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CONTINUOUS SHIFT COMMUTING MAPS BETWEEN ULTRAGRAPH SHIFT SPACES

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Abstract. Recently a generalization of shifts of finite type to the infinite alphabet case was proposed, in connection with the theory of ultragraph C*-algebras. In this work we characterize the class of continuous shift commuting maps between these spaces. In particular, we prove a Curtis-Hedlund-Lyndon type theorem and use it to completely characterize continuous, shift commuting, length preserving maps in terms of generalized sliding block codes.

1. **Introduction.** The generalization of the idea of a subshift of finite type to the case of a countable alphabet, called a countable-state topological Markov chain, is a natural one to make and comes up in various contexts, including problems in magnetic recording, see [22]. Countable-state topological Markov chains have also been studied in papers like [5, 6, 7, 8, 18], to mention a few. Although the subject of intense research, the development of results for infinite alphabet shift spaces, that parallel the symbolic dynamics of shifts over finite alphabets, has challenged researchers over the years. The lack of compactness (or local compactness) in the spaces considered account to many results in usual symbolic dynamics failing. For example, it is shown in [22, 23] that for shifts with a countable alphabet (defined via product topology) the entropy of a factor may increase.

In [21] Ott, Tomforde and Willis proposed a definition of a compact shift space that is related to C*-algebra theory. Building from these ideas, and on work of Webster (see [28]), a generalization of shifts of finite type to the infinite alphabet case was proposed recently in [12]. The construction proposed in [12] takes the shift space as the boundary path space of an ultragraph (ultragraphs are combinatorial objects that generalize direct graphs). The idea is that the boundary path space is the spectrum of a certain Abelian subalgebra of the ultragraph C*-algebra. In a similar way, in the finite alphabet case, a Markov shift is the spectrum of an abelian subalgebra of the associated Cuntz-Krieger algebra, see [4]. Although the theory of shift spaces defined in [12] is still in its infancy, there has been already applications to KMS states associated to ultragraph C*-algebras, see [2], and to the diagonal-preserving isomorphism problem of ultragraph C*-algebras, see [1, 12].

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Continuous shift commuting maps form the main class of maps studied in symbolic dynamics, since they correspond to topological conjugacies between shift spaces. For example, in [12], the existence of a shift commuting map between two ultragraph shift spaces is showed to be connected with the existence of an isomorphism between the associated ultragraph C*-algebras.

For shift spaces over finite alphabets, the Curtis-Hedlund-Lyndon Theorem gives a complete characterization of the class of continuous shift commuting maps: Such class of maps corresponds to the class of sliding block codes, that is, corresponds to the class of maps which have bounded local rules[1] (see [19, Chap. 6]).

For infinite-alphabet shift spaces (with the product topology) it was proved that continuous shift commuting maps correspond to generalized sliding block codes, that are maps which have local rules, but their local rules are not necessarily bounded (see [24]). In particular, uniformly continuous shift commuting maps correspond to sliding block codes in the classical sense of maps with bounded local rules (see [3]). In the Ott-Tomforde-Willis context, it was showed in [13] that there exist continuous shift commuting maps that are not generalized sliding block codes, and there exist generalized sliding block codes that are not continuous shift commuting maps. Furthermore, in [13] a complete characterization of the intersection of the class of continuous shift commuting maps with the class of generalized sliding block codes was given.

In this paper we provide a characterization of continuous shift commuting maps between the shift spaces defined in [12] (see Theorem 3.7). In particular, we describe the connection between continuous shift commuting maps and generalized sliding block codes (see Theorem 3.8). As a result we completely characterize continuous, shift commuting, length preserving maps in terms of generalized sliding block codes (see Corollary 4). Before we proceed to the main section (Section 3), we present a review of the ultragraph shift spaces given in [12] in Section 2 below.

2. Background.

In this section we recall some background on ultragraphs and the shift spaces associated to them. We also set notation. Throughout this paper $\mathbb{N}$ denotes the set of positive integers.

2.1. Ultragraphs.

Ultragraphs were introduced by Tomforde in [25] as the correct object to unify the study of graph and Cuntz-Krieger algebras (via ultragraph C*-algebras). Since their introduction ultragraphs have been used in connection with both dynamical systems and C*-algebra theory (see [9, 17, 26] for example). Recently ultragraphs have become a key object in the study of infinite alphabet shift spaces, see [11, 12]. In this section we recall the main definitions and set up notation, following closely the notions introduced in [20, 25].

Definition 2.1. An *ultragraph* is a quadruple $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ consisting of a countable set of vertexes G^0, a countable set of edges $\mathcal{G}^1$, a map $s : \mathcal{G}^1 \to G^0$, and a map $r : \mathcal{G}^1 \to P(G^0) \setminus \{\emptyset\}$, where $P(G^0)$ stands for the power set of G^0.

Definition 2.2. Let $\mathcal{G}$ be an ultragraph. Define $\mathcal{G}^0$ to be the smallest subset of $P(G^0)$ that contains $\{v\}$ for all $v \in G^0$, contains $r(e)$ for all $e \in \mathcal{G}^1$, and is closed under finite unions and non-empty finite intersections (a characterization of

[1]When sliding block codes are defined from a shift space onto itself they are named cellular automata and, as proposed by von Neumann (see [27]), are topological dynamical systems that serve as models for self-reproducing and self-organizing systems.

$\mathcal{G}^0$ in terms of intersections and unions of ranges of edges can be found in [25, Lemma 2.12]).

Let $\mathcal{G}$ be an ultragraph. A *finite path* in $\mathcal{G}$ is either an element of $\mathcal{G}^0$ or a sequence of edges $\alpha = (\alpha_i)_{i=1}^k$ in $\mathcal{G}^1$, where $s(\alpha_{i+1}) \in r(\alpha_i)$ for $1 \leq i \leq k$. The set of finite paths in $\mathcal{G}$ is denoted by $\mathcal{G}^*$.

If we write $\alpha = (\alpha_i)_{i=1}^k$, then the length $|\alpha|$ of α is just k. The length $|A|$ of a path $A \in \mathcal{G}^0$ is zero. We define $r(\alpha) = r(\alpha_k)$ and $s(\alpha) = s(\alpha_1)$. For $A \in \mathcal{G}^0$, we set $r(A) = A = s(A)$.

An *infinite path* in $\mathcal{G}$ is an infinite sequence of edges $\gamma = (\gamma_i)_{i\geq 1}$ in $\prod \mathcal{G}^1$, such that $s(\gamma_{i+1}) \in r(\gamma_i)$ for all i. The set of infinite paths in $\mathcal{G}$ is denoted by $\mathfrak{p}_{\mathcal{G}}^{\infty}$. The length $|\gamma|$ of $\gamma \in \mathfrak{p}_{\mathcal{G}}^{\infty}$ is defined to be ∞, and we define $s(\gamma) = s(\gamma_1)$. A vertex v in $\mathcal{G}$ is called a *sink* if $\left|s^{-1}(v)\right| = 0$ and is called an *infinite emitter* if $\left|s^{-1}(v)\right| = \infty$.

We set $\mathfrak{p}_{\mathcal{G}}^0 := \mathcal{G}^0$ and, for $n \geq 1$, we define $\mathfrak{p}_{\mathcal{G}}^n := \{(\alpha, A) : \alpha \in \mathcal{G}^*, |\alpha| = n, A \in \mathcal{G}^0, A \subseteq r(\alpha)\}$, and

$$\mathfrak{p}_{\mathcal{G}} := \bigcup_{n\geq 0} \mathfrak{p}_{\mathcal{G}}^n.$$

We specify that $(\alpha, A) = (\beta, B)$ if, and only if, $\alpha = \beta$ and $A = B$. We define the length of $(\alpha, A) \in \mathfrak{p}_{\mathcal{G}}$ as $|(\alpha, A)| := |\alpha|$. We call $\mathfrak{p}_{\mathcal{G}}$ the *ultrapath space* associated with $\mathcal{G}$ and the elements of $\mathfrak{p}_{\mathcal{G}}$ are called *ultrapaths*. Each $A \in \mathcal{G}^0$ is regarded as an ultrapath of length zero and can be identified with the pair (A, A). We embed the set of finite paths $\mathcal{G}^*$ in $\mathfrak{p}$ by sending α to $(\alpha, r(\alpha))$. We extend the range map r and the source map s to $\mathfrak{p}_{\mathcal{G}}$ by the formulas, $r((\alpha, A)) = A$, $s((\alpha, A)) = s(\alpha)$ and $r(A) = s(A) = A$.

