-x_k\rangle_{L^2}\,.
\end{aligned}$$

From (h3), we deduce that

$$\langle Bx_n-Bx_k,\ x_n-x_k\rangle_{L^2}\leq c\,\|x_n-x_k\|_{L^2}^2\,.$$

Then

$$\begin{aligned}\langle a_n x_n - a_k x_k,\ x_n - x_k\rangle_{L^2} &\leq (c-1)\|x_n - x_k\|_{L^2}^2 \\ &\quad + \langle n a_n A x_n - k a_k A x_k,\ x_n - x_k\rangle_{L^2}.\end{aligned}$$

Now we use the following identity, which in fact is true in any Hilbert space,

$$\begin{aligned}2\langle a_n x_n - a_k x_k,\ x_n - x_k\rangle_{L^2} &= (a_n + a_k)\|x_n - x_k\|_{L^2}^2 \\ &\quad + (a_n - a_k)\left(\|x_n\|_{L^2}^2 - \|x_k\|_{L^2}^2\right)\end{aligned}$$

to deduce that

$$\begin{aligned}(a_n + a_k + 2(1-c))\|x_n - x_k\|_{L^2}^2 &\leq (a_n - a_k)\left(\|x_k\|_{L^2}^2 - \|x_n\|_{L^2}^2\right) \\ &\quad + 2\langle n a_n A x_n - k a_k A x_k,\ x_n - x_k\rangle_{L^2}.\end{aligned}$$

Consequently

$$\begin{aligned}\|x_n - x_k\|_{L^2}^2 &\leq \left(\|x_k\|_{L^2}^2 - \|x_n\|_{L^2}^2\right)\frac{a_n - a_k}{a_n + a_k + 2(1-c)} \\ &\quad + \frac{2}{a_n + a_k + 2(1-c)}\|n a_n A x_n - k a_k A x_k\|_{L^2}\|\ x_n - x_k\|_{L^2}.\end{aligned}$$

Here $a_n - a_k$ and $\|n a_n A x_n - k a_k A x_k\|_{L^2}$ converge to zero as $n, k \to \infty$, while $\|x_k\|_{L^2}^2 - \|x_n\|_{L^2}^2$, $\|\ x_n - x_k\|_{L^2}$ and $2/(a_n + a_k + 2(1-c))$ are bounded. This implies that (x_n) is a Cauchy sequence.

Thus according to Theorem 2.5, T has a fixed point $x \in L^2\left(0,1;\mathbb{R}^N\right)$. Finally the function

$$u(t) = u^0 + \int_0^t x(s)\,ds$$

solves problem (8). □

Remark 1. (1^0) In (h1) we may take $p = 1$ if it is assumed that $a < 1$. Indeed, the 'a priori' boundedness of solutions still holds in this case as shows (11).

(2^0) If f is null, then condition (h3) is not necessary. Indeed, in this case

$$\|x_n - x_k\|_{L^2}^2 \leq \left(\|x_k\|_{L^2}^2 - \|x_n\|_{L^2}^2\right)\frac{a_n - a_k}{a_n + a_k},$$

where $(a_n - a_k)/(a_n + a_k)$ is bounded and the sequence $(\|x_n\|_{L^2})$ can be assumed to be convergent.

Remark 2. In connection with the second step in the proof of Theorem 5.1 and the Leray-Schauder condition, see also the early paper of S. Reich [18], and the more recent one by D. Reem, S. Reich and A.J. Zaslavski [16].

Acknowledgments. The authors would like to thank the referee for his/her kind words about their paper and for very useful remarks and suggestions.

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Received November 2018; revised January 2019.

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