EXPLORING DATA AND STATISTICS

13.7 Parametric Equations and Projectile Motion

What you should learn

GOAL 1 Use parametric equations to represent motion in a plane.

GOAL 2 Use parametric equations to represent projectile motion, as applied in **Example 4**.

Why you should learn it

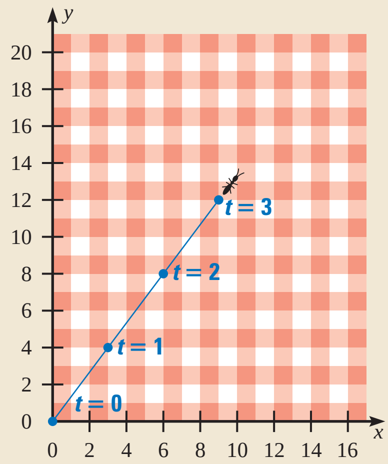
▼ To solve **real-life** problems, such as modeling the path of a leaping dolphin in **Exs. 36–38**.

REAL LIFE

GOAL 1 Using Parametric Equations

ACTIVITY Developing Concepts: Investigating Linear Motion

Suppose an ant starts at one corner of a picnic tablecloth and moves in a straight line, as shown. The ant's position (x, y) relative to the edges of the tablecloth is given for different times t (in seconds).



1. Write two equations: one that gives the ant's horizontal position x as a function of t, and one that gives the ant's vertical position y as a function of t.
2. What is the ant's position after 5 seconds?
3. How long will it take the ant to reach an edge of the tablecloth?

In the investigation you wrote a pair of equations that expressed x and y in terms of a third variable t. These equations, $x = f(t)$ and $y = g(t)$, are called **parametric equations**, and t is called the **parameter**.

EXAMPLE 1 *Graphing a Set of Parametric Equations*

Graph $x = 3t - 12$ and $y = -2t + 3$ for $0 \le t \le 5$.

SOLUTION

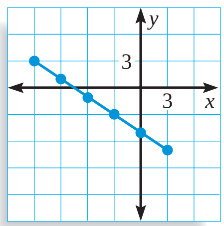
Begin by making a table of values.

t	0	1	2	3	4	5
x	−12	−9	−6	−3	0	3
y	3	1	−1	−3	−5	−7

Plot the points (x, y) given in the table:

$(-12, 3), (-9, 1), (-6, -1),$
$(-3, -3), (0, -5), (3, -7)$

Then connect the points with a line segment as shown.



EXAMPLE 2 *Eliminating the Parameter*

Write an xy-equation for the parametric equations in Example 1: $x = 3t - 12$ and $y = -2t + 3$ for $0 \leq t \leq 5$. State the domain for the equation.

SOLUTION

First solve one of the parametric equations for t.

$$x = 3t - 12 \qquad \text{Write original equation.}$$

$$x + 12 = 3t \qquad \text{Add 12 to each side.}$$

$$\frac{1}{3}x + 4 = t \qquad \text{Multiply each side by } \frac{1}{3}.$$

Then substitute for t in the other parametric equation.

$$y = -2t + 3 \qquad \text{Write original equation.}$$

$$y = -2\left(\frac{1}{3}x + 4\right) + 3 \qquad \text{Substitute for } t.$$

$$y = -\frac{2}{3}x - 5 \qquad \text{Simplify.}$$

This process is called *eliminating the parameter* because the parameter t is not in the final equation. When $t = 0$, $x = -12$ and when $t = 5$, $x = 3$. So, the domain of the xy-equation is $-12 \leq x \leq 3$.

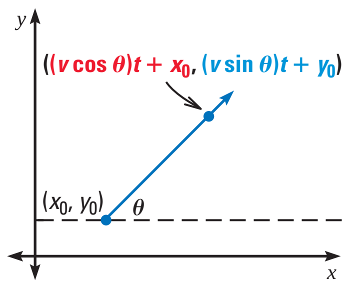
. . . .

Consider an object that is moving with constant speed v along a straight line that makes an angle θ measured counterclockwise from a line parallel to the x-axis. The position of the object at any time t can be represented by the parametric equations

$$x = (v \cos \theta)t + x_0$$

$$y = (v \sin \theta)t + y_0$$

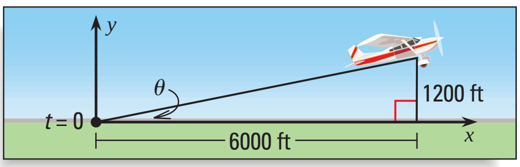
where (x_0, y_0) is the object's location when $t = 0$.