Given $\alpha = (\alpha_i)_{i=1}^k$ and $\beta = (\beta_i)_{i=1}^\ell$ in $\mathcal{G}^*$ with $s(\beta) \in r(\alpha)$ we define the concatenation of α with β as $\alpha\beta := (\alpha_1 \dots \alpha_k \beta_1 \dots \beta_\ell) \in \mathcal{G}^*$. Given $\alpha \in \mathcal{G}^*$ we say that $\alpha' \in \mathcal{G}^*$ is a prefix, or initial segment, of α if either $\alpha' = \alpha$ or $\alpha = \alpha'\beta$ for some $\beta \in \mathcal{G}^*$.

Given $x \in \mathfrak{p}_{\mathcal{G}}$ and $y \in \mathfrak{p}_{\mathcal{G}} \cup \mathfrak{p}_{\mathcal{G}}^{\infty}$ such that $s(y) \subseteq r(x)$ (if $|y| = 0$) or $s(y) \in r(x)$ (if $|y| \geq 1$), we define the concatenation of x and y (and denote it as xy) as follows:

$$\begin{array}{lcl} x = A & \Rightarrow & xy := y; \\ x = (\alpha, A) \text{ and } y = B & \Rightarrow & xy := (\alpha, B); \\ x = (\alpha, A) \text{ and } y = (\beta, B) & \Rightarrow & xy := (\alpha\beta, B); \\ x = (\alpha, A) \text{ and } y = (y_i)_{i\geq 1} \dots & \Rightarrow & xy := (\alpha_1 \dots \alpha_{|\alpha|} y_1 y_2 y_3 \dots) \end{array} \tag{1}$$

Given $x \in \mathfrak{p}_{\mathcal{G}} \cup \mathfrak{p}_{\mathcal{G}}^{\infty}$, we say that x has $x' \in \mathfrak{p}_{\mathcal{G}}$ as a prefix, or initial segment, if $x = x'y$, for some $y \in \mathfrak{p}_{\mathcal{G}} \cup \mathfrak{p}_{\mathcal{G}}^{\infty}$.

Definition 2.3. For each subset A of G^0, let $\varepsilon(A)$ be the set $\{e \in \mathcal{G}^1 : s(e) \in A\}$. We shall say that a set A in $\mathcal{G}^0$ is an *infinite emitter* whenever $\varepsilon(A)$ is infinite.

2.2. **Ultragraph shift spaces.** In this section we recall the definition of a shift space associated to an ultragraph, as introduced in [12]. Since [12] only deals with ultragraphs without sinks we make the same assumption here.

Throughout assumption. *From now on all ultragraphs in this paper are assumed to have no sinks.*

Before we define the topological space associated to an ultragraph we need the following definition.

Definition 2.4. Let $\mathcal{G}$ be an ultragraph and $A \in \mathcal{G}^0$. We say that A is a minimal infinite emitter if it is an infinite emitter that contains no proper subsets (in $\mathcal{G}^0$)

that are infinite emitters. For a finite path α in $\mathcal{G}$, we say that A is a minimal infinite emitter in $r(\alpha)$ if A is a minimal infinite emitter and $A \subseteq r(\alpha)$. We denote the set of all minimal infinite emitters in $r(\alpha)$ by M_α, and define

$$\mathfrak{p}_{\mathcal{G}min} := \{(\alpha, A) \in \mathfrak{p}_{\mathcal{G}} : A \in M_\alpha\},$$

and

$$\mathfrak{p}^0_{\mathcal{G}min} := \mathfrak{p}_{\mathcal{G}min} \cap \mathfrak{p}^0_{\mathcal{G}}.$$

To standardize the notation with previous work on sliding block codes (see [15, 13] for example) we let

$$X^{fin}_{\mathcal{G}} := \{(x_i)_{i\geq 1} : (x_i)_{i\geq 1} = (\alpha_1 \ldots \alpha_k AA \ldots) \text{ with } (\alpha_1 \ldots \alpha_k, A) \in \mathfrak{p}_{\mathcal{G}min}\};$$

and let $X^{inf}_{\mathcal{G}} := \mathfrak{p}^\infty_{\mathcal{G}}$.

Remark 1. Notice that $X^{fin}_{\mathcal{G}}$ can be embedded in $\mathfrak{p}_{\mathcal{G}}$, via the map ι that takes $(\alpha_1 \ldots \alpha_k AA \ldots)$ to $(\alpha_1 \ldots \alpha_k, A)$. So we can translate concepts defined in $\mathfrak{p}_{\mathcal{G}}$ to $X^{fin}_{\mathcal{G}}$. For example, $y \in \mathfrak{p}_{\mathcal{G}}$ is a prefix of x in $X^{fin}_{\mathcal{G}}$ iff it is a prefix of $\iota(x)$ in $\mathfrak{p}_{\mathcal{G}}$.

Definition 2.5. Let $\mathcal{G}$ be an ultragraph. We denote the set of sequences of the form $(AAA\ldots)$ in $X^{fin}_{\mathcal{G}}$ by $X^0_{\mathcal{G}}$. Elements of $X^0_{\mathcal{G}}$ are called 0-sequences and we set their length as zero. A sequence of the form $x = (\alpha_1 \ldots \alpha_n AA \ldots) \in X^{fin}_{\mathcal{G}}$ is called a finite-sequence, or an n-sequence, (and we set its length as $|x| := n$). Finally, a sequence of the form $x = (x_i)_{i\geq 1} \in X^{inf}_{\mathcal{G}}$ is called an infinite-sequence and we set its length as $|x| := \infty$.

As a topological space, the (shift) space associated to an ultragraph $\mathcal{G}$ is the set

$$X_{\mathcal{G}} := X^{fin}_{\mathcal{G}} \ \cup \ X^{inf}_{\mathcal{G}},$$

endowed with the topology generated by *generalized cylinders*, which are sets of the form:

$$D_{\mathbf{y},F} := \{x \in X_{\mathcal{G}} : \mathbf{y} \text{ is prefix of } x \text{ and } x_{|\mathbf{y}|+1} \notin F\}, \tag{2}$$

where $\mathbf{y} \in \mathfrak{p}_{\mathcal{G}}$ and F is a finite (possibly empty) subset of $\varepsilon(r(\mathbf{y}))$. When $F = \emptyset$ we use the short notation $D_{\mathbf{y}} := D_{\mathbf{y},F}$.

We remark that the generalized cylinders form a countable basis of clopen (but not necessarily compact) sets for a metrizable topology on $X_{\mathcal{G}}$ (see [12] for details, including conditions for local compactness of $X_{\mathcal{G}}$, and [16] for the definition of a metric on $X_{\mathcal{G}}$). Furthermore:

- If $x = (x_i)_{i\geq 1} \in X^{inf}_{\mathcal{G}}$ then a neighbourhood basis for x is given by
$$\{D_{(x_1 \ldots x_n, r(x_n))} : n \geq 1\};$$
- If $x = (\alpha AAA\ldots) \in X^{fin}_{\mathcal{G}}$ then a neighbourhood basis for x is given by
$$\{D_{(\alpha,A),F} : F \subset \varepsilon(A), |F| < \infty\};$$

For our work the description of convergence of sequences in $X_{\mathcal{G}}$ is important. We recall it below:

Proposition 1. *Let $\{x^n\}_{n=1}^\infty$ be a sequence of elements in $X_{\mathcal{G}}$, where $x^n = (\gamma_1^n \ldots \gamma_{k_n}^n, A_n)$ or $x^n = \gamma_1^n \gamma_2^n \ldots$, and let $x \in X_{\mathcal{G}}$.*

(a) If $|x| = \infty$, say $x = \gamma_1\gamma_2\ldots$, then $\{x^n\}_{n=1}^\infty$ converges to x if, and only if, for every $M \in \mathbb{N}$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|x^n| \geq M$ and $\gamma_i^n = \gamma_i$ for all $1 \leq i \leq M$.

(b) If $|x| < \infty$, say $x = (\gamma_1 \dots \gamma_k, A)$, then $\{x^n\}_{n=1}^{\infty}$ converges to x if, and only if, for every finite subset $F \subseteq \varepsilon(A)$ there exists $N \in \mathbb{N}$ such that $n > N$ implies that $x^n = x$ or $|x^n| > |x|$, $\gamma^n_{|x|+1} \in \varepsilon(A) \setminus F$, and $\gamma^n_i = \gamma_i$ for all $1 \le i \le |x|$.

