COMPUTATION OF SEQUENCES
MAXIMIZING LEAST COMMON MULTIPLES

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Abstract: Let k, n be positive integers in the intervals $1 \leq k \leq \pi(n)$, $1 \leq n \leq 100$, where $\pi(n)$ is the number of primes up to n. For each pair k, n we specify one of the k-sets of positive integers up to n, with the property that its least common multiple is maximum. For each n, the specification takes the form of a sequence of integers and "cancellative terms"; each k-set can immediately be read off once the sequence is given.

1. Introduction.

For any finite set S of positive integers, let $\lambda(S)$ denote the least common multiple of the integers in S, let $[1,n]$ denote the set of positive integers up to n, and let $L(n,k) = \max\{\lambda(S): S \subseteq [1,n], |S| = k\}$. A k-set $S \subseteq [1,n]$ will be called *optimal* for n if $\lambda(S) = L(n,k)$.

In [1] it was shown that $L(n,k)$ increases monotonically with k up to its maximum, which occurs when $k = \pi(n)$ if $n \geq 2$, where

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as usual $\pi(n)$ denotes the number of primes up to n. A concise method employing a sequence to specify an optimal k-set for fixed n and each k in the interval $1 \leq k \leq \pi(n)$ was also indicated. We outline the idea below, then give a precise definition of the sequence. The main purpose of this report is to present some computationally-oriented results concerned with these sequences. A number of more theoretical results about these and related sequences will be presented elsewhere.

2. Sequence specifying optimal k-sets for n.

We would like to list a sequence of integers from $[1,n]$ with the property that for each $k \leq \pi(n)$ the first k terms of the sequence constitute an optimal k-set for n. However this cannot quite be achieved in practice, since for suitable choices of n and k no optimal k-set for n has any proper subset which is also optimal for n. (For example, the only optimal 3-set for 12 is $\{11,10,9\}$, whereas $\{12\}$ and $\{12,11\}$ are the only smaller optimal sets for 12.) This complication is accommodated by allowing terms present in some initial segments of our sequence to be removed (cancelled) from larger initial segments. We shall now make these ideas precise.

For each positive integer n, the *cancellative sequence maximizing least common multiples up to* n is the sequence

$$A(n) = a_1, a_2, \ldots, a_\ell$$

with the following properties. Each term a_i is either an integer from $[1,n]$ or else is a *cancellative term*, written as a^{-1} where a is an integer occurring earlier in the sequence. For $1 \leq j \leq \ell$, a *residual term* of the jth initial segment of $A(n)$ is any integer m such that $a_i = m$ for some $i \leq j$ and $a_r \neq m^{-1}$ for $i < r \leq j$. For $1 < k \leq \pi(n)$, let $\nu(k)$ be the smallest integer for which the $\nu(k)$th initial segment of $A(n)$ contains k residual terms. The subset of $[1,n]$ comprising these k residual terms will be denoted by $A(n,k)$. The sequence $A(n)$ is chosen so that for each $k \leq \pi(n)$ the set $A(n,k)$ is optimal for n, and so that $\nu(k+1) - \nu(k)$ is minimal for each $k < \pi(n)$.

This completes the specification of the most important features of $A(n)$. However, to ensure uniqueness we must add several further requirements of a technical nature. For each $k \leq \pi(n)$, any cancellative terms in the segment $\{a_i : \nu(k) < i \leq \nu(k+1)\}$ precede any terms which are integers. Let the elements of the relative complement $A(n,k) \backslash A(n,k+1)$ be arranged in decreasing order: the corresponding terms in $A(n)$ are the cancellative terms in the segment $\{a_i : \nu(k) < i \leq \nu(k+1)\}$, and they occur in the same order. Similarly let the elements of the relative complement $A(n,k+1) \backslash A(n,k)$ be arranged in decreasing order: they occur in precisely this order as the integers in the segment $\{a_i : \nu(k) < i \leq \nu(k+1)\}$. Finally if S is an optimal $(k+1)$-set for n and $S \neq A(n,k+1)$ then either $|S \backslash A(n,k)| \geq |A(n,k+1) \backslash A(n,k)|$, or else the elements of $S \backslash A(n,k)$ arranged in decreasing order form a lexicographically later ("larger") sequence than the corresponding sequence obtained from $A(n,k+1) \backslash A(n,k)$.

