# OUKA 

Osaka University Knowledge Archive

| Title | On the structure of the augmentation quotients <br> relative to an $N_{p}$-series |
| :---: | :--- |
| Author(s) | Shinya, Kazunari |
| Citation | Osaka Journal of Mathematics. 21(4) P.707-P. 720 |
| Issue Date | 1984 |
| Text Version publisher |  |
| URL | https://doi.org/10.18910/5316 |
| D0I | $10.18910 / 5316$ |
| rights |  |
| Note |  |

Osaka University Knowledge Archive : OUKA
https://ir. Library. osaka-u.ac.jp/repo/ouka/all/

# ON THE STRUCTURE OF THE AUGMENTATION QUOTIENTS RELATIVE TO AN $\mathbf{N}_{p}$-SERIES 

Kazunari SHINYA

(Received January 30, 1984)

## 1. Introduction

Let $G$ be a group with lower central series $G=G_{1} \supseteqq G_{2} \supseteqq G_{3} \supseteq \cdots \supseteqq G_{n} \supseteq$ $G_{n+1} \supseteq \cdots$, and define

$$
W_{n}(G)=\sum{\underset{i=1}{n} S p_{i}^{a}\left(G_{i} / G_{i+1}\right), ~}_{\text {, }}
$$

where $\sum$ runs over all non-negative integers $a_{1}, a_{2}, \cdots, u_{n}$ such that $\sum i a_{i}=n$, and $S p^{a}\left(G_{i} / G_{i+1}\right)$ is the $a_{i}$-th symmetric power of the abelian group $G_{i} / G_{i+1}$. Let $I(G)$ be the augmentation ideal of $G$ in $\boldsymbol{Z} G$. We denote by $Q_{n}(G)$ the additive groups $I^{n}(G) / I^{n+1}(G)$ for $n \geqq 1$. Some results are known about the structure of $Q_{n}(G)$.

It is well known that $Q_{1}(G) \simeq W_{1}(G)$ for any group $G$. G. Losey [3] proved that $Q_{2}(G) \simeq W_{2}(G)$ for any finitely generated group $G$. Tahara [6], [7] proved that $Q_{3}(G) \simeq W_{3}(G) / R_{4}^{*}$ and $Q_{4}(G) \simeq W_{4}(G) / R_{5}^{*}$ hold for any finite group $G$, where $R_{4}^{*}$ and $R_{5}^{*}$ are precisely determined subgroups of $W_{3}(G)$ and $W_{4}(G)$. Furthermore Sandling and Tahara [5] proved that $Q_{n}(G) \simeq W_{n}(G)(n \geqq 1)$ if $G_{i} / G_{i+1}$ is free abelian for any $i \geqq 1$.

Let $p$ be a prime number. In the first half of this paper we restrict our attention to groups of exponent $p$, and prove that

$$
Q_{n}(G) \simeq W_{n}(G) / R_{n+1} \quad(n \geqq 1),
$$

where $R_{n+1}$ is a precisely determined subgroup of $W_{n}(G)$ (Theorem 8). As its corollaries we have a well known result 1), and a new result 2) as follows:

1) $D_{n}(G)=G_{n}$ for any such group $G$, where $D_{n}(G)$ is the $n$-th dimension subgroup of $G$ (Corollary 9).
2) Let $G$ be a finite group with lower central series

$$
G=G_{1} \supseteqq G_{2} \supseteq \cdots \supseteqq G_{c} \supseteqq G_{c+1}=1 .
$$

If this series is an $N_{p}$-series then $Q_{n}(G) \simeq W_{n}(G)$ for $n<p$ (Remark 12).
In the latter half we prove that $Q_{p}(G) \simeq W_{p}(G)$ if the lower central series of $G$ is an $N_{p}$-series (Theorem 13). Furthermore we construct a subgroup
$R_{p+2}$ of $W_{p+1}(G)$ for which $Q_{p+1}(G) \simeq W_{p+1}(G) / R_{p+2}$ holds if the lower central series of $G$ is an $N_{p}$-series (Theorem 14). As for dimension subgroup problem, we will show that $D_{n}(G)=G_{n}$ for all $n \geqq 1$, if the lower central series of $G$ is an $N_{p}$-series (Theorem 15).

## 2. Notations and definitions

Let $G$ be a finite $p$-group of order $p^{m}$, and let $\mathscr{S}$ be a fixed finite $N_{p}$-series

$$
G=H_{1} \supseteqq H_{2} \supseteq \cdots \supseteq H_{c} \supseteq H_{c+1}=1,
$$

that is $\left[H_{i}, H_{j}\right] \leqq H_{i+j}$ for all $i, j \geqq 1$, and $H_{i}^{p} \leqq H_{i p}$ for all $i \geqq 1$. The series $\mathfrak{S}$ defines a weight function $\omega$ of $G$ in the usual way; $\omega(g)=i$ if $g \in H_{i}-H_{i+1}$, $\omega(g)=\infty$ if $g=1$. Conditions of $N_{p}$-series imply that $\omega([g, h]) \geqq \omega(g)+\omega(h)$ for all $g, h \in G$, and $\omega\left(g^{p}\right) \geqq p \omega(g)$ for all $g \in G$. Since each factor $H_{i} / H_{i+1}$ is an elementary abelian $p$-group, we can put

$$
t_{i}=\operatorname{rank}\left(H_{i} / H_{i+1}\right), \quad i=1,2, \cdots, c
$$

We fix an ordered uniqueness basis $\Phi$ for $G$;

$$
\Phi=\left\{x_{1}, x_{2}, \cdots, x_{m}\right\}, \quad \omega\left(x_{1}\right) \leqq \omega\left(x_{2}\right) \leqq \cdots \leqq \omega\left(x_{m}\right)
$$

Let $\Lambda_{n}$ be the $\boldsymbol{Z}$-linear span in $\boldsymbol{Z} \boldsymbol{G}$ of all the elements

$$
\left(g_{1}-1\right)\left(g_{2}-1\right) \cdots\left(g_{k}-1\right), \quad \sum \omega\left(g_{i}\right) \geqq n
$$

Then

$$
I(G)=\Lambda_{1} \supseteqq \Lambda_{2} \supseteqq \cdots \supseteqq \Lambda_{n} \supseteqq \cdots
$$

is a series of ideals of $Z G$ with the property that $\Lambda_{i} \Lambda_{j} \subseteq \Lambda_{i+j}$ for all $i, j \geqq 1$. This filtration determines a family of $\boldsymbol{Z} G$-modules $Q_{n}(\mathfrak{E})=\Lambda_{n} / \Lambda_{n+1}$ for all $n \geqq 1$. These modules are called the augmentation quotients of $G$ relative to $\mathfrak{S}$.

A proper sequence $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ is an ordered $m$-tuple of non-negative integers $\alpha_{i} ; \alpha$ is basic if $0 \leqq \alpha_{i}<p$ for all $i$. The weight of $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ is $W(\alpha)=\sum \omega\left(x_{i}\right) \alpha_{i}$. Let $A_{n}$ be the set of all proper sequences of weight $n$. Corresponding to each proper (basic) sequence $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$, we have the proper (basic) product

$$
P(\alpha)=\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}} .
$$

We define $i_{\alpha}=\max \left\{i: \alpha_{i} \neq 0\right\}$ if $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right) \neq 0=(0,0, \cdots, 0)$ and $i_{0}=1$. We set $W_{n}(\mathfrak{E})=\sum \sum_{i=1}^{n} S p^{a_{i}}\left(H_{i} / H_{i+1}\right)$, where $\sum$ runs over all non-negative integers $a_{1}, a_{2}, \cdots, a_{n}$ such that $\sum i a_{i}=n$, and $S p^{a_{i}}\left(H_{i} / H_{i+1}\right)$ is the $a_{i}$-th symmetric power of the abelian group $H_{i} / H_{i+1}$. Define $m_{\infty}(n)$ to be the least non-negative integer such that $W(\alpha)+m_{\infty}(n)(p-1) \omega\left(x_{i_{a}}\right) \geqq n$.