We define the shift map $\sigma : X_{\mathcal{G}} \to X_{\mathcal{G}}$ in the usual way:

$$\sigma\big((x_i)_{i\ge 1}\big) = (x_{i+1})_{i\ge 1}.$$

The shift map is not continuous at points of length zero. This will play an important role in our results. In fact we have the following result regarding continuity of σ.

Proposition 2. *The shift map $\sigma : X_{\mathcal{G}} \to X_{\mathcal{G}}$ is continuous at all points of $X_{\mathcal{G}}$ with length greater than zero. Furthermore, if $\mathbf{y} \in \mathfrak{p}_{\mathcal{G}}$ and $|y| > 0$ then $\sigma(D_{\mathbf{y},F}) = D_{\sigma(\mathbf{y}),F}$ (but this is not necessarily true if $|\mathbf{y}| = 0$).*

Proof. The proof of the continuity of σ at all points of $X_{\mathcal{G}}$ with length greater than zero is given in Proposition 3.16 of [12].

To prove the second statement, suppose $\mathbf{y} \in \mathfrak{p}_{\mathcal{G}}$ with $|\mathbf{y}| > 0$, that is, $\mathbf{y} = (y_1 \dots y_n, A)$ with $A \subseteq r(y_n)$, and let $F \subset \varepsilon(r(\mathbf{y}))$ be a finite set. If $x = (x_i)_{i\ge 1} \in D_{(y_1\dots y_n,A),F}$ then $x_1 \dots x_n = y_1 \dots y_n$, $s(x_{n+1}) \in A$, and $x_{n+1} \notin F$. Therefore $\sigma(x) = (x_i)_{i\ge 2}$ belongs to $D_{(y_2\dots y_n,A),F}$, and thus $\sigma(D_{\mathbf{y},F}) \subset D_{\sigma(\mathbf{y}),F}$. On the other hand, if $z = (z_i)_{i\ge 1} \in D_{\sigma(\mathbf{y}),F}$ then $z_1 \dots z_{n-1} = y_2 \dots y_n$, $s(z_n) \in A$, and $z_n \notin F$. Notice that, since $\mathbf{y} \in \mathfrak{p}_{\mathcal{G}}$, we have $y_1 z \in D_{\mathbf{y},F}$. Furthermore, $z = \sigma(y_1 z) \in \sigma(D_{\mathbf{y},F})$ and hence $D_{\sigma(\mathbf{y}),F} \subset \sigma(D_{\mathbf{y},F})$ as desired.

To finish, suppose that $\mathbf{y} \in \mathfrak{p}_{\mathcal{G}}$ with $|\mathbf{y}| = 0$, that is, $\mathbf{y} = A$. Since $\sigma(A) = A$ we have that $D_{\sigma(\mathbf{y}),F} = D_{\mathbf{y},F}$, but in general $\sigma(D_{\mathbf{y},F}) \neq D_{\mathbf{y},F}$. □

Next we recall the definition of the shift space.

Definition 2.6. Let $\mathcal{G}$ be an ultragraph. The *one-sided shift space* associated to $\mathcal{G}$ is the pair $(X_{\mathcal{G}}, \sigma)$, where $X_{\mathcal{G}}$ and σ are as defined above (with $X_{\mathcal{G}}$ viewed as a topological space). We will often refer to the space $X_{\mathcal{G}}$ with the understanding that the map σ is attached to it.

For use in the next sections we introduce the following definition.

Definition 2.7. Let $\mathcal{G}$ be an ultragraph. The *alphabet* of the shift $X_{\mathcal{G}}$ is defined as the set $\mathbf{A}_{\mathcal{G}}$ of all the symbols that can appear in some sequence of $X_{\mathcal{G}}$, that is,

$$\mathbf{A}_{\mathcal{G}} := \mathcal{G}^1 \cup \mathfrak{p}^0_{\mathcal{G}min}.$$

3. **Continuous shift invariant maps.** The characterization of continuous, shift commuting maps is the main goal of this section (and of the paper). Before we prove our main results (in Subsection 3.3), we need to develop a few auxiliary results. As mentioned before, we are under the assumption that all ultragraphs have no sinks.

3.1. **Shift commuting maps.** In this subsection we study shift commuting maps between shift spaces. We give a characterization of such maps below.

Proposition 3. *Let $\mathcal{G}$ and $\mathcal{H}$ be ultragraphs, and let $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ be their respective associated ultragraph shifts. A map $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is shift commuting (i.e. $\Phi \circ \sigma = \sigma \circ \Phi$) if, and only if, there exists a family of sets $\{C_a\}_{a\in\mathbf{A}_{\mathcal{H}}}$, which is a partition of $X_{\mathcal{G}}$, such that for all $x \in X_{\mathcal{G}}$ and $n \ge 1$ we have*

$$\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x), \tag{3}$$

where $\mathbf{1}_{C_a}$ is the characteristic function of the set C_a and $\sum$ stands for the symbolic sum.

Proof. Suppose that Φ is shift commuting. Given $a \in \mathbf{A}_{\mathcal{H}}$, let $C_a := \Phi^{-1}\big(D_{(a,r(a))}\big)$ if $a \in \mathcal{H}^1$, and let $C_a := \Phi^{-1}\big(\{A\}\big)$ if $a = A \in \mathfrak{p}^0_{\mathcal{H}min}$. It is straightforward that $\big\{C_a\big\}_{a\in\mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$.

Let $x \in X_{\mathcal{G}}$. To determine $(\Phi(x))_1$ it is only necessary to know what set C_a contains x, that is, $\big(\Phi(x)\big)_1 = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a}(x)$. Therefore, since for each $n \geq 1$ we have $\Phi \circ \sigma^n = \sigma^n \circ \Phi$, it follows that

$$\big(\Phi(x)\big)_n = \big(\sigma^{n-1}(\Phi(x))\big)_1 = \big(\Phi(\sigma^{n-1}(x))\big)_1 = \sum_{a\in\mathcal{H}} a\mathbf{1}_{C_a}(\sigma^{n-1}(x)).$$

For the converse, suppose that Φ is given by (3). To check that Φ is shift commuting we just need to check that, for all $x \in X_{\mathcal{G}}$ and $n \geq 1$, we have $\Big(\Phi\big(\sigma(x)\big)\Big)_n = \Big(\sigma\big(\Phi(x)\big)\Big)_n$. This follows from the following computation.

$$\Big(\Phi\big(\sigma(x)\big)\Big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a}\circ\sigma^{n-1}\big(\sigma(x)\big) = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a}\circ\sigma^{n}(x) = \big(\Phi(x)\big)_{n+1} = \Big(\sigma\big(\Phi(x)\big)\Big)_n.$$

□

The following results will be useful in the next section.

Lemma 3.1. *Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a shift commuting map and $(AA\ldots) \in X^0_{\mathcal{G}}$. If $|\Phi(AA\ldots)| = 0$ then the image of every finite sequence $x = (x_1x_2\ldots x_nA\ldots) \in X^{fin}_{\mathcal{G}}$ under Φ is a finite sequence in $X_{\mathcal{H}}$ with length no greater than $|x|$.*

Proof. Let $(AA\ldots) \in X^0_{\mathcal{G}}$ and suppose that $\Phi(AA\ldots) = (BB\ldots) \in X^0_{\mathcal{H}}$. Let $x := (x_1x_2\ldots x_nA\ldots) \in X^{fin}_{\mathcal{G}}$. Then

$$\sigma^{|x|}\circ\Phi(x) = \Phi\circ\sigma^{|x|}(x) = \Phi(AA\ldots) = (BB\ldots).$$

□

Lemma 3.2. *Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a shift commuting map. Then for all $(x_1x_2x_3\ldots) \in X_{\mathcal{G}}$ it follows that $\phi(x_1x_2x_3\ldots) = a\phi(x_2x_3\ldots)$, where $a \in \mathbf{A}_{\mathcal{H}}$.*

The next two results follow as in Section 3.1 of [15].

Proposition 4. *If $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is a shift commuting map and $x \in X_{\mathcal{G}}$ is a sequence with period $p \geq 1$ (that is, such that $\sigma^p(x) = x$) then $\Phi(x)$ also has period p.*

Corollary 1. *If $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is a shift commuting map then, for all $(AA\ldots) \in X^0_{\mathcal{G}}$, we have that $\Phi(AA\ldots)$ is a constant sequence (that is, $\Phi(AA\ldots) = (ddd\ldots)$ for some $d \in \mathbf{A}_{\mathcal{H}}$).*

We end this section by proving that for a shift commuting map $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$, described in terms of characteristic functions of a partition $\big\{C_a\big\}_{a\in\mathbf{A}_{\mathcal{H}}}$ of $X_{\mathcal{H}}$ as in Proposition 3, the sets associated to the elements of length zero are shift invariant.

