KLEIN'S MODULAR FUNCTIONS.

FELIX KLEIN, *Vorlesungen über die Theorie der elliptischen Modulfunctionen*, ausgearbeitet und vervollständigt von Dr. ROBERT FRICKE. Erster Band. Grundlegung der Theorie. Leipzig, Teubner, 1890. 8vo, pp. xix + 764.

THE mathematical public is under great obligation to Professor Klein's former pupil, Dr. Robert Fricke, for his able presentation of the theory of the modular functions. His clearness of treatment and skillful grouping of the many intricate features of the subject have rendered this theory now thoroughly accessible. Beside the work of arrangement, in itself a labor of no small magnitude, Dr. Fricke has contributed many of the intermediate steps necessary to the symmetry and completeness of the subject. His task has been performed throughout with a highly creditable degree of conscientiousness and ability.

The theory of the modular functions and the allied branches has been one of the chief series of investigations to which Professor Klein has devoted himself in the period of some twenty years over which his scientific activity now extends. It is characteristic of these investigations that they are not included as a subordinate part in any of the great mathematical theories heretofore commonly so recognized. Their distinctive tendency is in the direction of the combination and unification of the latter into a broader method of research. This idea has been developed by Klein to an extent and with an elaboration which have long since entitled it to recognition as an independent, and in the highest degree productive mathematical point of view. In the present paper some attempt is made to sketch the general outlines of the new method, so far as it concerns the modular functions, and to illustrate it more definitely by the consideration of some of the more important details.

Historically, Klein's work has developed accurately along the lines of a thoroughly predigested plan, the bolder features of which are already sharply defined in his earliest publications.* On this ground, then, a brief semi-biographical, semi-scientific sketch of his career may properly find place here. It is to be observed that this sketch makes no pretension to completeness. It confines itself mainly on the scientific side to the development of the theory of the regular bodies and of the modular functions.

Klein's first productive activity dates from his relation to

* Cf. the preface to the "*Ikosaeder*," and the *Eintrittsprogramm* mentioned on the following page.

Julius Plücker, as the latter's assistant in physics at Bonn. On Plücker's death the preparation of the second volume of his posthumous work on line geometry * was entrusted to Klein, who was then at the age of nineteen. The first volume was edited by Clebsch. Having completed this task, and having taken the doctor's degree at Bonn, Klein studied in Berlin and in Paris until the outbreak of the Franco-German war, which compelled his return to Germany. Soon afterward he was appointed *Privat-Docent* at Göttingen, where Clebsch was approaching the close of his brilliant career. In 1872 he was called to the *ordinarius* professorship of mathematics at Erlangen. His *Eintrittsprogramm* † prepared on the occasion of assuming this chair is certainly a most remarkable production for a young man of twenty-three, containing, as it does, not merely a foreshadowing, but actually a systematic program, conceived with perfect maturity and definiteness, of the scientific work to which he has since devoted himself. It is with the theory of operations that he is here concerned; not the formal theory of operations in themselves, but entirely with reference to the *content* to which the operations are conceived to be applied, in particular when this content is a geometrical configuration. Two such theories were already in existence: the theory of invariants and covariants, which deals with the effect of the entire system of linear transformations of two or more homogeneous variables, and the theory of substitutions, in which the operations are the permutations of a finite system of elements. These two theories can be regarded as extreme types, between which an infinite series of others can be inserted. A definite complex of these intermediate types has furnished the field to which Klein's labors have thus far mainly been devoted. The *Eintrittsprogramm* appears as a preliminary survey of the general doctrine of operations, with reference to geometrical configurations. It involves not only the discontinuous operations, within which Klein's specific work has been included, but also the continuous systems, which belong with differential equations, and the theory of which has been mainly developed by his friend and fellow-student, Professor Sophus Lie.

In Erlangen Klein formed the acquaintance of Gordan, to whose personal friendship and scientific cooperation a high tribute is paid in the preface to the *Ikosaeder*. It was here and at Munich, to which city Klein was called in 1875, that

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* *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement,* von Dr. JULIUS PLÜCKER. Leipzig, Teubner, 1869.

† *Vergleichende Betrachtungen über neuere geometrische Forschungen.* von Dr. FELIX KLEIN, o. ö Professor der Mathematik an der Universität Erlangen.

the theory of the icosahedron and the other regular bodies was gradually developed in a series of papers in the *Mathematische Annalen,* of which Klein had become an editor in 1873. It may be noted that nearly all of his writings are published in this journal, which is, indeed, distinctively the organ of the school of which its editor is the leader.