REAL LIFE Aviation

EXAMPLE 3 *Modeling Linear Motion*

Write a set of parametric equations for the airplane shown, given that its speed is 306 feet per second.



SOLUTION

The angle of elevation is $\theta = \tan^{-1}\left(\frac{1200}{6000}\right) \approx 11.3°$.

Using $v = 306$, $\theta = 11.3°$, and $(x_0, y_0) = (0, 0)$, you can write the following.

$$x = (v \cos \theta)t + x_0 \qquad \text{and} \qquad y = (v \sin \theta)t + y_0$$

$$x \approx (306 \cos 11.3°)t + 0 \qquad\qquad y \approx (306 \sin 11.3°)t + 0$$

$$\approx 300t \qquad\qquad \approx 60t$$

FOCUS ON APPLICATIONS

REAL LIFE **PUMPKIN TOSSING**
In the annual Morton, Illinois, pumpkin tossing contest, contestants use machines they built to throw pumpkins. In 1998 an air cannon entrant set a record by throwing a pumpkin 4491 feet.

INTERNET DATA UPDATE
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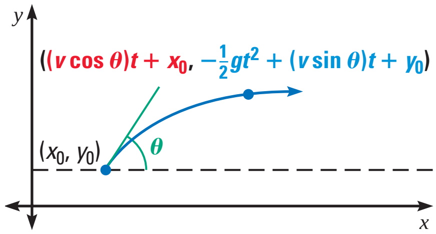
GOAL 2 MODELING PROJECTILE MOTION

Parametric equations can also be used to model nonlinear motion in a plane. For instance, consider an object that is projected into the air at an angle θ with an initial speed v. The object's parabolic path can be modeled with the parametric equations

$$x = (v \cos \theta)t + x_0$$

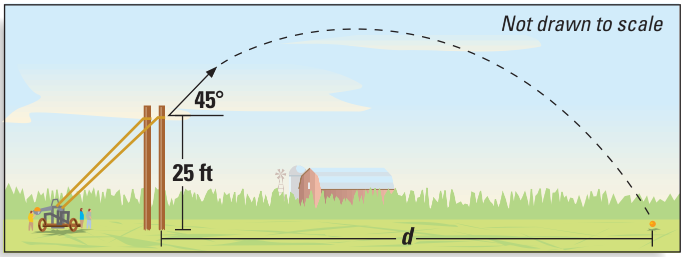
$$y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$



where (x_0, y_0) is the object's location when $t = 0$. The constant g is the acceleration due to gravity. Its value is 32 ft/sec^2 or 9.8 m/sec^2. (Note that this model neglects air resistance.)

EXAMPLE 4 *Modeling Projectile Motion*

PUMPKIN TOSSING In a pumpkin tossing contest in Morton, Illinois, a contestant won the catapult competition by using two telephone poles, huge rubber bands, and a power winch. Suppose the pumpkin was launched with an initial speed of 125 feet per second, at an angle of 45°, and from an initial height of 25 feet.



a. Write a set of parametric equations for the motion of the pumpkin.

b. Use the equations to find how far the pumpkin traveled.

SOLUTION

a. Using $v = 125$ ft/sec, $\theta = 45°$, and $(x_0, y_0) = (0, 25)$, you can write the following.

$$x = (v \cos \theta)t + x_0 \quad \text{and} \quad y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$

$$\approx 88.4t \qquad\qquad \approx -16t^2 + 88.4t + 25$$

b. The pumpkin hits the ground when $y = 0$.

$-16t^2 + 88.4t + 25 = y$ — **Write parametric equation for *y*.**

$-16t^2 + 88.4t + 25 = 0$ — **Substitute 0 for *y*.**

$t = \frac{-88.4 \pm \sqrt{(88.4)^2 - 4(-16)(25)}}{2(-16)}$ — **Use the quadratic formula to find *t*.**

$t \approx 5.8$ seconds — **Simplify and choose positive *t*-value.**

When $t = 5.8$ seconds, the pumpkin's location will have an x-value of $x = (88.4)(5.8) \approx 513$ feet. So, the pumpkin traveled about 513 feet.

STUDENT HELP

Look Back
For help with the quadratic formula, see p. 291.