Corollary 2. *Let $\mathcal{G}$ and $\mathcal{H}$ be two ultragraphs, and $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ be the associated ultragraph shifts, respectively. Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a shift commuting map and $\big\{C_a\big\}_{a\in\mathbf{A}_{\mathcal{H}}}$ be the partition of $X_{\mathcal{H}}$ given in Proposition 3. Then, for all $A \in \mathfrak{p}^0_{\mathcal{H}min}$, we have that $\sigma(C_A) \subset C_A$, that is, C_A is shift invariant.*

Proof. Given $A \in \mathfrak{p}^0_{\mathcal{H}min}$, let $x \in C_A$ and $y := \Phi(x) \in X_{\mathcal{H}}$. Since $y_1 = \big(\Phi(x)\big)_1 = A \in X_{\mathcal{H}}$ it follows that $y_i = A$ for all $i \geq 1$. Thus, for all $i \geq 1$, it follows that $A = y_i = \Big(\sigma^{i-1}\big(\Phi(x)\big)\Big)_1 = \Big(\Phi\big(\sigma^{i-1}(x)\big)\Big)_1$, which means that $\sigma^{i-1}(x) \in C_A$. □

3.2. **Generalized sliding block codes.** In this subsection we recall the concept of generalized sliding block codes, which rely on the notion of finitely defined sets. We also present examples in the ultragraph setting. We start with the definition of blocks.

Let $\mathcal{G}$ be an ultragraph. For each $n \geq 1$, let

$$B_n(X_{\mathcal{G}}) := \{(a_1 \dots a_n) \in (\mathbf{A}_{\mathcal{G}})^n : \exists x \in X_{\mathcal{G}}, \exists i \geq 1, \text{ s. t. } x_{i+j-1} = a_j \ \forall j = 1, \dots, n\}$$

be the set of all *blocks of length* n in $X_{\mathcal{G}}$.

The *language* of $X_{\mathcal{G}}$ is

$$B(X_{\mathcal{G}}) := \bigcup_{n \geq 1} B_n(X_{\mathcal{G}}). \tag{4}$$

Remark 2. Notice that while a finite sequence $(\alpha AAA \dots) \in X^{fin}_{\mathcal{G}}$ has length $|\alpha|$ the block $(\alpha A \dots A)$, where we repeat n times the symbol A, has length $|\alpha| + n$.

Before we can introduce finitely defined sets we need the notion of pseudo cylinders.

Definition 3.3. A *pseudo cylinder* in a shift space $X_{\mathcal{G}}$ is a set of the form

$$[b]^\ell_k := \{(x_i)_{i \in \mathbb{N}} \in X_{\mathcal{G}} : (x_k \dots x_\ell) = b\},$$

where $1 \leq k \leq \ell$ and $b \in B(X_{\mathcal{G}})$. We also assume that the empty set is a pseudo cylinder.

We remark that in the context of shift spaces with the product topology pseudo cylinders are equivalent to cylinders. On the other hand, for ultragraph shift spaces (and also in the context of the shift spaces studied in [13, 14] and [21]), a pseudo cylinder is not necessarily an open set. However, as we will see in Proposition 5, a generalized cylinder, and its complement, can always be written as union of pseudo cylinders, that is, a generalized cylinder is a finitely defined set, accordingly to the following:

Definition 3.4. Given $C \subset X_{\mathcal{G}}$, we say that C is a *finitely defined* in $X_{\mathcal{G}}$ if both C and C^c can be written as unions of pseudo cylinders. More precisely, C is finitely defined if there exist two collections of pseudo cylinders in $X_{\mathcal{G}}$, namely $\{[b^i]^{\ell_i}_{k_i}\}_{i \in I}$ and $\{[d^j]^{n_j}_{m_j}\}_{j \in J}$, such that

$$C = \bigcup_{i \in I} [b^i]^{\ell_i}_{k_i} \qquad \text{and} \qquad C^c = \bigcup_{j \in J} [d^j]^{n_j}_{m_j}.$$

Remark 3. Intuitively, a finitely defined set C in $X_{\mathcal{G}}$ is a set such that, given $x \in X_{\mathcal{G}}$, we can 'decide' whether it belongs (or not) to C by knowing a finite quantity of its coordinates.

The empty set and $X_{\mathcal{G}}$ itself are trivial examples of finitely defined sets in $X_{\mathcal{G}}$. Other examples are:

Example 1. Let $\mathcal{G}$ be an ultragraph and let Z be a subset of $X^0_{\mathcal{G}}$. Then Z is a finitely defined set. On the other hand, suppose that there exist $\gamma = e_1 e_2 \dots \in \mathfrak{p}^\infty_{\mathcal{G}}$ such that $|s(e_i)| \geq 2$ for each i. Then $\{\gamma\}^c$, which can be written as a countable union of generalized cylinder sets, is not finitely defined.

Proof. Notice that $Z = \bigcup_{AA\ldots\in Z} [AA]_1^2$ and $Z^c = \bigcup_{a\in \mathbf{A}_{\mathcal{G}}\setminus Z} [a]_1^1$ (where on the second union we use the identification of $AA\ldots$ in $X_{\mathcal{G}}^0$ with $A\in\mathfrak{p}_{\mathcal{G}min}^0$).

For the second part, notice that $\{\gamma\}$ can not be written as an union of pseudo cylinders. □

As we already mentioned, generalized cylinder sets are finitely defined. We prove this below.

Proposition 5. *Let $\mathcal{G}$ be an ultragraph and $X_{\mathcal{G}}$ be the associated ultragraph shift space. Then, for all $\boldsymbol{y}\in\mathfrak{p}_{\mathcal{G}}$ and all finite set $F\subset\varepsilon(r(\boldsymbol{y}))$, the generalized cylinder $D_{\boldsymbol{y},F}$ is a finitely defined set.*

Proof. Let $\mathbf{y}\in\mathfrak{p}_{\mathcal{G}}$, and let $F\subset\varepsilon(r(\mathbf{y}))$ be a finite set. If $\mathbf{y}=A\in\mathfrak{p}_{\mathcal{G}}^0$, then it follows that

$$D_{A,F} = \bigcup_{e\in\varepsilon(A)\setminus F} [e]_1^1 \;\bigcup\; \bigcup_{B\subset A,\ B\in\mathfrak{p}_{\mathcal{G}min}^0} [B]_1^1$$

and

$$D_{A,F}^c = \bigcup_{e\in\varepsilon(A)^c\cup F} [e]_1^1 \;\bigcup\; \bigcup_{B\not\subset A,\ B\in\mathfrak{p}_{\mathcal{G}min}^0} [B]_1^1.$$

If $\mathbf{y}=(\gamma_1\ldots\gamma_n,A)\in\mathfrak{p}_{\mathcal{G}}\setminus\mathfrak{p}_{\mathcal{G}}^0$, then

$$D_{(\gamma_1\ldots\gamma_n,A),F} = \bigcup_{e\in\varepsilon(A)\setminus F} [\gamma_1\ldots\gamma_n e]_1^{n+1} \;\bigcup\; \bigcup_{B\subset A,\ B\in\mathfrak{p}_{\mathcal{G}min}^0} [\gamma_1\ldots\gamma_n B]_1^{n+1}$$

and

$$D_{(\gamma_1\ldots\gamma_n,A),F}^c = \bigcup[\alpha_1\ldots\alpha_n]_1^n\bigcup[\gamma_1\ldots\gamma_n e]_1^{n+1}\bigcup[\gamma_1\ldots\gamma_n B]_1^{n+1},$$

where the unions in the right side range over $(\alpha_1\ldots\alpha_n)\neq(\gamma_1\ldots\gamma_n)$, $e\in F\cup(r(\gamma_n)\setminus\varepsilon(A))$, and $B\not\subset A,\ B\in\mathfrak{p}_{\mathcal{G}min}^0$, respectively. □

Following the same outline of the proof of Proposition 3.6 in [14], one can prove that:

Proposition 6. *Finite unions and finite intersections of finitely defined sets are also finitely defined.*

Remark 4. In general infinite unions or intersections of finitely defined are not finitely defined sets. Thus infinite unions of generalized cylinders need not be finitely defined sets.

Now that we have a good understanding of finitely defined sets we can define generalized sliding block codes.