In 1881 Klein was called from Munich to Leipzig, where he remained until 1886, when he was appointed to the chair in Göttingen, vacated by the death of Enneper, which he still holds. At Munich he numbered among his students Hurwitz, Rohn, and Dyck, all of whom have made a name among mathematicians. In Leipzig the first American students were admitted to his *Seminar,* Messrs. Irving Stringham and Henry B. Fine, now professors of mathematics at the University of California and at Princeton, the precursors of a numerous throng, among whom the writer had the good fortune to be included. During the stay in Leipzig negotiations were at one time pending toward inviting Klein to Sylvester's vacated chair at Johns Hopkins. Various considerations, relating mainly to his health, never very robust, led him to decide in favor of remaining in Germany. A circumstance which must have contributed greatly to his decision was the fact that he had already gathered about him a band of talented and mature young mathematicians the direction of whose vigorous development was a most gratifying task. Among the members of his *Seminar* in 1884–5 were Pick now at Prague, Hölder now at Göttingen, Study at Marburg, and Fricke the editor of the *Modulfunctionen.*

The management of the *Seminar* has always been exceptionally efficient, even among the German models. It is Klein's custom to distribute among his students certain portions of the broader field in which he himself is engaged, to be investigated thoroughly under his personal guidance and to be presented in final shape at one of the weekly meetings. An appointment to this work means the closest scientific intimacy with Klein, a daily or even more frequent conference, in which the student receives generously the benefit of the scholar's broad experience and fertility of resource, and is spurred and urged on with unrelenting energy to the full measure of his powers. When the several papers have been presented, the result is a symmetric theory to which each investigation has contributed its part. Each member of the *Seminar* profits by the others' points of view. It is a united attack from many sides on the same field. In this way a strong community of interest is maintained in the *Seminar,* in addition to the pleasure afforded by genuine creative work.

The theory of the icosahedron appeared in book form in

1884.* The investigations included under this title traverse a definite, self-limited field, identified on the one hand with the groups of rotations of the regular bodies and the corresponding finite groups of linear transformations of a complex variable, and on the other with the theory of the algebraic equations of the first five degrees and certain other special types. The close relation of the subject to the theory of the modular functions is also so far touched upon as to indicate the direction of the more extended theory which has culminated in Dr. Fricke's book. From the point of view of this relation the *Ikosaeder* appears as a first step in the systematic treatment of the modular functions, for which it is also to serve as a model. In the meantime Klein's lectures were forecasting the coming theory, already developed in many of its features in articles in the *Annalen.* Beginning with the general theory of functions, he treated in successive semesters the elliptic functions, the elliptic modular functions, the new geometry, the hyperelliptic functions of deficiency 2, and the Plücker line geometry. Of these lectures the first three, together with the later lectures on algebra at Göttingen, relate largely to the present theory, while the others were to a considerable extent preliminary to the theory of the general equations of the sixth and seventh degrees.

In all these investigations it is again the theory of operations that furnishes the guiding principle and the general outline of the subject. But the field of research being once mapped out, it is characteristic of Klein to bring to bear on it every instrument that modern mathematics can provide. The theory of functions, invariants and covariants, differential equations, modern geometry, in short every method is put under requisition and made to render its contribution to the symmetry of the result. No advantageous point of view is neglected, and not until the subject is traversed in every direction, and until its external and internal relations are clearly pictured to the mind, is the investigation to be regarded as complete. For example the *Ikosaeder* discusses in successive chapters the rotations of the regular bodies, the corresponding groups of linear transformations and their invariants, the actual solution of the problem by the aid of a class of differential equations, the algebraic phases of the subject, based on the theory of substitutions, the general position of the theory in reference to other correlated fields, its historical development, the coordinated geometrical problems, etc. The same plan obtains in the lectures and the *Seminar.*

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* F. KLEIN, *Vorlesungen über das Ikosaeder und die Auflösung der Gleichungen vom fünften Grade.* Leipzig, Teubner, 1884.
English translation by G. C. MORRICE. London, Trübner, 1888.