## G. Losey and N. Losey [3] proved the following:

Lemma 1. For any $n \geqq 1, \Lambda_{n}$ has af free $\boldsymbol{Z}$-basis

$$
B_{n}=\left\{p^{m_{\alpha}(n)} P(\alpha): \alpha \neq 0 \text { basic }\right\}
$$

## 3. The structure of $Q_{n}(\mathfrak{G})$ and its applications

In this section we deal only with groups of exponent $p$. Let $G$ be a finite $p$-group of order $p^{m}$ with exponent $p$. Then any $N$-series $\mathfrak{L}: G=H_{1} \supseteqq H_{2} \supseteq \cdots$ $\supseteq H_{c} \supseteq H_{c+1}=1$ is an $N_{p}$-series.

Definition 2.

1) Define the $p$-sequences of numbers $\left\{a_{k}^{0}\right\}_{k=0}^{\infty},\left\{a_{k}^{1}\right\}_{k=0}^{\infty}, \cdots,\left\{a_{k}^{p-1}\right\}_{k=0}^{\infty}$ as follows:

$$
\left(\begin{array}{c}
a_{0}^{0} \\
a_{0}^{1} \\
a_{0}^{2} \\
\vdots \\
a_{0}^{p-1}
\end{array}\right)=\left(\begin{array}{c}
1 \\
0 \\
0 \\
\vdots \\
0
\end{array}\right), \quad\left(\begin{array}{c}
a_{1}^{0} \\
a_{1}^{1} \\
a_{1}^{2} \\
\vdots \\
a_{1}^{p-1}
\end{array}\right)=\left(\begin{array}{c}
0 \\
1 \\
0 \\
\vdots \\
0
\end{array}\right)
$$

and

$$
\left(\begin{array}{c}
a_{k+1}^{0} \\
a_{k+1}^{1} \\
a_{k+1}^{2} \\
\vdots \\
\vdots \\
a_{k+1}^{p-1}
\end{array}\right)=\left(\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -\binom{p}{1} \\
0 & 1 & & & -\binom{p}{2} \\
& \ddots & 0 & & \vdots \\
& 0 & \ddots & & \vdots \\
0 & & & 1 & -\binom{p}{p-1}
\end{array}\right)\left(\begin{array}{l}
a_{k}^{0} \\
a_{k}^{1} \\
a_{k}^{2} \\
\vdots \\
a_{k}^{p-1}
\end{array}\right)
$$

for $k \geqq 1$.
Note that the next identity holds for any $x \in G$ of order $p$ and for any nonnegative integer $n$ :

$$
(x-1)^{n}=a_{n}^{0} \cdot 1+a_{n}^{1}(x-1)+\cdots+a_{n}^{p-1}(x-1)^{p-1}
$$

2) Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence and $\beta=\left(\beta_{1}, \beta_{2}, \cdots, \beta_{m}\right)$ be a basic sequence. We define the integer $C_{a}^{\beta}$ as $C_{a}^{\beta}=a_{\alpha_{1}}^{\beta_{1}} a_{\alpha_{2}}^{\beta_{2}} \cdots a_{a_{m}}^{\beta_{m}}$.

We can express $P(\alpha)$ as a $\boldsymbol{Z}$-linear combination of basic products by the following:

Lemma 3. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence with $W(\alpha)=n$, then 1) $P(\alpha)=\sum_{\beta: \text { basic }} C_{\alpha}^{\beta} P(\beta)$,
2) $p^{m_{\beta}(n)} \mid C_{\alpha}^{\beta}$ for any basic sequence $\beta$,
3) if $\alpha$ is basic then $C_{\alpha}^{\beta} \neq 0$ if and only if $\beta=\alpha$.

Proof. Expand each $\left(x_{i}-1\right)^{\infty}{ }_{i}$ as in Definition 2. Then we have

$$
\begin{aligned}
& P(\alpha)=\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}} \\
& =\left\{\sum_{\beta_{1}=0}^{p-1} a_{\alpha_{1}}^{\beta_{1}}\left(x_{1}-1\right)^{\beta_{1}}\right\}\left\{\sum_{\beta_{2}=0}^{p-1} a_{\alpha_{2}}^{\beta_{2}}\left(x_{2}-1\right)^{\beta_{2}}\right\} \ldots\left\{\sum_{\beta_{m}=0}^{p-1} a_{a_{m}}^{\beta_{m}}\left(x_{m}-1\right)^{\beta_{m}}\right\} \\
& =\sum_{\beta_{1}, \beta_{2}, \cdots, \beta_{m}=0}^{p-1} a_{\alpha_{1}}^{\beta_{1}} a_{\alpha_{2}}^{\beta_{2} \cdots a_{\alpha_{m}}^{\beta_{m}}\left(x_{1}-1\right)^{\beta_{1}}\left(x_{2}-1\right)^{\beta_{2}} \cdots\left(x_{m}-1\right)^{\beta_{m}}} \\
& =\sum_{\beta: \text { basic }} C_{\alpha}^{\beta} P(\beta) .
\end{aligned}
$$

Thus 1) is obtained. Since $\left\{p^{m_{\beta}^{-}(n)} P(\beta) \mid \beta \neq 0\right.$ : basic $\}$ is a basis system of $\Lambda_{n}, P(\alpha)$ is uniquely expressed as a $Z$-linear combination of $p^{m_{\beta}(n)} P(\beta)$ with $\beta \neq 0$ basic. On the other hand $\{P(\beta) \mid \beta$ : basic $\}$ is a basis system of $\boldsymbol{Z} G$. So $P(\alpha)$ is uniquely expressed as a $Z$-linear combination of $P(\beta), \beta$ basic. Then uniqueness of coefficients implies that $p^{m_{\beta}(n)} \mid C_{\alpha}^{\beta}$ for all basic sequence $\beta$. 3 ) is trivial from 1 ).

Definition 4. Let $\alpha$ be a proper sequence with $W(\alpha)=n$. For any basic sequence $\beta$, we put $D_{\alpha}^{\beta}=C_{\alpha}^{\beta} / p^{m_{\beta}(n)} \in Z$. Therefore

$$
P(\alpha)=\sum_{\beta: \text { basic }} D_{\alpha}^{\beta} p^{m_{\beta}(n)} P(\beta)
$$

Note that $D_{\beta}^{\beta}=1$ if $\beta$ is a basic sequence with $W(\beta) \geqq 1$.
Lemma 5 (Passi and Vermani [4]). Let $p$ be a prime number and $H=\langle a\rangle$ be a cyclic group of order $p^{m}$. Then

$$
p^{m-1}(a-1)^{(r+1)(p-1)+1} \equiv(-1)^{(r+1)} p^{m+r}(a-1) \quad \bmod I^{(r+1)(p-1)+2}(H)
$$

for all $r \geqq 0$.
Corollary 6. Let $x \in \Phi$, then

$$
(x-1)^{r(p-1)+1} \equiv(-1)^{r} p^{r}(x-1) \quad \bmod \Lambda_{[r(p-1)+2] \omega(x)}
$$

for all $r \geqq 0$.
Proof. We set $m=1$ in Lemma 5, then we have

$$
(x-1)^{(r+1)(p-1)+1} \equiv(-1)^{(r+1)} p^{(r+1)}(x-1) \quad \bmod I^{(r+1)(p-1)+2}(\langle x\rangle)
$$

for all $r \geqq 0$. This trivially holds for $r=-1$. Then we have

$$
(x-1)^{r(p-1)+1} \equiv(-1)^{r} p^{r}(x-1) \quad \bmod I^{r(p-1)+2}(\langle x\rangle) \quad \text { for } r \geqq 0 .
$$

Since $I^{r(p-1)+2}(\langle x\rangle)=(x-1)^{r(p-1)+2} \boldsymbol{Z}\langle x\rangle$, we have $I^{r(p-1)+2}(\langle x\rangle) \cong \Lambda_{(r(p-1)+2) \omega(x)}$. So the result follows.