As a first example, we give the cancellative sequence maximizing least common multiples up to 12:

$$A(12) = 12,\ 11,\ 12^{-1},\ 10,\ 9,\ 7,\ 8.$$

At the end of this report we tabulate $A(n)$ for $1 \leq n \leq 100$.

3. Bounds for terms of $A(n)$, with computational consequences.

Given $S \subseteq [1,n]$ an *elementary replacement* on S replaces any one integer in S by one from its complement. If S is not optimal for n, there need not exist a sequence of elementary replacements by which we can proceed from S and reach an optimal set of the same cardinality without incurring a decrease in least common multiple at some intermediate step. For example, if $6|n$ then $L(n,3) = (n-1)(n-2)(n-3)$, and $\{n-1,\ n-2,\ n-3\}$ is the only optimal 3-set for n if $n \geq 12$. However, this set is obtainable from $\{n,\ n-1,\ n-5\}$ by a sequence of elementary replacements only if λ is decreased at some intermediate step, for both sets are maximal members of the partially ordered set resulting from ordering the 3-subsets of $[1,n]$ so that $S \leq T$ just when T is obtainable from S by a sequence of elementary replacements none of which decreases λ.

This observation accounts for much of the difficulty involved in explicitly determining $A(n)$, since in general we cannot calculate $A(n,k+1)$ from $A(n,k)$ by first adjoining the largest available integer which is prime relative to $L(n,k)$, and then making elementary replacements which do not decrease λ. However, this approach does yield useful lower bounds on the terms of $A(n)$, as we shall now show.

Note that from any $S \subseteq [1,n]$ it is trivial to derive a set S' of coprime integers with $\lambda(S') = \lambda(S)$, merely by deleting excess prime-power factors from elements of S. The following theorem uses coprime optimal sets.

THEOREM. *For j, k satisfying $1 \leq j < k \leq \pi(n)$, suppose there exists a decreasing sequence $t_1 > t_2 > \ldots > t_j$ of j integers from $[1,n]$, each of which is prime relative to $L(n,k-j)$. Let $s_1 < s_2 < \ldots < s_k$ be any increasing sequence of k coprime integers from $[1,n]$ such that $s_1 s_2 \ldots s_k = L(n,k)$. Then $s_1 s_2 \ldots s_j \geq \lambda(t_1, t_2, \ldots, t_j)$*

Proof. For brevity, write $s = s_1 s_2 \ldots s_j$ and $t = \lambda(t_1, t_2, \ldots, t_j)$. Suppose $s < t$. Let T be a k-set formed by adjoining $t_1, t_2, \ldots, t_j$ to some $(k-j)$-set which is optimal for n. Then

$$s_1 s_2 \cdots s_k = s s_{j+1} \cdots s_k = L(n,k) \geq \lambda(T) = tL(n,k-j),$$

whence $s_{j+1} \cdots s_k \geq L(n,k-j)$. But $S = \{s_{j+1}, \ldots, s_k\}$ is a coprime $(k-j)$-set of integers from $[1,n]$, so $L(n,k-j) \geq \lambda(S) = s_{j+1} \cdots s_k$. This contradiction proves the theorem. □

For any integer $m \leq n$, let $[m,n]$ denote the set of integers from m to n, inclusive. The case $j = 1$ of the theorem yields

COROLLARY 1. *If* $1 < k \leq \pi(n)$ *and* S *is any* k*-set which is optimal for* n*, then* $S \subseteq [m,n]$*, where* m *is the largest integer in* $[1,n]$ *which is prime relative to* $L(n,k-1)$.