The value to the student of this breadth of treatment is simply inestimable. No one can study long under Klein without obtaining an intelligent comprehension of most of the great tendencies in mathematics. This is at present particularly desirable in view of the extreme degree of specialization which has come to prevail among mathematicians. Of all the great services which Klein has rendered to mathematics, there is none more valuable than his successful unification of its heretofore rapidly diverging branches. Lately he has turned his attention to mechanics, in which new field, under the application of the same principle, we have every reason to expect from him another series of brilliant results.

Turning to the *Modulfunctionen,* one cannot but admire the simplicity and perfect proportion with which Dr. Fricke has developed the subject. So far as possible, everything is traced from first principles. The network of interwoven theories is constructed with a painstaking elaboration of details and a rare geniality of method. A clearer and more scholarly presentation than that before us could hardly be imagined.

The work divides into three principal investigations : (I.) the theory of the modular functions in the narrower sense as a specific class of elliptic transcendents ; (II.) the formal definition of the general problem, as based on the doctrine of groups of operations ; (III.) the union of these two methods and the further development of the subject in connection with a class of Riemann's surfaces.

We turn our attention for the present to the second division of the subject. The operations with which we have to deal belong to that fertile field of modern mathematics, the linear transformations. The characteristic system on which the theory of the modular functions turns is composed of all the linear transformations of a complex variable

$$(1) \qquad z' = \frac{\alpha z + \beta}{\gamma z + \delta},$$

or, in homogeneous form, of the binary linear transformations

$$(2) \qquad \begin{aligned} z'_1 &= \alpha z_1 + \beta z_2, \\ z'_2 &= \gamma z_1 + \delta z_2, \end{aligned}$$

for which the constants α, β, γ, δ are real integers, subject to the further condition that

$$\alpha\delta - \beta\gamma = +1.$$

The equations (1) and (2), and other similar types, must be regarded throughout, as already indicated, as defining *operations;* namely, the operation of passing in each case from the initial values z to the transformed values z'. Restricting ourselves for the present to the general linear transformations of a single complex variable, $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$, if the values of z are represented in the ordinary manner by points in the complex plane, the transformation is to be conceived as carrying every point z to the corresponding position z'. The result of the transformation is therefore to effect a rearrangement of the position of the points of the plane, and this geometric conception, to be presently more fully developed, not only serves to picture the corresponding analytic formula, but may often with great advantage entirely replace it.

If now any transformation $z' = \frac{\alpha z + \beta}{\gamma z + \delta}$ is followed by a second $z'' = \frac{\alpha_1 z' + \beta_1}{\gamma_1 z' + \delta_1}$, the relation of the points z'' to the original points z is directly defined by the equation

$$(3) \qquad z'' = \frac{\alpha_1\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) + \beta_1}{\gamma_1\left(\frac{\alpha z + \beta}{\gamma z + \delta}\right) + \delta_1} = \frac{(\alpha_1\alpha + \beta_1\gamma)z + \alpha_1\beta + \beta_1\delta}{(\gamma_1\alpha + \delta_1\gamma)z + \gamma_1\beta + \delta_1\delta} = \frac{\alpha_2 z + \beta_2}{\gamma_2 z + \delta_2},$$

which is again linear. The combination, or "product," of two linear transformations of a complex variable is therefore itself a linear transformation of a complex variable. The total system of these transformations accordingly forms a "group," this name being applied to any system of operations of whatever kind such that the product of any two of them is itself an operation of the system. If, furthermore, we confine our consideration to those transformations for which α, β, γ, δ are real integers, this more limited system is clearly still a group, which in reference to the including general group just considered is designated as a "subgroup" of the latter. Again we obtain a subgroup of this subgroup by selecting from the latter all those operations for which the determinant $\alpha\delta - \beta\gamma = +1$. For, on referring to (3), we have at once for the product of any two of these operations

$$(4) \quad \alpha_2\delta_2 - \beta_2\gamma_2 = (\alpha_1\alpha + \beta_1\gamma)(\gamma_1\beta + \delta_1\delta) - (\alpha_1\beta + \beta_1\delta)(\gamma_1\alpha + \delta_1\gamma) = (\alpha_1\delta_1 - \beta_1\gamma_1)(\alpha\delta - \beta\gamma) = +1.$$

By way of contrast we may observe that those transformations with real integral coefficients for which

$$\alpha\delta - \beta\gamma = -1$$

do not form a group, since the product of any two of them has for its determinant $(-1)(-1) = +1$. If however we combine the two systems $\alpha\delta - \beta\gamma = \pm 1$, the result is again a group.