## Lemma 7.

1) $W_{n}(\mathfrak{S})$ is an elementary abelian $p$-group of order $p^{r}$, where $r=\sum \prod_{i=1}^{n} \times$ $\binom{a_{i}+t_{i}-1}{a_{i}}$, and $\sum$ runs over all non-negative integers $a_{1}, a_{2}, \cdots, a_{n}$ such that $\sum_{i=1}^{n} i a_{i}=n$.
2) Regard $W_{n}(\mathfrak{I})$ as vector space over $\boldsymbol{Z} \mid p \boldsymbol{Z}$, then $\left\{\stackrel{\alpha_{1}}{\otimes} x_{1} \stackrel{\alpha_{2}}{\unrhd} x_{2} \cdots \stackrel{\alpha_{m}}{\otimes} \bar{x}_{m}\right.$ : $\left.\alpha \in A_{n}\right\}$ is a basis system of $W_{n}(\mathfrak{l})$, where

$$
\text { and } \bar{x}_{i}=x_{i} H_{\omega\left(x_{i}\right)+1}
$$

Proof. Easy to prove.
For convenience we write $x_{i}$ instead of $x_{i}$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right) \neq 0$ be a basic sequence. Then we call $\alpha$ to be regular for $n$ if $W(\alpha)+m_{a}(n)(p-1)$ $\times \omega\left(x_{i_{\infty}}\right)=n$.

Theorem 8. Let $R_{n+1}$ be the submodule of $W_{n}(\mathfrak{C})$ generated by the elements of the form
where $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ runs over all elements of $A_{n}$. Then $\Lambda_{n} / \Lambda_{n+1}$ is isomorphic to $W_{n}(\mathfrak{S}) / R_{n+1}$ for all $n \geqq 1$.

Proof. We shall divide the proof in the following four steps.
Step 1. We define a homomorphism $\psi_{n}$ from $\Lambda_{n}$ to $W_{n}\left(\mathscr{S}_{\mathcal{S}}\right) / R_{n+1}$ which is defined on the basis of $\Lambda_{n}$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a basic sequence with $W(\alpha) \geqq 1$. Then

$$
P(\alpha)=\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{i_{\omega}-1}-1\right)^{\alpha_{i \alpha}}-1\left(x_{i_{\omega}}-1\right)^{\alpha_{i}} .
$$

Define the image of $p^{m_{\alpha}(n)} P(\alpha)$ under $\psi_{n}$ as follows:

1) If $\alpha$ is regular for $n$ then
2) If $\alpha$ is not regular for $n$ then

$$
\psi_{n}\left(p^{m_{\alpha}(n)} P(\alpha)\right)=R_{n+1}
$$

Then we shall show that $\psi_{n}\left(\Lambda_{n+1}\right)=R_{n+1}$ and hence $\psi_{n}$ induces a homomorphism $\psi_{n}^{*}$ from $\Lambda_{n} / \Lambda_{n+1}$ to $W_{n}(\mathcal{L}) / R_{n+1}$.

It suffices to prove it on the $Z$-basis of $\Lambda_{n+1}$. Let $p^{m_{\alpha}(n+1)} P(\alpha) \in B_{n+1}$. By the definition of $m_{\alpha}(n)$ we have $m_{\infty}(n) \leqq m_{a}(n+1) \leqq m_{\alpha}(n)+1$. If $m_{a}(n+1)=$ $m_{\alpha}(n)$ then $\alpha$ is not regular for $n$ since $W(\alpha)+m_{a}(n)(p-1) \omega\left(x_{i_{\alpha}}\right)=W(\alpha)+$ $m_{a}(n+1)(p-1) \omega\left(x_{i_{a}}\right) \geqq n+1$. Therefore by the definition of $\psi_{n}$ we have

$$
\psi_{n}\left(p^{m_{\alpha}(n+1)} P(\alpha)\right)=\psi_{n}\left(p^{m_{\alpha}(n)} P(\alpha)\right)=R_{n+1} .
$$

If $m_{\infty}(n+1)=m_{\infty}(n)+1$ then

$$
\psi_{n}\left(p^{m_{\alpha}(n+1)} P(\alpha)\right)=p \psi_{n}\left(p^{m_{\alpha}(n)} P(\alpha)\right)=R_{n+1},
$$

since $W_{n}(\mathfrak{C})$ is an elementary abelian $p$-group. So the result follows.
Step 2. We define a linear transformation $\phi_{n}$ from $W_{n}(\mathcal{S})$ to $\Lambda_{n} / \Lambda_{n+1}$ as follows: By Lemma $7\left\{\bigotimes_{1}^{\alpha_{1}} x_{1} \stackrel{\alpha}{2}^{\alpha_{2}} \cdots x_{2} x_{m} ; \alpha \in A_{n}\right\}$ is a basis system of $W_{n}(\mathfrak{F})$. Note that G. Losey and N. Losey proved that $\Lambda_{n}!\Lambda_{n+1}$ is an ele-
 the element

$$
\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{n+1},
$$

and extend it $\boldsymbol{Z} \mid p \boldsymbol{Z}$-linearly.
Then we shall show that $\phi_{n}\left(R_{n+1}\right)=\Lambda_{n+1}$, so $\phi_{n}$ induces a homomorphism $\phi_{n}^{*}$ from $W_{n}(\mathfrak{C}) / R_{n+1}$ to $\Lambda_{n} / \Lambda_{n+1}$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence with $W(\alpha)=n$. Then

$$
\begin{aligned}
& \phi_{n}\left(\stackrel{\alpha_{1}}{\otimes} x_{1} \stackrel{\alpha_{2}}{\square} x_{2} \cdots \stackrel{\alpha_{m}}{\boxtimes} x_{m}-\sum_{\beta: \text { regular for } n} D_{a}^{\beta}(-1)^{m_{\beta}(n)} \stackrel{\beta_{1}}{\unrhd} x_{1} \stackrel{\beta_{2}}{\square} x_{2}\right.
\end{aligned}
$$

$$
\begin{aligned}
& =\left(x_{1}-1\right)^{\alpha_{1}}\left(x_{2}-1\right)^{\alpha_{2}} \cdots\left(x_{m}-1\right)^{\alpha_{m}}-\sum_{\beta: \text { regular for } n} D_{\alpha}^{\beta}(-1)^{m_{\beta}(n)}\left(x_{1}-1\right)^{\beta_{1}}\left(x_{2}-1\right)^{\beta_{2}} \\
& \cdots\left(x_{i_{\beta}-1}-1\right)^{\beta_{i_{\beta}}-1}\left(x_{i_{\beta}}-1\right)^{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)}+\Lambda_{n+1} \\
& =\sum_{\gamma: \text { basic }} D_{\alpha}^{\gamma} p^{m_{\gamma}(n)}\left(x_{1}-1\right)^{\gamma_{1}}\left(x_{2}-1\right)^{\gamma_{2}} \cdots\left(x_{i \gamma-1}-1\right)^{\gamma_{i \gamma}-1}\left(x_{i \gamma}-1\right)^{\gamma_{i \gamma}} \\
& -\sum_{\beta: \text { regular for } n} D_{\alpha}^{\beta}(-1)^{m_{\beta}(n)}\left(x_{1}-1\right)^{\beta_{1}}\left(x_{2}-1\right)^{\beta_{2}} \ldots \\
& \left(x_{i_{\beta}-1}-1\right)^{\beta_{i_{\beta}}-1}\left(x_{i_{\beta}}-1\right)^{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)}+\Lambda_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
= & \sum_{\beta: \text { regular for } n} D_{a}^{\beta}\left(x_{1}-1\right)^{\beta_{1}}\left(x_{2}-1\right)^{\beta_{2}} \cdots\left(x_{i_{\beta}-1}-1\right)^{\beta_{i_{\beta}-1}}\left\{p^{m_{\beta}(n)}\left(x_{i_{\beta}}-1\right)^{\beta_{i}}\right. \\
& \left.-(-1)^{m_{\beta}(n)}\left(x_{i_{\beta}}-1\right)^{\beta i_{\beta}+m_{\beta}(n)(p-1)}\right\}+\Lambda_{n+1} .
\end{aligned}
$$