More generally, for any $S \subseteq [1,n]$ define the *lexic sequence for* S *and* n to be the decreasing sequence $t_1 > t_2 > \ldots$ in which each term t_j is the largest integer in $[1,n]$ which is prime relative to $\lambda(S)$ and to each t_i with $i < j$. Note that the lexic sequence terminates with 1. If S is optimal for n, no term in the lexic sequence for S and n exceeds the least integer in S, for otherwise an elementary replacement of the least integer in S by the first term in the lexic sequence would increase λ. The theorem immediately implies

COROLLARY 2. *For* j, k *satisfying* $1 \leq j < k \leq \pi(n)$*, let* T *be an optimal* $(k-j)$*-set for* n*, and suppose the lexic sequence for* T *and* n *has at least* j *terms,* $t_1 > t_2 > \ldots > t_j$*. Let* S *be the optimal* k*-set containing the integers* $s_1 < s_2 < \ldots < s_k$*. Then* $\lambda(s_1, s_2, \ldots, s_j) \geq t_1 t_2 \ldots t_j$.

The proof of the theorem shows that $s = t$ can only hold if we have $s_{j+1} \ldots s_k = L(n,k-j)$, so the case $j = 1$ yields

COROLLARY 3. *For* $1 < k \leq \pi(n)$*, let* m *be the largest integer in* $[1,n]$ *which is prime relative to* $L(n,k-1)$*. Then either every coprime optimal* k*-set for* n *is of the form* $S = T \cup \{m\}$*, where* T *is an optimal* $(k-1)$*-set, or else every optimal* k*-set comprises integers from* $[m+1,n]$.

These results have numerous consequences for the calculation of $A(n)$; we shall conclude this report by pointing out several of them. If m is the largest integer in $[1,n]$ prime relative to $L(n,k-1)$, it follows from Corollary 3 that either $A(n,k) = A(n,k-1) \cup \{m\}$ or else $A(n,k) \subseteq [m+1,n]$. The latter is precisely the case which necessitates the presence of one or more cancellative terms in $A(n)$. Next note that if $m > n/p^{\alpha}$, where p^{α} is a power of a prime, Corollary 1 shows that $A(n,k)$ can contain at most one multiple of p^{α}, by considering any coprime k-set derived from $A(n,k)$. Another useful result which follows at once from the theorem is that $A(p,k) = A(p-1,k-1) \cup \{p\}$ for every prime p and $1 < k \leq \pi(p)$.

Reference

[1] R. B. Eggleton, *Maximizing least common multiples: A report*, Proc. Fourth Manitoba Conf. on Numerical Math., 1974, pp. 217-222.

TABLE: The cancellative sequences $A(n)$, for $1 \leq n \leq 100$.