The group composed of the transformations (1) or (2) is called simply the modular group, and is denoted by Γ. The two forms (1) and (2) are distinguished as the non-homogeneous and the homogeneous groups Γ respectively. We note that under the condition $\alpha\delta - \beta\gamma = +1$, a simultaneous change of sign of all the coefficients is admissible, but that the coefficients cannot otherwise be multiplied by a common factor. The change of sign is of no effect on the form (1), but alters the form (2). It appears therefore that the operations of (2) are precisely twice as numerous as those of (1).

The modular group has itself a great variety of subgroups, and it is precisely the theory of these subgroups which determines the formal character of the entire theory of the modular functions. The problem of establishing all these subgroups presents extreme difficulties and is not yet solved. Much is to be hoped, however, from the powerful general method of attacking the subject, devised by Klein and based on the theory of Riemann's surfaces.* The known subgroups are, with a few elementary exceptions, the "congruence groups," and their theory is exhaustively developed in the *Modulfunctionen*. These groups are defined by the additional condition that

$$(5) \qquad \alpha \equiv \delta \equiv \pm 1, \quad \beta \equiv \gamma \equiv 0, \quad (\text{mod. } n),$$

where n is any integer.† Under this condition we have, referring again to (3),

$$\begin{matrix} \alpha_1\alpha + \beta_1\gamma \equiv \gamma_1\beta + \delta_1\delta \equiv \pm 1, \\ \alpha_1\beta + \beta_1\delta \equiv \gamma_1\alpha + \delta_1\gamma \equiv 0, \end{matrix} \qquad (\text{mod. } n),$$

from which the group character is verified. In the case of the non-homogeneous transformations it is plainly sufficient to employ only the upper algebraic sign of ± 1. It is also a fact of great interest that those substitutions of the homoge-

* *Modulfunctionen*, II., 5.

† More correctly, this is the definition of the *Hauptcongruenzgruppen*. For other cases *cf. Modulfunctionen*, II., 7, § 6.

neous congruence group for which the upper sign holds form a subgroup of the latter, which therefore agrees operation for operation with the non-homogeneous group.

Of the various characteristics of a group its *order, i.e.* the number of operations which it contains, is of prime importance. In the present case both the modular group and the congruence subgroups are of infinite order, and the question therefore presents itself here in a modified form, *viz.* it requires the determination of the ratio of the order of the entire group to that of the respective subgroups. This ratio is termed the "index" of the subgroup and for the modular n is denoted by $\mu(n)$, the corresponding subgroup being designated by $\Gamma_{\mu(n)}$. The value of the function $\mu(n)$ is deducible from purely arithmetical considerations. If we define as congruent (mod. n) all those transformations for which either of the relations hold

$$(6)\quad \begin{matrix} \alpha' \equiv \alpha, & \beta' \equiv \beta, & \gamma' \equiv \gamma, & \delta' \equiv \delta, \\ \alpha' \equiv -\alpha, & \beta' \equiv -\beta, & \gamma' \equiv -\gamma, & \delta' \equiv -\delta, \end{matrix} \quad (\text{mod. } n),$$

the value of $\mu(n)$ is equal to the number of incongruent (mod. n) systems of solutions of

$$\alpha\delta - \beta\gamma = +1.$$

If n is a prime number p, it is readily found that

$$\mu(n) = p\,\frac{(p^2-1)}{2}.$$

For a compound $n = q_1^{\nu_1} \cdot q_2^{\nu_2} \cdot q_3^{\nu_3} \ldots$ the calculation is more complicated. The result is found to be*

$$(7)\qquad \mu(n) = \frac{n^3}{2}\left(1-\frac{1}{q_1^2}\right)\left(1-\frac{1}{q_2^2}\right)\left(1-\frac{1}{q_3^2}\right)\ldots$$

Leaving the specific theory of the modular group at this point for the moment, we have next to consider the position of the present investigations relatively to the general theory of linear transformation.† If we regard n elements z_1, z_2, $\ldots$ z_n as coordinates in an $(n-1)$–dimensional space or manifoldness, the projective geometry of this space is identical on the formal side with the theory of the general group of linear transformations

* *Modulfunctionen*, II., 7, § 4.
† *Cf. Ikosaeder*. Chap. V.