By Corollary 6 we have

$$
p^{m_{\beta}(n)}\left(x_{i_{\beta}}-1\right)^{\beta_{i}}-(-1)^{m_{\beta}(n)}\left(x_{i_{\beta}}-1\right)^{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)} \in \Lambda_{\left\{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)+1\right\} \omega\left(x_{i_{\beta}}\right)} .
$$

Therefore
belongs to $\Lambda_{r}$, where $r=W(\beta)+m_{\beta}(n)(p-1) \omega\left(x_{i_{\beta}}\right)+\omega\left(x_{i_{\beta}}\right) \geqq n+1$. Thus we have

$$
\begin{aligned}
& \phi_{n}\left(\stackrel{\alpha_{1}}{\left(\square x_{1} \stackrel{\alpha_{2}}{\bigotimes} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}-\sum_{\beta: \text { regular for } n} D_{a}^{\beta}(-1)^{m_{\beta}(n)} \stackrel{\beta_{1}}{\bigotimes} x_{1} \stackrel{\beta_{2}}{\bigotimes} x_{2}\right.}\right. \\
& \left.\cdots \bigotimes^{\beta_{i_{\beta}-1}} x_{i_{i_{\beta}-1}}{ }^{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)}{ }^{(1)} x_{i_{\beta}}\right)=\Lambda_{n+1} .
\end{aligned}
$$

Consequently we have $\phi_{n}\left(R_{n+1}\right)=\Lambda_{n+1}$, and so $\phi_{n}$ induces a homomorphism $\phi_{n}^{*}$ from $W_{n}(\mathfrak{f}) / R_{n+1}$ to $\Lambda_{n} / \Lambda_{n+1}$.

Step 3. We shall prove that $\psi_{n}^{*} \circ \phi_{n}^{*}$ is the identity map on $W_{n}(\mathcal{S}) / R_{n+1}$. Since $W_{n}(\mathfrak{S}) / R_{n+1}$ is generated by $\left\{\bigotimes x_{1} \boxtimes x_{2} \cdots \stackrel{\alpha_{2}}{\alpha_{m}} x_{m}+R_{n+1}: \alpha \in A_{n}\right\}$, it suffices to prove

$$
\psi_{n}^{*} \circ \phi_{n}^{*}\left(\stackrel{\alpha_{1}}{\oslash} x_{1} \stackrel{\alpha_{2}}{\bigotimes} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{n+1}\right)=\stackrel{\alpha_{1}}{\emptyset} x_{1} \stackrel{\alpha_{2}}{\oslash} x_{2} \cdots \stackrel{\alpha_{m}}{\oslash x_{m}}+R_{n+1}
$$

for any $\alpha \in A_{n}$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence with $W(\alpha)=n$, namely $\alpha \in A_{n}$. Then we have

$$
\begin{aligned}
& \psi_{n}^{*} \circ \phi_{n}^{*}\left(\stackrel{\alpha_{1}}{\bigotimes} x_{1} \stackrel{\alpha_{2}}{\bigotimes} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{n+1}\right) \\
& =\psi_{n}^{*}\left(\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{n+1}\right) \\
& =\psi_{n}^{*}\left(\sum_{\beta: \text { regular for } n} D_{\alpha}^{\beta} p^{m_{\beta}^{(n)}}\left(x_{1}-1\right)^{\beta_{1}}\left(x_{2}-1\right)^{\beta_{2}} \ldots\right. \\
& \left.\left(x_{i_{\beta}-1}-1\right)^{\beta_{i}-1}\left(x_{i_{\beta}}-1\right)^{\beta_{i}}+\Lambda_{n+1}\right) \\
& =\sum_{\beta: \text { regular for } n} D_{a}^{\beta}(-1)^{m_{\beta}(n)} \stackrel{\beta_{1}}{\unrhd} x_{1} \stackrel{\beta_{2}}{\otimes} x_{2} \cdots \\
& \beta_{i_{\beta}-1} \quad \beta_{i_{\beta}}+m_{\beta}(n)(p-1) \\
& \text { (1) } x_{i_{\beta}-1} \quad \text { ( ) } \quad x_{i_{\beta}}+R_{n+1} \\
& =\stackrel{\alpha_{1}}{\oslash} x_{1} \stackrel{\alpha_{2}}{\square} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{n+1} .
\end{aligned}
$$

Step 4. Finally we shall prove that $\phi_{n}^{*} \circ \psi_{n}^{*}$ is the identity map on $\Lambda_{n} / \Lambda_{n+1}$. Since $\Lambda_{n} / \Lambda_{n+1}$ is generated by $\left\{p^{m_{\alpha}(n)} P(\alpha)+\Lambda_{n+1} \mid \alpha\right.$ : regular for $\left.n\right\}$, it suffices to prove

$$
\phi_{n}^{*} \circ \psi_{n}^{*}\left(p^{m_{\alpha}(n)} P(\alpha)+\Lambda_{n+1}\right)=p^{m_{\alpha}(n)} P(\alpha)+\Lambda_{n+1}
$$

for such an $\alpha$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a sequence regular for $n$. Then we have

$$
\begin{aligned}
& \phi_{n}^{*} \circ \psi_{n}^{*}\left(p^{m_{\alpha}(n)}\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{i_{\omega_{\alpha}}-1}-1\right)^{\alpha_{i_{\omega}}-1}\left(x_{i_{\omega}}-1\right)^{\alpha_{i_{i}}}+\Lambda_{n+1}\right)
\end{aligned}
$$

$$
\begin{aligned}
& =(-1)^{m_{\alpha}(n)}\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{i_{\alpha}-1}-1\right)^{\omega_{i_{\alpha}}-1}\left(x_{i_{\alpha}}-1\right)^{\omega_{i_{\alpha}}}+m_{\alpha}(n)(p-1) \quad+\Lambda_{n+1} \\
& =p^{m_{\alpha}(n)}\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{i_{\infty}-1}-1\right)^{\alpha_{i_{\infty}}-1}\left(x_{i_{a}}-1\right)^{\alpha_{i}}+\Lambda_{n+1}
\end{aligned}
$$

by using Corollary 6.
Step $1 \sim$ Step 4 imply that $\Lambda_{n} / \Lambda_{n+1} \simeq W_{n}(\mathfrak{F}) / R_{n+1}$ for all $n \geqq 1$.
Corollary 9 (P.M. Cohn [1]). Let G be a group of prime exponent p. Let $\left\{H_{j}\right\}$ be an $N$-series for $G$ and $\left\{\Lambda_{j}\right\}$ the canonical filtration of $I(G)$ relative to $\left\{H_{j}\right\}$. Then $D\left(\Lambda_{n}\right)=H_{n}$ for all $n \geqq 1$.