GUIDED PRACTICE

Vocabulary Check ✓

1. Complete this statement: Parametric equations express variables like x and y in terms of another variable such as t. In this case, t is called the __?__.

Concept Check ✓

2. For an object moving in a straight line at a constant speed v, what do you need to know in order to write parametric equations describing the object's motion?

3. In this lesson you studied two parametric models for describing motion:

$$x = (v \cos \theta)t + x_0 \qquad\qquad x = (v \cos \theta)t + x_0$$
$$\text{and}$$
$$y = (v \sin \theta)t + y_0 \qquad\qquad y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$

Under what circumstances would you use each model?

Skill Check ✓

Graph the parametric equations.

4. $x = 2t$ and $y = t$ for $0 \le t \le 4$

5. $x = 3t + 4$ and $y = t - 3$ for $0 \le t \le 5$

6. $x = (20 \cos 60°)t$ and $y = (20 \sin 60°)t$ for $2 \le t \le 6$

Write an *xy*-equation for the parametric equations. State the domain.

7. $x = 7t$ and $y = 3t - 2$ for $0 \le t \le 5$

8. $x = -4t + 2$ and $y = 5t - 4$ for $0 \le t \le 6$

9. $x = (11.5 \cos 72.1°)t$ and $y = (11.5 \sin 72.1°)t + 3$ for $0 \le t \le 10$

10. **SOFTBALL CONTEST** At a softball throwing contest, you throw a softball with an initial speed of 60 feet per second, at an angle of 50°, and from an initial height of 5.5 feet. Write parametric equations for the softball's motion.

PRACTICE AND APPLICATIONS

STUDENT HELP

Extra Practice to help you master skills is on p. 959.

GRAPHING **Graph the parametric equations.**

11. $x = 2t - 2$ and $y = -t + 3$ for $0 \le t \le 5$

12. $x = 5 - 5t$ and $y = 3t - 2$ for $0 \le t \le 5$

13. $x = 2t - 6$ and $y = t - 3$ for $3 \le t \le 8$

14. $x = 30t + 10$ and $y = 60t - 20$ for $0 \le t \le 4$

15. $x = (80.6 \cos 7.1°)t$ and $y = (80.6 \sin 7.1°)t$ for $0 \le t \le 5$

ELIMINATING THE PARAMETER **Write an *xy*-equation for the parametric equations. State the domain.**

STUDENT HELP

HOMEWORK HELP
Example 1: Exs. 11–15
Example 2: Exs. 16–20
Example 3: Exs. 24–32
Example 4: Exs. 33–40

16. $x = 2t$ and $y = -4t$ for $0 \le t \le 5$

17. $x = t + 1$ and $y = 2t - 3$ for $0 \le t \le 5$

18. $x = 3t + 6$ and $y = 5t - 1$ for $0 \le t \le 20$

19. $x = (14.14 \cos 45°)t$ and $y = (14.14 \sin 45°)t$ for $0 \le t \le 10$

20. $x = (111.8 \cos 63.43°)t$ and $y = (111.8 \sin 63.43°)t$ for $0 \le t \le 10$

STUDENT HELP

INTERNET HOMEWORK HELP Visit our Web site www.mcdougallittell.com for help with Exs. 21–23.

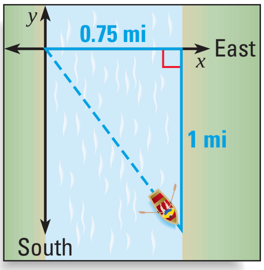
DESCRIBING LINEAR MOTION **Use the given information to write parametric equations describing the linear motion.**

21. An object is at (0, 0) at time $t = 0$ and then at (19, 57) at time $t = 3$.

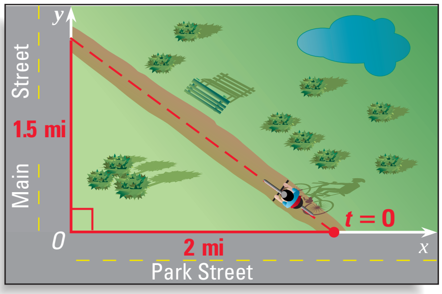
22. An object is at (18, 8) at time $t = 4$ and then at (40.8, 19.0) at time $t = 9$.