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(8) \qquad \begin{aligned}
z'_1 &= a_{11} z_1 + a_{12} z_2 + \dots + a_{1n} z_n, \\
z'_2 &= a_{21} z_1 + a_{22} z_2 + \dots + a_{2n} z_n, \\
&\quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot , \\
z'_n &= a_{n1} z_1 + a_{n2} z_2 + \dots + a_{nn} z_n.
\end{aligned}
$$

It is a principal problem of this theory to determine the full system of invariant, covariant, and other concomitant forms belonging to any configuration of the space, defined by any given set of equations

$$
\begin{aligned}
f_1 (z_1, z_2, \dots z_n) &= 0, \\
f_2 (z_1, z_2, \dots z_n) &= 0, \\
\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot &\quad \cdot , \\
f_n (z_1, z_2, \dots z_n) &= 0.
\end{aligned}
$$

Prominence is also given to the determination of the identities which may exist among these concomitants. The theory of the subgroups of the general group of transformations is, however, not usually considered.

On the other hand, the theory of substitutions deals with the permutations (substitutions) of n given elements $z_1, z_2, \dots z_n$. Such a substitution is commonly and most conveniently written in the cycle notation

$$(z_a \, z_b \, z_c \dots) \; (z_\alpha \, z_\beta \, z_\gamma \dots) \dots$$

the effect of the substitution being precisely to permute the elements of each parenthesis cyclically. Written, however, in the form

$$z'_1 = z_{i_1}, \quad z'_2 = z_{i_2}, \; \dots \; z'_n = z_{i_n},$$

(where the subscripts $i_1, i_2, \dots i_n$ are identical, apart from their order, with $1, 2, \dots n$), the substitution is obviously interpretable as a collineation of an $(n-1)$-dimensional space. From this point of view, we may regard the theory of substitutions as a special field within a general projective geometry; in other words, the groups of substitutions of n elements may be considered as subgroups of the general group of linear transformations of n coordinates. An important characteristic of the substitution groups is the fact that they are all of finite order, the latter being, in fact, always a divisor of $n\,!$ On the other hand, every substitution group, like the general linear group, possesses a system of invariants. These are the "functions belonging to the group," *i.e.* such rational integral functions of the n elements as are unchanged in value by all and by only the substitutions of the group. The invariants belonging to any substitution group G are all rational integral functions of any arbitrary one among them, with coefficients which are symmetrical in the n elements z. Every such group possesses, therefore, only a single independent invariant.

Again suppose H to be any subgroup of G with an invariant ψ. If all the substitutions of G are applied to ψ, the latter will take a series of values ψ_1 $(= \psi)$, ψ_2, ψ_3, $\ldots$ ψ_k, their number k being the (always integral) ratio of the order of G to that of H. To every one of these values belongs a group of the same order as H and similar to H. These k groups are the "conjugates" of H with respect to G. In special cases they may all coincide; H is then a "self-conjugate" group (*ausgezeichnete Untergruppe*), and every ψ is a rational function of every other. A case of especial importance is that for which H reduces to the identical operation alone. H is then obviously self-conjugate, since it is the only group of order 1. In the general case, if we apply the substitution of G to the k values ψ, the effect is simply a permutation of these values. In the particular case where $H = 1$ we have then on the one hand a group of substitutions of the ψ's of the same order as G, and on the other, as an algebraic equivalent, an equal number of rational processes by which the ψ's proceed from one another. These processes also form a group. By a proper choice of ψ, it happens in certain cases that these rational relations become linear. The group G, which originally represented a system of collineations in an $(n-1)$-dimensional space, appears then under a new form as identified with an equal group of linear transformations in the complex plane. Such a reduction in the dimension (as it may be called) of the group is obviously a step of the greatest importance. In the *Ikosaeder* and the *Modulfunctionen* this problem is reversed, the groups being taken at the start in their reduced form.*

Within the general theory of groups of linear transformations the projective geometry and the theory of substitutions appear as special cases, which possess the advantage of historical precedence and a correspondingly high degree of elaboration. To these Klein has now added in the *Modulfunctionen* a third system of certainly comparable interest and importance. There still remain an unlimited number of other special types which will undoubtedly in the future furnish one of the most fertile fields of mathematical research. The general problem has hardly yet been touched upon. In systematic form it requires the determination of all the binary, then the ternary, quaternary, and higher groups. In each dimension the groups of finite order naturally attract immediate attention. The

* *Cf. Ikosaeder*, I., 4, §§ 3–4, and I., 5, § 5; also *Math. Ann.* XV.