Proof. We prove it by induction on $n$. By standard reduction arguments we may assume that $H_{n+1}=1, D\left(\Lambda_{n}\right)=H_{n}$ and $G$ is finite. Define the homomorphism $f$ from $H_{n}$ to $\Lambda_{n} / \Lambda_{n+1}$ by $f(x)=(x-1)+\Lambda_{n+1}$. Then $D\left(\Lambda_{n+1}\right)=$ $\operatorname{ker} f$. Let $x \in H_{n}$ be an element of $D\left(\Lambda_{n+1}\right)$. Write $x$ as $x=\Pi x_{j}^{c} j\left(0 \leqq c_{j}<p\right)$ using elements of uniqueness basis of weight $n$. Then $f(x)=\sum c_{j}\left(x_{j}-1\right)+\Lambda_{n+1}$ and $\psi_{n}^{*}(f(x))=\sum c_{j} x_{j}+R_{n+1}$. Since $f(x)=\Lambda_{n+1}, \sum c_{j} x_{j}$ can be expressed as a $\boldsymbol{Z}$-linear combination of generators of $R_{n+1}$. But the elements of uniqueness basis of weight $n$ do not appear in the generators of $R_{n+1}$. We shall prove it. If an element of uniqueness basis of weight $n$ is in the generators of $R_{n+1}$, there must exist some proper sequence $\alpha=(0, \cdots, 0,1,0, \cdots, 0)$ of weight $n$ such that

$$
x_{k}-\sum_{\beta: \text { regular for } n} D_{\infty}^{\beta}(-1)^{m_{\beta}(n)} \stackrel{\beta_{1}}{\bigotimes} x_{1} \stackrel{\beta_{2}}{\bigotimes} x_{2} \cdots \bigotimes^{\beta_{i_{\beta}-1}} x_{i_{\beta}-1}^{\beta_{i_{\beta}}+m_{\beta}(n)(p-1)} \bigotimes^{i_{\beta}} \neq 0 .
$$

Now $\alpha$ is a basic sequence, so by Lemma $3 D_{\infty}^{\beta} \neq 0$ if and only if $\beta=\alpha$. Trivially $m_{\omega}(n)=0$ and $D_{a}^{\alpha}=1$, so

$$
x_{k}-\sum_{\beta: \text { regular for } n} D_{a}^{\beta}(-1)^{m_{\beta}(n)} \stackrel{\beta_{1}}{\bigotimes} x_{1} \beta_{2} x_{2} \cdots \bigotimes^{\beta_{i_{\beta}-1}} x_{i_{\beta}-1} \beta_{i_{\beta}+m_{\beta}(n)(p-1)}^{\emptyset} x_{i_{\beta}}=0
$$

Thus any element of uniqueness basis of weight $n$ does not appear in the generators of $R_{n+1}$. If some $c_{j} \neq 0, \sum c_{j} x_{j}$ is not able to be expressed as a $\boldsymbol{Z}$-linear combination of generators of $R_{n+1}$. This implies $c_{j}=0$ for all $j$, and $x=\prod_{j} x_{j}^{c} j$ $=1$. Therefore the result follows.

Passi and Vermani [4] proved the following
 $k>1$. Then $I^{n}(G) / I^{n+1}(G) \simeq S p^{n}(G)$ if and only if $n \leqq p+r(p-1)$.

As a special case of this result we have that if $G$ is an elementary abelian $p$-group of order $\geqq p^{2}$ then $I^{n}(G) / I^{n+1}(G) \simeq S p^{n}(G)$ if and only if $n \leqq p$. Our method is available for non-abelian $p$-group of exponent $p$ and we have a similar result as follows.

Corollary 11. Let $G$ be a finite $p$-group of exponent $p$ with $N$-series $\mathfrak{S}$ : $G=H_{1} \supseteq H_{2} \supseteq \cdots \supseteq H_{c} \supseteq H_{c+1}=1$ with $\left|H_{1}\right| H_{2} \mid \geqq p^{2}$. Then $\Lambda_{n} / \Lambda_{n+1} \simeq W_{n}(\mathfrak{l})$ if and only if $n \leqq p$.

Proof. Let $\Phi=\left\{x_{1}, x_{2}, \cdots, x_{m}\right\}$ be the uniqueness basis for $G$ relative to $\mathfrak{S}$. By Theorem $8 \Lambda_{n} / \Lambda_{n+1} \simeq W_{n}(\mathfrak{S}) / R_{n+1}$. We shall prove that $R_{n+1}=0$ for $n \leqq p$ and $R_{n+1} \neq 0$ for $n>p$.

Case 1. $n<p$.
Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence of weight $n$. Then $\alpha$ is a basic sequence. By Lemma 3, $D_{a}^{\beta} \neq 0$ if and only if $\beta=\alpha$. Trivially $m_{a}(n)=0$ and $D_{a}^{\omega}=1$. These conditions imply that

$$
\begin{aligned}
& \stackrel{\alpha_{1}}{\otimes} x_{1} \stackrel{\alpha_{2}}{\oslash} x_{2} \cdots \stackrel{\alpha_{m}}{\otimes} x_{m}-\sum_{\beta: \text { regular for } n} D_{n}^{\beta}(-1)^{m_{\beta}^{(n)} \stackrel{\beta_{1}}{\otimes} x_{1} \stackrel{\beta_{2}}{\oslash} x_{2}}
\end{aligned}
$$

Therefore $R_{n+1}=0$ for $n<p$.
Case 2. $n=p$.
Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence of weight $p$. If $\alpha$ is a basic sequence it follows as above that

$$
\begin{aligned}
& \stackrel{\alpha_{1}}{\otimes} x_{1} \stackrel{\alpha_{2}}{\oslash} x_{2} \cdots \stackrel{\alpha_{m}}{\otimes} x_{m}-\sum_{\beta: \text { regular for } p} D_{\alpha}^{\beta}(-1)^{m_{\beta}(p)} \stackrel{\beta_{1}}{\otimes} x_{1} \stackrel{\beta_{2}}{\otimes} x_{2} \\
& \cdots \bigotimes^{\beta_{i_{\beta}-1}} x_{i_{\beta}-1} \overbrace{i_{\beta}}+m_{\beta}(p)(p-1) \quad x_{i_{\beta}}=0 .
\end{aligned}
$$

If $\alpha$ is not a basic sequence then $\alpha$ has the form $\alpha=\left(0, \cdots, 0, p_{j}, 0, \cdots, 0\right)$ for
 $(k \neq j)$. Let $\beta_{0}$ be a basic sequence of the form $\beta_{0}=(0, \cdots, 0,1,0, \cdots, 0)$. If $\beta$ is any basic sequence different from $\beta_{0}$, then $C_{\alpha}^{\beta}=0$ or $\beta$ is not regular for $p$. Clearly $m_{\beta_{0}}(p)=1$ and $D_{\alpha_{0}}^{\beta_{0}}=a_{p}^{1} / p=-1$. Therefore

Thus we have $R_{p+1}=0$.