1

2

3 2

4 3

5 4 3

6 5 4

7 6 5 4

8 7 5 3

9 8 7 5

10 9 7 8

11 10 9 7 8

12 11 12^{-1} 10 9 7 8

13 12 11 12^{-1} 10 9 7 8

14 13 11 9 5 8

15 14 13 11 8 9

16 15 13 11 7 9

17 16 15 13 11 7 9

18 17 18^{-1} 16 15 13 11 7 9

19 18 17 18^{-1} 16 15 13 11 7 9

20 19 17 13 20^{-1} 16 15 11 7 9

21 20 19 17 13 11 16 9

22 21 19 17 13 20 16 9

23 22 21 19 17 13 20 16 9

24 23 24^{-1} 22 21 19 17 13 20 16 9

25 24 23 24^{-1} 22 21 19 17 13 16 9

26 25 23 21 19 17 11 16 9

27 26 25 23 19 17 11 16 7

28 27 25 23 19 17 13 11 16

29 28 27 25 23 19 17 13 11 16

30 29 30^{-1} 28 27 25 23 19 17 13 11 16

31 30 29 30^{-1} 28 27 25 23 19 17 13 11 16

32 31 29 27 25 23 19 17 13 11 7

33 32 31 29 25 23 19 17 13 27 7

34 33 31 29 25 23 19 32 13 27 7

35 34 33 31 29 23 19 32 13 27 25

36 35 36^{-1} 34 33 31 29 23 19 32 13 27 25

37 36 35 36^{-1} 34 33 31 29 23 19 32 13 27 25

38 37 35 33 31 29 23 17 32 13 27 25

39 38 37 35 31 29 23 17 32 11 27 25

40 39 37 40^{-1} 38 35 31 29 23 17 32 11 27 25

TABLE: The cancellative sequences $A(n)$-*continued*

41 40 39 37 40^{-1} 38 35 31 29 23 17 32 11 27 25

42 41 42^{-1} 40 39 37 40^{-1} 38 35 31 29 23 17 32 11 27 25

43 42 41 42^{-1} 40 39 37 40^{-1} 38 35 31 29 23 17 32 11 27 25

44 43 41 39 37 35 31 29 23 19 17 27 32 25

45 44 43 41 37 31 45^{-1} 39 35 29 23 19 17 27 32 25

46 45 43 41 37 31 45^{-1} 39 35 29 19 17 32 11 27 25

47 46 45 43 41 37 31 45^{-1} 39 35 29 19 17 32 11 27 25

48 47 48^{-1} 46 45 43 41 37 31 45^{-1} 39 35 29 19 17 32 11 27 25

49 48 47 48^{-1} 46 45 43 41 37 31 29 44 45^{-1} 39 25 19 17 27 32

50 49 47 43 50^{-1} 46 45 41 37 31 29 44 45^{-1} 39 25 19 17 27 32

51 50 49 47 43 41 37 31 29 23 44 19 13 27 32

52 51 49 47 43 41 37 31 29 25 23 19 11 27 32

53 52 51 49 47 43 41 37 31 29 25 23 19 11 27 32

54 53 54^{-1} 52 51 49 47 43 41 37 31 29 25 23 19 11 27 32

55 54 53 54^{-1} 52 51 49 47 43 41 37 31 29 23 19 27 32 25

56 55 53 51 47 56^{-1} 52 49 43 41 37 31 29 23 19 27 32 25

57 56 55 53 47 56^{-1} 52 49 43 41 37 31 29 23 17 27 32 25

58 57 55 53 49 47 43 41 37 31 52 23 17 27 32 25

59 58 57 55 53 49 47 43 41 37 31 52 23 17 27 32 25

60 59 60^{-1} 58 57 55 53 49 47 43 41 37 31 52 23 17 27 32 25

TABLE: The cancellative sequences $A(n)$ *-oontinued*

61 60 59 60^{-1} 58 57 55 53 49 47 43 41 37 31 52 23 17 27 32 25

62 61 59 57 55 53 49 47 43 41 37 29 52 23 17 27 32 25

63 62 61 59 55 53 47 63^{-1} 57 49 43 41 37 29 52 23 17 27 32 25

64 63 61 59 55 53 47 63^{-1} 57 49 43 41 37 31 29 23 17 13 27 25

65 64 63 61 59 53 47 63^{-1} 57 49 43 41 37 31 29 23 17 11 27 25

66 65 66^{-1} 64 63 61 59 53 47 63^{-1} 57 49 43 41 37 31 29 23 17 11 27 25

67 66 65 66^{-1} 64 63 61 59 53 47 63^{-1} 57 49 43 41 37 31 29 23 17 11 27 25

68 67 65 63 61 59 53 47 63^{-1} 57 49 43 41 37 31 29 23 64 11 27 25

69 68 67 65 61 59 53 49 47 43 41 37 31 29 19 64 11 27 25

70 69 67 70^{-1} 68 65 61 59 53 49 47 43 41 37 31 29 19 64 11 27 25

71 70 69 67 70^{-1} 68 65 61 59 53 49 47 43 41 37 31 29 19 64 11 27 25

72 71 72^{-1} 70 69 67 70^{-1} 68 65 61 59 53 49 47 43 41 37 31 29 19 64 11 27 25