researches of Poincaré, Jordan, and Klein have shown that the number of finite groups is surprisingly small, and this field is accordingly narrowly limited. The problem of the finite binary groups is completely solved in the *Ikosaeder*. Of the finite ternary groups which are not reducible to binary forms, one of order 432 belongs with the theory of the points of inflection of the plane cubic, and has been repeatedly discussed from this point of view. Another of order 168 is treated in the *Modulfunctionen*.* The Borchardt quaternary group belongs with the theory of the general equations of the sixth and seventh degrees, as the icosahedron group does with that of the fifth degree.† On the other hand, of the infinite binary groups, which naturally succeed the finite cases, the simplest instance is precisely the present modular group.

Returning to the modular group in particular, we can now, from analogy with the theory of substitutions, state briefly the nature of the problem involved. It requires the determination of all the subgroups of the modular group and of the corresponding invariants, together with the systematic examination of the functional relations between their invariants, particularly when these relations are algebraic. An important distinction from the theory of substitutions lies in the fact that, as the groups under consideration are for the most part of infinite order, their invariants are no longer rational, but belong to the family of integral transcendental functions with linear transformations into themselves. It must also be noted here that the *homogeneous* groups here considered have in every case not one invariant, but three, which are then connected by an identity, precisely as in the *Ikosaeder*. In regard to the entire theory we observe further that from Klein's point of view it is not to be regarded as an isolated subject, but is to be connected as closely as possible with other mathematical fields. It is to be examined from every possible side, and, in particular, it is to be made tangible by the aid of geometric representation.

At the outstart the theory of the elliptic functions is put under requisition. To obtain the invariant of the modular group the direct construction is not necessary. Such an invariant is already at hand. It is known that the periods of the elliptic integral of the first species, which we write throughout in the Weierstrass form

$$u = \int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}},$$

* *Modulfunctionen*, III., 7.

† KLEIN, *Math. Ann.* XXVIII., *Zur Theorie der allgemeinen Gleichungen sechsten und siebenten Grades.*

are all linear combinations of two among them, ω_1 and ω_2,

$$\omega'_1 = \alpha\omega_1 + \beta\omega_2,$$
$$\omega'_2 = \gamma\omega_1 + \delta\omega_2.$$

Again ω_1 and ω_2 can be expressed in the same way in terms of ω'_1 and ω'_2 if and only if $\alpha\delta - \beta\gamma = \pm 1$. Accordingly all the systems

$$\begin{matrix} \alpha\omega_1 + \beta\omega_2 \\ \gamma\omega_1 + \delta\omega_2 \end{matrix} \qquad (\alpha\delta - \beta\gamma = \pm 1)$$

furnish primitive period pairs. If we consider the ratio of such a pair, all such ratios are expressed in terms of any one of them by

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta} \qquad \left(\alpha\delta - \beta\gamma = \pm 1,\ \omega = \frac{\omega_1}{\omega_2},\ \omega = \frac{\omega'_1}{\omega'_2}\right).$$

On the other hand, we have at once in the three invariants g_2, g_3, and the discriminant $\Delta = g_2^3 - 27g_3^2$ of the binary biquadratic form $4z_1^3z_2 - g_2z_1z_2^3 - g_3z_2^4$ the three invariants of the homogeneous modular group, while the absolute invariant $J = \frac{g_2^3}{\Delta}$ is the invariant of the non-homogeneous group. The periods ω_1, ω_2 and their ratio ω are also invariants of the biquadratic form, the latter being like J an absolute invariant. In reference to the modular group these quantities are again invariants belonging to the identical subgroup. Their calculation from g_2, g_3 and from J respectively are already furnished by the theory of the elliptic functions. Of the congruence subgroups (5) an invariant is also directly known in the case of modulus 2. This is the anharmonic ratio λ of the roots of the biquadratic form, or for the homogeneous group the three finite roots e_1, e_2, e_3 (where, in agreement with the general principle above stated, $e_1 + e_2 + e_3 = 0$). Under the operation of the modular group λ assumes the six familiar values

$$\lambda,\ \frac{1}{\lambda},\ 1 - \lambda,\ \frac{\lambda}{\lambda - 1},\ \frac{\lambda - 1}{\lambda},\ \frac{1}{1 - \lambda},$$

which furnish again a linear group. The latter is in fact a dihedron group. The six values of λ are connected with J by the equation

$$J : J - 1 : 1 = 4(\lambda^3 - \lambda + 1)^3 : (\lambda + 1)^2(\lambda - 2)^2(2\lambda - 1)^2 : 27\lambda^2(\lambda - 1)^2.$$