Definition 3.5. Let $\mathcal{G}$ and $\mathcal{H}$ be two ultragraphs and let $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ be the associated ultragraph shift spaces, respectively. We say that a map $\Phi: X_{\mathcal{G}}\to X_{\mathcal{H}}$ is a *generalized sliding block code* if for all $x\in X_{\mathcal{G}}$ and $n\geq 1$ it follows that

$$\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a}\circ\sigma^{n-1}(x),$$

where $\{C_a\}_{a\in\mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$ by finitely defined sets.

Note that, from Proposition 3, generalized sliding block codes are shift commuting maps. Notice also that a generalized sliding block code can be interpreted as a map with a (possible unbounded) local rule, that is, a map such that to determine $\big(\Phi(x)\big)_j$ one just need to know the configuration of x in a finite window around x_j (but this window can vary). When the local rule is bounded, in the sense that the window around x_j is always the same, the classical notion of sliding block codes is recovered.

We also remark that in the case of classical shift spaces over a finite alphabet, generalized sliding block codes are the same as classical sliding block codes (see [24]) and hence they coincide with the continuous shift commuting maps (see [19]). In the case of shift spaces over an infinite alphabet, with the product topology, generalized sliding block codes also always coincide with the continuous shift commuting maps [24]. For Ott-Tomforde-Willis shift spaces, it was showed in [13] that there exists generalized sliding block codes that are not continuous, and sufficient and necessary condition under which generalized sliding block codes coincide with continuous shift-invariant maps were presented. In our setting, if $X_{\mathcal{G}}^{fin}=\emptyset$ (for example, in the case of a row finite graph), then the topology on $X_{\mathcal{G}}$ coincides with the product topology and hence generalized sliding block codes coincide with continuous, shift commuting maps (as in [24]). In the next section we characterize continuous, shift commuting maps in $X_{\mathcal{G}}$.

3.3. **Continuous shift commuting maps and generalized sliding block codes.** In this section we study continuous, shift commuting maps and their connection with generalized sliding block codes. We start by proving a result regarding continuity of shift commuting maps on $X_{\mathcal{G}}^{\text{inf}}$ (for which we need the following lemma).

Lemma 3.6. *Let C be a finitely defined set in $X_{\mathcal{G}}$. If $x\in C\cap X_{\mathcal{G}}^{inf}$ then there exists a generalized cylinder D such that $x\in D$ and $D\subseteq C$.*

Proof. Let $x\in C\cap X_{\mathcal{G}}^{inf}$, where C is finitely defined in $X_{\mathcal{G}}$. Since $|x|=\infty$, C must contain a pseudo cylinder of the form $[x_k\dots x_l]_k^l$, where $x_j\in\mathcal{G}^1$ for $j=l,\dots,k$. But pseudo cylinders of the aforementioned type can be written as an union of generalized cylinders and hence the result follows. $\square$

Proposition 7. *Let $\Phi:X_{\mathcal{G}}\to X_{\mathcal{H}}$ be a shift commuting map, characterized in terms of partitions $\{C_a\}_{a\in\mathbf{A}_{\mathcal{H}}}$, as in Proposition 3. Suppose that Φ is continuous on $X_{\mathcal{G}}^{inf}\cap\Phi^{-1}(X_{\mathcal{H}}^0)$. Furthermore, suppose that for all $a\in\mathbf{A}_{\mathcal{H}}\setminus\mathfrak{p}^0_{\mathcal{H}min}$, and all $x\in C_a\cap X_{\mathcal{G}}^{inf}$, there exists a cylinder D such that $x\in D\subseteq C_a$. Then ϕ is continuous on $X_{\mathcal{G}}^{inf}$.*

Proof. Let $x\in X_{\mathcal{G}}^{inf}\setminus\Phi^{-1}(X_{\mathcal{H}}^0)$, say $x=\alpha_1\alpha_2\dots$, and let (x^n) be a sequence in $X_{\mathcal{G}}$ converging to x.

First suppose that $|\Phi(x)|=\infty$, say $\Phi(x)=\beta_1\beta_2\dots$. Given $K>0$ we have to show that there exists $N>0$ such that $\Phi(x^n)_j=\Phi(x)_j$, for all $j=1,\dots,K$, and $n>N$.

Notice that, for $j=1,\dots,K$, $\sigma^{j-1}(x)\in C_{\beta_j}\cap X_{\mathcal{G}}^{inf}$. Hence, by hypothesis, there exists a cylinder D_j such that $\sigma^{j-1}(x)\in D_j\subseteq C_{\beta_j}$. Since $\sigma^j(x^n)$ converges to $\sigma^j(x)$, there exists N_j such that $\sigma^{j-1}(x^n)\in C_{\beta_j}$ for all $n>N_j$. Therefore $\Phi(x^n)_j=\Phi(x)_j$ for all $j=1,\dots,K$ and $n>\max\{N_1\dots N_K\}$.

Now suppose that $|\Phi(x)| < \infty$, say $\Phi(x) = (\beta_1 \dots \beta_k BB \dots) \in X_{\mathcal{H}}^{fin}$. Since $\sigma^{i-1}(x) \in C_{\beta_i}$ for each $i = 1, \dots, k$, there exists $N_1 > 0$ such that, for all $n > N_1$, we have $\sigma^{i-1}(x^n) \in C_{\beta_i}$. Hence $(\Phi(x^n))_i = \beta_i$ for all $i = 1, \dots, k$, and $n > N_1$. Notice that $\sigma^k(x) \in C_B \cap X_{\mathcal{G}}^{inf}$. Since $\sigma^k(x^n)$ converges to $\sigma^k(x)$, Φ is shift commuting, and by hypothesis Φ is continuous on $\sigma^k(x)$, it follows that $\Phi(x^n)$ converges to $\Phi(x)$. □

Corollary 3. *If $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is a generalized sliding block code then it is shift commuting and continuous on $X_{\mathcal{G}}^{inf}$.*

Proof. It follows from Proposition 3 that Φ is shift commuting.

Let $\{C_a\}_{a \in \mathbf{A}_{\mathcal{H}}}$ be the partition that defines Φ, as in Definition 3.5. Let $x \in X_{\mathcal{G}}^{inf}$. By Lemma 3.6 if $x \in C_a$, for some $a \in \mathbf{A}_{\mathcal{H}} \setminus \mathfrak{p}^0_{\mathcal{H}min}$, there exists a cylinder D_a such that $x \in D_a \subseteq C_a$. If $x \in \Phi^{-1}(X_{\mathcal{H}}^0)$ then $x \in C_B$, for some $B \in \mathfrak{p}^0_{\mathcal{H}min}$. Since C_B is a finitely defined set, Lemma 3.6 implies again that there exists a cylinder D such that $x \in D \subseteq C_B$. So Φ is locally constant in x and hence continuous on x. Continuity of Φ on $X_{\mathcal{G}}^{inf}$ now follows from Proposition 7. □

Next we characterize continuous shift commuting maps.

Theorem 3.7. *Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a map. If Φ is continuous and shift commuting then Φ is a map given by*

$$\big(\Phi(x)\big)_n = \sum_{a \in \mathbf{A}_{\mathcal{H}}} a \mathbf{1}_{C_a} \circ \sigma^{n-1}(x), \text{ for all } n \geq 1,$$

where

i. $\{C_a\}_{a \in \mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$ such that, for each $a \in \mathbf{A}_{\mathcal{H}} \setminus \mathfrak{p}^0_{\mathcal{H}min}$, the set C_a is a (possibly empty) union of generalized cylinders of $X_{\mathcal{G}}$;

ii. if $\Phi(\alpha_1 \dots \alpha_k AA \dots) = (\beta_1 \dots \beta_l BB \dots)$, for some $(\alpha_1 \dots \alpha_k AA \dots) \in X_{\mathcal{G}}^{fin}$ (in particular $l \leq k$ by Lemma 3.1), then for every neighbourhood $D_{B,F}$ of B there exists a cylinder $D_{(\sigma^l(\alpha_1 \dots \alpha_k), A), F'}$ such that $\Phi(D_{(\sigma^l(\alpha_1 \dots \alpha_k), A), F'}) \subseteq D_{B,F}$;

iii. if $|\Phi(AA \dots)| > 0$ for some $(AA \dots) \in X_{\mathcal{G}}^0$, say $\Phi(AA \dots) = (ddd \dots)$, then for all $M > 0$ there exists a cylinder $D_{A,F}$ such that $\sigma^i(D_{A,F}) \subseteq C_d$ for all $i = 0, 1, \dots, M$.

Under the additional hypothesis that Φ is continuous on $X_{\mathcal{G}}^{inf} \cap \Phi^{-1}(X_{\mathcal{H}}^0)$ the converse of the statement above also holds.

Proof. Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a continuous and shift commuting map.