Case 3. $n>p$.
Since $\left|H_{1}\right| H_{2} \mid \geqq p^{2}$, there exists a proper sequence $\alpha=(n-1,1,0, \cdots, 0)$ in $A_{n}$. If $C_{\omega}^{\beta}=a_{n}^{\beta_{1}} a_{1}^{\beta_{2}} a_{0}^{\beta_{3}} \cdots a_{0}^{\beta_{m}} \neq 0$ for a basic sequence $\beta=\left(\beta_{1}, \beta_{2}, \cdots, \beta_{m}\right)$, then $\beta_{2}=1$ and $\beta_{3}=\beta_{4}=\cdots=\beta_{m}=0$. Moreover if $\beta=\left(\beta_{1}, 1,0, \cdots, 0\right)$ is regular for $n$, then
since $\beta_{1}<p \leqq n-1$. Thus

$$
\begin{aligned}
& =\stackrel{n-1}{\varnothing} x_{1} \vee x_{2}-\sum_{\beta: \text { regular for } n} D_{\alpha}^{\beta}(-1)^{m_{\beta}^{(n)}} \stackrel{\beta_{1}}{\oslash} x_{1} \bigotimes^{n-\beta_{1}} x_{2} \neq 0,
\end{aligned}
$$

and hence $R_{n+1} \neq 0$ for all $n>p$. Therefore the result follows.
Remark 12. When we determine the structure of $Q_{n}(\mathfrak{F})(n<p)$ for an $N_{p}$-series $\mathfrak{S}: G=H_{1} \supseteq H_{2} \supseteq \cdots \supseteq H_{c} \supseteq H_{c+1}=1$, we may assume that $H_{p}=1$. So we may assume that $G$ has exponent $p$. Then by Corollary 11 we have $Q_{n}(\mathfrak{S}) \simeq W_{n}(\mathfrak{S})(n<p)$ for any $N_{p}$-series of a finite $p$-group $G$. (It is easy to see $R_{n+1}=0$ for $n<p$ if $H_{1} / H_{2}$ is a cyclic group of order $p$.)

## 4. The structure of $\Lambda_{p} / \Lambda_{p+1}$ and $\Lambda_{p+1} / \Lambda_{p+2}$

In the previous section we proved that $\Lambda_{n} / \Lambda_{n+1} \simeq W_{n}(\mathfrak{F})$ holds for $n<p$ and for any $N_{p}$-series $\mathscr{S}$ of the finite $p$-group $G$. In this section we determine the structure of $\Lambda_{p} / \Lambda_{p+1}$ and $\Lambda_{p+1} / \Lambda_{p+2}$.

Theorem 13. Let $G$ be a finite $p$-group with $N_{p}$-series $\mathfrak{F}$, and $\left\{\Lambda_{j}\right\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{S}$. Then $\Lambda_{p} / \Lambda_{p+1}$ is isomorphic to $W_{p}(\mathfrak{I})$.

Proof. The proof is similar to that of Theorem 8. Since $\left\{p^{m_{\alpha}(n)} P(\alpha)\right.$ $+\Lambda_{p+1} \mid \alpha$ : regular for $\left.p\right\}$ is a basis system of the vector space $\Lambda_{p} / \Lambda_{p+1}$, we can define a linear transformation $\psi: \Lambda_{p} / \Lambda_{p+1} \rightarrow W_{p}(\mathfrak{E})$ as follows: Let $\alpha$ be a regular sequence for $p$. Then $p^{m_{a}(n)} P(\alpha)$ is either $p\left(x_{i}-1\right)$ with $\omega\left(x_{i}\right)=1$, or $P(\alpha)$ with $W(\alpha)=p$. We define to be $\psi\left(p\left(x_{i}-1\right)+\Lambda_{p+1}\right)=-\stackrel{p}{\boxtimes} x_{i}+x_{i}^{p}$ and $\psi\left(P(\alpha)+\Lambda_{p+1}\right)=\psi\left(\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{p+1}\right)=\stackrel{\alpha_{1}}{\emptyset} x_{1} \stackrel{\alpha_{2}}{\emptyset} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}$. Next we define a linear transformation $\phi: W_{p}(\mathfrak{K}) \rightarrow \Lambda_{p} / \Lambda_{p+1}$ by just the same way as $\phi_{p}$ which we defined in Step 2 of the proof of Theorem 8. Then we can easily show that $\psi_{0} \phi$ and $\phi_{0} \psi$ are the identity maps on $W_{p}(\mathscr{S})$ and $\Lambda_{p} / \Lambda_{p+1}$ respectively, and hence $\Lambda_{p} / \Lambda_{p+1} \simeq W_{p}(\mathfrak{E})$.

Theorem 14. Let $G$ be a finite p-group with $N_{p}$-series $\mathfrak{S}=\left\{H_{j}\right\},\left\{\Lambda_{j}\right\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{S}_{2}$ and $\Phi=\left\{x_{1}, x_{2}, \cdots, x_{m}\right\}$ its uniqueness basis. Then $\Lambda_{p+1} / \Lambda_{p+2}$ is isomorphic to $W_{p+1}(\mathfrak{S}) / R_{p+2}$ where $R_{p+2}$ is generated by the elements
$x_{i} \stackrel{p}{\otimes} x_{j}-\stackrel{p}{\otimes} x_{i} \vee x_{j}-x_{i} \otimes x_{j}^{p}+x_{j} \otimes x_{i}^{p}+\left[x_{i}^{p}, x_{j}\right], i<j$ and $\omega\left(x_{i}\right)=\omega\left(x_{j}\right)=1$.
Proof. Tahara [6] proved that $\Lambda_{3} / \Lambda_{4} \simeq W_{3}(\mathfrak{G}) / R_{4}^{*}$ holds for any $N$-series $\mathfrak{S}$ of the finite group $G$, where $R_{4}^{*}$ is the submodule of $W_{3}(\mathfrak{S})$ generated by the elements

$$
\begin{aligned}
& \frac{d(j)}{d(i)}\binom{d(i)}{2} \stackrel{2}{\bigotimes} x_{1 i} \vee x_{1 j}-\binom{d(j)}{2} \stackrel{2}{x_{1 i} \stackrel{\unrhd}{x_{1 j}}+\bar{x}_{1 i} \otimes x_{1 j}^{d(j)}} \\
& -\frac{d(j)}{d(i)}\left\{x_{1 j} \otimes x_{1 i}^{d(i)}\right\}-\frac{d(j)}{d(i)} \overline{\left[x_{1 i}^{d(i)}, x_{1 j}\right]}, i<j .
\end{aligned}
$$

The case $p=2$ of Theorem 14 is directly obtained by this Theorem. Let $p$ be an odd prime. We shall divide the proof in the following 4 steps.

Step 1. $B_{p+1}=\left\{p^{m_{\alpha}(p+1)} P(\alpha) \mid \alpha \neq 0\right.$ : basic $\}$ is classified into following three subsets a) $\sim \mathrm{c}$ ), and we define a homomorphism $\psi$ from $\Lambda_{p+1}$ to $W_{p+1}(\mathcal{E}) / R_{p+2}$ as follows:
a) $p\left(x_{i}-1\right)\left(x_{j}-1\right), i \leqq j$ and $\omega\left(x_{i}\right)=\omega\left(x_{j}\right)=1$,

$$
\psi\left(p\left(x_{i}-1\right)\left(x_{j}-1\right)\right)=-x_{i} \stackrel{p}{\otimes} x_{j}+x_{i} \otimes x_{j}^{p}+R_{p+2},
$$

b) $P(\alpha)=\left(x_{1}-1\right)^{a_{1}}\left(x_{2}-1\right)^{\alpha_{2}} \cdots\left(x_{m}-1\right)^{\alpha_{m}}, W(\alpha)=p+1$,

$$
\psi(P(\alpha))=\stackrel{\alpha_{1}}{\boxtimes x_{1} \boxtimes x_{2} \cdots \otimes x_{m}+R_{p+2}, ~}
$$

c) $p^{m_{\alpha}(p+1)} P(\alpha), \alpha$ not regular for $p+1, \psi\left(p^{m_{\alpha}(p+1)} P(\alpha)\right)=R_{p+2}$.