To obtain a comprehension of functional relations which in the present state of advancement can be regarded as in any way satisfactory, recourse must be had to Riemann's surfaces. Perhaps no more brilliant exemplification of the value of this geometric instrument exists than the theory of the modular functions, as Klein has created it.* Beginning with the period ratio ω, regarded as a function of J, we suppose the values of ω to be laid off in one complex plane and those of J in another. Since however to every value of J correspond an infinite number of primitive period pairs, and consequently an infinite number of values of ω, we must, in order to secure a one to one correspondence between the ω and the J points, suppose the J plane to consist of an infinite number of leaves which are connected in cycles about certain junctions (*Verzweigungspunkte*). In the present case, as in the *Ikosaeder*, these junctions are three in number and lie at $J = 0$, $J = 1$, and $J = \infty$. The leaves are joined at these points in cycles of three, two, and an infinite number respectively. If now we suppose the J point to pass along the upper side of the real axis from $-\infty$ to $+\infty$, the corresponding ω points describe in every case three circular arcs bounding a curvilinear triangle. To every upper half leaf of the J plane corresponds the interior of one of these triangles, in the sense that if the ω point takes successively every position in such a triangle, the corresponding J point will take successively every position in an upper half leaf. The ω triangles then produced fill just half of the upper half leaf of the ω plane. Between them lie an infinite number of empty spaces, of the same triangular form, and these new triangles correspond to the lower half leaves of the J plane. The triangles become infinitely small and are crowded infinitely close together as we approach the real axis, which is in fact a "natural boundary" beyond which the function $\omega(J)$ cannot be extended.

An immediate connection presents itself here with the theory of the modular group.† The effect of the latter on the systems of triangles is obviously merely to interchange the two sets corresponding to upper and lower half leaves each among themselves. Given any one of the triangles we can obtain every other belonging to the same system by applying to the former all the modular operations. We observe that it is only those operations

$$\omega' = \frac{\alpha\omega + \beta}{\gamma\omega + \delta}$$

for which $\alpha\delta - \beta\gamma = +1$ that are here admissible; those

* *Cf. Modulfunctionen*, II. † *Ibid.*, II., 2.

for which $\alpha\delta - \beta\gamma = -1$ simply convert the upper half leaf of the ω plane into the similarly divided lower half leaf. This again agrees with the fact that the product of two of the latter operations belongs not to these but to the modular group.

Having now obtained a means of generating all the triangles of either system from a single one among them, the question naturally presents itself how the one system can be obtained from the other. We have seen that the real axis of the J plane corresponds to the circular arcs bounding an ω triangle. By a linear transformation $\omega' = \frac{a\omega + b}{c\omega + d}$ we can convert any circle in the ω plane, for example, one of the sides of the ω triangle into the real ω axis. The function $\omega'(J)$ has then an infinite series of real values corresponding to real values of J. Consequently conjugate imaginary values of J correspond to conjugate imaginary values of $\omega'(J)$. The ω' triangle corresponding to the lower J half leaf is therefore the reflection of that corresponding to the upper J half leaf on the ω real axis. Retransforming now to the original ω triangle, the reflection becomes the operation of *inversion* on the corresponding circular boundary; *i.e.* given the one triangle, if we construct the inverse of every point within it with respect to one of its sides we have a triangle of the second system. In fact every triangle of the ω plane can be obtained from any one by a series of such reflections on bounding lines.

The preceding considerations furnish an entirely new point of departure for the present and for a more general theory. We may suppose any triangle bounded by circular arcs to be *a priori* given, as corresponding to a half plane of the companion Riemann's surface, the analytic connection of the two variables being for the present purpose left out of immediate consideration. From the given triangle we then construct all possible others by the operation of reflection. In order that these may just fill the complex plane without overlapping, the angles of the given triangle must all be submultiples of 2π: $\frac{2\pi}{\nu_1}, \frac{2\pi}{\nu_2}, \frac{2\pi}{\nu_3}$. Three cases are distinguished according as

$$\frac{2\pi}{\nu_1} + \frac{2\pi}{\nu_2} + \frac{2\pi}{\nu_3} \begin{matrix}<\\=\\>\end{matrix} 2\pi, \ i.e.\ \frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} \begin{matrix}<\\=\\>\end{matrix} 1.$$