From Proposition 3 we have that, for all $x \in X_{\mathcal{G}}$ and $n \geq 1$,

$$\big(\Phi(x)\big)_n = \sum_{a \in \mathbf{A}_{\mathcal{H}}} a \mathbf{1}_{C_a} \circ \sigma^{n-1}(x),$$

where for each $e \in \mathcal{H}^1$ we have $C_e := \Phi^{-1}(D_{(e, r(e))})$, and for each $B \in \mathfrak{p}^0_{\mathcal{H}min}$ we have $C_B = \Phi^{-1}(B)$.

Notice that $\{C_a\}_{a \in \mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$. Furthermore, notice that each C_e is clopen and, since the generalized cylinders in $X_{\mathcal{G}}$ form a countable basis, each C_e can be written as a countable union of generalized cylinder sets. Therefore Item i. is satisfied.

To check that Item ii. holds notice that if $\Phi(\alpha_1 \dots \alpha_k AA \dots) = \beta_1 \dots \beta_l BB \dots$ then, by the continuity of Φ, for every neighbourhood $D_{B,F}$ of B there exists a cylinder $D_{(\alpha_1\ldots\alpha_k,A),F'}$ such that $\Phi(D_{(\alpha_1\ldots\alpha_k,A),F'}) \subseteq D_{(\beta,B),F}$. Hence, since Φ is shift commuting, we get that $\Phi(D_{(\sigma^l(\alpha_1\ldots\alpha_k),A),F'}) \subseteq D_{B,F}$.

Next suppose that there exists $(AA\ldots) \in X^0_{\mathcal{G}}$ such that $|\Phi(AA\ldots)| > 0$, say $\Phi(AA\ldots) = (ddd\ldots)$. Notice that $|\Phi(AA\ldots)| = \infty$. Let $M > 0$ and $\alpha = (d\ldots d)$ be a block of length $M+1$. Then $(AA\ldots) \in \Phi^{-1}(D_{(\alpha,r(d))})$ and $\Phi^{-1}(D_{(\alpha,r(d))})$ is open. Therefore there exists a cylinder $D_{A,F} \subseteq \Phi^{-1}(D_{(\alpha,r(d))})$. Let $1 \leq i \leq M$ and $x \in D_{A,F}$. Then $\Phi(\sigma^i(x)) = \sigma^i(\Phi(x))$ and, since $\Phi(x) \in D_{(\alpha,r(d))}$, we get that $\sigma^i(x) \in C_d$ and Item iii. is proved.

Now suppose that Φ is continuous on $X^{inf}_{\mathcal{G}} \cap \Phi^{-1}(X^0_{\mathcal{H}})$. Under this condition we show the converse of the theorem.

Assume that Φ is given by $\big(\Phi(x)\big)_n = \sum_{a\in A_{\mathcal{G}_2}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$, and Items i. to iii. above are satisfied. By Proposition 3 we have that Φ is shift commuting. We prove that it is also continuous.

Notice that, by Proposition 7, Φ is continuous on $X^{inf}_{\mathcal{G}}$. Therefore we only need to show continuity on $X^{fin}_{\mathcal{G}}$.

Let (x^n) be a sequence in $X_{\mathcal{G}}$ that converges to $x \in X^{fin}_{\mathcal{G}}$. We divide the proof in two cases.

Case 1. If $|x| = 0$.

Then $x = (AA\ldots)$ for some $(AA\ldots) \in X^0_{\mathcal{G}}$. If $\Phi(AA\ldots) = (BB\ldots)$ for some $(BB\ldots) \in X^0_{\mathcal{H}}$ then, by Item ii., $\Phi(x^n)$ converges to $\Phi(x)$. Suppose that $\Phi(AA\ldots) = (ddd\ldots)$, with $|\Phi(AA\ldots)| = \infty$. Let $M \in \mathbb{N}$. By Item iii. there exists a cylinder $D_{A,F}$ such that $\sigma^i(D_{A,F}) \subseteq C_d$ for all $i = 0,1,\ldots,M-1$. Since x_n converges to A, there exists $N > 0$ such that $x_n \in D_{A,F}$ for every $n > N$. Therefore $\Phi(x_n)_i = d$ for all $i = 1,\ldots,M$ and hence $\Phi(x^n)$ converges to $\Phi(x)$.

Case 2. If $0 < |x| < \infty$, say $x = \alpha_1\ldots\alpha_k AA\ldots$.

By the description of converge of sequences in $X_{\mathcal{G}_1}$ we may assume, without loss of generality, that $|x^n| \geq k$ for all n.

Suppose that $\Phi(AA\ldots) = (B'B'\ldots)$, where $(B'B'\ldots) \in X^0_{\mathcal{H}}$. By Lemma 3.1 we have that $\Phi(x) = (\beta_1\ldots\beta_l BB\ldots)$, where $l \leq k$. Notice that $\Phi(AA\ldots) = \phi(\sigma^k(x)) = \sigma^k(\Phi(x)) = (BB\ldots)$ and hence $B = B'$. Fix a natural number j such that $1 \leq j \leq l$. Note that $\sigma^{j-1}(x) \in C_{\beta_j}$, and hence there is a generalized cylinder D_j such that $\sigma^{j-1}(x) \in D_j \subseteq C_{\beta_j}$. Since $(\sigma^{j-1}(x^n))$ converges to $\sigma^{j-1}(x)$, there exists an N_j such that, for all $n > N_j$, $\sigma^{j-1}(x^n) \in D_j$ and hence $\big(\Phi(x^n)\big)_j = \beta_j$ (so the j-entry of $\Phi(x^n)$ is β_j). Now, let $D_{(\beta_1\ldots\beta_l,B),F}$ be a generalized cylinder set containing $(\beta_1\ldots\beta_l, B)$. Then $D_{B,F}$ is a generalized cylinder set containing B. Pick a cylinder $D_{(\sigma^l(\alpha),A),F'}$ such that $\Phi(D_{(\sigma^l(\alpha),A),F'}) \subseteq D_{B,F}$ (from item ii. of hypothesis). By Proposition 2, we have that $(\sigma^l(x^n))$ converges to $(\sigma^l(\alpha)AA\ldots)$ and hence there exists N_{l+1} such that, for all $n > N_{l+1}$, $\sigma^l(x^n) \in D_{(\sigma^l(\alpha),A),F'}$. Taking N as the maximum among $N_1,\ldots N_{l+1}$, and using Lemma 3.2, we have that $\Phi(x^n) \in D_{(\beta_1\ldots\beta_l,B),F}$ for all $n > N$. Therefore $\Phi(x_n)$ converges to $\Phi(x)$.

Now suppose that $\Phi(AA\ldots) = (ddd\ldots)$, with $|\Phi(AA\ldots)| = \infty$ (so $d \in \mathcal{H}^1$). By Lemma 3.2 we have that $\Phi(x) = \beta_1\beta_2\ldots$, where $\beta_i \in \mathcal{H}^1$ for $i = 1..|x|$, and $\beta_i = d \in \mathcal{H}^1$ for $i > |x|$. Notice that $\sigma^{j-1}(x) \in C_{\beta_j}$ for each $j \in \mathbb{N}$, and hence, by Item i., there are generalized cylinders D_j such that $\sigma^{j-1}(x) \in D_j \subseteq C_{\beta_j}$ for all $j \leq |x|$. Since x^n converges to x we have that $\sigma^j(x_n)$ converges to $\sigma^j(x)$ for all $j \leq |x|$. Therefore we can find N_1 such that, for all $n > N_1$ and for all $j = 1,\ldots,|x|$, it holds that $\sigma^{j-1}(x^n) \in D_j$ and hence $\big(\Phi(x^n)\big)_j = \beta_j$. Let $M > |x|$. Take a cylinder

$D_{A,F}$ as in Item iii., that is, such that $\sigma^i(D_{A,F}) \subseteq C_d$ for all $i = 0, 1, \ldots, M - |x|$. Since $\sigma^{|x|}(x_n)$ converges to $\sigma^{|x|}(x) = (AA\ldots)$ we have that there exists $N > N_1$ such that, for all $n > N$, $\sigma^{|x|}(x_n) \in D_{A,F}$. Hence $\sigma^{i+|x|}(x_n) \in D_{A,F}$ for all $i = 0, 1, \ldots, M - |x|$ and therefore $\big(\Phi(x^n)\big)_j = \beta_j$ for $j = 1 \ldots M$. We conclude that $\Phi(x_n)$ converges to $\Phi(x)$. $\square$

Remark 5. When $X_{\mathcal{G}}^{inf} \cap \Phi^{-1}(X_{\mathcal{H}}^0)$ is empty, the above theorem is a complete characterization of shift commuting maps. This is the case of maps such that $\Phi(X_{\mathcal{G}}^{inf}) \subseteq X_{\mathcal{H}}^{inf}$ or that preserve length (when dealing with infinite alphabet shift spaces the hypothesis that shift commuting maps preserve length is common, see for example [10, 11, 12, 21]).