23. An object is at (3, 2) at time $t = 0$ and then at (14.3, 66.1) at time $t = 5$.

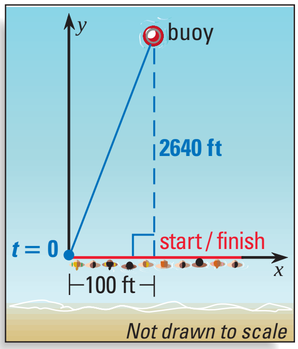
24. **ROWBOAT** You are trying to row a boat due east across a river that is 0.75 mile wide and flows due south. You reach the other side in 15 minutes, but the current has pulled you 1 mile downstream. Write a set of parametric equations to describe the path you traveled. Then write an *xy*-equation for the parametric equations. State the domain of the *xy*-equation.



25. **BIKE PATH** A bike trail connects Park Street and Main Street as shown. You enter the trail 2 miles from the intersection of the streets and bike at a speed of 10 miles per hour. You reach Main Street 1.5 miles from the intersection. Write a set of parametric equations to describe your path.



SWIMMING **In Exercises 26–28, use the following information.**
You are swimming in a race across a lake and back. Swimmers must swim to, and then back from, a buoy placed 2640 feet from the center of the start/finish line. You start the race 100 feet from the center of the start/finish line as shown.



26. You swim to the buoy at a steady rate of 0.7 foot per second. Write a set of parametric equations for your path.

27. Use the equations to determine how long it takes you to reach the buoy.

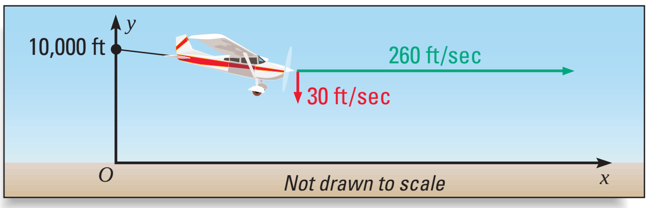
28. If you continue to swim at a steady rate of 0.7 foot per second straight back to the center of the start/finish line, how long will it take you to complete the race?

FOCUS ON APPLICATIONS

REAL LIFE INSTRUMENT PANEL In addition to gauges that give speed and altitude readings, the instrument panel of an airplane also contains an *attitude indicator.* This instrument tells how the airplane is tilted in relation to Earth's horizon.

INTERNET APPLICATION LINK www.mcdougallittell.com

LANDING A PLANE **In Exercises 29–32, use the following information.**
You are flying in a small airplane at an altitude of 10,000 feet. When you descend to land the plane, your horizontal air speed will be 260 feet per second (177 miles per hour) and your rate of descent will be 30 feet per second.



29. Write a set of parametric equations for the plane's descent.

30. What is the angle of descent?

31. How long will it take for the plane to land?

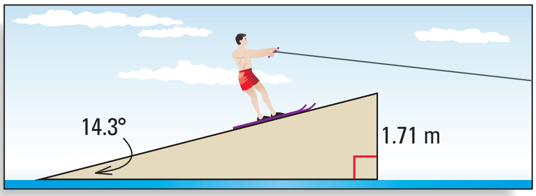
32. How far from the airport should you begin the descent?

WATER SKIING **In Exercises 33–35, use the following information.**
A water skier jumps off a ramp at a speed of 17.9 meters per second. The ramp's angle of elevation is 14.3°, and the height of the end of the ramp above the surface of the water is 1.71 meters. ▶ Source: American Water Ski Association

33. Write a set of parametric equations for the water skier's jump.

34. For how many seconds is the water skier in the air?

35. How far from the ramp does the water skier land?



LEAPING DOLPHIN **In Exercises 36–38, use the following information.**
A dolphin is performing in a show at an oceanic park and makes a leap out of the water. The dolphin leaves the water traveling at a speed of 32 feet per second and at an angle of 48° with the surface of the water.

36. Write a set of parametric equations for the dolphin's motion.

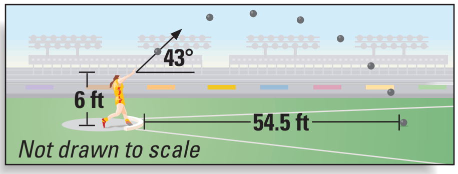
37. For how many seconds is the dolphin in the air?

38. How far across the water does the dolphin travel in the air?

SHOT PUT **In Exercises 39 and 40, use the following information.**
A shot put is thrown a distance of 54.5 feet at a high school track and field meet. The shot put was released from a height of 6 feet and at an angle of 43°.