In the case $\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} > 1$ only a small number of systems of integral values ν_1, ν_2, ν_3 are possible. These lead to the *finite* linear groups. The number of triangles is in this case also finite, and they cover the entire plane. On the other hand, if

$\frac{1}{\nu_1} + \frac{1}{\nu_2} + \frac{1}{\nu_3} < 1$ an infinite number of solutions are possible. The three circular boundaries of the given triangle have in this case a common real orthogonal circle. This circle is moreover the orthogonal circle of every triangle of the system. The latter all lie within this circle and are crowded more and more closely together as they approach the circumference. In the case of the modular group the circumference is exactly the real axis. This group is distinguished among other types by the criterion that $\nu_1 = 2$, $\nu_2 = 3$, $\nu_3 = \infty$. The remaining case, where the sum of the three angles of the triangle is 2π, leads to the theory of the periods of the elliptic function.*

Turning now to the subgroups of the modular group, we observe that these too have in each case a "fundamental domain" (*Fundamental-Bereich*). This is composed of a system of the ω triangles equal in number to the index of the group. This fundamental region, like the double ω triangle, has the property that from its points every other point in the complex plane can be obtained by the operations of the corresponding subgroup, and that it is the smallest region which has this property. Every ω triangle can be converted by the subgroup into one and only one triangle of the fundamental region. If now we suppose every ω triangle to be represented by its "equivalent" triangle in the fundamental region, the effect of all the operations of the modular group is simply to permute these representative triangles among themselves. These permutations again form a group, the group $G_{\mu(n)}$.† In accordance with the entire tendency of the subject, the question at once presents itself whether quantities referred to the several triangles of the fundamental region can be found such that the group $G_{\mu(n)}$ transforms them linearly. This is actually the case, and in fact for $n = 2, 3, 4, 5$ the corresponding linear groups are identical with the dihedron group of order 6, the tetrahedron, the octahedron, and the icosahedron groups respectively.

For these cases, which exhaust the possibility of binary groups, the "deficiency" of the fundamental region is 0. For the next important case $n = 7$, the deficiency is 3 and the corresponding linear group is ternary. It is in fact the group repeatedly treated by Klein in connection with the Gordan plane curve of the fourth order ‡

$$x_1^3 x_2 + x_2^3 x_3 + x_3^3 x_1 = 0.$$

* *Cf.* throughout *Ikosaeder*, I., 5 and *Modulfunctionen*, I., 3.
† *Modulfunctionen*, II., 4, §6.
‡ *Ibid.*, III., 7.

With this brief and imperfect account we must now regretfully leave the subject, consoling ourselves with the reflection that Dr. Fricke's book contains in itself that which will most certainly attract deserved attention to this most beautiful of Klein's creations.

F. N. COLE.

ANN ARBOR, *December* 30, 1891.

PERTURBATIONS OF THE FOUR INNER PLANETS.

Periodic Perturbations of the Longitudes and Radii Vectores of the Four Inner Planets of the First Order as to the Masses. Computed under the direction of SIMON NEWCOMB. Washington, Navy Department, 1891; 4to, pp. 180.

THIS work forms the concluding part of volume III. of a series of astronomical researches, published under the general title, "*Astronomical Papers, prepared for the use of the American Ephemeris and Nautical Almanac.*"

During the past twelve years, one of the principal works which has been in progress at the office of the Nautical Almanac is that of collecting and discussing data for new tables of the planets. The most recent existing tables, which are now used in all European Ephemerides, are those of Leverrier, the construction of which was the greatest work ever undertaken by that celebrated astronomer. The first tables published, those of the Sun, were issued in 1858; those of Uranus and Neptune appeared about 18 years later. The whole work probably took about 25 years in preparation and publication. Yet the number of observations on which the tables were actually based was only a few hundred in the case of each planet, about 500 being used for Venus, 800 for Mars, and probably yet fewer in the cases of the other planets. The results were not completely discussed, and, in consequence, different data were employed in different tables, making it extremely difficult for future astronomers to derive the results of comparing them with future observations. None except those of the Sun and Mercury, which were the first issued, have shown a satisfactory agreement with subsequent observations. The error in the geocentric place of Venus at the time of the recent transit was surprisingly great, amounting to no less than nine seconds in longitude.

The actual number of observations now available for each of the principal planets is several thousand. The recent ones