Next we connect shift commuting maps with generalized sliding block codes.

Theorem 3.8. *Let $X_{\mathcal{G}}$ and $X_{\mathcal{H}}$ be two ultragraph shift spaces. Suppose that $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is a map such that for each $B \in \mathfrak{p}^0_{\mathcal{H}min}$ the set $C_B := \Phi^{-1}(BBB\ldots)$ is a finitely defined set. Then Φ is continuous and shift commuting if, and only if, Φ is a generalized sliding block code given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$ where:*

i. For any $a \in \mathcal{H}^1$, the set C_a is a (possibly empty) union of generalized cylinders of $X_{\mathcal{G}}$;

ii. If $(\bar{x}_1 \ldots \bar{x}_{|\bar{x}|}AAA\ldots) \in X_{\mathcal{G}}^{fin}$ is such that $\Phi(\bar{x}_1 \ldots \bar{x}_{|\bar{x}|}AAA\ldots) = (BBB\ldots) \in X_{\mathcal{H}}^0$, then:

a. There exists a finite subset $F \subseteq \varepsilon(A)$ such that, for all $e \in \varepsilon(A)\setminus F$, if $x \in X_{\mathcal{G}}$ satisfies $x_i = \bar{x}_i$ for all $i = 1, \ldots, |\bar{x}|$, and $x_{|\bar{x}|+1} = e$, then $\big(\Phi(x)\big)_1 = B$ or $\big(\Phi(x)\big)_1 \in \varepsilon(B)$, i.e., $\Phi(x) \in D_B$;

b. For all $x \in X_{\mathcal{G}}$ with $x_i = \bar{x}_i$ for $i = 1, \ldots, |\bar{x}|$, $x_{|\bar{x}|+1} \in \varepsilon(A)$, and $\big(\Phi(x)\big)_1 \in \varepsilon(B)$, the set
$A_x := \{g \in \varepsilon(A) :$ there exists $y \in X_{\mathcal{G}}$ with $y_i = \bar{x}_i$ for $i = 1, \ldots, |\bar{x}|$, $y_{|\bar{x}|+1} = g$, and $\big(\Phi(y)\big)_1 = \big(\Phi(x)\big)_1\}$
is finite;

iii. If $(AAA\ldots) \in X_{\mathcal{G}}^0$ is such that $\Phi(AAA\ldots) = (ddd\ldots) \in X_{\mathcal{H}}^{inf}$, then for all $M \geq 1$ there exists a cylinder $D_{A,F}$ such that $\sigma^i(D_{A,F}) \subseteq C_d$ for all $i = 0, 1, \ldots, M$.

Proof. Let $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ be a map such that, for all $B \in \mathfrak{p}^0_{\mathcal{H}min}$, the set $C_B := \Phi^{-1}(BBB\ldots)$ is a finitely defined set. Then, by Lemma 3.6, Φ is continuous on $X_{\mathcal{G}}^{inf} \cap \Phi^{-1}(\mathfrak{p}^0_{\mathcal{H}min})$ and hence both the forward implication and the converse of Theorem 3.7 are valid.

Suppose first that Φ is continuous and shift commuting. By Theorem 3.7, Φ is given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$, where $\{C_a\}_{a\in\mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$, and Items i. and iii. above are satisfied. We need to check that Φ is a generalized sliding block code and Item ii. above holds.

Notice that, for all $a \in \mathbf{A}_{\mathcal{H}}$, the sets C_a and $C_a^c = \bigcup_{b\in\mathbf{A}_{\mathcal{H}}\setminus\{a\}} C_b$ are unions of pseudo cylinders, which means that each C_a is a finitely defined set. Hence Φ is a generalized sliding block code.

Next we check Item ii.. Suppose that $\Phi(\bar{x}_1 \ldots \bar{x}_{|\bar{x}|}AAA\ldots) = (BBB\ldots) \in X_{\mathcal{H}}^0$. Consider the cylinder D_B. By Theorem 3.7 (Item ii.), there exists a cylinder $D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F}$ such that $\Phi(D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F}) \subseteq D_B$. Then the finite set F is such that Item ii.a. is satisfied. To check Item ii.b., let $x \in X_{\mathcal{G}}$ be such that $x_i = \bar{x}_i$ for $i = 1, \ldots, |\bar{x}|$, $x_{|\bar{x}|+1} \in \varepsilon(A)$, and $\big(\Phi(x)\big)_1 \in \varepsilon(B)$. Let $F = \{(\Phi(x))_1\}$.

Then, by Theorem 3.7 (Item ii.), there exists a cylinder $D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F'}$ such that $\Phi(D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F'}) \subseteq D_{B,F}$. Hence $A_x \subseteq F'$.

For the converse, suppose that Φ is a generalized sliding block code given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$ satisfying Items i., ii., and iii. above. All we need to do is verify Item ii. in Theorem 3.7.

Suppose that $\Phi(\alpha_1\ldots\alpha_k AA\ldots) = \beta_1\ldots\beta_l BB\ldots$, for some $\alpha_1\ldots\alpha_k AA\ldots \in X_{\mathcal{G}}^{fin}$. By Lemma 3.1 we have $l \leq k$. Then $\Phi(\sigma^l(\alpha_1\ldots\alpha_k AA\ldots)) = BB\ldots$. Denote $\sigma^l(\alpha_1\ldots\alpha_k AA\ldots)$ by $\bar{x} := \bar{x}_1\ldots\bar{x}_{|\bar{x}|}AA\ldots$ (notice that $|\bar{x}|$ can be zero). Then $\Phi(\bar{x}_1\ldots\bar{x}_{|\bar{x}|}AA\ldots) = (BBB\ldots)$.

Suppose, by contradiction, that there exists a generalized cylinder $D_{B,F''}$ such that, for every generalized cylinder $D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F'}$, we have that $\Phi(D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F'})$ is not contained in $D_{B,F''}$.

Take F as in Item ii.a., so that $\Phi(D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F}) \subseteq D_B$. Let $x^1 \in D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F}$ be such that $\Phi(x^1) \notin D_{B,F''}$. Then $(\Phi(x^1))_1 \in F''$. Let $D_2 := D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F\cup\{(x^1)_{|\bar{x}|+1}\}}$, and $x^2 \in D_2$ be such that $\Phi(x^2) \notin D_{B,F''}$ (so that $(\Phi(x^2))_1 \in F''$). Let $D_3 := D_{(\bar{x}_1\ldots\bar{x}_{|\bar{x}|},A),F\cup\{(x^1)_{|\bar{x}|+1},(x^2)_{|\bar{x}|+1}\}}$, and $x^3 \in D_3$ be such that $(\Phi(x^3))_1 \in F''$. Proceed by induction to define x^n, for all $n \in \mathbb{N}$. Since F'' is finite, there exists $e \in F$ and, a subsequence (x^{n_k}), such that $(\Phi(x^{n_k}))_1 = e$ for all k. Since the elements of (x^n) are distinct this implies that $A_{x^{n_1}}$ is infinite, a contradiction. Hence Item ii. in Theorem 3.7 is verified and it follows that Φ is continuous and shift commuting. $\square$

As we mentioned before, when dealing with infinite alphabet shift spaces it is common to require that a continuous shift commuting map $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ preserves length. The next corollary characterizes continuous, shift commuting, length-preserving maps.

Corollary 4. *A map $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ is continuous, shift commuting, and preserves length, if and only if it is a generalized sliding block code given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$ where:*

i. For each $a \in \mathbf{A}_{\mathcal{H}} \setminus \mathfrak{p}^0_{\mathcal{H}min}$, the set C_a is a (possibly empty) union of generalized cylinders of $X_{\mathcal{G}}$;

ii. $\displaystyle\bigcup_{B\in\mathfrak{p}^0_{\mathcal{H}min}} C_B = \mathfrak{p}^0_{\mathcal{G}min}$;

iii. If $\Phi(AAA\ldots) = (BBB\ldots) \in X^0_{\mathcal{H}}$ then:

a. There exists a finite subset $F \subseteq \varepsilon(A)$ such that, for all $e \in \varepsilon(A)\setminus F$, if $x \in X_{\mathcal{G}}$ and $x_1 = e$, then $\big(\Phi(x)\big)_1 = B$ or $\big(\Phi(x)\big)_1 \in \varepsilon(B)$, i.e., $\Phi(x) \in D_B$;

b. For all $x \in X_{\mathcal{G}}$ with $x_1 \in \varepsilon(A)$, and $\big(\Phi(x)\big)_1 \in \varepsilon(B)$, the set

$$A_x := \{g \in \varepsilon(A) : \text{ there exists } y \in X_{\mathcal{G}} \text{ with } y_1 = g, \text{ and } \big(\Phi(y)\big)_1 = \big(\Phi(x)\big)_1\}$$

is finite.