Then in the same way as in Step 1 of the proof of Theorem 8, we can easily show $\psi\left(\Lambda_{p+2}\right)=R_{p+2}$ and hence $\psi$ induces the homomorphism $\psi^{*} ; \Lambda_{p+1} / \Lambda_{p+2} \rightarrow$ $W_{p+1}(\mathfrak{W}) / R_{p+2}$.

Step 2. We define a linear transformation $\phi$ from $W_{p+1}(\mathfrak{E})$ to $\Lambda_{p+1} / \Lambda_{p+2}$ by defining it on the basis of $W_{p+1}(\mathfrak{F})$ as follows:

$$
\phi\left(\stackrel{\alpha_{1}}{\bigotimes} x_{1} \stackrel{\alpha_{2}}{\bigotimes} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}\right)=\left(x_{1}-1\right)^{\alpha_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{p+2},
$$

where $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right) \in A_{p+1}$. Then we shall prove that $\phi\left(R_{p+2}\right)=\Lambda_{p+2}$ and $\phi$ induces the linear transformation $\phi^{*}$ from $W_{p+1}(\mathfrak{S}) / R_{p+2}$ to $\Lambda_{p+1} / \Lambda_{p+2}$.

Since

$$
\left(x_{i}-1\right)^{p}\left(x_{j}-1\right)=-\sum_{k=1}^{p-1}\binom{p}{k}\left(x_{i}-1\right)^{k}\left(x_{j}-1\right)+\left(x_{i}^{p}-1\right)\left(x_{j}-1\right)
$$

and

$$
\begin{aligned}
& \left(x_{i}^{p}-1\right)\left(x_{j}-1\right)=\left(x_{j}-1\right)\left(x_{i}^{p}-1\right)+\left(\left[x_{i}^{p}, x_{j}\right]-1\right)+\left(x_{j}-1\right)\left(\left[x_{i}^{p}, x_{j}\right]-1\right) \\
& \quad+\left(x_{i}^{p}-1\right)\left(\left[x_{i}^{p}, x_{j}\right]-1\right)+\left(x_{j}-1\right)\left(x_{i}^{p}-1\right)\left(\left[x_{i}^{p}, x_{j}\right]-1\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
& \left.\quad \stackrel{p}{\phi} x_{i} x_{j}-\stackrel{p}{\boxtimes} x_{i} \vee x_{j}-x_{i} \otimes x_{j}^{p}+x_{j} \otimes x_{i}^{p}+\left[x_{i}^{p}, x_{j}\right]\right) \\
& = \\
& \quad\left(x_{i}-1\right)\left(x_{j}-1\right)^{p}-\left(x_{i}-1\right)^{p}\left(x_{j}-1\right)-\left(x_{i}-1\right)\left(x_{j}^{p}-1\right) \\
& \quad \quad+\left(x_{j}-1\right)\left(x_{i}^{p}-1\right)+\left(\left[x_{i}^{p}, x_{j}\right]-1\right)+\Lambda_{p+2} \\
& = \\
& =-\sum_{k=2}^{p-1}\binom{p}{k}\left(x_{i}-1\right)\left(x_{j}-1\right)^{k}+\sum_{k=2}^{p-1}\binom{p}{k}\left(x_{i}-1\right)^{k}\left(x_{j}-1\right)+\Lambda_{p+2} \\
& = \\
& \Lambda_{p+2} .
\end{aligned}
$$

Thus $\phi\left(R_{p+2}\right)=\Lambda_{p+2}$ and $\phi^{*}$ is induced.
Step 3. We shall prove that $\phi^{*} \circ \psi^{*}$ is the identity map on $\Lambda_{p+1} / \Lambda_{p+2}$. It suffices to prove it on $\left\{p^{m_{\alpha}(p+1)} P(\alpha)+\Lambda_{p+2} \mid \alpha\right.$ : regular for $\left.p+1\right\}$. If $p^{m_{\alpha}(p+1)} P(\alpha)=p\left(x_{i}-1\right)\left(x_{j}-1\right)$ with $i \leqq j$ and $\omega\left(x_{i}\right)=\omega\left(x_{j}\right)=1$, then

$$
\begin{aligned}
& \phi^{*} \circ \psi^{*}\left(p\left(x_{i}-1\right)\left(x_{j}-1\right)+\Lambda_{p+2}\right)=\phi^{*}\left(-x_{i} \stackrel{p}{\otimes} x_{j}+x_{i} \otimes x_{j}^{p}+R_{p+2}\right) \\
& =-\left(x_{i}-1\right)\left(x_{j}-1\right)^{p}+\left(x_{i}-1\right)\left(x_{j}^{p}-1\right)+\Lambda_{p+2} \\
& =\sum_{k=1}^{p-1}\binom{p}{k}\left(x_{i}-1\right)\left(x_{j}-1\right)^{k}+\Lambda_{p+2}, \\
& =p\left(x_{i}-1\right)\left(x_{j}-1\right)+\Lambda_{p+2} .
\end{aligned}
$$

If $p^{m_{a}(p+1)} P(\alpha)=\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}$ where $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ and $W(\alpha)=p+1$, then

$$
\begin{aligned}
& \phi^{*} \circ \psi^{*}\left(\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{p+2}\right) \\
& =\phi^{*}\left(\triangleq x_{1} \stackrel{\alpha_{1}}{\otimes}{ }_{2} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{p+2}\right) \\
& =\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\omega_{2}} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{p+2} \text {. }
\end{aligned}
$$

Now our assertion is proved.
Step 4. Finally we shall show that $\psi^{*} \circ \phi^{*}$ is the identity map on $W_{p+1}(\mathfrak{S}) / R_{p+2}$. Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right) \in A_{p+1}$. Clearly $\alpha$ has one of the following 4 forms:
a) $\alpha=(0, \cdots, 0, p+1,0, \cdots, 0)$ and $\omega\left(x_{i}\right)=1$,
b) $\alpha=(0, \cdots, 0, p, 0, \cdots, 0,1,0, \cdots, 0)$ and $\omega\left(x_{i}\right)=\omega\left(x_{j}\right)=1$,
c) $\alpha=\left(0, \cdots, 0, \frac{1}{i}, 0, \cdots, 0, p, 0, \cdots, 0\right)$ and $\omega\left(x_{i}\right)=\omega\left(x_{j}\right)=1$,
d) $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right), \alpha$ basic and $W(\alpha)=p+1$.

Let $\alpha=\left(\alpha_{1}, \alpha_{2}, \cdots, \alpha_{m}\right)$ be a proper sequence of type a). Since

$$
\begin{aligned}
& \left(x_{i}-1\right)^{p+1}+\Lambda_{p+2}=\left(x_{i}-1\right)\left\{-\sum_{k=1}^{p-1}\binom{p}{k}\left(x_{i}-1\right)^{k}+\left(x_{i}^{p}-1\right)\right\}+\Lambda_{p+2} \\
& =-p\left(x_{i}-1\right)^{2}+\left(x_{i}-1\right)\left(x_{i}^{p}-1\right)+\Lambda_{p+2}
\end{aligned}
$$

we have

$$
\begin{aligned}
& \psi^{*} \circ \phi^{*}\left(\stackrel{p+1}{\emptyset} x_{i}+R_{p+2}\right)=\psi^{*}\left(\left(x_{i}-1\right)^{p+1}+\Lambda_{p+2}\right) \\
& =\psi^{*}\left(-p\left(x_{i}-1\right)^{2}+\left(x_{i}-1\right)\left(x_{i}^{p}-1\right)+\Lambda_{p+2}\right) \\
& =\stackrel{p+1}{\oslash} x_{i}-x_{i} \otimes x_{i}^{p}+x_{i} \otimes x_{i}^{p}+R_{p+2} \\
& =\stackrel{p+1}{\oslash} x_{i}+R_{p+2} .
\end{aligned}
$$