39. Write a set of parametric equations for the path of the shot put.

40. Use the equations to determine the speed of the shot put at the time of release.



Test Preparation

41. **MULTIPLE CHOICE** Which equation is an *xy*-equation for the parametric equations $x = 3t + 12$ and $y = 12t - 8$ where $0 \le t \le 20$?

Ⓐ $y = 9x + 4;\ 0 \le x \le 20$ Ⓑ $y = 36x + 132;\ -8 \le x \le 132$

Ⓒ $y = 4x - 56;\ 12 \le x \le 72$ Ⓓ $y = 15x + 4;\ 4 \le x \le 304$

42. **MULTIPLE CHOICE** An airplane takes off at an angle of 10.6° with the ground and travels at a constant speed of 324 miles per hour. Which set of parametric equations describes the airplane's ascent?

Ⓐ $x = 4670t,\ y = 874t$ Ⓑ $x = 318t,\ y = 60t$

Ⓒ $x = 5000t,\ y = 80t$ Ⓓ $x = 800t,\ y = 150t$

★ **Challenge**

43. **CRITICAL THINKING** Write the following pairs of equations in the form $y = f(x)$.

$$x = (v \cos \theta)t + x_0 \qquad x = (v \cos \theta)t + x_0$$
$$y = (v \sin \theta)t + y_0 \qquad y = -\frac{1}{2}gt^2 + (v \sin \theta)t + y_0$$

In which case is the path of the moving object *not* affected by changing the speed *v*? Explain why this makes sense.

EXTRA CHALLENGE
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MIXED REVIEW

GRAPHING **Graph the function.** (Review 5.1, 7.5, 8.1 for 14.1)

44. $y = 9x^2$ **45.** $y = -10x^2$ **46.** $y = 7\sqrt{x}$

47. $y = -6\sqrt{x}$ **48.** $y = 7 \cdot 2^x$ **49.** $y = -\frac{3}{4} \cdot 3^x$

FINDING SUMS **Find the sum of the series.** (Review 11.1, 11.4)

50. $\sum_{i=1}^{10} -3i$ **51.** $\sum_{i=1}^{27} i^2$ **52.** $\sum_{n=1}^{\infty} 20\left(\frac{4}{5}\right)^{n-1}$ **53.** $\sum_{n=1}^{\infty} -\frac{1}{6}\left(-\frac{1}{2}\right)^{n-1}$

NORMAL DISTRIBUTIONS **Find the probability that a randomly selected *x*-value is in the given interval.** (Review 12.7)

54. to the left of the mean

55. between the mean and 1 standard deviation to the left of the mean

56. between 2 and 3 standard deviations from the mean

57. more than 3 standard deviations to the right of the mean

QUIZ 3

Self-Test for Lessons 13.5–13.7

Solve $\triangle ABC$. (Lessons 13.5 and 13.6)

1. $B = 70°, b = 30, c = 25$ **2.** $B = 10°, C = 100°, a = 15$

3. $A = 40°, B = 110°, b = 30$ **4.** $A = 122°, a = 9, c = 13$

5. $a = 45, b = 32, c = 24$ **6.** $A = 107°, b = 15, c = 28$

Find the area of $\triangle ABC$. (Lessons 13.5 and 13.6)

7. $B = 95°, a = 12, c = 30$ **8.** $C = 103°, a = 41, b = 25$

9. $A = 117°, b = 16, c = 8$ **10.** $a = 7, b = 7, c = 5$

11. $a = 89, b = 55, c = 71$ **12.** $a = 40, b = 21, c = 32$

Graph the parametric equations. (Lesson 13.7)

13. $x = 4 - 2t$ and $y = 3t + 1$ for $0 \le t \le 5$

14. $x = 2t - 5$ and $y = 4t - 3$ for $3 \le t \le 7$

15. $x = (10.5 \cos 45°)t$ and $y = (10.5 \sin 45°)t + 4$ for $0 \le t \le 5$

Write an *xy*-equation for the parametric equations. State the domain. (Lesson 13.7)

16. $x = -5t + 3$ and $y = t - 6$ for $0 \le t \le 5$

17. $x = (10 \cos 35°)t$ and $y = (10 \sin 35°)t$ for $0 \le t \le 30$

18. **SOCCER** You are a goalie in a soccer game. You save the ball and then drop kick it as far as you can down the field. Your kick has an initial speed of 26 feet per second and starts at a height of 2 feet. If you kick the ball at an angle of 45°, how far down the field does the ball hit the ground? (Lesson 13.7)