73 72 71 72^{-1} 70 69 67 70^{-1} 68 65 61 59 53 49 47 43 41 37 31 29 19 64 11 27 25

74 73 71 69 67 65 61 59 53 49 47 43 41 68 31 29 19 64 11 27 25

75 74 73 71 67 61 75^{-1} 69 65 59 53 49 47 43 41 68 31 29 19 64 11 27 25

76 75 73 71 67 61 75^{-1} 69 65 59 53 49 47 43 41 37 31 29 17 64 11 27 25

77 76 75 73 71 67 61 75^{-1} 69 65 59 53 47 43 41 37 31 29 17 64 27 49 25

78 77 78^{-1} 76 75 73 71 67 61 75^{-1} 69 65 59 53 47 43 41 37 31 29 17 64 27 49 25

79 78 77 78^{-1} 76 75 73 71 67 61 75^{-1} 69 65 59 53 47 43 41 37 31 29 17 64 27 49 25

80 79 77 73 80^{-1} 76 75 71 67 61 75^{-1} 69 65 59 53 47 43 41 37 31 29 17 64 27 49 25

TABLE: The cancellative sequences $A(n)$ *-continued*

81 80 79 77 73 71 67 80^{-1} 76 65 61 59 53 47 43 41 37 31 29 23 17 64 49 25

82 81 79 77 73 71 67 65 61 59 53 47 43 76 37 31 29 23 17 64 49 25

83 82 81 79 77 73 71 67 65 61 59 53 47 43 76 37 31 29 23 17 64 49 25

84 83 84^{1} 82 81 79 77 73 71 67 65 61 59 53 47 43 76 37 31 29 23 17 64 49 25

85 84 83 84^{1} 82 81 79 77 73 71 67 61 59 53 47 43 76 37 31 29 23 64 13 49 25

86 85 83 81 79 77 73 71 67 61 59 53 47 41 76 37 31 29 23 64 13 49 25

87 86 85 83 79 77 73 71 67 61 59 53 47 41 76 37 31 81 23 64 13 49 25

88 87 85 83 79 88^{-1} 86 77 73 71 67 61 59 53 47 41 76 37 31 81 23 64 13 49 25

89 88 87 85 83 79 88^{-1} 86 77 73 71 67 61 59 53 47 41 76 37 31 81 23 64 13 49 25

90 89 90^{-1} 88 87 85 83 79 88^{-1} 86 77 73 71 67 61 59 53 47 41 76 37 31 81 23 64 13 49 25

91 90 89 90^{-1} 88 87 85 83 79 73 71 67 61 59 53 47 43 41 37 31 81 23 19 64 49 25

92 91 89 87 85 83 79 73 71 67 61 59 53 47 43 41 37 31 81 88 19 64 49 25

93 92 91 89 85 83 79 73 71 67 61 59 53 47 43 41 37 29 81 88 19 64 49 25

94 93 91 89 85 83 79 73 71 67 61 59 53 92 43 41 37 29 81 88 19 64 49 25

95 94 93 91 89 83 79 73 71 67 61 59 53 92 43 41 37 29 81 88 17 64 49 25

96 95 96^{-1} 94 93 91 89 83 79 73 71 67 61 59 53 92 43 41 37 29 81 88 17 64 49 25

97 96 95 96^{-1} 94 93 91 89 83 79 73 71 67 61 59 53 92 43 41 37 29 81 88 17 64 49 25

98 97 95 93 89 98^{-1} 94 91 83 79 73 71 67 61 59 53 92 43 41 37 29 81 88 17 64 49 25

99 98 97 95 89 98^{-1} 94 91 83 79 73 71 67 61 59 53 92 43 41 37 31 29 99^{-1} 88 81 17 64 49 25

100 99 97 100^{-1} 98 95 89 98^{-1} 94 91 83 79 73 71 67 61 59 53 92 43 41 37 31 29 99^{-1} 88 81 17 64 49 25