Let $\alpha$ be a proper sequence of type b). Then

$$
\begin{aligned}
& \psi^{*} \circ \phi^{*}\left(\stackrel{p}{\bigotimes} x_{i} \vee x_{j}+R_{p+2}\right)=\psi^{*}\left(\left(x_{i}-1\right)^{p}\left(x_{j}-1\right)+\Lambda_{p+2}\right) \\
& =\psi^{*}\left(-p\left(x_{i}-1\right)\left(x_{j}-1\right)+\left(x_{i}^{p}-1\right)\left(x_{j}-1\right)+\Lambda_{p+2}\right) \\
& =x_{i} \stackrel{\unrhd}{\bigotimes} x_{j}-x_{i} \otimes x_{j}^{p}+x_{j} \otimes x_{i}^{p}+\left[x_{i}^{p}, x_{j}\right]+R_{p+2} \\
& =\stackrel{\square}{\bigotimes} x_{i} \vee x_{j}+R_{p+2} .
\end{aligned}
$$

Let $\alpha$ be a proper sequence of type c ). Then

$$
\begin{aligned}
& \psi^{*} \circ \phi^{*}\left(x_{i} \stackrel{p}{\bigotimes} x_{j}+R_{p+2}\right)=\psi^{*}\left(\left(x_{i}-1\right)\left(x_{j}-1\right)^{p}+\Lambda_{p+2}\right) \\
& =\psi^{*}\left(-p\left(x_{i}-1\right)\left(x_{j}-1\right)+\left(x_{i}-1\right)\left(x_{j}^{p}-1\right)+\Lambda_{p+2}\right) \\
& =x_{i} \stackrel{p}{\bigotimes} x_{j}+R_{p+2} .
\end{aligned}
$$

Let $\alpha$ be a basic sequence of type d). Then

$$
\begin{aligned}
& \left.\psi^{*} \circ \phi^{*} \stackrel{\alpha_{1}}{\otimes} x_{1} \stackrel{\alpha_{2}}{\bigotimes} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{p+2}\right) \\
& =\psi^{*}\left(\left(x_{1}-1\right)^{\omega_{1}}\left(x_{2}-1\right)^{\left.\alpha_{2} \cdots\left(x_{m}-1\right)^{\omega_{m}}+\Lambda_{p+2}\right)}\right. \\
& =\alpha_{1} \stackrel{\alpha_{2}}{\boxtimes} x_{1} x_{2} \cdots \stackrel{\alpha_{m}}{\bigotimes} x_{m}+R_{p+2} .
\end{aligned}
$$

Step $1 \sim$ Step 4 imply that $\Lambda_{p+1} / \Lambda_{p+2} \simeq W_{p+1}(\mathfrak{S}) / R_{p+2}$.
Using Theorem 14 we can easily show that $D\left(\Lambda_{p+2}\right)=H_{p+2}$ for any $N_{p^{-}}$ series of the finite $p$-group $G$. But we can get more powerful result as follows.

Theorem 15. Let $G$ be a finite $p$-group with $N_{p}$-series $\mathfrak{S}=\left\{H_{i}\right\}$, and $\left\{\Lambda_{i}\right\}$ its canonical filtration of $I(G)$ with respect to $\mathfrak{S}$. Then $D\left(\Lambda_{n}\right)=H_{n}$ for all $n \geqq 1$.

Proof. We prove it by induction on $n$. We may assume $D\left(\Lambda_{n}\right)=H_{n}$, and $H_{n+1}=1$. We fix an ordered uniqueness basis $\Phi=\left\{x_{1}, x_{2}, \cdots, x_{m}, y_{1}, \cdots, y_{s}\right\}$ $\left.\omega\left(x_{1}\right) \leqq \omega\left(x_{2}\right) \leqq \cdots \leqq\left(x_{m}\right)<n, \omega\left(y_{1}\right)=\omega\left(y_{2}\right)=\cdots=\omega\left(y_{s}\right)=n\right\}$ for $G$. Let $x \in H_{n}$ be an element of $D\left(\Lambda_{n+1}\right)$. Write $x$ as $x=\Pi y_{j}^{c} j\left(0 \leqq c_{j}<p\right)$. Then

$$
\begin{aligned}
x-1 & =\prod_{j} y_{j}^{c_{j}}-1=\prod_{j}\left\{\left(y_{j}-1\right)+1\right\}^{c_{j}-1} \\
& =\prod_{j}\left\{\sum_{k_{j}=0}^{c_{j}}\binom{c_{j}}{k_{j}}\left(y_{j}-1\right)^{k_{j}}\right\}-1 \\
& =\sum_{k_{1}=0}^{c_{1}} \cdots \sum_{k_{s}=0}^{c_{s}}\binom{c_{1}}{k_{1}}\binom{c_{2}}{k_{2}} \cdots\binom{c_{s}}{k_{s}}\left(y_{1}-1\right)^{k_{1} \cdots\left(y_{s}-1\right)^{k_{s}}-1} \\
& =c_{1}\left(y_{1}-1\right)+\cdots+c_{s}\left(y_{s}-1\right)+\text { higher terms } .
\end{aligned}
$$

Note that each $\left(y_{1}-1\right)^{k_{1}}\left(y_{2}-1\right)^{k_{2}} \cdots\left(y_{s}-1\right)^{k_{s}}$ is basic product. As $x-1$ belongs to $\Lambda_{n+1}, x-1$ is expressed as a $Z$-linear combination of $p^{m_{\alpha}(n+1)} P(\alpha) \alpha \neq 0$ basic. Write $x-1$ as follows:

$$
x-1=\sum_{\omega: \text { basic }} a_{a} p^{m_{\alpha}(n+1)} P(\alpha) \quad\left(a_{\infty} \in \boldsymbol{Z}\right)
$$

Let $\beta_{j}$ be a basic sequence such that $P\left(\beta_{j}\right)=\left(y_{j}-1\right)$. By uniqueness of coefficients we have $a_{\beta_{j}} p^{m_{\beta_{j}}{ }^{(n+1)}}=c_{j}$ for all $j$. Since $m_{\beta_{j}}(n+1)=1, c_{j}$ is a multiple of $p$. This gives $c_{j}=0$ for all $j$, because $0 \leqq c_{j}<p$. So $x=\prod_{j} y_{j}^{c}=1$. Thus we have $D\left(\Lambda_{n+1}\right)=H_{n+1}$.

Remark 16. Corollary 9 is also obtained from this theorem.
Acknowledgment. The author wishes to express his appreciation to Professors H. Nagao and K. Tahara for their kind suggestions.

## References

[1] P.M. Cohn: Generalization of a theorem of Magnus, Proc. London Math. Soc. (3) 2 (1952), 297-310 (see Correction at the end of the same volume).
[2] G. Losey: On the structure of $Q_{2}(G)$ for finitely generated groups, Canad. J. Math. 25 (1973), 353-359.
[3] G. Losey and N. Losey: Augmentation quotients of some non-abelian finite groups, Math. Proc. Cambridge Philos. Soc. 85 (1979), 261-270.
[4] I.B.S. Passi and L.R. Vermani: The associated graded ring of an integral group ring, Math. Proc. Cambridge Philos. Soc. 82 (1977), 25-33.
[5] R. Sandling and K. Tahara: Augmentation quotients of group rings and symmetric powers, Math. Proc. Cambridge Philos. Soc. 85 (1979), 247-252.
[6] K. Tahara: On the structure of $Q_{3}(G)$ and the fourth dimension subgroups, Japan. J. Math. New Ser. 3 (1977), 381-394.
[7] K. Tahara: The augmentation quotients of group rings and the fifth dimension subgroups, J. Algebra 71 (1981), 141-173.