Proof. Suppose that Φ is continuous, shift commuting and length preserving. By Proposition 3 we have that Φ is given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_{\mathcal{H}}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$, where $\{C_a\}_{a\in\mathbf{A}_{\mathcal{H}}}$ is a partition of $X_{\mathcal{G}}$. Since Φ is length preserving Item ii. above is satisfied. Furthermore, for each $B \in \mathfrak{p}^0_{\mathcal{H}min}$, the set $C_B := \Phi^{-1}(BBB\ldots)$ is a countable union of elements of length zero in $X_{\mathcal{G}}$. By Example 1 we have that C_B is finitely defined. Items i. and iii. now follow from Theorem 3.8.

For the converse, let Φ be a generalized sliding block code such that Items i. to iii. above hold. Notice that Item ii. implies that Φ is length preserving and hence, for all $B \in \mathfrak{p}^0_{\mathcal{H}min}$, $\Phi^{-1}(B)$ is a finitely defined set. Now Items i. and iii. above imply that all conditions of Theorem 3.8 are satisfied and hence Φ is continuous and shift commuting. $\square$

We end the paper presenting some examples.

Example 2.

a) Let $\mathcal{G}$ be the graph with only one vertex, say $G^0 := \{w\}$, and edge set given by $\mathcal{G}^1 := \{d, f_1, f_2, \ldots\}$ (so all edges are loops). Let $\mathcal{H}$ be the graph with only one vertex, say $H^0 := \{v\}$, and edge set given by $\mathcal{H}^1 := \{e_1, e_2, \ldots\}$. It follows that the ultragraph shifts $X_\mathcal{G}$ and $X_\mathcal{H}$ have alphabets $\mathbf{A}_\mathcal{G} = \{A\} \cup \mathcal{G}^1$ with $A := G^0$ and $\mathbf{A}_\mathcal{H} = \{B\} \cup \mathcal{H}^1$ with $B := H^0$, respectively ($X_\mathcal{G}$ and $X_\mathcal{H}$ coincide with Ott-Tomforde-Willys full shifts).

Let $C_B = [A]^1_1 \cup \{(ddd\ldots)\}$ and, for all j, let $C_{e_j} = [f_j]^1_1 \cup [df_j]^2_1 \cup [ddf_j]^3_1 \cup [dddf_j]^4_1 \cup \ldots$. This partition of $X_\mathcal{G}$ defines a shift commuting map Φ given by $\big(\Phi(x)\big)_n = \sum_{a\in\mathbf{A}_\mathcal{H}} a\mathbf{1}_{C_a} \circ \sigma^{n-1}(x)$ which is not continuous (notice that $\Phi^{-1}(D_{B,\{e_1\}})$ is not open, since every open neighbourhood of $(ddd\ldots)$ contains elements of C_{e_1}). We remark that in this case C_B is not finitely defined.

b) Let $\mathcal{G}$ be the graph with only one vertex, say $G^0 := \{w\}$, and edge set given by $\mathcal{G}^1 := \{0\} \cup \mathbb{N}$. Let $X_\mathcal{G}$ be the correspondent ultragraph shift (which, as before, has alphabet $\mathbf{A}_\mathcal{G} = \{A\} \cup \mathcal{G}^1$ with $A := G^0$). Consider the map $\Phi : X_\mathcal{G} \to X_\mathcal{G}$ given, for all $x \in X_\mathcal{G}$ and $n \in \mathbb{N}$, by

$$\big(\Phi(x)\big)_n = \begin{cases} x_n & if \quad x_n \neq 0 \ and\ x_n \neq A, \\ A & if \quad x_n = A \ or\ x_{n+j} = 0\ \forall j \geq 0, \\ k & if \quad x_{n+j} = 0 \ for\ 0 \leq j \leq k,\ and\ x_{n+k+1} \neq 0. \end{cases}$$

We have that Φ is continuous and shift commuting, but it is not a generalized sliding block code, since $C_A = [A]^1_1 \cup \{(000\ldots)\}$ is not a finitely defined set.

c) In this example we consider again the ultragraph shifts of example a). From Theorem 3.8, a map $\Phi : X_\mathcal{G} \to X_\mathcal{H}$, where $\Phi^{-1}(BBB\ldots)$ is a finitely defined set, is continuous and shift commuting if and only if: either $\Phi(AAA\ldots) = (BBB\ldots)$ and for all $a \in \mathcal{H}^1$ the set C_a is a finite union of generalized cylinders; or $\Phi(AAA\ldots) = (e_je_je_j\ldots)$ for some $e_j \in \mathcal{H}^1$, there are just a finite number of nonempty sets C_a, and for all M there exists a finite $F_M \subset \mathbf{A}_\mathcal{G}$ such that $\sigma^{n-1}(D_{A,F_M}) \subset C_{e_j}$ for all $1 \leq n \leq M$.

Recall that $X_\mathcal{G}$ and $X_\mathcal{H}$ coincide with Ott-Tomforde-Willys full shifts, and therefore we can alternatively apply Theorems 3.16 and 3.17 of [13] to obtain the above result.

d) In this example we use $\mathbb{Z}^*$ to denote the set of all non-zero integers. Let $\mathcal{G}$ be the ultragraph with vertex set $G^0 := \{v_k : k \geq 0\}$, edge set $\mathcal{G}^1 := \{e_k : k \geq 0\}$, and the source $s_\mathcal{G} : \mathcal{G}^1 \to G^0$ and the range $r_\mathcal{G} : \mathcal{G}^1 \to P(G^0) \setminus \{\emptyset\}$ given by

$$s_\mathcal{G}(e_k) := v_k, \qquad \forall k \geq 0,$$

and

$$r_\mathcal{G}(e_k) := \begin{cases} \{v_\ell : \ell \geq 0\} & if \quad k = 0, \\ & \\ \{v_0,\ v_k\} & if \quad k \geq 1. \end{cases}$$

Note that the unique minimal infinite emitter of $\mathcal{G}$ is the set $A := G^0$.

Let $\mathcal{H}$ be the ultragraph with vertex set $H^0 := \{w_k : k \in \mathbb{Z}^*\}$, edge set $\mathcal{H}^1 := \{f_k : k \in \mathbb{Z}^*\}$, and source $s_{\mathcal{H}} : \mathcal{H}^1 \to H^0$ and range $r_{\mathcal{H}} : \mathcal{H}^1 \to P(H^0) \setminus \{\emptyset\}$ maps given by

$$s_{\mathcal{H}}(f_k) := w_k \qquad \forall k \in \mathbb{Z}^*,$$

and

$$r_{\mathcal{H}}(f_k) := \begin{cases} \{w_{k+1}\} & if \quad k \leq -2, \\ \\ \{w_\ell : \ell \geq 1\} & if \quad k = -1, \\ \\ \{w_k\} \cup \{w_\ell : \ell \leq -1\} & if \quad k \geq 1. \end{cases}$$

We notice that the minimal infinite emitters of $\mathcal{H}$ are the sets $P := \{w_\ell : \ell \leq -1\}$ and $Q := \{w_\ell : \ell \geq 1\}$.

Now consider the map $\Phi : X_{\mathcal{G}} \to X_{\mathcal{H}}$ given, for all $x \in X_{\mathcal{G}}$ and $n \geq 1$, by

$$\big(\Phi(x)\big)_n = \begin{cases} P & if \quad x_{n+j} = e_0 \ \forall j \geq 0, \\ \\ f_{-k} & if \quad x_{n+j} = e_0, \ 0 \leq j \leq k-1, \text{ and } x_{n+k} \neq e_0, \\ \\ f_k & if \quad x_n = e_k \ for \ k \neq 0, \\ \\ Q & if \quad x_n = A. \end{cases}$$

It follows that Φ is an invertible continuous and shift commuting map, but it is not a generalized sliding block code (since $C_P := \{(e_0e_0e_0\ldots)\}$). On the other hand, Φ^{-1} is a generalized sliding block code.

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